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HOLOMORPHIC ORBI-DISCS AND LAGRANGIAN FLOER COHOMOLOGY OF SYMPLECTIC TORIC ORBIFOLDS

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Abstract

We develop Floer theory of Lagrangian torus fibers in compact symplectic toric orbifolds. We first classify holomorphic orbi-discs with boundary on Lagrangian torus fibers. We show that there exists a class of basic discs such that we have one-to-one correspondences between a) smooth basic discs and facets of the moment polytope, and b) between basic orbi-discs and twisted sectors of the toric orbifold. We show that there is a smooth Lagrangian Floer theory of these torus fibers, which has a bulk deformation by fundamental classes of twisted sectors of the toric orbifold. We show by several examples that such bulk deformation can be used to illustrate the very rigid Hamiltonian geometry of orbifolds. We define its potential and bulk-deformed potential, and develop the notion of leading order potential. We study leading term equations analogous to the case of toric manifolds by Fukaya, Oh, Ohta, and Ono.

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1. Introduction

Floer theory of Lagrangian torus fibers of symplectic toric manifolds has been studied very extensively in the last decade, starting from the case of $\mathbb{CP}^n$ in [**C1**] and the toric Fano case in [**CO**]. These are based on the Lagrangian Floer theory ([**Fl**], [**O1**],[**O2**]), whose general construction was developed by Fukaya, Oh, Ohta, and Ono [**FOOO**]. More recent works [**FOOO2**], [**FOOO3**], [**FOOO4**] have used the (bulk) deformation theory developed in [**FOOO**], bringing deep understanding of the theory in toric manifolds, and providing beautiful pictures of (homological) mirror symmetry and symplectic dynamics.

We develop an analogous theory for compact symplectic toric orbifolds in this paper. Namely, this paper can be regarded as an orbifold generalization of [**C1**], [**CO**], [**FOOO2**], [**FOOO3**]. We will see that the main framework is very similar, but that the characteristics of the resulting Floer theory for toric orbifolds are somewhat different than those of toric manifolds.

The main new ingredient is the orbifold J-holomorphic disc (called orbi-disc). These are J-holomorphic discs with orbifold singularity in the interior. The study of toric manifolds has illustrated that understanding holomorphic discs is a crucial step in developing Lagrangian Floer theory. The holomorphic discs can be used to define the potential, corresponding to the Landau-Ginzburg superpotential for the mirror and the potential essentially computes the Lagrangian Floer cohomology of the torus fibers. Holomorphic discs (which are non-singular) were classified in [**CO**]. We find a classification for holomorphic orbi-discs in section 6.

One of the main observations of this paper is that the orbifold Lagrangian Floer theory should be considered in the following way. Let us consider a Lagrangian submanifold L which lies in the smooth locus of a symplectic orbifold $\mathcal{X}$. There is a Lagrangian Floer theory of L which only considers maps from smooth (non-orbifold) (stable) bordered Riemann surfaces. (Here smooth means that there is no orbifold singularity, but it could be a nodal Riemann surface.) Namely, there is a version of Lagrangian Floer cohomology, and A_∞-algebra of L, by considering smooth J-holomorphic discs and strips, which we call smooth Lagrangian Floer cohomology. We remark that a smooth J-holomorphic disc can meet orbifold locus if it has correct multiplicity around the orbifold point, as will be seen in the basic discs later.

The new ingredients such as orbifold J-holomorphic strips and discs enter into the theory in the form of bulk deformation of the smooth Floer theory. Bulk deformation was introduced in [**FOOO**] to deform the given Lagrangian Floer theory by an ambient cycle in the symplectic manifold. Orbifold J-holomorphic strips and discs can be considered to give bulk deformations from the fundamental cycles of twisted sectors of the symplectic orbifold $\mathcal{X}$. In the case of manifolds, bulk deformation utilizes the already existing J-holomorphic discs in Floer theory. On the other hand, for orbifolds the orbifold strips and discs do not exist in the smooth Floer theory. We observe that the mechanism of bulk deformation by orbifold maps captures the very rigid Hamiltonian geometry of symplectic orbifolds.

As noted in [**Wo**], [**WW**], [**ABM**], the dynamics of Hamiltonian vector fields in symplectic orbifolds are quite restrictive. This is because the induced Hamiltonian diffeomorphism should preserve the isomorphism type of the points in the given orbifold. This phenomenon can be easily seen in the example of a teardrop orbifold, which will be explained later in this introduction. For example, in [**FOOO2**] or [**FOOO3**], Fukaya, Oh, Ohta, and Ono find locations of non-displaceable Lagrangian torus fibers in toric manifolds, which turn out to be always codimension one or higher in the corresponding moment polytopes. For toric orbifolds, already in the case of the teardrop orbifold, we find codimension 0 locus of non-displaceable fibers, and we will find in Proposition 15.2 that if all the points in the toric divisors have orbifold singularity, then in fact all the Lagrangian torus fibers are non-displaceable. It is quite remarkable that this phenomenon can be explained as a flexibility to choose the bulk deformation coefficient in the leading order potential, which is essentially due to the fact that the orbifold discs and strips do not appear in the smooth Lagrangian Floer theory.

We remark that the non-displaceability of torus orbits in toric orbifolds such as those discussed in our examples has been recently proved by Woodward [**Wo**] and Wilson-Woodward [**WW**] using gauged Floer theory, which is somewhat different from our methods. Their work is roughly based on holomorphic discs in $\mathbb{C}^m$ and gauged theory for symplectic reduction. But note that the actual bulk orbi-potentials defined in this paper cannot be defined using their methods, as orbifold discs with more than one orbifold marked point do not come from discs in $\mathbb{C}^m$. Also the formalism of bulk deformation developed in this paper seems to give more intuitive understanding of these non-displaceability results in orbifolds, which should generalize to a non-toric setting. We also remark that Lagrangian Floer theory in a different orbifold setting has been considered by Seidel [**Se**] for the global quotient of the genus two curve (see also [**Sh**]).

Beyond the symplectic dynamics of the toric orbifolds, the development of this theory can be meaningful in the following aspects. First, it provides basic ingredients to study (homological) mirror symmetry ([**Ko**]) of toric orbifolds. In [**FOOO4**], mirror symmetry of toric manifolds has been proved using Lagrangian Floer theory of toric manifolds. It is easy to see that similar formalism may be used to explain mirror symmetry of toric orbifolds, which we leave for future research.

Second, the study of orbifold J-holomorphic discs provides a new approach to study the crepant resolution conjecture, which relates the invariants of certain orbifolds and their crepant resolutions. In a joint work of the first author with K. Chan, S.C. Lau, and H.H. Tseng [**CCLT**], we formulate an open version of the crepant resolution conjecture for toric orbifolds, and we find a geometric explanation for the change of variable in the Kähler moduli spaces of a toric orbifold and its crepant resolution. Also, this provides a natural explanation of specialization to the root of unity in the crepant resolution conjecture, in terms of associated potential functions.

Now we explain the basic setting and results of the paper in more detail. Compact symplectic toric orbifolds have one-to-one correspondence with labeled polytopes $(P, \vec{c})$, as explained by Lerman and Tolman [**LT**]. Here P is a simple rational polytope equipped with positive integer labels $\vec{c}$ for each facet. Also, the underlying complex orbifold may be obtained from the stacky fan of Borisov, Chen, and Smith [**BCS**]. The stacky fan is a simplicial fan in a finitely generated $\mathbb{Z}$-module N with a choice of lattice vectors in one-dimensional cones. A stacky fan corresponds to a toric orbifold when the module N is freely generated. The moment map μ_T exists for the Hamiltonian torus action on a symplectic toric orbifold, and each Lagrangian T^n orbit is given by $L(u) = \mu_T^{-1}(u)$ for an interior point $u \in P$.

We recall that orbifolds are locally given as quotients of Euclidean space by a finite group action, and the study of Gromov-Witten theory has been extended to the case of orbifolds in the last decade, starting from the work of Chen and Ruan in [**CR**]. In particular, they have introduced J-holomorphic maps from orbi-curves to an almost complex orbifold and have shown that a moduli space of such J-holomorphic maps of a fixed type has a Kuranishi structure and a virtual fundamental cycle, and hence can be used to define Gromov-Witten invariants.

To find holomorphic orbi-discs with boundary on $L(u)$, we first define what we call the desingularized Maslov index for J-holomorphic orbi-discs (Definition 3.1). This is done using the desingularization of the pull-back orbi-bundle introduced in [**CR**]. The standard Maslov index cannot be defined here since the pull-back tangent bundle is not a vector bundle but an orbi-bundle. (See [**CS**] for related results.) We then establish a desingularized Maslov index formula in terms of intersection numbers with toric divisors (analogous to [**C1**], [**CO**]) in Theorem 5.2. Using the formula, we prove a classification theorem of orbi-discs in toric orbifolds (Theorem 6.2).

There is a class of holomorphic (orbi-)discs which plays the role of Maslov index two discs in the smooth cases. We call them basic discs, and they are either smooth holomorphic discs of Maslov index two, or holomorphic orbi-discs with one orbifold marked point, of desingularized

Maslov index zero. These basic discs are relevant for the computation of Floer cohomology of Lagrangian torus fibers. We find that there exist holomorphic orbi-discs corresponding to each non-trivial twisted sector of the toric orbifold, which are basic:

Theorem 1.1 (Corollary 6.3)**.** *The holomorphic orbi-discs with one interior singularity and desingularized Maslov index 0 (modulo T^n-action and automorphisms of the source disc) correspond to the twisted sectors $\nu \in Box'$ of the toric orbifold.*

In addition, we find the area formula of the basic orbi-discs in section 7 and prove their Fredholm regularity in section 8.

We can use *smooth* J-holomorphic discs to set up A_∞-algebra for a Lagrangian torus fiber L, and its potential function $PO(b)$ for bounding cochains $b \in H^1(L; \Lambda_0)$ in the same way as in [**FOOO2**], which we call smooth potential function.

The leading order smooth potential function $PO_0(b)$ of toric orbifold can be defined combinatorially as

$$PO_0(b) := \sum_{j=1}^{m} T^{\ell_j(u)} (y_1)^{b_{j1}} \cdots (y_n)^{b_{jn}}, \tag{1.1}$$

the j-th term of which corresponds to stacky vector $\boldsymbol{b}_j$ from the classification of smooth Maslov index two discs in Corollary 6.4. By introducing the variables z_j as

$$z_j = T^{\ell_j(u)} (y_1)^{b_{j1}} \cdots (y_n)^{b_{jn}}, \tag{1.2}$$

the leading order potential function can be written as $PO_0(b) = z_1 + \cdots + z_m$. This leading order potential of $PO_0(b)$ is usually called the Hori-Vafa Landau-Ginzburg superpotential of the mirror [**HV**].

The full potential $PO(b)$ is given as follows:

Theorem 1.2 (Theorem 11.3)**.** 1) *$PO(b)$ can be written as*

$$PO(b) = \sum_{i=1}^{m} z_j + \sum_{k=1}^{N} T^{\lambda_k} P_k(z_1, \ldots, z_m) \tag{1.3}$$

for $N \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $\lambda_k \in \mathbb{R}_{>0}$. If $N = \infty$, then $\lim_{k\to\infty} \lambda_k = \infty$. Here $P_k(z_1, \ldots, z_m)$ are monomials of $z_1, \ldots, z_m$ with coefficients in Λ_0.

2) *If $\mathcal{X}$ is Fano then $P_k = 0$.*
3) *The above formula* (11.13) *is independent of u and depends only on $\mathcal{X}$.*

This potential $PO(b)$ can be used to compute smooth Lagrangian Floer cohomology for L, by considering its critical points.

Now, as explained above, we can use orbifold J-holomorphic discs and strips to set up bulk deformation of the above smooth Lagrangian Floer theory, following [**FOOO3**]. The bulk deformed A_∞-algebra gives rise to a bulk potential $PO^{\mathfrak{b}}(b)$, which is a bulk deformation of the potential $PO(b)$ above. The leading order potential of $PO^{\mathfrak{b}}(b)$, which we denote by $PO^{\mathfrak{b}}_{orb,0}(b)$, can be explicitly written down from the classification results on basic (orbi-)discs.

More precisely, consider the bulk deformation term $\mathfrak{b} = \mathfrak{b}_{sm} + \mathfrak{b}_{orb}$ given by

$$\begin{cases} \mathfrak{b}_{sm} = \sum_{i=1}^m \mathfrak{b}_i D_i & \mathfrak{b}_i \in \Lambda_+ \\ \mathfrak{b}_{orb} = \sum_{\nu \in Box'} \mathfrak{b}_\nu 1_{\mathcal{X}_\nu} & \mathfrak{b}_\nu \in \Lambda_+. \end{cases} \tag{1.4}$$

Here, D_i are toric divisors, and $1_{\mathcal{X}_\nu}$ are fundamental classes of twisted sectors. (One may consider bulk deformation by other cohomology classes of twisted sectors, but we do not do so here to simplify the exposition.)

Definition 1.1 (Definition 12.7)**.** Leading order bulk potential $PO^{\mathfrak{b}}_{orb,0}$ is explicitly defined as

$$PO^{\mathfrak{b}}_{orb,0} = z_1 + \cdots + z_m + \sum_{\nu \in Box'} \mathfrak{b}_\nu z^\nu. \tag{1.5}$$

Here $\nu = \sum_{i=1}^m c_i \boldsymbol{b}_i \in N$ so that z^ν is a well-defined Laurent polynomial of $y_1, \ldots, y_n$, $y^{-1}, \ldots, y_n^{-1}$.

It is important to note that the leading order potential $PO(b)_0$, in the case of toric manifolds, is independent of bulk parameter $\mathfrak{b}$, in the sense that it can be read off from the toric polytope. In the orbifold cases, the above definition of $PO^{\mathfrak{b}}_{orb,0}$ *depends* on the choice of $\mathfrak{b}_\nu$. Later, we will use this freedom to choose $\mathfrak{b}_\nu$ so that we can prove non-displaceability results. Note that different choices of $\mathfrak{b}_\nu$ (on different energy levels) change the leading term equation (Definition 13.1) that is obtained from the leading order equation.

Note that the full bulk potential $PO^{\mathfrak{b}}(b)$ is difficult to compute, but the leading order potential $PO^{\mathfrak{b}}_{orb,0}(b)$ given in (1.4) can be used to determine non-displaceable Lagrangian torus fibers, by studying the corresponding leading term equation of $PO^{\mathfrak{b}}_{orb,0}(b)$ as in Theorem 4.5 of [**FOOO3**] for toric manifolds:

Theorem 1.3 (Theorem 13.2)**.** *The following conditions on u are equivalent:*

1) *The leading term equation of $PO^{\mathfrak{b}}_{orb,0}(u)$ has a solution $y_{l,s} \in R \setminus \{0\}$ $(l = 1, \ldots, K, s = 1, \ldots, d(l))$.*
2) *There exists $\widetilde{\mathfrak{b}} \in H(\Lambda_+)$ such that $\widetilde{\mathfrak{b}}_{orb,0} = \mathfrak{b}_{orb,0}$ and $PO^{\widetilde{\mathfrak{b}}}(u)$ has a critical point on $(\Lambda_0 \setminus \Lambda_+)^n$.*
3) *There exists $\widetilde{\mathfrak{b}} \in H(\Lambda_+)$ such that $\widetilde{\mathfrak{b}}_{orb,0} = \mathfrak{b}_{orb,0}$ and $y_{l,s} \in R \setminus \{0\}$ $(l = 1, \ldots, K, s = 1, \ldots, d(l))$ in the item (1) above is a critical point of $PO^{\widetilde{\mathfrak{b}}}(u)$.*

The construction of A_∞-algebra and its bulk deformation, and those of A_∞-bimodules for a pair of Lagrangian submanifolds, and the related isomorphism between Floer cohomology of bimodule and of A_∞-algebra are almost the same as that of [**FOOO2**] and [**FOOO3**]. To keep the size of the paper reasonable, we only explain how to adapt their constructions in our cases.

To illustrate the results of this paper, we explain the conclusions of the paper in the case of a teardrop orbifold $\mathbb{P}(1,3)$. The teardrop example is explained in more detail in section 15.1.

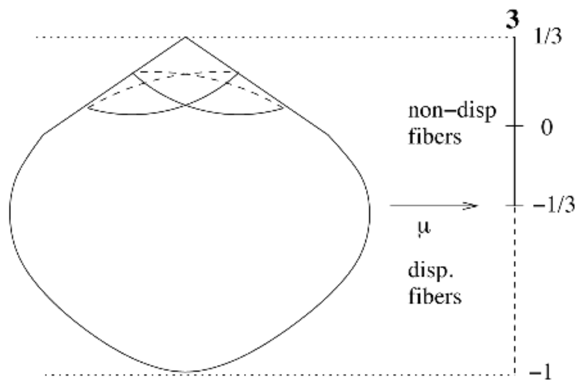

Figure 1. Teardrop

The teardrop orbifold has an orbifold singularity at the north pole N, with isotropy group $\mathbb{Z}/3$. The moment map μ_T has an image given by an interval $[-1,1/3]$, and we put integer label 3 at the vertex $1/3$. The inverse image $\mu_T^{-1}((0,1/3])$ defines an open neighborhood U_N of the north pole N, with a uniformizing cover $\widetilde{U}_N \cong D^2$ with $\mathbb{Z}/3$-action. The inverse image of $\mu_T^{-1}([-1,0))$ defines a neighborhood U_S of the south pole S. The length for $\mu_T(U_N)$ is one third that of U_S, since the symplectic area of U_N should be considered as one third of that of the uniformizing cover D^2. A Hamiltonian function H is an invariant function on $\widetilde{U}_N$ near N, and hence N is a critical point of such H. Thus any Hamiltonian flow fixes N, and a nearby circle fiber cannot be displaced from itself, as illustrated in the figure. But the fiber $\mu_T^{-1}(u)$ for $u < -1/3$ can be displaced in the open set $\mathbb{P}(1,3) \setminus \{N\}$. The fibers $\mu_T^{-1}(u)$ for $u \in [-1/3, 1/3]$ are non-displaceable as shown in [**Wo**]. This is a prototypical example of Hamiltonian rigidity in symplectic orbifolds.

As explained in section 15.1, such non-displaceability can be proved using our methods. Smooth potential function of $\mathbb{P}(1,3)$ has two terms corresponding to two smooth discs:

$$PO(b) = PO(b)_0 = T^{1-3u}y^{-3} + T^{1+u}y.$$

When $u = 0$, two areas of smooth holomorphic discs equal ($1 - 3u = 1 + u = 1$) since a smooth disc wraps around U_N three times, which then has the same symplectic area as the smooth disc covering U_S. One can check that the critical point equation, when $u = 0$, becomes $y^4 = 1/3$, which has non-trivial solutions. Hence, this shows that the (central) fiber $\mu_T^{-1}(0)$ is non-displaceable.

If we introduce bulk deformation by one of the twisted sectors, say $\nu = \frac{1}{3}$, then we can show that fibers $\mu_T^{-1}(u)$ for $u \in (-1/3, 1/3)$ are indeed non-displaceable. Namely, instead of cancelling smooth discs covering the upper and lower hemisphere, we can cancel the smooth discs of smaller area with one of the orbi-discs of N. Their symplectic areas do not match, but as the orbi-discs appear as bulk deformations, we can adjust the coefficient $\mathfrak{b}_\nu$ to match their areas, using our freedom to choose such coefficients as follows:

In this case, the leading order bulk potential is

$$PO^{\mathfrak{b}}_{orb,0} = T^{1-3u}y^{-3} + T^{1+u}y + \mathfrak{b}_\nu T^{1/3-u}y^{-1}.$$

For $-1/3 < u < 0$, we have $1/3 - u < 1 + u < 1 - 3u$, and we set $\mathfrak{b}_\nu = T^{1+u-(1/3-u)} = T^{2/3+2u}$; then the leading term equation becomes $\frac{\partial}{\partial y}(y^{-1} + y) = 0$, which has a non-trivial solution in $\mathbb{C}^*$.

Now, for $0 < u < 1/3$, we have $1/3 - u < 1 - 3u < 1 + u$, and we set $\mathfrak{b}_\nu = T^{1-3u-(1/3-u)} = T^{2/3-2u}$, then the leading term equation becomes $\frac{\partial}{\partial y}(y^{-3} + y^{-1}) = 0$ which has a non-trivial solution in $\mathbb{C}^*$. Hence, from Theorem 13.2, we can prove the non-displaceability results for $\mu_T^{-1}(u)$ for $u \in (-1/3, 1/3)$.

Note that this method does not work for the fibers with $u \in (-1, -1/3)$ since the areas of orbi-discs are bigger than that of the smooth disc of minimal area, and since $\mathfrak{b}_\nu$ should lie in Λ_+.

Notation. Throughout the paper, $\mathcal{X}$ is an orbifold and X is the underlying topological space. We denote by $I\mathcal{X}$ the inertia orbifold, and denote by $T = \{0\} \cup T'$ the index set of inertia components. We denote by ι_ν the rational number called *age* or *degree shifting number* associated to each connected component $\mathcal{X}_\nu$. For toric orbifolds, we will identify T and denote it as *Box* in Definition 4.1.

The lattice $N \cong \mathbb{Z}^n$ parametrizes the one parameter subgroups of the group $(\mathbb{C}^*)^n$. Let M be its dual lattice. Σ is a rational simplicial polyhedral fan in $N_\mathbb{R}$, and $P \subset M_R$ is a rational convex polytope.

The minimal lattice vectors perpendicular to the facets of P, pointing toward the interior, are denoted by $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m$. Certain integral multiples $\boldsymbol{b}_j = c_j \boldsymbol{v}_j$ will be called stacky vectors.

For $u \in M_\mathbb{R}$, let

$$\ell_j(u) = \langle u, \boldsymbol{b}_j \rangle - p_j. \tag{1.6}$$

Then the moment polytope P and its boundary are given by

$$P = \{u \in M_{\mathbb{R}} \mid \ell_i(u) \geq 0, i = 1, \ldots, m\}, \;\; \partial_i P = \{u \in M_{\mathbb{R}} \mid \ell_i(u) = 0\}.$$

Here $\partial_i P$ is the i-th facet of the polytope P.

Let $\mu_T : X \to P$ be the moment map. Consider $u \in Int(P)$ and denote $L(u) = \mu_T^{-1}(u)$. We may write L instead of $L(u)$ to simplify notations.

We will consider the coefficient ring R to be $\mathbb{R}$ (as we work in the de Rham model of A_∞-algebra) or $\mathbb{C}$ (when finding the critical point of the potential). To emphasize the choice of coefficient ring R in the Novikov ring below, we may write Λ^R, Λ_0^R instead of Λ, Λ_0.

Universal Novikov ring Λ and Λ_0 is defined as

$$\Lambda = \{\sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid a_i \in R, \lambda_i \in \mathbb{R}, \lim_{i\to\infty} \lambda_i = \infty\}, \tag{1.7}$$

$$\Lambda_0 = \{\sum_{i=1}^{\infty} a_i T^{\lambda_i} \in \Lambda \mid \lambda_i \in \mathbb{R}_{\geq 0}\}, \tag{1.8}$$

$$\Lambda_+ = \{\sum_{i=1}^{\infty} a_i T^{\lambda_i} \in \Lambda \mid \lambda_i > 0\}. \tag{1.9}$$

In [**FOOO**], the universal Novikov ring $\Lambda_{0,nov}$ is defined as

$$\Lambda_{0,nov} = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} e^{n_i} \;\middle|\; a_i \in R, n_i \in \mathbb{Z}, \lambda_i \in \mathbb{R}_{\geq 0}, \lim_{i\to\infty} \lambda_i = \infty \right\}. \tag{1.10}$$

This is a graded ring by defining $\deg T = 0$, $\deg e = 2$. Λ_{nov} and $\Lambda_{0,nov}^+$ can be similarly defined. By forgetting e from $\Lambda_{0,nov}$ and working with Λ_0, we can only work in a $\mathbb{Z}_2$ graded complex.

We define a valuation $\mathfrak{v}_T$ on Λ by

$$\mathfrak{v}_T(\sum_{i=0}^{\infty} a_i T^{\lambda_i}) = \inf\{\lambda_i \mid a_i \neq 0\}.$$

It is unfortunate, but due to the convention, three b's, written as $b, \mathfrak{b}, \boldsymbol{b}$, will be used throughout the paper, each of which has a different meaning. Here $\boldsymbol{b}_j$ is the stacky normal vector to the j-th facet of the polytope, $b = \sum x_i e_i$ denotes a bounding cochain in $H^1(L, \Lambda_0)$, and $\mathfrak{b}$ denotes the bulk bounding cochain.

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2. J-holomorphic discs and moduli space of bordered stable maps

In this section, we discuss moduli spaces of isomorphism classes of stable maps from a genus 0 prestable bordered Riemann surface with Lagrangian boundary condition together with interior orbifold marked points of a fixed type. Let $(\mathcal{X}, \omega, J)$ be a symplectic orbifold with a compatible almost complex structure. Let $L \subset \mathcal{X}$ be a Lagrangian submanifold (in the smooth part of $\mathcal{X}$).

An *orbifold Riemann surface* $\mathbf{\Sigma}$ is a Riemann surface Σ (with complex structure j) together with the orbifold points $z_1, \ldots, z_k \in \Sigma$ such that each orbifold point $z_i \in \Sigma$ has a disc neighborhood U of z_i which is uniformized by a branched covering map $br : z \to z^{m_i}$. We set $m = 1$ for smooth points $z \in \Sigma$. If Σ has a non-trivial boundary, we always assume that $\partial\Sigma$ is smooth, and that orbifold points lie in the interior of Σ, and such $\mathbf{\Sigma}$ will be called a *bordered orbifold Riemann surface*. Hence $\mathbf{\Sigma}$ can be written as $(\Sigma, \vec{z}, \vec{m})$ for short.

Definition 2.1. Let $\mathbf{\Sigma}$ be a (bordered) orbifold Riemann surface.

A continuous map $f : \Sigma \to \mathcal{X}$ is called pseudo-holomorphic if for any $z_0 \in \Sigma$, the following holds:

1) There is a disc neighborhood of z_0 with a branched covering $br : z \to z^m$.
2) There is a local chart $(V_{f(z_0)}, G_{f(z_0)}, \pi_{f(z_0)})$ of $\mathcal{X}$ at $f(z_0)$ and a local lifting $\widetilde{f}_{z_0}$ of f in the sense that $f \circ br = \pi_{f(z_0)} \circ \widetilde{f}_{z_0}$.
3) $\widetilde{f}_{z_0}$ is pseudo-holomorphic.
4) If $\partial\Sigma \neq 0$, the map f satisfies the boundary condition $f(\partial X) \subset L$.

We need a few technical lemmas following [**CR**] regarding orbifold maps, and we refer readers to the Appendix or [**CR**] for more details.

Definition 2.2. A C^∞ map $\mathbf{f} : \mathbf{\Sigma} \to \mathcal{X}$ (see Definition 16.10) is called regular if the underlying continuous map f has the following property: $f^{-1}(\mathcal{X}_{reg})$ is an open dense and connected subset of Σ.

Lemma 2.1. *If $\mathbf{\Sigma}$ is a bordered orbifold Riemann surface and $\mathbf{f}$: $\mathbf{\Sigma} \to \mathcal{X}$ is regular and pseudo-holomorphic with Lagrangian boundary condition, then it is the unique germ of C^∞ liftings of f. Moreover, $\mathbf{f}$ is good with a unique isomorphism class of compatible systems.*

Lemma 2.1 may be proved using the main idea of Lemma 4.4.11 in [**CR**] together with the result on the local behavior of a pseudo-holomorphic map from a Riemann surface near a singularity in the image, given in Lemma 2.1.4 of [**CR**]. This latter result yields unique continuity of a local lift of a pseudo-holomorphic map near a singularity in the target.

Lemma 2.2. *Suppose $f : \mathbf{\Sigma} \to \mathcal{X}$ is a pseudo-holomorphic map with interior marked points $\vec{z}^+ = (z_1^+, \ldots, z_k^+)$, such that $f(\partial\Sigma)$ does not intersect the singular set of $\mathcal{X}$. Then there exist a finite number of orbifold structures on Σ with singular set contained in $\vec{z}^+$, for which there are good C^∞ maps covering f. Moreover, for each such orbifold structure there exist a finite number of pairs $(\mathbf{f}, \xi)$ where $\mathbf{f}$ is a good map lifting f and ξ is an associated isomorphism class of compatible systems. The number of such pairs is bounded above by a constant that depends on $\mathcal{X}$, the genus of Σ and k only.*

The proof of the above lemma is very similar to Chen-Ruan's proof in the case without boundary; see Proposition 2.2.1 in [**CR**]. Simply note that the homomorphisms θ_{ξ_0,ξ_1} and θ_ξ of [**CR**] are well defined in our case by an application of Lemma 16.1.

The construction of the moduli space is a combination of the construction of Fukaya, Oh, Ohta, and Ono [**FOOO**] regarding the Lagrangian boundary condition, and that of Chen-Ruan [**CR**] regarding the interior orbifold singularities.

We recall the definition of genus 0 prestable bordered Riemann surfaces from [**FOOO**].

Definition 2.3. Σ is called a *genus 0 prestable bordered Riemann surface* if Σ is a possibly singular Riemann surface with boundary $\partial\Sigma$ such that the double $\Sigma \cup_{\partial\Sigma} \overline{\Sigma}$ is a connected and simply connected compact singular Riemann surface whose singularities are only nodes. $(\Sigma, \vec{z}, \vec{z}^+)$ is called a *genus 0 prestable bordered Riemann surface with* (k, l) *marked points* if Σ is a genus 0 prestable bordered Riemann surface and $\vec{z} = (z_0, \ldots, z_{k-1})$ are boundary marked points on $\partial\Sigma$ away from the nodes, and $\vec{z}^+ = (z_1^+, \ldots, z_l^+)$ are interior marked points in $\Sigma \setminus \partial\Sigma$ away from nodal points.

A genus 0 prestable bordered Riemann surface $(\Sigma, \vec{z}, \vec{z}^+)$ is said to be stable if each sphere component has three special (nodal or marked) points and each disc component Σ_ν satisfies $2l_\nu + k_\nu \geq 3$ where l_ν is the number of interior special points and k_ν is the number of boundary special points. We denote by $\mathcal{M}_{k,l}$ the space of isomorphism classes of genus 0 stable bordered Riemann surfaces with (k,l) marked points. From the cyclic ordering of the boundary marked points, $\mathcal{M}_{k,l}$ has $(k-1)!$ connected components. The main component $\mathcal{M}_{k,l}^{main}$ is defined by considering a subset of curves in $\mathcal{M}_{k,l}$ whose boundary marked points are ordered in a cyclic counterclockwise way.

We give the definition of a genus 0 prestable bordered orbi-curve following [**CR**] and [**FOOO**].

Definition 2.4. $(\mathbf{\Sigma}, \vec{z}, \vec{z}^+)$ is called a *genus* 0 *prestable bordered orbi-curve (with interior singularity)* with (k,l) marked points if $(\Sigma, \vec{z}, \vec{z}^+)$ is a genus 0 prestable bordered Riemann surface with (k,l) marked points with the following properties:

1) Orbifold points are contained in the set of interior marked points and interior nodal points.
2) A disc neighborhood of an interior orbifold marked point z_i^+ is uniformized by a branched covering map $z \mapsto z^{m_i}$.
3) A neighborhood of an interior nodal point (which is away from $\partial\Sigma$) is uniformized by $(X(0,r_j), \mathbb{Z}_{n_j})$.

Recall from [**CR**] that the local model of the interior orbifold nodal point, $(X(0,r_j), \mathbb{Z}_{n_j})$, is defined as follows: For real numbers $t \geq 0, r > 0$, set $X(t,r) = \{(x,y) \in \mathbb{C}^2 |\ ||x||, ||y|| < r, xy = t\}$. Fix an action of $\mathbb{Z}_n$ on $X(t,r)$ for any $n > 1$ by $e^{2\pi i/n} \cdot (x,y) = (e^{2\pi i/n}x, e^{-2\pi i/n}y)$. The (branched) covering map $X(t,r) \to X(t^n, r^n)$ given by $(x,y) \to (x^n, y^n)$ is $\mathbb{Z}_n$-invariant. So $(X(t,r), \mathbb{Z}_n)$ can be regarded as a uniformizing system of $X(t^n, r^n)$. Here m_i, n_j are allowed to take the value one, in which case the corresponding orbifold structure is trivial. Hence, the data of a genus 0 prestable bordered orbi-curve includes the numbers m_i, n_j but we do not write them for simplicity. A notion of isomorphism and the group of automorphisms of genus 0 prestable bordered orbi-curves with interior singularity are defined in a standard way, and omitted.

Now we define the notion of an orbifold stable map to be used in this paper. We write $\Sigma = \bigcup_\nu \Sigma_\nu$ where each Σ_ν is an irreducible component.

Definition 2.5. A genus 0 stable map from a bordered orbi-curve with (k,l) marked points is a pair $\big((\mathbf{\Sigma}, \vec{z}, \vec{z}^+), w, \xi\big)$ satisfying the following properties:

1) $(\mathbf{\Sigma}, \vec{z}, \vec{z}^+)$ is a genus 0 prestable bordered orbi-curve.
2) $w : (\Sigma, \partial\Sigma) \to (\mathcal{X}, L)$ is a pseudo-holomorphic map (see Definition 2.1). (Here, we say that w is pseudo-holomorphic (resp. good) if each w_ν is pseudo-holomorphic (resp. good) and induces a continuous map $w : \Sigma \to X$.)
3) w is a C^∞ good map with an isomorphism class ξ of compatible systems (see Definition 16.11).
4) w is representable, i.e. it is injective on local groups.

5) The set of all $\phi : \Sigma \to \Sigma$ satisfying the following properties is finite:
   a) ϕ is biholomorphic.
   b) $\phi(z_i) = z_i, \phi(z_i^+) = z_i^+$.
   c) $w \circ \phi = w$.

Definition 2.6. Two stable maps $\big((\mathbf{\Sigma}_1, \vec{z}_1, \vec{z}_1^+), w_1, \xi_1\big)$ and $\big((\mathbf{\Sigma}_2, \vec{z}_2, \vec{z}_2^+), w_2, \xi_2\big)$ are equivalent if there exists an isomorphism $h : (\mathbf{\Sigma}_1, \vec{z}_1, \vec{z}_1^+) \to (\mathbf{\Sigma}_2, \vec{z}_2, \vec{z}_2^+)$ such that $w_2 \circ h = w_1$ and $\xi_1 = \xi_2 \circ h$, i.e. the isomorphism class ξ_2 pulls back to the class ξ_1 via h.

Definition 2.7. An automorphism of a stable map $\big((\mathbf{\Sigma}, \vec{z}, \vec{z}^+), w, \xi\big)$ is a self equivalence. The automorphism group is denoted by $Aut\big((\mathbf{\Sigma}, \vec{z}, \vec{z}^+), w, \xi\big)$.

Given a stable map $\big((\mathbf{\Sigma}, \vec{z}, \vec{z}^+), w, \xi\big)$, we associate a homology class $w_*([\Sigma]) \in H_2(X, L)$. Note that for each interior marked point z_j^+ (on Σ_ν), ξ_ν determines by the group homomorphism at z_j^+ a conjugacy class (g_j), where $g_j \in G_{w(z_j)}$.

Let $I\mathcal{X}$ be the inertia orbifold of $\mathcal{X}$. Denote by $T = \{0\} \cup T'$ the index set of inertia components, and for $(g) \in T$, call the corresponding component $\mathcal{X}_{(g)}$. Here $\mathcal{X}_{(0)}$ is $\mathcal{X}$ itself, and elements of $x \in \mathcal{X}_{(g)}$ are written as (x, g).

We thus have a map ev_i^+ sending each (equivalence class of) stable map into $I\mathcal{X}$ by

$$ev_i^+ : \big((\mathbf{\Sigma}, \vec{z}, \vec{z}^+), w, \xi\big) \to (w(z_i^+), g_i).$$

Denote by $\underline{l} = \{1, \ldots, l\}$ and consider the map $\boldsymbol{x} : \underline{l} \to T$ describing the inertia component for each (orbifold) marked point. A stable map $\big((\mathbf{\Sigma}, \vec{z}, \vec{z}^+), w, \xi\big)$ is said to be of type $\boldsymbol{x}$ if for $i = 1, \ldots, l$,

$$ev_i^+\big((\mathbf{\Sigma}, \vec{z}, \vec{z}^+), w, \xi\big) \in \mathcal{X}_{\boldsymbol{x}(i)}.$$

Definition 2.8. Given a homology class $\beta \in H_2(X, L)$, we denote by $\mathcal{M}_{k,l}(L, J, \beta, \boldsymbol{x})$ the moduli space of isomorphism classes of genus 0 stable maps to $\mathcal{X}$ from a bordered orbi-curve with (k, l) marked points of type $\boldsymbol{x}$ and with $w_*([\Sigma]) = \beta$. We denote by $\mathcal{M}_{k,l}^{main}(L, J, \beta, \boldsymbol{x})$ the sub-moduli space with $(\mathbf{\Sigma}, \vec{z}, \vec{z}^+) \in \mathcal{M}_{k,l}^{main}$.

Remark 2.9. We follow the notations of [**FOOO**] and denote by $\mathcal{M}$ the compactified moduli space, and by $\mathcal{M}^{reg}$ the moduli space before compactification.

We can give a topology on the moduli space $\mathcal{M}_{k,l}^{main}(L, J, \beta, \boldsymbol{x})$ in a way similar to [**FO**], [**FOOO**], and [**CR2**] (definition 2.3.7). As it is standard, we omit the details. Following Proposition 2.3.8 and Lemma 2.3.9 of [**CR**], we have

Lemma 2.3. *The moduli space $\mathcal{M}^{main}_{k,l}(L, J, \beta, \boldsymbol{x})$ is compact and metrizable.*

The symplectic area of elements in $\mathcal{M}^{main}_{k,l}(L, J, \beta, \boldsymbol{x})$ only depends on the homology class β and the symplectic form ω.

The orientation issues can be dealt with exactly in the same way as in [**FOOO**].

Theorem 2.4. *Let L be relatively spin. Then a choice of relative spin structure of $L \subset X$ canonically induces an orientation of $\mathcal{M}^{main}_{k,l}(L, J, \beta, \boldsymbol{x})$.*

We will consider the moduli space $\mathcal{M}^{main}_{k,l}(L, \beta, \boldsymbol{x})$ (with $J = J_0$ the standard complex structure of the toric orbifolds) in more detail later, but the virtual dimension of the moduli space is given as follows:

Lemma 2.5. *The virtual dimension of the moduli space $\mathcal{M}^{main}_{k,l}(L, \beta, \boldsymbol{x})$ is*

$$n + \mu^{de}(\beta, \boldsymbol{x}) + k + 2l - 3 \;=\; n + \mu_{CW}(\beta) + k + 2l - 3 - 2\iota(\boldsymbol{x}).$$

In the next section, we will explain μ^{de}, which is the desingularized Maslov index of $(\beta, \boldsymbol{x})$, and $\mu_{CW}(\beta)$, which is the Chern-Weil Maslov index of [**CS**]. Let $\iota(\boldsymbol{x}) = \sum_i \iota_{(\boldsymbol{x}(i))}$, where $\iota_{(\boldsymbol{x}(i))}$ is the degree shifting number defined by Chen-Ruan [**CR2**] (see the next section). We remark that the desingularized Maslov index depends on $\boldsymbol{x}$ as we need to desingularize the pull-back tangent orbi-bundle, which depends on $\boldsymbol{x}$.

3. Desingularized Maslov index

Maslov index is related to the (virtual) dimension of moduli spaces in Lagrangian Floer theory (Lemma 2.5). For orbifolds, the standard definition of Maslov index does not have natural extension, since the pull-back tangent bundle under a good map is usually an orbi-bundle which is not a trivial bundle over the bordered orbi-curve.

In this section, we define what we call the desingularized Maslov index, and provide computations of several examples of holomorphic orbi-discs, which will appear in later sections. On the other hand, recently, the first author and H.-S. Shin [**CS**] gave the Chern-Weil definition of Maslov index, which is given by curvature integral of an orthogonal connection. This Chern-Weil definition naturally extends to the orbifold setting, and the relation between Chern-Weil and desingularized Maslov indices has been discussed in [**CS**]. We give a brief explanation at the end of this section.

3.1. Definition of the desingularized Maslov index. Chen and Ruan [**CR**] have shown that for orbifold holomorphic map $u : \boldsymbol{\Sigma} \to \mathcal{X}$ from a closed orbi-curve without boundary to an orbifold, the Chern number $c_1(T\mathcal{X})([\mathbf{\Sigma}])$ (defined via Chern-Weil theory) is in general a rational number and by suitable subtraction of degree shifting number for each orbifold point, one obtains the Chern number of a desingularized bundle which is an honest bundle. Hence the corresponding number is an integer. It is related to the Fredholm index for the moduli spaces.

A similar phenomenon occurs for orbi-discs (discs with interior orbifold singularities). We will mainly work with a Maslov index of a desingularized orbi-bundle and such an index will be calleda desingularized Maslov index for short, and this will be an integer.

Let us first recall the standard definition of Maslov index for a smooth disk with Lagrangian boundary condition. If $w : (D^2, \partial D^2) \to (X, L)$ is a smooth map of pairs, we can find a unique symplectic trivialization (up to homotopy) of the pull-back bundle $w^*TX \cong D^2 \times \mathbb{C}^n$. This trivialization defines a map from $S^1 = \partial D^2$ to $U(n)/O(n)$, the set of Lagrangian planes in $\mathbb{C}^n$, and the Maslov index is the rotation number of the composition of this map with the map $det^2 : (U(n)/O(n)) \to U(1)$ (see [**Ar**]). For bordered Riemann surfaces with several boundary components, one can define the Maslov index similarly by taking the sum of Maslov indices along $\partial\Sigma$ using the fact that a symplectic vector bundle over a Riemann surface with boundary Σ is always trivial. The data of a symplectic vector bundle over Σ, and a Lagrangian subbundle over $\partial\Sigma$, is called a bundle pair, and one can define the Maslov index for any bundle pair in the same way.

Next, we recall the desingularization of an orbi-bundle on an orbifold Riemann surface by Chen and Ruan ([**CR2**]), which plays a key role.

Consider a closed (complex) Riemann surface Σ, with distinct points $\vec{z} = (z_1, \ldots, z_k)$ paired with multiplicity $\vec{m} = (m_1, \ldots, m_k)$. We consider the orbifold structure at z_i which is given by the ramified covering $z \to z^{m_i}$. For simplicity we denote it as $\mathbf{\Sigma} = (\Sigma, \vec{z}, \vec{m})$, which is a closed, reduced, 2-dimensional orbifold.

Consider a complex orbi-bundle E over $\mathbf{\Sigma}$ of rank n. Then, at each singular point z_i, E gives a representation $\rho_i : \mathbb{Z}_{m_i} \to Aut(\mathbb{C}^n)$ so that over a disc neighborhood D_i of z_i, E is uniformized by $(D_i \times \mathbb{C}^n, \mathbb{Z}_{m_i}, \pi)$ where the action of $\mathbb{Z}_{m_i}$ on $D_i \times \mathbb{C}^n$ is defined as

$$e^{2\pi i/m_i} \cdot (z, w) = \big(e^{2\pi i/m_i} z, \rho_i(e^{2\pi i/m_i}) w\big) \tag{3.1}$$

for any $w \in \mathbb{C}^n$. Note that ρ_i is uniquely determined by integers $(m_{i,1}, \ldots, m_{i,n})$ with $0 \le m_{i,j} < m_i$, as it is given by the matrix

$$\rho_i(e^{2\pi i/m_i}) = diag(e^{2\pi i m_{i,1}/m_i}, \ldots, e^{2\pi i m_{i,n}/m_i}). \tag{3.2}$$

The sum $\sum_{j=1}^{n} \frac{m_{i,j}}{m_i}$ is called the degree shifting number ([**CR**]).

Over the punctured disc $D_i \setminus \{z_i\}$ at z_i, E is given a specific trivialization from $(D_i \times \mathbb{C}^n, \mathbb{Z}_{m_i}, \pi)$ as follows: Consider a $\mathbb{Z}_{m_i}$-equivariant map $\Psi_i : D \setminus \{0\} \times \mathbb{C}^n \to D \setminus \{0\} \times \mathbb{C}^n$ defined by

$$(z, w_1, w_2, \ldots, w_n) \to (z^{m_i}, z^{-m_{i,1}} w_1, \ldots, z^{-m_{i,n}} w_n), \tag{3.3}$$

where the $\mathbb{Z}_{m_i}$ action on the target $D \setminus \{0\} \times \mathbb{C}^n$ is trivial. Hence Ψ_i induces a trivialization $\Psi_i : E_{D_i \setminus \{0\}} \to D_i \setminus \{0\} \times \mathbb{C}^n$. We may extend the smooth complex vector bundle $E_{\mathbf{\Sigma} \setminus \{z_1, \ldots, z_k\}}$ over $\mathbf{\Sigma} \setminus \{z_1, \ldots, z_k\}$ to a smooth complex vector bundle over $\mathbf{\Sigma}$ by using these trivializations Ψ_i for each i. The resulting complex vector bundle is called the desingularization of E and denoted by $|E|$.

The essential point as observed in [**CR2**] is that the sheaf of holomorphic sections of the desingularized orbi-bundle and the orbi-bundle itself are the same.

Proposition 3.1 ([**CR2**], Proposition 4.2.2)**.** *Let E be a holomorphic orbifold bundle of rank n over a compact orbi-curve $(\mathbf{\Sigma}, \boldsymbol{z}, \boldsymbol{m})$ of genus g. $\mathcal{O}(E)$ equals $\mathcal{O}(|E|)$, where $\mathcal{O}(E)$ and $\mathcal{O}(|E|)$ are sheaves of holomorphic sections of E and $|E|$.*

As the local group action on the fibers of the desingularized orbi-bundle $|E|$ is trivial, one can think of it as a smooth vector bundle on Σ which is analytically the same as $\mathbf{\Sigma}$ (in other words, there exists a canonically associated vector bundle $|E|$ over the smooth Riemann surface Σ). Hence, for the bundle $|E|$, the ordinary index theory can be applied, which provides the required index theoretic tools for the orbi-bundle E.

Now we give a definition of the desingularized Maslov index, which determines the virtual dimension of the moduli space of J-holomorphic orbi-discs.

Definition 3.1. Let $\mathbf{\Sigma} = (\Sigma, \vec{z}, \vec{m})$ be a bordered orbi-curve with $(0, k)$ marked points. Let $u : \mathbf{\Sigma} \to X$ be an orbifold stable map. Then, u^*TX is a complex orbi-bundle over $\mathbf{\Sigma}$, with Lagrangian subbundle $u|_{\partial \mathbf{\Sigma}}^* TL$ at $\partial \mathbf{\Sigma}$. Let $|u^*TX|$ be the desingularized bundle over $\mathbf{\Sigma}$ (or Σ), which still has the Lagrangian subbundle at the boundary from $u|_{\partial \Sigma}^* TL$. The Maslov index of the bundle pair $(|u^*TX|, u|_{\partial \Sigma}^* TL)$ over $(\Sigma, \partial \Sigma)$ is called the *desingularized Maslov index* of u, and denoted by $\mu^{de}(u)$. Note that this index is well-defined as it is independent of the choice of compatible system for u, within the same isomorphism class, by Lemma 16.1.

3.2. Examples of computations of the index. Here we give a few examples of computations of the desingularized Maslov index. Consider the orbifold disc $\mathcal{D}$ with $\mathbb{Z}_p$ singularity at the origin, and the orbifold complex plane $\boldsymbol{C}$ with $\mathbb{Z}_p$ singularity at the origin. Let the unit circle $L = S^1 \in \boldsymbol{C}$ be a Lagrangian submanifold. Consider the natural inclusion $u : \mathcal{D} \to \boldsymbol{C}$.

Lemma 3.2. *The desingularized Maslov index of u equals* 0.

Proof. Consider the tangent orbi-bundle $T\mathcal{D}$ over $\mathcal{D}$, and its uniformizing chart $D^2 \times \mathbb{C} = \{(z,w)|z \in D^2, w \in \mathbb{C}\}$ with the $\mathbb{Z}_p$ action given by

$$e^{2\pi i/p} \cdot (z,w) = \big(e^{2\pi i/p}z, e^{2\pi i/p}w\big). \tag{3.4}$$

Then, the subbundle TL at $z \in S^1$ is given by $\mathbb{R} \cdot iz \subset \mathbb{C}$. We consider its image under the desingularization map $\Psi : D^2 \times \mathbb{C} \to D^2 \times \mathbb{C}$ defined as $\Psi(z,w) = (z^p, z^{-1}w)$. The image of TL via Ψ at the point $\alpha \in D^2$ with $\alpha = z^p$ is given by $\mathbb{R} \cdot z^{-1}iz = i \cdot \mathbb{R} \subset \mathbb{C}$.

The desingularization provides a desingularized vector bundle over the orbi-disc $\mathcal{D}$, which is a trivial vector bundle, and the loop of Lagrangian subspaces at the boundary is a constant loop. Therefore the desingularized Maslov index is zero. q.e.d.

We now consider a more general case: Consider the orbifold disc $\mathcal{D}$ with $\mathbb{Z}_m$ singularity at the origin, and the complex plane $\boldsymbol{C}$ with $\mathbb{Z}_{mn}$ singularity at the origin, and the unit circle $L = S^1 \in \boldsymbol{C}$ as a Lagrangian submanifold. Consider the uniformizing cover D^2 of $\mathcal{D}$, with coordinate $z \in D^2$. Consider the uniformizing cover $\mathbb{C}$ of $\boldsymbol{C}$, with coordinate $y \in \mathbb{C}$.

Lemma 3.3. *Consider the map $u : \mathcal{D} \to \boldsymbol{C}$, induced from the map $\widetilde{u} : D^2 \to \mathbb{C}$ defined by $\widetilde{u}(z) = z^k$. Here we assume that k and m are relatively prime to ensure that the group homomorphism is injective. Then, the desingularized Maslov index of u equals $2[k/m]$ where $[k/m]$ is the largest integer $\leq k/m$.*

Proof. Consider the tangent orbi-bundle $T\boldsymbol{C}$ over $\boldsymbol{C}$ whose uniformizing chart is given by $\mathbb{C} \times \mathbb{C} = \{(y,w)|y,w \in \mathbb{C}\}$ with the $\mathbb{Z}_{mn}$ action given by the diagonal action. Then, the subbundle TL at $y \in S^1$ is given by $\mathbb{R} \cdot iy \subset \mathbb{C}$. We consider the pull-back orbi-bundle $u^*T\boldsymbol{C}$ whose uniformizing chart is $\mathbb{C} \times \mathbb{C} = \{(z,w)|z,w \in \mathbb{C}\}$ with the $\mathbb{Z}_m$ action given by

$$e^{2\pi i/m} \cdot (z,w) = \big(e^{2\pi i/m}z, e^{2\pi k i/m}w\big). \tag{3.5}$$

In this chart, the subbundle $(u|_{\partial\mathcal{D}})^*TL$ is given by $(z, \mathbb{R} \cdot z^k i)$ for $z \in \partial D^2$. Now, we consider its image under the desingularization map $\Psi : D^2 \setminus \{0\} \times \mathbb{C} \to D^2 \setminus \{0\} \times \mathbb{C}$ defined as $\Psi(z,w) = (z^m, z^{-k'}w)$, where $k' = k - [k/m]m$. The image of TL via Ψ at the point $\alpha \in D^2$ with $\alpha = z^m$ is given by $\mathbb{R} \cdot z^{-k'}iz^k = z^{[k/m]m}i \cdot \mathbb{R} \subset \mathbb{C}$.

Hence we obtain a trivialized desingularized vector bundle $|E|$ over $\mathcal{D}$ (and hence D^2), and from the above computation, the loop of Lagrangian subspaces along the boundary is given by $z^{[k/m]m}i \cdot \mathbb{R}$. But also note that the coordinate on D^2 is in fact z^m, and hence the desingularized Maslov index of u is $2[k/m]$. q.e.d.

Remark 3.2. Note that in the case that k and m are not relatively prime, say $d = gcd(k,m)$, then instead of the map from the above orbifold disc, we consider a domain with simpler singularity, say $\mathcal{D}$ with $\mathbb{Z}_{m/d}$ singularity at the origin, and the map given by $x \mapsto x^{k/d}$. The Maslov index of this orbifold holomorphic disc is still $2[k/m] = 2[(k/d)/(m/d)]$.

The following computations of indices will be used later in the paper. We compute desingularized Maslow indices for orbi-discs in $\mathcal{X} = \mathbb{C}^n/G$. Consider the orbifold disc $\mathcal{D}$ with $\mathbb{Z}_m$ singularity at the origin, and the orbifold $\mathcal{X}$ defined by the complex vector space $\mathbb{C}^n$ with an action of a finite abelian group G. Consider the uniformizing cover D^2 of $\mathcal{D}$, with coordinate $z \in D^2$.

Lemma 3.4. *Consider the holomorphic orbi-disc $u : (\mathcal{D}, \partial\mathcal{D}) \to (\mathcal{X}, L)$, induced from an equivariant map $\widetilde{u} : D^2 \to \mathbb{C}^n$ given by*

$$(a_1 z^{d_1}, \ldots, a_k z^{d_k}, a_{k+1}, \ldots, a_n), \tag{3.6}$$

where $a_i \in U(1), d_i \geq 0$ for all i. We set $d_{k+1} = \cdots = d_n = 0$ and $L = (S^1)^n \in \mathbb{C}^n$. Then, the desingularized Maslov index of u equals $2\sum_i [d_i/m]$.

Proof. Consider the tangent orbi-bundle $T\mathcal{X}$ over $\mathcal{X}$ whose uniformizing chart is given by $\mathbb{C}^n \times \mathbb{C}^n = \{(\vec{y}, \vec{w}) | \vec{y}, \vec{w} \in \mathbb{C}^n\}$ with the group G acting diagonally. Then, the fiber of TL at $\vec{y}$ is given by $(\mathbb{R}\cdot iy_1, \ldots, \mathbb{R}\cdot iy_n) \in \mathbb{C}^n$. We consider the pull-back orbi-bundle, $u^*T\mathcal{X}$, whose uniformizing chart is given by $\mathbb{C}^n \times \mathbb{C}^n = \{(z, \vec{w}) | z, \vec{w} \in \mathbb{C}^n\}$ with the $\mathbb{Z}_m$ action given by

$$e^{2\pi i/m} \cdot (z, \vec{w}) = \big(e^{2\pi i/m} z, e^{2\pi d_1 i/m} w_1, \ldots, e^{2\pi d_n i/m} w_n\big). \tag{3.7}$$

In this chart, the subbundle $(u|_{\partial\mathcal{D}})^*TL$ is given by

$$(z, \mathbb{R} \cdot a_1 z^{d_1} i, \ldots, \mathbb{R} \cdot a_n z^{d_n} i).$$

Now, we consider its image under the desingularization map $\Psi : D^2 \setminus \{0\} \times \mathbb{C}^n \to D^2 \setminus \{0\} \times \mathbb{C}^n$ defined by $\Psi(z, w) = (z^m, x^{-d_1'} w_1, \ldots, x^{-d_n'} w_n)$, where $d_i' = d_i - [d_i/m]m$. We have $d_{k+1}' = \cdots, d_n' = 0$. The image of TL via Ψ at the point $\alpha \in D^2$ with $\alpha = z^m$ is given by

$$(\ldots, \prod_i \mathbb{R} \cdot z^{-d_i'} i z^{d_i}, \ldots) = (\ldots, z^{[d_i/m]m} i \cdot \mathbb{R}, \ldots) \subset \mathbb{C}^n.$$

Hence we obtain a trivialized desingularized vector bundle $|E|$ over $\mathcal{D}$, and the Maslov index of the loop of Lagrangian subspaces over uniformizing cover D^2 is $\sum 2[d_i/m]m$ and hence the Maslov index for the orbi-disc u is $\sum 2[d_i/m]$. q.e.d.

3.3. Relation to the Chern-Weil Maslov index. Now, we explain the Chern-Weil construction of the Maslov index for orbifold from [**CS**] and its relationship with the desingularized Maslov index defined in this section.

By bundle pair $(\mathcal{E},\mathcal{L})$ over Σ, we mean a symplectic vector bundle $\mathcal{E}\to\Sigma$ equipped with compatible almost complex structure, together with Lagrangian subbundle $\mathcal{L}\to\partial\Sigma$ over the boundary of Σ. Let ∇ be a unitary connection of E, which is orthogonal with respect to $\mathcal{L}$: This means that ∇ preserves $\mathcal{L}$ along the boundary $\partial\Sigma$. See Definition 2.3 of [**CS**] for the precise definition.

Definition 3.3. The Maslov index of the bundle pair $(\mathcal{E},\mathcal{L})$ is defined by

$$\mu_{CW}(\mathcal{E},\mathcal{L}) = \frac{\sqrt{-1}}{\pi}\int_\Sigma tr(F_\nabla)$$

where $F_\nabla \in \Omega^2(\Sigma, End(\mathcal{E}))$ is the curvature induced by ∇.

It is proved in [**CS**] that this Chern-Weil definition agrees with the usual topological definition of the Maslov index. But the above definition of the Maslov index has an advantage over the topological one in that it extends more readily to the orbifold case, as observed in [**CS**]. In the orbifold case, $\mathcal{E}$ is assumed to be a symplectic orbi-bundle over an orbifold Riemann surface $\mathbf{\Sigma}$ and the Maslov index is defined by considering orthogonal connections which are, in addition, invariant under local group actions. Thus, the Maslov index of the bundle pair $(\mathcal{E},\mathcal{L})$ over orbifold Riemann surface with boundary is defined as the curvature integral as in Definition 3.3. It is shown in [**CS**] that the Maslov index $\mu_{CW}(\mathcal{E},\mathcal{L})$ is independent of the choice of orthogonal unitary connection ∇ and also independent of the choice of an almost complex structure.

Finally, we recall Proposition 6.10 of [**CS**] relating the Maslov index with the desingularized Maslov index:

Proposition 3.5. *Suppose $\mathbf{\Sigma}$ has k interior orbifold marked points of order $m_1,\ldots,m_k$. Then*

$$\mu_{CW}(\mathcal{E},\mathcal{L}) = \mu^{de}(\mathcal{E},\mathcal{L}) + 2\sum_{i=1}^{k}\sum_{j=1}^{n}\frac{m_{i,j}}{m_i},$$

where $m_{i,j}$ are defined as in (3.2).

4. Toric orbifolds

In this paper, we consider compact toric orbifolds. These are more general than compact simplicial toric varieties, in that their orbifold singularities may not be fully captured by the analytic variety structure. In fact, we are mainly interested in a subclass called symplectic toric orbifolds. These have been studied by Lerman and Tolman [**LT**], and correspond to polytopes with a positive integer label on each facet. In algebraic geometry, Borisov, Chen, and Smith [**BCS**] considered toric DM stacks that correspond to stacky fans. The vectors of such a stacky fan take values in a finitely generated abelian group N. A toric DM stack is a toric orbifold when N is free and in this case the stabilizer of a generic point is trivial.

4.1. Compact toric orbifolds as complex quotients. Combinatorial data called a *complete fan of simplicial rational polyhedral cones*, Σ, are used to describe compact toric manifolds (see [**Co**] or [**Au**]). For the definitions of rational simplicial polyhedral cone σ and fan Σ, we refer to Fulton's book [**Ful**]. If the minimal lattice generators of one-dimensional edges of every top dimensional cone $\sigma \in \Sigma$ form a $\mathbb{Z}$-basis of N, then the fan is called smooth and the corresponding toric variety is nonsingular. Otherwise, such a fan defines a simplicial toric variety (which are orbifolds). The toric orbifolds to be considered here are more general than simplicial toric varieties. They need additional data of multiplicity for each 1-dimensional cone, or equivalently, a choice of lattice vectors in them.

Let N be the lattice $\mathbb{Z}^n$, and let $M = Hom_{\mathbb{Z}}(N, \mathbb{Z})$ be the dual lattice. Let $N_{\mathbb{R}} = N \otimes \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes \mathbb{R}$. The set of all k-dimensional cones in Σ will be denoted by $\Sigma^{(k)}$. We label the minimal lattice generators of cones in $\Sigma^{(1)}$ as $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\} := G(\Sigma)$, where $\boldsymbol{v}_j = (v_{j1}, \ldots, v_{jn}) \in N$. For $\boldsymbol{v}_j$, consider a lattice vector $\boldsymbol{b}_j \in N$ with $\boldsymbol{b}_j = c_j \boldsymbol{v}_j$ for some positive integer c_j. We call $\boldsymbol{b}_j$ a stacky vector, and denote $\vec{\boldsymbol{b}} = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_m)$. For a simplicial rational polyhedral fan Σ, the stacky fan $(\Sigma, \vec{\boldsymbol{b}})$ defines a toric orbifold as follows.

We call a subset $\mathcal{P} = \{\boldsymbol{v}_{i_1}, \ldots, \boldsymbol{v}_{i_p}\} \subset G(\Sigma)$ a *primitive collection* if $\mathcal{P}$ does not generate a p-dimensional cone in Σ, while for all k $(0 \le k < p)$, each k-element subset of $\mathcal{P}$ generates a k-dimensional cone in Σ.

Let $\mathcal{P} = \{\boldsymbol{v}_{i_1}, \ldots, \boldsymbol{v}_{i_p}\}$ be a primitive collection in $G(\Sigma)$. Denote

$$\mathbb{A}(\mathcal{P}) = \{(z_1, \ldots, z_m) \mid z_{i_1} = \cdots = z_{i_p} = 0\}.$$

Define the closed algebraic subset $Z(\Sigma)$ in $\mathbb{C}^m$ as $Z(\Sigma) = \cup_{\mathcal{P}} \mathbb{A}(\mathcal{P})$, where $\mathcal{P}$ runs over all primitive collections in $G(\Sigma)$ and we put $U(\Sigma) = \mathbb{C}^m \setminus Z(\Sigma)$.

Consider the map $\pi : \mathbb{Z}^m \to \mathbb{Z}^n$ sending the basis vectors e_i to $\boldsymbol{b}_i$ for $i = 1, \ldots, m$. Note that the $\mathbb{K} := Ker(\pi)$ is isomorphic to $\mathbb{Z}^{m-n}$ and that π may not be surjective for toric orbifolds. However, by tensoring with $\mathbb{R}$, we obtain the following exact sequences:

$$0 \to \mathfrak{k} \to \mathbb{R}^m \xrightarrow{\pi} \mathbb{R}^n \to 0, \tag{4.1}$$

$$0 \to K \to T^m \xrightarrow{\pi} T^n \to 0, \tag{4.2}$$

$$0 \to K_{\mathbb{C}} \to (\mathbb{C}^*)^m \xrightarrow{\pi'} (\mathbb{C}^*)^n \to 0. \tag{4.3}$$

Here $T^m = \mathbb{R}^m/\mathbb{Z}^m$ and the map π' is defined as

$$\pi'(\lambda_1, \ldots, \lambda_m) = (\prod_j \lambda_j^{b_{j1}}, \ldots, \prod_j \lambda_j^{b_{jn}}).$$

Here, even though $\mathbb{K}$ is free, K may have a non-trivial torsion part. For a complete stacky fan $(\Sigma, \boldsymbol{b})$, $K_{\mathbb{C}}$ acts effectively on $U(\Sigma)$ with finite isotropy groups. The global quotient orbifold

$$X_\Sigma = U(\Sigma)/K_{\mathbb{C}}$$

is called the *compact toric orbifold* associated to the complete stacky fan $(\Sigma, \boldsymbol{b})$. We refer readers to [**BCS**] for more details.

There exists an open covering of $U(\Sigma)$ by affine algebraic varieties: Let σ be a k-dimensional cone in Σ generated by $\{\boldsymbol{v}_{i_1}, \ldots, \boldsymbol{v}_{i_k}\}$. Define the open subset $U(\sigma) \subset \mathbb{C}^m$ as

$$U(\sigma) = \{(z_1, \ldots, z_m) \in \mathbb{C}^m \mid z_j \neq 0 \text{ for all } j \notin \{i_1, \ldots, i_k\}\}.$$

Then the open sets $U(\sigma)$ have the following properties:

1) $U(\Sigma) = \cup_{\sigma \in \Sigma} U(\sigma)$;
2) if $\sigma \prec \sigma'$, then $U(\sigma) \subset U(\sigma')$;
3) for any two cone $\sigma_1, \sigma_2 \in \Sigma$, one has $U(\sigma_1) \cap U(\sigma_2) = U(\sigma_1 \cap \sigma_2)$; in particular,

$$U(\Sigma) = \bigcup_{\sigma \in \Sigma^{(n)}} U(\sigma).$$

We define the open set $U_\sigma := U(\sigma)/K_{\mathbb{C}}$. For toric orbifolds, U_σ may not be smooth.

The following lemma is elementary (see the case of smooth toric manifold in [**B1**] together with the considerations of the orbifold case in [**BCS**]).

Lemma 4.1. *Let σ be a n-dimensional cone in Σ, with a choice of lattice vectors $\boldsymbol{b}_\sigma = (\boldsymbol{b}_{i_1}, \ldots, \boldsymbol{b}_{i_n})$ from its one-dimensional cones. Suppose that $\boldsymbol{b}_\sigma$ spans the sublattice $N_{\boldsymbol{b}_\sigma}$ of the lattice N. Consider the dual lattice $M_{\boldsymbol{b}_\sigma} \supset M$ of $N_{\boldsymbol{b}_\sigma}$, and the dual $\mathbb{Z}$-basis $(\boldsymbol{u}_{i_1}, \ldots, \boldsymbol{u}_{i_n})$ in $M_{\boldsymbol{b}_\sigma}$ defined by*

$$\langle \boldsymbol{b}_{i_k}, \boldsymbol{u}_{i_l} \rangle = \delta_{k,l}.$$

Recall that σ with the lattice $N_{\boldsymbol{b}_\sigma}$ (resp. N) gives rise to a space U'_σ (resp. U_σ), and the abelian group $G_{\boldsymbol{b}_\sigma} = N/N_{\boldsymbol{b}_\sigma}$ acts on U'_σ to give

$$U'_\sigma / G_{\boldsymbol{b}_\sigma} = U_\sigma.$$

In terms of the homogeneous coordinates $z_1, \ldots, z_m$ on $\mathbb{C}^m$, the coordinate functions $x_1^\sigma, \ldots, x_n^\sigma$ of the uniformizing open set U'_σ are given *by*

$$\begin{cases} x_1^\sigma = z_1^{\langle \boldsymbol{b}_1, \boldsymbol{u}_{i_1}\rangle} \cdots z_m^{\langle \boldsymbol{b}_m, \boldsymbol{u}_{i_1}\rangle} \\ \quad\vdots \\ x_n^\sigma = z_1^{\langle \boldsymbol{b}_1, \boldsymbol{u}_{i_n}\rangle} \cdots z_m^{\langle \boldsymbol{b}_m, \boldsymbol{u}_{i_n}\rangle} \end{cases} \tag{4.4}$$

The $G_{\boldsymbol{b}_\sigma}$-action on U'_σ for $g \in N/N_{\boldsymbol{b}_\sigma}$ is given by

$$g \cdot x_j^\sigma = e^{2\pi i \langle g, \boldsymbol{u}_{i_j}\rangle} x_j^\sigma. \tag{4.5}$$

Now, we discuss $\mathbb{C}^*$-action on U'_σ and U_σ. In what follows there is a complication because there exists a $\mathbb{C}^*$-action on the quotient (coming from the $\mathbb{C}^*$-action on the disc) which does not lift to the $\mathbb{C}^*$-action on the uniformizing cover.

Lemma 4.2. *For any lattice vector $w \in N_{\boldsymbol{b}_\sigma}$ there is an associated $\mathbb{C}^*$-action on U'_σ given by*

$$\lambda_w(z) \cdot x_j^\sigma = z^{\langle w, \boldsymbol{u}_{i_j}\rangle} x_j^\sigma. \tag{4.6}$$

Proof. For any $w \in N_{\boldsymbol{b}_\sigma}$, there exists an associated $\mathbb{C}^*$ action: Let $z \in \mathbb{C}^*$, and $\boldsymbol{u} \in M_{\boldsymbol{b}_\sigma}$. Toric structure provides action λ_w of w on the function $\chi^{\boldsymbol{u}}$ on U'_σ by $\lambda_w(z) \cdot (\chi^{\boldsymbol{u}}) = z^{\langle w, \boldsymbol{u}\rangle}\chi^{\boldsymbol{u}}$. The lemma follows by writing this formula in terms of the coordinates $(x_1^\sigma, \ldots, x_n^\sigma)$. q.e.d.

Lemma 4.3. *For a lattice vector $v \in N$, there is an associated $\mathbb{C}^*$-action on the quotient U_σ as in* (4.6)*. Furthermore, such a $\mathbb{C}^*$-action induces a morphism $\mathbb{C} \to U_\sigma$, if v lies in the cone σ.*

Proof. We write $v = \sum_j c_j \boldsymbol{b}_{i_j}$ where c_j's are rational numbers. Hence, (4.6) does not provide a $\mathbb{C}^*$-action of v on U'_σ. But there exists a $\mathbb{C}^*$-action of $v \in N$ on the quotient U'_σ/G. We define the action $\lambda_{\boldsymbol{v}}(z)$ by the formula (4.6). Then possible values of $(z^{\langle v, \boldsymbol{u}_{i_1}\rangle}, \ldots, z^{\langle v, \boldsymbol{u}_{i_n}\rangle})$ for different choices of branch cuts differ by multiplication of $(e^{2\pi i a\langle v, \boldsymbol{u}_{i_1}\rangle}, \ldots, e^{2\pi i a\langle v, \boldsymbol{u}_{i_n}\rangle})$ for some integer $a \in \mathbb{Z}$. Therefore the difference is exactly given by the G-action (4.5).

The $\mathbb{C}^*$-action corresponding to v defines a map from $\mathbb{C}^*$ to the principal $(\mathbb{C}^*)^n$ orbit of the toric variety. If v lies in the cone σ, we have $\langle v, \boldsymbol{u}_{i_j}\rangle \geq 0$ for all j. In this case the above map extends to a map from $\mathbb{C}$ to U_σ (see [**Ful**], chapter 2.3). q.e.d.

Definition 4.1. Let σ be a d-dimensional cone in Σ with a choice of lattice vectors $\boldsymbol{b}_\sigma = (b_{i_1}, \ldots, b_{i_d})$. Let $N_{\boldsymbol{b}_\sigma}$ be the submodule of N generated by these lattice vectors. Define

$$Box_{\boldsymbol{b}_\sigma} = \{\nu \in N \mid \nu = \sum_{k=1}^d c_k \boldsymbol{b}_{i_k}, c_k \in [0,1)\}.$$

This set has one-to-one correspondence with the group

$$G_{\boldsymbol{b}_\sigma} = ((N_{\boldsymbol{b}_\sigma} \otimes_{\mathbb{Z}} \mathbb{Q}) \cap N)/N_{\boldsymbol{b}_\sigma}. \tag{4.7}$$

This generalizes the definition of $G_{\boldsymbol{b}_\sigma} = N/N_{\boldsymbol{b}_\sigma}$ given in Lemma 4.1 for n-dimensional cones. It is easy to observe that if $\sigma \prec \sigma'$, then $Box_{\boldsymbol{b}_\sigma} \subset Box_{\boldsymbol{b}_{\sigma'}}$.

Define

$$Box^\circ_{\boldsymbol{b}_\sigma} = Box_{\boldsymbol{b}_\sigma} - \bigcup_{\tau \prec \sigma} Box_{\boldsymbol{b}_\tau}.$$

Define

$$Box = \bigcup_{\sigma \in \Sigma^{(n)}} Box_{\boldsymbol{b}_\sigma} = \bigsqcup_{\sigma \in \Sigma} Box^\circ_{\boldsymbol{b}_\sigma}. \tag{4.8}$$

We set $Box = \{0\} \sqcup Box'$. Box is the index set T of the components of the inertia orbifold of the toric orbifold corresponding to $(\Sigma, \vec{\boldsymbol{b}})$. To every $\nu \in Box^\circ_{\boldsymbol{b}_\sigma} \cap Box'$, there corresponds a twisted sector $\mathcal{X}_\nu$ which is isomorphic to the orbit closure $\bar{O}_\sigma$ as an analytic variety. However, it has a specific orbifold structure that includes the trivial action of $G_{\boldsymbol{b}_\sigma}$. In particular, the fundamental class of $\mathcal{X}_\nu$ is $\frac{1}{o(G_{\boldsymbol{b}_\sigma})}[\bar{O}_\sigma]$.

Remark 4.2. We would like to point out here that there is a natural orbifold structure on the variety $\bar{O}_\tau$. This comes from considering it as a toric orbifold with the fan $star(\tau)$ as described in section 3.1 of [**Ful**]: Let L be the submodule of N generated by $\tau \cap N$ and $N(\tau) = N/L$. Then $star(\tau)$ is the set of cones containing τ, realized as a fan in $N(\tau)$. The projection of stacky lattice vectors $\boldsymbol{b}_j$ to $N(\tau)$ gives $\bar{O}_\tau$ the desired orbifold structure. This structure induces an inclusion of $\bar{O}_\tau$ into $\mathcal{X}$ as a suborbifold.

This orbifold structure is in general different from the orbifold structure of $\bar{O}_\tau$ as an analytic variety. For instance, when $\dim(\tau) = n-1$, the variety $\bar{O}_\tau$ is a smooth sphere whereas the above structure may involve orbifold singularities. On the other hand, this structure is also different from the orbifold structure of $\bar{O}_\tau$ as a twisted sector. It precisely misses the trivial action of $G_{\boldsymbol{b}_\tau}$ corresponding to the group actions in the normal bundle of $\bar{O}_\tau$ in $\mathcal{X}$. The orbifold structure of $\bar{O}_\tau$ as a twisted sector induces a different inclusion of it into $\mathcal{X}$ as a suborbifold.

4.2. Symplectic toric orbifolds. Recall that a symplectic toric manifold is a symplectic manifold that admits Hamiltonian action of a half dimensional compact torus. Delzant polytopes, which are rational simple smooth convex polytopes, classify compact symplectic toric manifolds up to equivariant symplectomorphism. Here we review the generalization to labeled polytope, a polytope together with a positive integer label attached to each of its facets, by Lerman and Tolman [**LT**]. Labeled polytopes classify compact symplectic toric orbifolds. We recall briefly the explicit construction of symplectic toric orbifolds from a labeled polytope following [**LT**] (see Audin [**Au**], for instance, for the smooth case).

Definition 4.3. A convex polytope P in $M_{\mathbb{R}}$ is called *simple* if there are exactly n facets meeting at every vertex. A convex polytope P is called *rational* if a normal vector to each facet P can be given by a lattice vector. A simple polytope P is called *smooth* if for each vertex, the n normal vectors to the facets meeting at the given vertex form a $\mathbb{Z}$-basis of N.

Let P be a simple rational convex polytope in $\mathbb{R}^n$ with m facets, with a positive integer assigned to each facet of P.

Definition 4.4. We denote by $\boldsymbol{v}_j$ the inward normal vector to the j-th facet of P, which is primitive and integral, for $j = 1, \dots m$. Let c_j be a positive integer label to the j-th facet of P for each j. Set $\boldsymbol{b}_j = c_j \boldsymbol{v}_j$.

The polytope P may be described as follows by choosing suitable $p_j \in \mathbb{R}$:

$$P = \bigcap_{j=1}^{m} \{u \in M_{\mathbb{R}} \mid \langle u, \boldsymbol{b}_j \rangle \geq p_j\}. \tag{4.9}$$

If we denote (as in (1.6))

$$\ell_j(u) = \langle u, \boldsymbol{b}_j \rangle - p_j,$$

then the polytope P may be defined as

$$P = \{u \in M_{\mathbb{R}} \mid \ell_j(u) \geq 0, j = 1, \dots, m\}.$$

From a polytope P, there is a standard procedure to get a simplicial fan $\Sigma(P)$. Then the stacky fan $(\Sigma(P), \vec{\boldsymbol{b}})$ defines a toric orbifold in the sense of complex orbifolds as explained in the last subsection. In this paper we are only concerned with toric orbifolds derived from labeled polytopes.

We recall a theorem by Lerman and Tolman.

Theorem 4.4. [**LT**] *Let (M, ω) be a compact symplectic toric orbifold, with moment map $\mu_T : M \to (\mathbb{R}^n)^*$. Then $P = \mu_T(M)$ is a rational simple convex polytope. For each facet F_j of P, there exists a positive integer c_j, the label of F_j, such that the structure group of every $p \in \mu_T^{-1}(int(F_j))$ is $\mathbb{Z}/c_j\mathbb{Z}$.*

Two compact symplectic toric orbifolds are equivariantly symplectomorphic if and only if their associated labeled polytopes are isomorphic. Moreover, every labeled polytope P arises from some compact symplectic toric orbifold (M_P, ω_P).

Before we recall the explicit construction of symplectic toric orbifolds, we remark that the isotropy group of each point $p \in M_P$ can be easily seen from the polytope (Lemma 6.6 of [**LT**]): First, the points p with $\mu_T(p) \in int(P)$ have trivial isotropy group. If $\mu_T(p)$ lies in the interior of a facet F, which has a label c_F, the isotropy group is $\mathbb{Z}/c_F\mathbb{Z}$. For the points p with $\mu_T(p)$ lying in the interior of a face F, which is the intersection of facets, say $F_1, \dots F_j$, the isotropy group at p is isomorphic to A_p/A_p': Here, consider the subtorus $H_p \subset T^n$ whose Lie algebra h_p is generated by $\boldsymbol{v}_i \otimes 1 \in N_{\mathbb{R}}$ for $i = 1, \dots, j$. Let A_p be the lattice of the circle subgroups of H_p. Let A_p' be the sublattice generated by $\{c_i \boldsymbol{v}_i\}$. We remark that even when $c_1 = \cdots = c_m = 1$, there can be orbifold singularities as $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_j\}$ may not form a $\mathbb{Z}$-basis of N.

Note that the face F corresponds to a j-dimensional cone σ in the fan $\Sigma(P)$ with stacky vectors $\{c_i \boldsymbol{v}_i : i = 1, \dots, j\}$. Then the group A_p' is the same as the group $N_{\boldsymbol{b}_\sigma}$ (see Definition 4.1), and A_p is the same as $N_{\boldsymbol{b}_\sigma} \otimes_{\mathbb{Z}} \mathbb{Q}$. Therefore the isotropy group A_p/A_p' is identical to $G_{\boldsymbol{b}_\sigma}$.

We briefly recall the construction of the symplectic toric orbifold (M_P, ω_P) from the labeled simple rational polytope P.

Recall from Equation (4.1) that for the standard basis $(e_1, \dots, e_m)$ of $\mathbb{R}^m$, the map π is defined by

$$\pi : \mathbb{R}^m \to \mathbb{R}^n \ \text{ by } \ \pi(e_j) = c_j \boldsymbol{v}_j, \ j = 1, \dots, m \tag{4.10}$$

producing the following exact sequences:

$$0 \to \mathfrak{k} \xrightarrow{\iota} \mathbb{R}^m \xrightarrow{\pi} \mathbb{R}^n \to 0 \quad \text{and its dual} \quad 0 \to (\mathbb{R}^n)^* \xrightarrow{\pi^*} (\mathbb{R}^m)^* \xrightarrow{\iota^*} \mathfrak{k}^* \to 0.$$

Note that $\mathfrak{k}$ is the Lie algebra of K defined in (4.2).

Consider $\mathbb{C}^m$ with its standard symplectic form

$$\omega_0 = \frac{i}{2} \sum dz_k \wedge d\overline{z}_k.$$

The standard action of T^m on $\mathbb{C}^m$ is Hamiltonian whose moment map is given by

$$\mu_{\mathbb{C}^m}(z_1, \dots, z_m) = \frac{1}{2}(|z_1|^2, \dots, |z_m|^2).$$

Hence K acts on $\mathbb{C}^m$ with the moment map

$$\mu_K = \iota^* \circ \mu_{\mathbb{C}^m} : \mathbb{C}^m \to \mathfrak{k}^*.$$

For the constant vector $p = (p_1, \dots, p_m)$ defining the polytope (4.9), define $\pi_p^* : (\mathbb{R}^n)^* \to (\mathbb{R}^m)^*$ by $\pi_p^*(\xi) = \pi^*\xi - p$. Then,

$$\pi_p^*(P) = \{\xi \in (\mathbb{R}^m)^* | \xi \in Im(\pi_p^*) \text{ and } \xi_i \geq 0 \text{ for all } i\} \tag{4.11}$$

$$= \{\xi \in (\mathbb{R}^m)^* | \xi \in (\iota^*)^{-1}(\iota^*)(-p) \text{ and } \xi_i \geq 0 \text{ for all } i\}. \tag{4.12}$$

Then, take $X = \mu_K^{-1}(\iota^*(-p))/K$ to be the symplectic quotient, which is the desired (Kähler) toric orbifold. Since the action of T^m commutes with K, there exists an induced T^m action on X and the T^m action descends to T^m/K action on X, and provides the moment map $\mu_T = (\pi_p^*)^{-1} \circ \mu_{\mathbb{C}^m}$ on X.

5. Desingularized Maslov index formula for toric orbifolds

We first recall the Maslov index formula for holomorphic discs in toric manifolds in terms of intersection numbers.

Theorem 5.1 ([**C1**], [**CO**]). *For a symplectic toric manifold $X_{\Sigma(P)}$, let L be a Lagrangian T^n orbit. Then the Maslov index of any holomorphic disc with boundary lying on L is twice the sum of the intersection multiplicities of the image of the disc with the divisors D_j corresponding to $\boldsymbol{v}_j \in \Sigma^{(1)}$, over all $j = 1, \ldots, m$.*

Here the divisor D_j is a complex codimension one submanifold, which can be defined using the principal bundle $(U(\Sigma) \xrightarrow{\pi} X_{\Sigma(P)})$ as $D_j = \pi(\{z_j = 0\}) = \{z_j = 0\}/K_{\mathbb{C}}$. For a toric orbifold X, the divisor D_j can be defined similarly as a suborbifold of X by $D_j = \{z_j = 0\}/K_{\mathbb{C}}$.

In this section, we find a similar formula for toric orbifolds. Consider an orbi-disc $\mathcal{D}$ with interior marked points z_i^+ with orbifold singularity $\mathbb{Z}/d_i\mathbb{Z}$ where $1 \leq i \leq k$. (Here $d_i = 1$ for smooth marked points.)

Note that intersections of holomorphic orbi-discs with divisors are discrete and there are only finitely many of them because the map is holomorphic. The multiplicity of such an intersection is given by the ordinary intersection number in the uniformizing cover (or in homogeneous coordinates of $U(\Sigma)$), divided by the order of local group of the orbi-disc at the intersection point. Here is the desingularized Maslov index theorem for toric orbifolds.

Theorem 5.2. *For the symplectic toric orbifold X corresponding to $(\Sigma(P), \boldsymbol{b})$, let L be a Lagrangian T^n orbit and let $(\mathcal{D}, (z_1^+, \ldots, z_k^+))$ be an orbi-disc with $\mathbb{Z}/d_i\mathbb{Z}$ singularity at z_i^+. Consider a holomorphic orbi-disc $w : (\mathcal{D}, \partial\mathcal{D}) \to (X, L)$ intersecting the divisor D_j with multiplicity $d_{i,j}/d_i$ at each marked point z_i^+, which does not intersect divisors away from marked points. Then the desingularized Maslov index of w is given as*

$$2 \sum_i \sum_j (\lfloor d_{i,j}/d_i \rfloor).$$

Here $\lfloor r \rfloor$ denotes the largest integer equal to or less than r.

Proof. Recall that in [**C1**] and [**CO**], the Maslov index was computed as a sum of local contributions near each intersection with divisors. A similar scheme still works in this setting. The local contribution at each intersection point has been computed in Lemma 3.4. Hence it remains to show how to modify the general scheme in the setting of toric orbifolds.

Without loss of generality, we discuss what happens in the neighborhood of z_1^+ only. The point $w(z_1^+)$ may lie in the intersection of several divisors D_j. Suppose that

$$w(z_1^+) \in D_{i_1} \cap \cdots \cap D_{i_k}, \tag{5.1}$$

with no intersection with any other toric divisor. As $D_{i_1} \cap \cdots \cap D_{i_k} \neq 0$, there are lattice vectors $\boldsymbol{v}_{i_{k+1}}, \ldots, \boldsymbol{v}_{i_n}$ so that $\langle \boldsymbol{v}_{i_1}, \ldots, \boldsymbol{v}_{i_n} \rangle$ defines an n-dimensional cone σ in Σ.

We may consider the map w in a uniformizing neighborhood $U_\epsilon(z_1^+)$ of z_1^+. We consider its uniformizing cover $D_\epsilon(z_1^+) \to U_\epsilon(z_1^+)$, which is the d_1-fold branch cover branched at the origin. By the definition of an orbifold holomorphic map, we can consider its equivariant lift $\widetilde{w} : D_\epsilon(z_1^+) \to U'_\sigma$ for the uniformizing chart U'_σ, as in Lemma 4.1. The intersection multiplicity $d_{1,j}$ can be defined as the order of zero at z_1^+ of the coordinate x_j^σ in U'_σ for $1 \leq j \leq k$. As $w(z_1^+)$ do not intersect divisors corresponding to $\boldsymbol{v}_{i_{k+1}}, \ldots, \boldsymbol{v}_{i_n}$, the coordinate functions x_j^σ for $\widetilde{w}$ are non-vanishing near z_1^+ when $j \geq k+1$.

We note that this multiplicity also can be seen in the homogeneous coordinates of $\mathbb{C}^m$. From Lemma 4.1, for the dual basis $\{\boldsymbol{u}_{i_1}, \ldots, \boldsymbol{u}_{i_n}\}$ of the linearly independent vectors $\{\boldsymbol{b}_{i_1}, \ldots, \boldsymbol{b}_{i_n}\}$, the affine coordinate function x_j^σ of U'_σ is given as

$$\begin{aligned} x_j^\sigma &= z_1^{\langle \boldsymbol{b}_1, \boldsymbol{u}_{i_j} \rangle} \cdots z_m^{\langle \boldsymbol{b}_m, \boldsymbol{u}_{i_j} \rangle} \\ &= C(z) \cdot z_{i_j}^{\langle \boldsymbol{b}_{i_j}, \boldsymbol{u}_{i_j} \rangle} = C(z) \cdot z_{i_j} \end{aligned}$$

where $C(z)$ is a function nonvanishing near $\widetilde{w}(z_1^+)$. Hence the order of zero of z_{i_j} equals that of x_j^σ.

We write the lift $\widetilde{w} : D_\epsilon(z_1^+) \to U'_\sigma$ in affine coordinates as

$$(a_1 z^{c_1} + \mathcal{O}(z^{c_1+1}), \ldots, a_k z^{c_k} + \mathcal{O}(z^{c_k+1}), a_{k+1} + \mathcal{O}(z), \ldots, a_n + \mathcal{O}(z)),$$

where $z = 0$ corresponds to the point z_1^+.

The lift $\widetilde{w}$ is equivariant and hence the dominating term

$$(a_1 z^{c_1}, \ldots, a_k z^{c_k}, a_{k+1}, \ldots, a_n) \tag{5.2}$$

is also equivariant in $D_\epsilon(z_1^+)$.

Now we are in a similar situation as in the smooth case [**C1**], [**CO**] and analogously we smoothly deform the map $\widetilde{w}$ in $D_\epsilon(z_1^+)$ in an equivariant way, without changing it near the boundary of this disc, so that the deformed map $\widetilde{w}$ satisfies

$$\widetilde{w}|_{\partial D_{\epsilon/2}(z_1^+)} \subset L. \tag{5.3}$$

Let $|x_i^\sigma| = r_i,\ 1 \leq i \leq n$, be the defining equations of L in these coordinates. We can make the deformation so that the map $\widetilde{w}$ on $D_{\epsilon/2}(z_1^+)$ is given by

$$\Big(\frac{r_1 a_1 z^{c_1}}{|a_1| (\frac{\epsilon}{2})^{c_1}}, \ldots, \frac{r_k a_k z^{c_k}}{|a_k| (\frac{\epsilon}{2})^{c_k}}, \frac{r_{k+1} a_{k+1}}{|a_{k+1}|}, \ldots, \frac{r_n a_n}{|a_n|} \Big). \tag{5.4}$$

We perform the same kind of deformations for $z_2^+, z_3^+, \ldots, z_k^+$ inside the uniformizing neighborhoods $D_\epsilon(z_2^+), \ldots, D_\epsilon(z_k^+)$ for sufficiently small ϵ and write the resulting map as $\widetilde{w}'$ and the corresponding map of orbifolds as w'. Over the punctured disc

$$S = \mathcal{D} \setminus (U_\epsilon(z_1^+) \cup \cdots U_\epsilon(z_k^+)),$$

the deformed map w' does not intersect with the toric divisors, and it intersects the Lagrangian torus L along the boundaries of the punctured disc.

Lemma 5.3. *The desingularized Maslow indices of w and w' are equal to each other: $\mu^{de}(w) = \mu^{de}(w')$.*

Proof. As the desingularized complex vector bundles of w and w' will be isomorphic as a bundle pair, they have the same desingularized Maslov index. q.e.d.

Hence, it is enough to compute $\mu(w')$. Since every intersection with the toric divisors occurs inside the balls $D_{\epsilon/2}$, $w'|_S$ does not meet the toric divisors. So it can be considered as a map into the cotangent bundle of L. Therefore we have

$$\mu(w'|_S) = 0. \tag{5.5}$$

On the other hand, the Maslov index of the map $w'|_S$ is given by the sum of the Maslov indices along the components of ∂S after fixing the trivialization.

Now consider the map $w' : \mathcal{D} \to \mathcal{X}$ and the pull-back bundle w'^*TX and its desingularization $(w'^*TX)^{de}$. We fix a trivialization Φ of $(w'^*TX)^{de}$. When restricted to S, Φ gives a trivialization Φ_S of $((w'|_S)^*TX)^{de}$ restricted over S, which does not contain any orbifold point. In this trivialization, it is easy to see that the Maslov index along the boundary $\partial\mathcal{D}$ in ∂S is the desingularized Maslow index $\mu(w) = \mu(w')$. Along the rest of the boundaries $\partial U_{\epsilon/2}(z_i)$ of S, which are oriented in the opposite way, the Maslov indices equal the negatives of the local contributions of desingularized Maslov indices and hence are $-2\sum[d_{i,j}/d_i]$ for each i by Lemma 3.4. This proves the theorem. q.e.d.

6. Orbifold holomorphic discs in toric orbifolds

In this section, we classify all holomorphic discs and orbi-discs in toric orbifolds with boundary on $L(u)$. We find a one-to-one correspondence between non-trivial twisted sectors in Box' and orbifold holomorphic discs with a single interior orbifold singularity (modulo T^n-action). We also find a one-to-one correspondence between the stacky vectors $\boldsymbol{b}_j$ of the fan and smooth holomorphic discs of Maslov index two (modulo T^n-action).

These two types of discs will be called *basic discs* for simplicity: namely, Maslov index two smooth holomorphic discs and holomorphic orbi-discs having one interior orbifold singularity and desingularized Maslov index zero. Basic discs will be used to define Landau-Ginzburg potentials PO_0 and $PO^{\mathfrak{b}}_{orb,0}$, and will be used for computing Lagrangian Floer cohomology of torus fibers.

6.1. Classification theorem. We first recall the corresponding theorem for holomorphic discs in toric manifolds.

Theorem 6.1 (Classification theorem [**C1**], [**CO**]). *Let $\widetilde{L} \subset \mathbb{C}^m \setminus Z(\Sigma)$ be a fixed orbit of the real m-torus $(S^1)^m$. Any holomorphic map $w : (D^2, \partial D^2) \to (X_{\Sigma(P)}, L)$ can be lifted to a holomorphic map*

$$\widetilde{w} : (D^2, \partial D^2) \to (U(\Sigma), \widetilde{L})$$

so that each of the homogeneous coordinate functions $\widetilde{w} = (\widetilde{w}_1, \ldots, \widetilde{w}_m)$ are given by Blaschke products with constant factors.

$$i.e.\ \ \widetilde{w}_j = a_j \cdot \prod_{s=1}^{\mu_j} \frac{z - \alpha_{j,s}}{1 - \overline{\alpha}_{j,s} z}$$

where $a_j \in \mathbb{C}^*$, μ_j is a non-negative integer for each $j = 1, \ldots, m$, and $\alpha_{j,s} \in int(D^2)$. In particular, there is no non-constant holomorphic disc of non-positive Maslov index.

We start by explaining the new basic factors of holomorphic orbi-discs (in addition to the factor $\frac{z-\alpha}{1-\overline{\alpha}z}$ used in the smooth cases above).

Consider an n-dimensional stacky cone $(\sigma, \boldsymbol{b}_\sigma)$ with $\boldsymbol{b}_\sigma = \{\boldsymbol{b}_{i_1}, \ldots, \boldsymbol{b}_{i_n}\}$. Take an element $\nu = c_1 \boldsymbol{b}_{i_1} + \cdots + c_n \boldsymbol{b}_{i_n} \in N$, where $0 \leq c_j < 1$ for $j = 1, \ldots, n$. Write each rational number c_j as p_j/q_j where p_j and q_j are relatively prime integers. Let $m_1 = l.c.m.(q_1, \ldots, q_n)$ be the least common multiple of the denominators, which is the order of ν in $G_{\boldsymbol{b}_\sigma}$.

Let $\mathcal{D}$ be a disc D^2 with orbifold marked point $z_1^+ \in \mathcal{D}$ with $\mathbb{Z}/m_1$ singularity. We find an explicit formula for a holomorphic orbi-disc w from $\mathcal{D}$ such that the generator of $\mathbb{Z}/m_1$ maps to $\nu \in G_{\boldsymbol{b}_\sigma} = G_{w(z_1^+)}$. We denote by $\phi_{z_1^+} : \mathbb{Z}/m_1 \to G_{\boldsymbol{b}_\sigma}$ the injective group homomorphism sending the generator 1 to ν.

Consider the open set U'_σ and its coordinate functions $x_1^\sigma, \ldots, x_n^\sigma$. In these coordinates, choose a point $(a_1, \ldots, a_n)$ in the Lagrangian fiber L. We consider the expression

$$\big(a_1 (\frac{z - z_1^+}{1 - \overline{z}_1^+ z})^{c_1}, \ldots, a_n (\frac{z - z_1^+}{1 - \overline{z}_1^+ z})^{c_n}\big). \tag{6.1}$$

As c_i's are rational numbers, an expression such as z^{c_i} for $z \in D^2$ is not well-defined, and depends on the choice of a branch cut. But recall that $G_{\boldsymbol{b}_\sigma}$ acts on U'_σ by (4.5), and the difference from the choice of a branch cut is given by this action (see the proof of Lemma 4.3). Hence, the expression (6.1) is well-defined in $U_\sigma = U'_\sigma / G_{\boldsymbol{b}_\sigma}$. It is not hard to check that the image of $z = z_1^+$ of (6.1) has ν as a stabilizer. From (4.4), one can easily lift (6.1) to the homogeneous coordinates of the toric orbifold. This will be the new basic factor in the classification of holomorphic (orbi-)discs. This is a holomorphic orbi-disc, which is a good map.

Now, we state the classification theorem of holomorphic (orbi-)discs in toric orbifolds.

Theorem 6.2. *Let $\mathcal{X}$ be a toric orbifold corresponding to $(\Sigma(P), \boldsymbol{b})$, and L be a Lagrangian torus fiber. Let $\widetilde{L} \subset U(\Sigma)$ be the corresponding orbit of the real m-torus $(S^1)^m$. A holomorphic map $w : (\mathcal{D}, \partial\mathcal{D}) \to (\mathcal{X}, L)$ with orbifold singularity at marked points $z_1^+, \ldots, z_k^+$ can be described as follows.*

1) *For each orbifold marked point z_i^+, the map w associates to it a twisted sector X_{ν^i} where $\nu^i = \sum_j c_{ij} \boldsymbol{b}_{i_j} \in Box$.*
2) *For analytic coordinate z of $D^2 = |\mathcal{D}|$, w can be written as a map*
$$\widetilde{w} : (D^2, \partial D^2) \to (U(\Sigma)/K_{\mathbb{C}}, \widetilde{L}/K)$$
so that the homogeneous coordinates functions (modulo $K_{\mathbb{C}}$-action) $\widetilde{w} = (\widetilde{w}_1, \ldots, \widetilde{w}_m)$ are given as
$$\widetilde{w}_j = a_j \cdot \prod_{s=1}^{d_j} \frac{z - \alpha_{j,s}}{1 - \overline{\alpha}_{j,s} z} \prod_{i=1}^{k} \Big(\frac{z - z_i^+}{1 - \overline{z}_i^+ z}\Big)^{c_{ij}} \tag{6.2}$$
where $a_j \in \mathbb{C}^*$, d_j is a non-negative integer for each $j = 1, \ldots, m$, $\alpha_{j,s} \in int(D^2)$ and c_{ij}'s are rational numbers as in (1).
3) *The desingularized Maslov index of the map $\widetilde{w}$, given in (6.2), is $\sum_{j=1}^m 2d_j$. The CW Maslov index of $\widetilde{w}$ is $\sum_{j=1}^m 2d_j + 2\sum_{i=1}^k \iota_{(\nu^i)}$.*
4) *$\widetilde{w}$ is holomorphic in the sense of Definition 2.1.*

Remark 6.1. $K_{\mathbb{C}}$ is defined in (4.3). Note that the expression is not well-defined as a map to $U(\Sigma)$, since c_{ij}'s are rational numbers. But it is well-defined up to $K_{\mathbb{C}}$-action.

Proof. We first claim the above expression (6.2) defines a holomorphic map in the sense of Definition 2.1. The first factor of (6.2) is obviously holomorphic, and we may assume that the map $\widetilde{w}$ is given by
$$\widetilde{w}_j = a_j \cdot \prod_{t=1}^{k} \Big(\frac{z - z_i^+}{1 - \overline{z}_i^+ z}\Big)^{c_{ij}}. \tag{6.3}$$
Note that $(\frac{z - z_i^+}{1 - \overline{z}_i^+ z})^{c_{ij}}$ is holomorphic in D^2 away from z_i^+ in the sense of Definition 2.1. Thus, it suffices to consider the map $(\frac{z - z_i^+}{1 - \overline{z}_i^+ z})^{c_{ij}}$ near z_i^+.

Let r be the order of $\sum_j c_{ij}\boldsymbol{b}_j$, which is the least common multiple of the denominators of the rational numbers $c_{i1}, \ldots, c_{im}$. By the automorphism $\psi_{z_i^+} : D^2 \to D^2$, $\psi_{z_i^+} = \frac{z - z_i^+}{1 - \overline{z}_i^+ z}$, and its inverse $\psi_{-z_i^+}$, we may only consider the case that $z_i^+ = 0$. Then consider the branch covering map at z_i^+, $br : B_\epsilon(0) \to B_{\epsilon^r}(0)$, which is defined by $br(\widetilde{z}) = (\widetilde{z})^r$. Here, we denote the coordinate on the cover by $\widetilde{z}$ with the relation $z = \widetilde{z}^r$. Thus, it is easy to see that the map $z^{c_{ij}} = \widetilde{z}^{rc_{ij}}$ is holomorphic. Thus the lift as a map of $\widetilde{z}$ is holomorphic, as required by Definition 2.1.

Now, we prove the classification results. The idea of the proof is similar to that of [**C1**] and [**CO**]. Namely, given a holomorphic smooth or orbifold disc, we consider its intersection with toric divisors, and by dividing by the basic factors, we remove the intersection with toric divisors to obtain a map which does not intersect any toric divisors. Then, it is easy to see that the resulting smooth disc whose image lies in one of the uniformizing charts $(\mathbb{C}^n, (S^1)^n)$ of the toric orbifold has vanishing Maslov index. By classical classification of smooth holomorphic discs, it is in fact a constant map.

Let $w : (\mathcal{D}, \partial\mathcal{D}) \to (\mathcal{X}, L)$ be a holomorphic good orbi-disc. Choose an interior orbifold marked point z_i^+ with $\mathbb{Z}_{m_i}$ singularity. Denote by $\phi_{z_i^+}$ the injective group homomorphism $\mathbb{Z}_{m_i} \to G_{w(z_i^+)}$ associated to the good map w at z_i^+. Take a toric open set U_σ containing $w(z_i^+)$, and denote the stacky vectors generating σ (over $\mathbb{Q}$) by $\boldsymbol{b}_{i1}, \ldots, \boldsymbol{b}_{in}$. Then, the image of the generator under $\phi_{z_i^+}$ can be written as

$$\phi_{z_i^+}(1) =: \nu^i = c_{i1}\boldsymbol{b}_{i1} + \cdots + c_{in}\boldsymbol{b}_{in} \in N$$

with $0 \leq c_{ij} < 1$ for $j = 1, \ldots, n$. Write each c_{ij} as a rational number p_{ij}/q_{ij} with relatively prime p_{ij}, q_{ij}. Observe that as $\phi_{z_i^+}$ is injective, we have $m_i = l.c.m.(q_{i1}, \ldots, q_{in})$, which is the order of ν^i in $G_{\boldsymbol{b}_\sigma}$. For simplicity, we assume that $z_i^+ = 0 \in D^2$. Consider the branch cover $br : B_\epsilon(0) \to B_{\epsilon^r}(0)$ defined by $br(\widetilde{z}) = \widetilde{z}^{m_i}$. The map w restricted on $B_{\epsilon^{m_i}}(0)$ has a lift (by definition) $\widetilde{w} : B_\epsilon(0) \to U'_\sigma$, which is holomorphic in $\widetilde{z}$. Note that the image of $z_i^+ = 0$, $\widetilde{w}(0)$ has ν^i in its stabilizer. Hence, in terms of the coordinates $(x_1^\sigma, \ldots, x_n^\sigma)$ on U'_σ, the j-th coordinate of $\widetilde{w}(0)$ vanishes if $c_{ij} \neq 0$. We denote the vanishing order (multiplicity) of $\widetilde{w}(0)$ at the j-th coordinate by d_{ij}. (Here $d_{ij} = 0$ if it does not vanish.)

We set

$$d_{ij} = d'_{ij} m_i + r_{ij}, \text{ where } 0 \leq r_{ij} < m_i.$$

By equivariance of $\widetilde{w}$, we have

$$\frac{r_{ij}}{m_i} = c_{ij}.$$

Thus $\widetilde{w}$ can be written near 0 in these coordinates as

$$(\widetilde{z}^{d_{i1}}\widetilde{w}'_1, \ldots, \widetilde{z}^{d_{in}}\widetilde{w}'_n)$$

with $\widetilde{w}'_j(0) \neq 0$. Or, in the coordinate $z = \widetilde{z}^{m_i}$, we have

$$(z^{d'_{i1}+c_{i1}}\widetilde{w}'_1, \ldots, z^{d'_{in}+c_{in}}\widetilde{w}'_n).$$

For the general z_i^+ (when $z_i^+ \neq 0$), similarly we have

$$((\frac{z - z_i^+}{1 - \overline{z}_i^+ z})^{d'_{i1}+c_{i1}}\widetilde{w}'_1, \ldots, (\frac{z - z_i^+}{1 - \overline{z}_i^+ z})^{d'_{in}+c_{in}}\widetilde{w}'_n). \tag{6.4}$$

We multiply the reciprocals $(\frac{1-\overline{z}_i^+ z}{z - z_i^+})^{d'_{ij}+c_{ij}}$ to the above to remove the intersection with toric divisors at z_i^+. Such a multiplication can be done via toric action. Namely, from Lemma 4.3, we have a $\mathbb{C}^*$-action, corresponding to the lattice vector $-\sum_j (d'_{ij} + c_{ij})\boldsymbol{b}_j \in N$ on $\mathcal{X}$. More precisely, this action corresponds to the multiplication in (homogeneous) coordinates of $\mathbb{C}^m$ by the following expression:

$$(1, \ldots, (\frac{1 - \overline{z}_i^+ z}{z - z_i^+})^{d'_{i1}+c_{i1}}, 1, \ldots, (\frac{1 - \overline{z}_i^+ z}{z - z_i^+})^{d'_{in}+c_{in}}, 1, \ldots, 1).$$

We denote the resulting holomorphic orbi-disc by $w_1 : (\mathcal{D}', \partial\mathcal{D}') \to (\mathcal{X}, L)$, which is obtained after such multiplication where $\mathcal{D}'$ is an orbifold disc obtained from $\mathcal{D}$ by removing the orbifold marked point z_i^+.

It is easy to see that the map w_1 still satisfies the Lagrangian boundary condition, and importantly, the intersection with the toric divisor at z_i^+ has been removed.

The case of w intersecting the toric divisor at smooth point (which is not a marked point) can be done as in [**CO**] and the analogous modified map has less intersection with toric divisors. By repeating this process, we obtain a map w_d which does not meet any toric divisor. This map is now smooth, and has Maslov index 0 from the Maslov index formula of Theorem 5.2. It is easy to see that the map w_d is indeed a constant map. Thus the formula of the original map w can be written as in the statement of the theorem by tracing backwards.

The index formula (part (3)) follows from Theorem 5.2. However, a more intuitive way to think about it is as follows: Note that μ_{CW} is homotopy invariant and so is μ^{de} as long as we do not change the twisted sector data $\boldsymbol{x}$. Especially, when the disc splits into several discs, the sum of μ_{CW} remains the same. Hence, given an expression (6.2), we consider the degeneration of the holomorphic disc by sending each $\alpha_{j,s}$ to the boundary ∂D^2. In this case, a disc bubble appears, and the component $\frac{z-\alpha_{j,s}}{1-\overline{\alpha}_{j,s}z}$ disappears from (6.2). Note that if $|\alpha| = 1$, then $\frac{z-\alpha}{1-\overline{\alpha}z} = -\alpha$. The bubble is the standard Maslov index two disc, and hence has $\mu_{CW} = 2$. Similarly, we can bubble off each orbifold marked point to obtain an orbifold disc bubble, and for each z_i^+, the corresponding Chern Weil Maslov index is $\mu_{CW} = 2\iota_{(\nu^i)}$. By adding them up, we obtain (3). q.e.d.

6.2. Classification of basic discs. In this subsection, we discuss the classification of basic discs.

Now, we find holomorphic orbi-discs of desingularized Maslov index 0 with one interior orbifold marked point and show that they are in one-to-one correspondence with twisted sectors.

Corollary 6.3. *The holomorphic orbi-discs with one interior singularity and desingularized Maslov index 0 (modulo T^n-action and automorphisms of the source disc) correspond to the twisted sectors $\nu \in Box'$ of the toric orbifold.*

Proof. Let w be a holomorphic orbi-disc with one orbifold marked point $z_1^+ \in \mathcal{D}$ with $\mu^{de} = 0$. Let $\nu = \sum_j c_j \boldsymbol{b}_j$ be the element of Box associated to the pair (w, z_1^+) as in part (1) of Theorem 6.2. Injectivity of the homomorphism $\phi_{z_1^+}$ implies that $\nu \in Box'$.

By the classification theorem, w can be written as

$$(a_1 z^{c_1}, a_2 z^{c_2}, \ldots, a_m z^{c_m}).$$

And this representation is unique up to T^n-action if we impose the condition that $a_i = 1$ whenever $c_i = 0$. Conversely, given an element of Box', we can easily construct such an orbi-disc as above. q.e.d.

We give another way to understand the above correspondence between basic orbi-discs and elements of Box'. Such a holomorphic orbi-disc $w : \mathcal{D} \to \mathcal{X}$ (with orbifold marked point at $0 \in \mathcal{D}$) with desingularized Maslov index 0 has an image in a open set U_σ for some n-dimensional cone σ. For its uniformizing chart $U'_\sigma \cong \mathbb{C}^n$, w has an equivariant lift to the uniformizing charts, $\widetilde{w} : D^2 \to \mathbb{C}^n$, which may be written as

$$\widetilde{w}(\widetilde{z}) = (a'_1 \widetilde{z}^{d_1}, \ldots, a'_n \widetilde{z}^{d_n}) = (a'_1 \widetilde{z}^{c_1 m_\nu}, \ldots, a'_n \widetilde{z}^{c_n m_\nu}) \tag{6.5}$$

where each d_i is a nonnegative integer. Here D^2 is the uniformizing chart of $\mathcal{D}$ which is a branch cover of degree m_ν, the order of ν.

From the explicit expression of $\widetilde{w}$ in (6.5), note that the image of such a holomorphic orbi-disc is invariant under S^1 action. More precisely, if one defines $\mathbb{C}^*$ action by

$$t \cdot (z_1, \ldots, z_n) = (t^{d_1} z_1, \ldots, t^{d_n} z_n), \text{for } t \in \mathbb{C}^*, \tag{6.6}$$

the image of (6.5) equals the image of $\mathbb{C}^*_{\leq 1}$-action on the point $(a'_1, \ldots, a'_n) \in L$, where $\mathbb{C}^*_{\leq 1} = \{z \in \mathbb{C}^* \| z| \leq 1\}$. This exactly corresponds to Lemma 4.3 about $\mathbb{C}^*$-actions on toric orbifolds, which extend to morphisms $\mathbb{C} \to \mathcal{X}$.

Summarizing the above discussion, we have seen that the image of a basic orbi-disc corresponds to the image of a $\mathbb{C}^*_{\leq 1}$-action which extends to a morphism $\mathbb{C} \to \mathcal{X}$. Such $\mathbb{C}^*$-actions are restricted to those corresponding to elements of Box'.

Now, we consider holomorphic discs of Maslov index two without orbifold marked points. We first note that the images of maps from smooth discs can intersect fixed loci of the orbifold. The definition of an orbifold map requires that the map from a smooth disc locally lifts to a map to the uniformizing chart, and hence can intersect the fixed loci.

We also illustrate another important point by the following example: Consider an orbifold map w from orbi-disc $\mathcal{D}$ with $\mathbb{Z}/m\mathbb{Z}$ singularity in the origin to $\mathcal{D}'$ with $\mathbb{Z}/mn\mathbb{Z}$ singularity in the origin, whose lift between uniformizing covers is given by $w(z) = z^k$. Then, if $m|k$, then w may be considered as a smooth disc $w' : D^2 \to \mathcal{D}'$ with the lifted map $\widetilde{w}' : D^2 \to D^2$ as given by $\widetilde{w}'(z) = z^{k/m}$.

Hence, given an orbifold holomorphic map $f : \mathcal{D} \to \mathcal{X}$, and a local lift $\widetilde{f}$, the related group homomorphism sometimes cannot be injective, if $\widetilde{f}$ has high multiplicities. In such a case, the orbifold structure of $\mathcal{D}$ has to be (and can be) replaced by less singular or sometimes smooth ones. The correspondence below is best understood in this sense.

Corollary 6.4. *The (smooth) Maslov index two holomorphic discs (modulo T^n-action) are in one-to-one correspondence with the stacky vectors $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_m$.*

Proof. This follows directly from the classification theorem. Namely, let $w : D^2 \to \mathcal{X}$ be a smooth holomorphic disc of Maslov index two. From the classification theorem, up to automorphism of D^2, such a holomorphic disc is given by $(a_1, \ldots, z, \ldots, a_m)$ in $\mathbb{C}^m$. In the form of expression (6.5), this corresponds to the case that $c_j = 1, m_\nu = 1$, and all the other $c_i = 0$ for $i \neq j$. This implies the corollary. q.e.d.

7. Areas of holomorphic orbi-discs

In this section, we compute the area of holomorphic orbi-discs. The method to compute them is somewhat different from that of [**CO**] and is more elementary.

We first illustrate how the moment map measures the area of a standard orbifold disc. Let $D := D^2 \subset \mathbb{C}$ be the standard disc with the standard symplectic structure. Let $\mathcal{D}$ be the orbifold disc obtained as the quotient orbifold $[D^2/(\mathbb{Z}_n)]$ where the generator $1 \in \mathbb{Z}_n$ acts on D^2 by multiplication of a primitive n-th root of unity. D and $\mathcal{D}$ have the following S^1-actions. Let $t \in S^1$ and $z \in \mathcal{D}$. Let $w \in D^2$ be the coordinate on the uniformizing cover of $\mathcal{D}$. Then the actions are

$$t \cdot z = tz, \quad t \cdot w = t^{1/n} w.$$

Note that the S^1 action is not well-defined on the uniformizing cover D^2, but well-defined on the quotient orbifold $\mathcal{D}$. If we compute the moment maps for D and $\mathcal{D}$, the length of the moment map image of D is n-times the length of the moment map image of $\mathcal{D}$. This is because the vector fields generated by S^1-actions have such a relation. Also, we point out that the symplectic area of D is also n-times the symplectic area of $\mathcal{D}$. In general, the area of a holomorphic orbi-disc w with one interior singular point can be obtained by taking the symplectic area of the lift $\widetilde{w} : D^2 \to U'_\sigma$ and dividing it by the order of orbifold singularity of $\mathcal{D}$.

Recall that symplectic areas are topological invariants. Hence, it is enough to find symplectic areas of generators of $H_2(X, L)$. From Lemma 9.1, it is enough to find symplectic areas of the basic discs. We denote the homology class of a disc corresponding to $\boldsymbol{b}_i$ (resp. $\nu \in Box'$) by β_i (resp. β_ν). Note that for $\nu \in Box'$, if we have $\nu = c_1\boldsymbol{b}_{i_1} + \ldots + c_n\boldsymbol{b}_{i_n}$, then the symplectic area for β_ν is given as the same linear combination of the symplectic areas of β_{i_j}'s. Thus, it suffices to find symplectic areas of β_{i_j}'s, which are those of smooth holomorphic discs corresponding to stacky vectors.

Recall that symplectic form on the toric orbifold is obtained from the standard symplectic form of $\mathbb{C}^m$ via symplectic reduction. The strategy is to find a lift of the holomorphic map to $U(\Sigma) \subset \mathbb{C}^m$ and compute the area there using the standard symplectic form.

As in the classification theorem, the smooth holomorphic discs which are basic can be easily obtained as follows. For simplicity, we state it for β_1. Let $\widetilde{w}_1 : D^2 \to \mathbb{C}^m$ be a map given by

$$\widetilde{w}_1(z) = (a_1 z, a_2, \ldots, a_m),$$

where $(a_1, \ldots, a_m) \in \widetilde{L}$ as in Theorem 6.2. Then if we compose it with the projection $\pi : U(\Sigma) \to X$, we have $w_1 = \pi \circ \widetilde{w}_1 : (D^2, \partial D^2) \to (X, L)$, which defines a smooth holomorphic disc of homology class β_1.

Consider $u = (u_1, \ldots, u_n) \in (\mathbb{R}^n)^*$. If L is defined by $\mu_T^{-1}(u)$, then, considering the map $\pi_p^* : (\mathbb{R}^n)^* \to (\mathbb{R}^m)^*$ defined by $\pi_p^*(\xi) = \pi^*\xi - p$, the image of $\widetilde{L}$ under the map $\mu_{\mathbb{C}^m} : \mathbb{C}^m \to (\mathbb{R}^m)^*$ corresponds to the point $\pi_p^*(u)$. In fact, $\pi_p^*(u)$ is given by

$$(\langle u, \boldsymbol{b}_1\rangle - p_1, \ldots, \langle u, \boldsymbol{b}_m\rangle - p_m) = (\ell_1(u), \ldots, \ell_m(u)).$$

But recall that for the standard moment map, the j-th coordinate of $\mu_{\mathbb{C}^m}$ is given by $|z_j|^2/2$. Hence, with the standard symplectic form, the symplectic area of the lift of $\widetilde{w}_j$ in $\mathbb{C}^m$ is just πr^2, which is $2\pi(\ell_j(u))$. Hence the area of w_j is given by $2\pi\ell_j(u)$.

In fact, due to the difference of complex and symplectic construction of toric orbifolds, we also need the following argument in the above computation. Note that the holomorphic disc $\widetilde{w}$ does not exactly lie on the level set $\mu_K^{-1}(\iota^*(-p))$ for the symplectic quotient. In fact, when we say holomorphic disc $\widetilde{w}$ in symplectic orbifold, we mean the following deformed disc which lies in the level set $\mu_K^{-1}(\iota^*(-p))$: From a general argument due to Kirwan [**Ki**], one can consider negative gradient flow of the function $||\mu_K - \iota^*(-p)||^2$ inside $U(\Sigma) = \mathbb{C}^m \setminus Z(\Sigma)$. Negative gradient flow will reach critical points, and in this case the only critical point set is $\mu_K^{-1}(\iota^*(-p))$. As the torus $\widetilde{L}$ already lies in the level set, points on $\widetilde{L}$ do not move under the homotopy. Thus, given a holomorphic disc in $\widetilde{w}$, it can be flowed into $\mu_K^{-1}(\iota^*(-p))$ with boundary image fixed, which gives precisely the holomorphic disc in the symplectic quotient. Then, the simple argument using Stoke's theorem tells us that the symplectic area of the corresponding disc obtained by flowing to the level set $\mu_K^{-1}(\iota^*(-p))$ is the same as that of $\widetilde{w}$. This proves the desired result.

By adding up homology classes, we obtain

Lemma 7.1. *For a smooth holomorphic disc of homotopy class β_j, its symplectic area is given by $2\pi\ell_j$.*

For a lattice vector $\nu = c_1\boldsymbol{b}_{i_1} + \ldots + c_n\boldsymbol{b}_{i_n}$, define

$$\ell_\nu = \sum_{j=1}^{n} c_i \ell_{i_j}. \tag{7.1}$$

Then the area of the holomorphic orbi-disc corresponding to ν is given by $2\pi\ell_\nu(u)$.

8. Fredholm regularity

In this section, we justify the use of the standard complex structure in the computation of Floer cohomology in this paper.

8.1. The case of smooth holomorphic discs in toric orbifolds. The first author, with Yong-Geun Oh, has shown the following Fredholm regularity results for toric manifolds:

Theorem 8.1. [**C1**], [**CO**] *Non-singular holomorphic discs of a toric manifold M with boundary on L are Fredholm regular, i.e. linearization of the $\overline{\partial}$ operator at each map is surjective.*

This implies that the moduli spaces of holomorphic discs (before compactification) are smooth manifolds of expected dimensions. Since the standard complex structure is integrable, the linearized operator D_w for a holomorphic disc w is complex linear and exactly the Dolbeault derivative $\overline{\partial}$.

We briefly recall the main arguments for the proof of regularity in [**CO**]. The exact sequence (4.1) induces the exact sequence of complex vector spaces

$$0 \to \mathbb{C}^{\mathfrak{k}} \to \mathbb{C}^m \xrightarrow{\pi} \mathbb{C}^n \to 0 \tag{8.1}$$

via tensoring with $\mathbb{C}$ where $\mathbb{C}^{\mathfrak{k}}$ is the $m-n$ dimensional subspace of $\mathbb{C}^m$ spanned by $\mathfrak{k} \subset \mathbb{R}^m$. Note that this exact sequence is equivariant under the natural actions by the associated complex tori.

Given a holomorphic disc $w : (D^2, \partial D^2) \to (M, L)$, denote

$$E = w^*TM, \quad F = (\partial w)^* TL.$$

Using the sheaf of local holomorphic sections of the bundle pair (E, F), one can define the sheaf cohomology group $H^q(D^2, \partial D^2; E, F)$. Note that the surjectivity of the linearization of w is equivalent to the vanishing result

$$H^1(D^2, \partial D^2; E, F) = \{0\}. \tag{8.2}$$

Denote by $\widetilde{w} : (D^2, \partial D^2) \to (\mathbb{C}^m, \widetilde{L})$ the lifting of w, whose boundary lies on

$$\widetilde{L} = (S^1)^m \cdot (c_1, \ldots, c_m) \subset \pi^{-1}(L) \subset \mathbb{C}^m.$$

We denote by

$$\begin{array}{rcl} (E,F) & = & (w^*TM, (\partial w)^*TL) \\ (\widetilde{E}, \widetilde{F}) & = & (D^2 \times \mathbb{C}^m, (\partial \widetilde{w})^*(T\widetilde{L})) \\ (E_{\mathfrak{k}}, F_{\mathfrak{k}}) & = & ((\widetilde{w})^*(TOrb_{\mathbb{C}^{\mathfrak{k}}}), (\partial \widetilde{w})^*(TOrb_{\mathfrak{k}})) \end{array}$$

and by

$$(\mathcal{E}, \mathcal{F}), \quad (\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}), \quad (\mathcal{E}_{\mathfrak{k}}, \mathcal{F}_{\mathfrak{k}})$$

the corresponding sheaves of local holomorphic sections.

Lemma 8.2 ([**CO**], Lemma 6.3)**.** *The natural complex of sheaves*

$$0 \to (\mathcal{E}_{\mathfrak{k}}, \mathcal{F}_{\mathfrak{k}}) \to (\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}) \to (\mathcal{E}, \mathcal{F}) \to 0 \tag{8.3}$$

is exact.

In [**CO**], Lemma 6.4, the vanishing $H^1(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}) = 0$ is proved by checking the Fredholm regularity of the trivial bundle pair. The above exact sequence then proves the desired Fredholm regularity for holomorphic discs for the case of toric manifolds.

Now, consider the case of smooth holomorphic discs in toric orbifolds. Note that the exact sequence (8.1) remains true in the case of toric orbifolds. For smooth discs in orbifolds, the pull-back bundle is a smooth vector bundle. We have shown in section 6 that smooth holomorphic discs admit holomorphic liftings to $\mathbb{C}^m$. Thus, exactly the same argument as in the case of manifolds proves the following:

Proposition 8.3. *Smooth (non-singular) holomorphic discs of a toric orbifold with boundary on L are Fredholm regular.*

8.2. The case of orbi-discs. We only discuss the case of holomorphic orbi-discs with one interior orbifold marked point. In the case of several orbifold marked points, stable map compactifications may contain constant orbi-sphere bubbles, which may not be Fredholm regular; hence one may expect that the moduli spaces of orbi-discs with several orbifold marked points are in general obstructed. But for the orbi-discs in the classification theorem whose domains do not have nodal singularity, we conjecture that they are Fredholm regular, yet we do not know how to prove it in this generality. (One can check that at least the dimension matches with the expected dimension.)

Suppose $\mathcal{D}$ is an orbifold disk D^2 with $\mathbb{Z}_m$ orbifold singularity at the origin, and boundary $\partial\mathcal{D}$. For a good orbifold map $w : (\mathcal{D}, \partial\mathcal{D}) \to (\mathcal{X}, L)$ to a toric orbifold, $\boldsymbol{E} = w^*T\mathcal{X}$ defines an orbifold holomorphic vector bundle with $F = (\partial w)^*TL$ a Lagrangian subbundle along the boundary. Namely, if we let $\pi : D^2 \to \mathcal{D}$ be its uniformizing chart, then the vector bundle $\boldsymbol{E}$ may be understood as a holomorphic vector bundle $E \to D^2$ with effective $\mathbb{Z}_m$ action on E, which acts linearly on the fibers. In addition, $F|_{\partial D^2} \subset E|_{\partial D^2}$ have induced $\mathbb{Z}_m$ action from E.

Denote by $\mathcal{E}$ (resp. $(\mathcal{E}, \mathcal{F})$) the sheaf of local holomorphic sections of E over D^2 (resp. with values in F on ∂D^2). Denote by $\mathcal{E}^{inv}$ (resp. $(\mathcal{E}, \mathcal{F})^{inv}$) the sheaf of local holomorphic sections of $\boldsymbol{E}$ over $\mathcal{D}$ (resp. $(\boldsymbol{E}, F)$ over $(\mathcal{D}, \partial\mathcal{D})$), which by definition is the sheaf of local holomorphic invariant sections of $E \to D^2$ (resp. $(E, F) \to (D^2, \partial D^2)$) under $\mathbb{Z}_m$-action.

Lemma 8.4. *Suppose $\mathcal{E}$ has a fine resolution*

$$0 \to \mathcal{E} \to \mathcal{H}_0 \xrightarrow{h} \mathcal{H}_1 \to 0,$$

where $\mathcal{H}_i$ $(i = 0, 1)$ are given an effective $\mathbb{Z}_m$ action so that all arrows are equivariant maps.

Then, $\mathcal{E}^{inv}$ also admits a fine resolution

$$0 \to \mathcal{E}^{inv} \to \mathcal{H}_0^{inv} \xrightarrow{h} \mathcal{H}_1^{inv} \to 0.$$

Analogous statements for $(\mathcal{E}, \mathcal{F})$ also hold true.

Proof. This is a standard fact, since taking invariants is an exact functor up to torsion. But we give a proof of it for the reader's convenience in the case of $\mathcal{E}$. First we recall that any open cover of an orbifold consisting of uniformized open subsets admits a partition of unity on X subordinate to it ([**CR**], Lemma 4.2.1). Hence, if $\mathcal{H}_i$ is a fine sheaf, then $\mathcal{H}_i^{inv}$ is also a fine sheaf. The resulting complex is exact: The injectivity of the first arrow is obvious. To prove the surjectivity of the last arrow, first take a preimage in $\mathcal{H}_0$, and its average over $\mathbb{Z}_m$ action still maps to the same element due to equivariance of the map. The exactness in the middle can be proved similarly. q.e.d.

Now, sheaf cohomology of $\mathcal{E}^{inv}$ over $\mathcal{D}$, or $(\mathcal{E}, \mathcal{F})^{inv}$ over $(\mathcal{D}, \partial\mathcal{D})$, can be introduced by taking a global section functor as before. Then the above lemma on taking invariant functor implies the following lemma:

Lemma 8.5. *We have*

$$H^0(\mathcal{D}, \mathcal{E}^{inv}) = H^0(D^2, \mathcal{E})^{inv}, H^1(\mathcal{D}, \mathcal{E}^{inv}) = H^1(D^2, \mathcal{E})^{inv},$$
$$H^0(\mathcal{D}, \partial\mathcal{D}; (\mathcal{E}, \mathcal{F})^{inv}) = H^0(D^2, \partial D^2; \mathcal{E}, \mathcal{F})^{inv},$$

$$H^1(\mathcal{D}, \partial\mathcal{D}; (\mathcal{E}, \mathcal{F})^{inv}) = H^1(D^2, \partial D^2; \mathcal{E}, \mathcal{F})^{inv}.$$

In particular, if $H^1(D^2, \mathcal{E}) = 0$, then $H^1(\mathcal{D}, \mathcal{E}^{inv}) = 0$ also.

Now, this enables us to prove the regularity for basic orbifold discs with only one singular point in the interior, by using the results of the first author and Oh on the Fredholm regularity of holomorphic discs. Namely, given an orbifold holomorphic disc $w : (\mathcal{D}, \partial\mathcal{D}) \to (\mathcal{X}, L)$, by definition, we have a lift $\widetilde{w} : (D^2, \partial D^2) \to (\mathcal{X}, L)$, which defines a smooth holomorphic disc to a toric orbifold. From the Fredholm regularity of smooth holomorphic discs in the previous section, we thus have the vanishing of $H^1(D^2, \partial D^2; \mathcal{E}, \mathcal{F})$, which implies $H^1(\mathcal{D}, \partial\mathcal{D}; (\mathcal{E}, \mathcal{F})^{inv}) = 0$. This proves:

Proposition 8.6. *Basic holomorphic (orbi-)discs are Fredholm regular.*

9. Moduli spaces of basic holomorphic discs in toric orbifolds

In this section, we find properties of moduli spaces of basic holomorphic (orbi-)discs.

9.1. Homology classes in $H_2(X, L; \mathbb{Z})$. For a toric manifold Q and a Lagrangian torus fiber L, recall that we have the exact sequence

$$0 \to Ker(\pi) \to \mathbb{Z}^m \xrightarrow{\pi} \mathbb{Z}^n \to 0,$$

where π sends the standard generator e_i of $\mathbb{Z}^m$ to $\boldsymbol{v}_i$. This exact sequence is isomorphic to the homotopy (or homology) exact sequence ([**FOOO2**]) and in particular the lattice N may be identified with $H_1(L; \mathbb{Z})$.

$$0 \to \pi_2(Q) \to \pi_2(Q, L) \to \pi_1(L) \to 0 \tag{9.1}$$

$$0 \to H_2(Q; \mathbb{Z}) \to H_2(Q, L; \mathbb{Z}) \to H_1(L; \mathbb{Z}) \to 0 \tag{9.2}$$

One may make an identification of the lattice N with $H_1(L; \mathbb{Z})$.

For a toric orbifold X, the situation is more complicated. For example, the natural map $\pi : \mathbb{Z}^m \to \mathbb{Z}^n$ sending e_i to $\boldsymbol{b}_i$ is not surjective in general but only $\pi \otimes 1_{\mathbb{Q}} : \mathbb{Q}^m \to \mathbb{Q}^n$ is surjective, and also $\pi_2(X, L)$ has additional classes corresponding to orbifold discs.

First, we consider the case of a stacky n-dimensional cone. Let $(\sigma, \boldsymbol{b}_\sigma)$ be an n-dimensional stacky cone with stacky vectors $\boldsymbol{b}_\sigma = \{\boldsymbol{b}_{i_1}, \ldots, \boldsymbol{b}_{i_n}\}$ where $\boldsymbol{b}_{i_j}$'s lie on one-dimensional cones of σ. Denote by $N_{\boldsymbol{b}_\sigma}$ the sublattice of N generated by stacky vectors $\boldsymbol{b}_\sigma$. Denote $N/N_{\boldsymbol{b}_\sigma}$ by $G_{\boldsymbol{b}_\sigma}$ as before. Denote by L a non-singular torus fiber.

We compute $H_2(X_{\sigma,\boldsymbol{b}_\sigma}, L; \mathbb{Z})$ where $X_{\sigma,\boldsymbol{b}_\sigma}$ is the underlying quotient space. Here, L may be replaced by $(\mathbb{C}^*)^n$, which is the semi-free orbit of $X_{\sigma,\boldsymbol{b}_\sigma}$. Since σ is an n-dimensional cone, it is easy to observe that

$$\pi_1(X_{\sigma,\boldsymbol{b}_\sigma}) = \pi_2(X_{\sigma,\boldsymbol{b}_\sigma}) = 0, \; \pi_1(X_{\sigma,\boldsymbol{b}_\sigma}, (\mathbb{C}^*)^n) = 0.$$

Thus in this case,

$$H_1(X_{\sigma,\boldsymbol{b}_\sigma}) = H_2(X_{\sigma,\boldsymbol{b}_\sigma}) = 0,\ H_1(X_{\sigma,\boldsymbol{b}_\sigma}, (\mathbb{C}^*)^n) = 0.$$

From the homotopy exact sequence and the Hurewicz theorem, we have

$$\pi_2(X_{\sigma,\boldsymbol{b}_\sigma}, (\mathbb{C}^*)^n) \cong \pi_1((\mathbb{C}^*)^n) \cong \mathbb{Z}^n \cong H_2(X_{\sigma,\boldsymbol{b}_\sigma}, (\mathbb{C}^*)^n; \mathbb{Z}) \cong H_1(L;\mathbb{Z}).$$

In fact, we can find generators of the above explicitly. Elements of $\mathbb{Z}^n$ above correspond to points of the lattice N. Finding generators of $\mathbb{Z}^n$ corresponds to finding that of the lattice N.

In the previous sections, we have found holomorphic discs corresponding to the stacky vectors $\boldsymbol{b}_\sigma = \{\boldsymbol{b}_{i_1}, \dots, \boldsymbol{b}_{i_n}\}$. We denote the homology class of a disc corresponding to $\boldsymbol{b}_i$ by β_i. Also, we have found holomorphic orbi-discs corresponding to elements of Box', and we denote the homology class of a disc corresponding to $\nu \in Box'$ by β_ν.

The lattice N is generated by stacky vectors in $\boldsymbol{b}_\sigma$ together with $Box_{\boldsymbol{b}_\sigma}$. Thus $H_2(X_{\sigma,\boldsymbol{b}_\sigma}, L : \mathbb{Z})$ is generated by β_i's and β_ν's. These correspond to the basic discs explained earlier.

In the general case of toric orbifolds, by applying the Mayer-Vietoris sequence of a pair, we obtain the following result.

Lemma 9.1. *For a toric orbifold X, and a Lagrangian torus fiber L, $H_2(X, L;\mathbb{Z})$ is generated by the homology classes of basic discs, β_i for $i = 1, \dots, m$ together with β_ν for $\nu \in Box'$.*

We have the following short exact sequence:

$$0 \to \pi_2(X_{\Sigma,\boldsymbol{b}}) \to \pi_2(X_{\Sigma,\boldsymbol{b}}, L) \to \pi_1(L) \to 0,$$

and from the fact that the map $H_2(L) \to H_2(X)$ is trivial, the five lemma gives

$$\pi_2(X, L) \cong H_2(X, L;\mathbb{Z}).$$

Thus, $\pi_2(X_{\Sigma,\boldsymbol{b}}, L)$ is generated by homotopy classes of smooth and orbifold holomorphic discs (or that of basic discs) and elements of $\pi_2(X_{\Sigma,\boldsymbol{b}})$ correspond to homotopy classes of orbi-spheres in toric orbifolds.

The following lemma (based on ideas on page 48 of [**Ful**]) shows that for an n-dimensional stacky cone, we can choose exactly n holomorphic (orbi)discs which generate $H_2(X_{\sigma,\boldsymbol{b}_\sigma}, L;\mathbb{Z})$.

Lemma 9.2. *Let σ be any n-dimensional simplicial rational polyhedral cone in $\mathbb{R}^n$. Then we can find an integral basis of the lattice $N = \mathbb{Z}^n$, all of whose vectors lie in σ.*

Proof. Let σ be an n-dimensional simplicial cone with primitive integral generators $v_1, \dots, v_n$ of its one-dimensional faces. Let N_σ be the submodule of N generated by $v_1, \dots, v_n$. Let $G_\sigma = N/N_\sigma$. Since σ is simplicial, N_σ has rank n and G_σ is finite. Let $\mathrm{mult}(\sigma) = o(G_\sigma)$.

Let $B = [v_1 \dots v_n]$ be the matrix with the v_i's as columns. Consider B as a linear operator $B : N_\sigma \to N$ and G_σ as the cokernel of B. Then from the Smith normal form of B and the corresponding decomposition of the finite abelian group G_σ into a direct product of cyclic groups, we conclude that $\mathrm{mult}(\sigma) = |\det(B)|$.

If $\mathrm{mult}(\sigma) = 1$, then we are done as $v_1, \ldots, v_n$ form a basis of N in this case. Assume $\mathrm{mult}(\sigma) > 1$. Then there exists an integral vector $v \in N$ which does not belong to N_σ. Therefore $v = \sum_{i=1}^n t_i v_i$ where not every t_i is an integer. By adding suitable integral multiples of the v_i's to v, we may assume that each $t_i \in [0,1)$ and not every t_i is zero. Without loss of generality assume that $1, \ldots, k$ are the values of i for which $t_i \neq 0$. Then v belongs to the relative interior of the face of σ generated by $v_1, \ldots, v_k$. Suppose that v/d is a primitive integral vector, where d is a positive integer.

We subdivide the cone σ into n-dimensional cones σ_i, $1 \leq i \leq k$. Here σ_i is generated by $\{v_1, \ldots, \widehat{v}_i, \ldots, v_n, v/d\}$. It is easy to check using determinants that $\mathrm{mult}(\sigma_i) = \frac{t_i}{d}\mathrm{mult}(\sigma)$. Therefore $\mathrm{mult}(\sigma_i) < \mathrm{mult}(\sigma)$. Note that the generators of one-dimensional faces of σ_i belong to $\sigma \cap N$.

Iterating the above process (if necessary), we obtain an n-dimensional cone τ having multiplicity one whose one-dimensional generators belong to $\sigma \cap N$. These generators give the required basis of N. q.e.d.

Here it is important that the basic lattice vectors lie in the cone σ, since then they correspond to holomorphic (orbi-)discs in $X_{\sigma, \boldsymbol{b}_\sigma}$.

9.2. Moduli spaces of smooth holomorphic discs. In this subsection, we discuss the moduli spaces of holomorphic discs *without interior orbifold marked points*.

Recall from Corollary 6.4 that we have a one-to-one correspondence between stacky vectors $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_m\}$ and smooth holomorphic discs of Maslov index two (modulo the T^n-action).

We denote by $\beta_i \in H_2(X, L(u); \mathbb{Z})$ $(i = 1, \ldots, m)$ the homology class of discs corresponding to $\boldsymbol{b}_i$. Note that we have $\mu(\beta_i) = 2$, and the intersection number of β_i with the j-th toric divisor is 1 if $i = j$ and 0 otherwise. (Here the intersection number may be counted either in the uniformizing chart or in $\mathbb{C}^m$.)

For each $\beta \in H_2(X, L; \mathbb{Z})$, consider the moduli space $\mathcal{M}^{main}_{k+1,0}(L(u), \beta)$ of stable maps from bordered genus zero Riemann surfaces with $k+1$ boundary marked points of homotopy class β. We denote by $\mathcal{M}^{main,reg}_{k+1,0}(L(u), \beta)$ its subset whose domain is a single disc. For the orientation of the moduli spaces, we use the spin structure of $L(u)$, which is induced from the T^n-action; it is the same as the case of toric manifolds (see [**C1**], [**CO**], and [**FOOO**] for more details).

In the following proposition, we do not consider interior marked points. Hence only holomorphic discs without orbifold marked points are allowed, and the Maslov index μ is defined as usual. We also emphasize that the moduli spaces discussed here are not perturbed.

Proposition 9.3. *Let β be a homology class in $H_2(X, L(u); \mathbb{Z})$.*

1) *The moduli space $\mathcal{M}^{main,reg}_{k+1,0}(L(u), \beta)$ is Fredholm regular for any β. Moreover, evaluation map*

$$ev_0 : \mathcal{M}^{main,reg}_{k+1,0}(L(u), \beta) \to L(u) \tag{9.3}$$

is submersion.

2) *For β with $\mu(\beta) < 0$, or $\mu(\beta) = 0, \beta \neq 0$, $\mathcal{M}^{main,reg}_{k+1,0}(L(u), \beta)$ is empty.*
3) *$\mathcal{M}^{main,reg}_{k+1,0}(L(u), \beta)$ is empty if $\mu(\beta) = 2$ and $\beta \neq \beta_1, \ldots, \beta_m$.*
4) *If $\mathcal{M}^{main}_{k+1,0}(L(u), \beta)$ is non-empty, then there exist $k_i \in \mathbb{Z}_{\geq 0}$ and $\alpha_j \in H_2(X; \mathbb{Z})$ such that*

$$\beta = \sum_i k_i \beta_i + \sum_j \alpha_j \tag{9.4}$$

and α_j is the homology class of a holomorphic sphere. If $\beta \neq 0$, at least one k_i is non-zero.

5) *For each $i = 1, \ldots, m$, we have*

$$\mathcal{M}^{main,reg}_{1,0}(L(u), \beta_i) = \mathcal{M}^{main}_{1,0}(L(u), \beta_i). \tag{9.5}$$

Hence, the moduli space $\mathcal{M}^{main}_{1,0}(L(u), \beta_i)$ is Fredholm regular and the evaluation map ev_0 becomes an orientation preserving diffeomorphism.

Proof. The proof follows from the classification theorem in section 6, in the same way as Theorem 11.1 of [**FOOO2**] follows from the classification theorem of [**CO**]. For (1), Fredholm regularity for holomorphic discs was proved in Proposition 8.3. The evaluation map ev_0 is a submersion since T^n acts on $L(u)$ and the moduli spaces in such a way that ev_0 becomes a T^n-equivariant map.

For (2), if $\mathcal{M}^{main,reg}_{1,0}(L(u), \beta)$ is non-empty, then since ev_0 is a submersion, we have

$$\dim \mathcal{M}^{main,reg}_{1,0}(L(u), \beta) = n + \mu(\beta) - 2 \geq n$$

for $\beta \neq 0$. This implies that $\mu(\beta) \geq 2$.

(3) is a direct consequence of the classification theorem.

For (4), consider a map $[h] \in \mathcal{M}^{main}_{k+1,0}(L(u), \beta)$. If the domain of h is a single disc, then the statement follows from the classification theorem, in which case $\alpha_j = 0$. In general, the domain of h is decomposed into irreducible components, which are discs or spheres. As holomorphic discs are already classified, the claim follows.

For (5), observe that the first statement can be proved as in [**FOOO2**]. Let $[h] \in \mathcal{M}^{main}_{1,0}(L(u), \beta_{i_0})$. By (4), we can write

$$\beta_{i_0} = \sum_i k_i \beta_i + \sum_j \alpha_j, \quad \partial \beta_{i_0} = \sum_i k_i \partial \beta_i. \tag{9.6}$$

We need to show that there exists no sphere bubble α_j and $k_i = 0$ if $i \neq i_0$ and $k_{i_0} = 1$. Since the symplectic area $\alpha_j \cap \omega > 0$, it follows that

$$\beta_{i_0} \cap \omega \geq \sum_i k_i \beta_i \cap \omega.$$

It suffices to show that

$$\beta_{i_0} \cap \omega \leq \sum_i k_i \beta_i \cap \omega,$$

and that equality holds only if $k_i = 0$ if $i \neq i_0$ and $k_{i_0} = 1$.

From (1.6) and from the second equation of (9.6), we have

$$\ell_{i_0}(u) = \sum_{i=1}^{m} k_i \ell_i(u) + c$$

for some constant c. This is because $\partial(\beta_{i_0} - \sum_i k_i \beta_i) = 0$, and hence its symplectic area $2\pi c$ is independent of u.

By evaluating at $u \in \partial_{i_0} P$, we have $c \leq 0$, since $\ell_i \geq 0$ on P. But since $\ell_i(u) = \beta_i \cap \omega$, this implies the desired inequality. Therefore, the equality $\ell_{i_0} = \sum_i k_i \ell_i$ holds. If there exists $i \neq j$ with $k_i, k_j > 0$, then since $u' \in P$ satisfies $u' \in \partial_j P$ if $\ell_j(u') = 0$, the above equality implies that $\partial_{i_0} P \subset \partial_i P \cap \partial_j P$, which is a contradiction since $\partial_{i_0} P$ is codimension one.

The second statement of (5) follows from the torus action and the orientation analysis of [**C1**], as in the case of smooth toric manifolds. But there is a little subtlety, which is different from the manifold case, which we now explain.

Given a smooth holomorphic disc $w : (D^2, \partial D^2) \to (\mathcal{X}, L)$, with marked point $z_0 \in \partial D^2$, the equivalence relation (Definition 2.6) implies that if an automorphism of the disc $\rho : D^2 \to D^2$ satisfies $w \circ \rho = w$, then the holomorphic disc $((D^2, z_0), w)$ is identified with $((D^2, \rho(z_0)), w)$.

We illustrate this phenomenon by an example, which explains what happens for a basic smooth disc. Consider a map $w : (D^2, \partial D^2) \to \mathbb{C}^m/G$ given by $z \mapsto (z, 1, \ldots, 1)$, where G is a finite group $G = \mathbb{Z}/k\mathbb{Z}$ acting by rotation on the first coordinate of $\mathbb{C}^m$ (so that the image of w is invariant under the G-action). Denote by ρ the multiplication by the k-th root of unity on D^2. Then, $w \circ \rho = w$ as w is a map to the quotient space. Hence, the marked point z_0 and $\rho(z_0)$ are identified.

Hence in the moduli space of smooth holomorphic discs containing the above map w, we may regard that the marked point z_0 moves only along the arc from 1 to $e^{2\pi i/k}$ of ∂D^2. The smooth disc wraps around the orbifold point with multiplicity k in the above example, but due to the identification of the boundary marked points as above, the evaluation image of ev_0 of the moduli space of discs only covers the boundary once. The rest is the same as in [**FOOO2**], and we leave the details to the reader. q.e.d.

9.3. Moduli spaces of holomorphic orbi-discs. In this subsection, we allow interior marked points, and in particular, interior orbifold marked points. Let $\beta \in H_2(X, L; \mathbb{Z})$ and let $\mathcal{M}^{main}_{k+1,l}(L(u), \beta, \boldsymbol{x})$ be the moduli space of good representable stable maps from bordered orbifold Riemann surfaces of genus zero with $k+1$ boundary marked points, and l interior (orbifold) marked points in the homology class β of type $\boldsymbol{x}$ where $\boldsymbol{x} = (\mathcal{X}_{(g_1)}, \ldots, \mathcal{X}_{(g_l)})$. We denote

$$\mathcal{M}^{main}_{k+1,l}(L(u), \beta) = \bigsqcup_{\boldsymbol{x}} \mathcal{M}^{main}_{k+1,l}(L(u), \beta, \boldsymbol{x}).$$

The problem of orientation of the moduli spaces is similar to that of the smooth discs, and we omit the details.

In Corollary 6.3, we found a one-to-one correspondence (modulo the T^n-action) between elements $\nu \in Box'$ and holomorphic orbi-discs with one orbifold marked point that satisfy $\mu^{de} = 0$. We have denoted the homotopy class of such orbi-discs by $\beta_v \in H_2(X, L; \mathbb{Z})$. In particular, such $\nu \in Box'$ can be written as $\nu = c_{i_1}\boldsymbol{b}_{i_1} + \ldots + c_{i_n}\boldsymbol{b}_{i_n} \in N$ with $0 \le c_{i_j} < 1$. Then it is easy to see that β_ν satisfies the following:

$$\partial\beta_\nu = \nu \in N \cong \mathbb{Z}^n, \ \mu^{de}(\beta_\nu, \mathcal{X}_\nu) = 0, \ \ \beta_\nu \cap [\pi^{-1}(\partial_j P)] = c_j.$$

Proposition 9.4. 1) *Suppose $\mu^{de}(\beta, \boldsymbol{x}) < 0$. Then, $\mathcal{M}^{main,reg}_{k+1,l}$ $(L(u), \beta, \boldsymbol{x})$ is empty.*

2) *If $\mu^{de}(\beta, \boldsymbol{x}) = 0$, and if $\beta \neq \beta_\nu$ for any $\nu \in Box$, then $\mathcal{M}^{main,reg}_{k+1,1}$ $(L(u), \beta, \boldsymbol{x})$ is empty.*

3) *The moduli space $\mathcal{M}^{main,reg}_{k+1,1}(L(u), \beta)$ is Fredholm regular for any β. Moreover, the evaluation map $ev_0 : \mathcal{M}^{main,reg}_{k+1,1}(L(u), \beta) \to L(u)$ is a submersion.*

4) *If $\mathcal{M}^{main}_{k+1,l}(L(u), \beta)$ is non-empty, then there exist $k_\nu, k_i \in \mathbb{N}$, $\alpha_j \in H_2(X; \mathbb{Z})$ such that*

$$\beta = \sum_{\nu \in Box'} k_\nu \beta_\nu + \sum_i k_i \beta_i + \sum_j \alpha_j$$

where α_j is realized by a holomorphic orbi-sphere, and at least one k_ν or k_i is non-zero.

If $\mathcal{M}^{main}_{1,1}(L(u), \beta)$ is not empty and if $\partial\beta \notin N_{\boldsymbol{b}} := \mathbb{Z}\langle \boldsymbol{b}_1, \ldots, \boldsymbol{b}_m \rangle$, then there exists $\nu \in Box'$ such that

$$\beta = \beta_\nu + \sum_i k_i \beta_i + \sum_j \alpha_j.$$

5) *For $\nu \in Box'$, we have*

$$\mathcal{M}^{main,reg}_{1,1}(L(u), \beta_\nu) = \mathcal{M}^{main}_{1,1}(L(u), \beta_\nu).$$

The moduli space $\mathcal{M}_{1,1}^{main}(L(u),\beta_\nu)$ is Fredholm regular and the evaluation map ev_0 becomes an orientation preserving diffeomorphism.

Proof. Part (1) follows from the desingularized Maslov index formula for holomorphic orbi-discs. And (2) follows from the classification results in section 6.

For (3), Fredholm regularity is already proved. The complex structure is invariant under T^n-action and $L(u)$ is a T^n-orbit. It follows that T^n acts on the moduli space $\mathcal{M}_{k,1}^{main,reg}(L(u),\beta)$ and ev_0 becomes a T^n-equivariant map. Hence ev_0 is a submersion.

For (4), the first statement follows from the structure of the stable map. For the second statement, consider a map $h \in \mathcal{M}_{1,1}(L(u),\beta)$. If the domain of h is a single (orbi-)disc, then the theorem follows from the classification theorem, in which case $\alpha_j = 0$. Otherwise, the domain of h has several irreducible components, which are (orbi-)discs and (orbi-)spheres. Since $\partial\beta \notin N_{\boldsymbol{b}}$, one of the disc components has to be a holomorphic orbi-disc, and as we allow only one interior marked point, there cannot be any other orbifold disc. Then the claim follows from the classification theorem.

For (5), let $h \in \mathcal{M}_1^{main}(L(u),\beta_\nu)$. By (4), we can write

$$\beta_\nu = \beta_{\nu'} + \sum_i k_i\beta_i + \sum_j \alpha_j, \tag{9.7}$$

for some $\nu' \in Box$. By considering their boundaries, we have

$$\partial\beta_\nu = \partial\beta_{\nu'} + \sum_i k_i\partial\beta_i$$

or equivalently,

$$\nu = \nu' + \sum k_i\boldsymbol{b}_i.$$

By the definition of Box, this implies that $\nu = \nu'$ since the coefficients of ν as a linear combination of $\boldsymbol{b}_i$'s should lie in the interval $[0,1)$ and since $k_i \in \mathbb{Z}_{\geq 0}$.

Thus, we have $\sum_i k_i\beta_i + \sum_j \alpha_j = 0$. As their symplectic areas are positive unless trivial, we have $k_i = 0$ for all i, and $\alpha_j = 0$ for all j. This proves the first statement, and the second statement follows as in the proof of Proposition 9.3. q.e.d.

10. Moduli spaces and their Kuranishi structures

In this section, we discuss the T^n-equivariant Kuranishi structures of moduli spaces $\mathcal{M}_{k+1,l}(L(u),\beta)$ of holomorphic (orbi-)discs. Recall that the T^n-equivariant Kuranishi structure of the moduli spaces in smooth toric manifolds has been constructed in [**FOOO2**]. And also recall that the Kuranishi structure of the moduli space of stable maps from orbi-curves (without boundary) has been established in the work of Chen and Ruan [**CR**]. We also recall that the Fredholm setup and gluing analysis for J-holomorphic discs has been carefully discussed in the foundational work of [**FOOO**], and the case with bulk insertion is discussed in [**FOOO3**].

For our case of toric orbifolds, the moduli spaces $\mathcal{M}_{k+1,l}(L(u),\beta)$ of holomorphic (orbi-)discs also have T^n-equivariant Kuranishi structure, as most of the construction of [**FOOO**] and [**FOOO2**] can be easily extended to these cases in a straightforward way by combining the work of Chen and Ruan [**CR**] regarding interior orbifold marked points. But we give brief explanations on some of the issues for readers who are not familiar with them.

10.1. Fredholm index. Let us explain the virtual dimension of the moduli spaces. First, we recall the case of closed J-holomorphic orbi-curves from Chen-Ruan [**CR**]. Let Σ be a closed Riemann surface, with complex vector bundle E on it. The index of the first order elliptic operator $\overline{\partial}$ is given by Riemann-Roch formula

$$index(\overline{\partial}) = 2c_1(E)[\Sigma] + 2n(1-g_\Sigma),$$

where $2n$ is the rank of E, and g_Σ the genus of Σ.

Let $\boldsymbol{\Sigma}$ be a closed orbi-curve with orbifold marked points $z_1,\ldots,z_k$ (with underlying Riemann surface Σ), and E is orbifold complex vector bundle, with degree shifting number ι_i at i-th marked point. Then the index of $\overline{\partial}$ is given by (Lemma 3.2.4 of [**CR**])

$$index(\overline{\partial}) = 2c_1(|E|)[\Sigma] + 2n(1-g_\Sigma) = 2c_1(E)[\Sigma] + 2n(1-g_\Sigma) - \sum_{i=1}^{k} 2\iota_i.$$

Here, $|E|$ is the desingularization of E explained in section 3, and the second identity is from Proposition 4.1.4 of [**CR2**], which follows from the curvature computation in Chern-Weil theory. The desingularized bundle $|E|$ can be used for index computations, as local holomorphic sections of E and $|E|$ can be identified (see Proposition 4.2.2 of [**CR2**]), and hence they have the same indices. Note that the desingularized orbi-bundle over an orbi-curve has trivial fiber-wise action near the orbifold point. Hence $|E|$ gives an honest vector bundle over Σ and we can apply the usual index theorem, and obtain the above equality.

The moduli space of stable maps from genus g orbi-curves with k marked points mapping to $\boldsymbol{x}$, of class $A \in H_2(X)$, is denoted as $\mathcal{M}_{g,k}$ $(X,J,A,\boldsymbol{x})$. Applying the above index formula to the pull-back orbi-bundle, the dimension of $\mathcal{M}_{g,k}(X,J,A,\boldsymbol{x})$ is given by (Lemma 3.2.4 of [**CR**])

$$2c_1(TX)[A] + 2n(1-g_\Sigma) - 6 - \sum_{i=1}^{k} 2\iota_{(\boldsymbol{x}(i))}.$$

Exactly the same argument applies to our case. Let Σ be a bordered Riemann surface, with complex vector bundle $E \to \Sigma$, a Lagrangian subbundle $\mathcal{L} \to \partial\Sigma$. Recall that (see [**KL**] for example) the index of $\overline{\partial}$ is given by Riemann-Roch formula

$$index(\overline{\partial}) = \mu(E, \mathcal{L}) + n \cdot e(\Sigma),$$

where $2n$ is the rank of E, and $e(\Sigma)$ the Euler characteristic of Σ.

Let Σ be a bordered orbi-curve with interior orbifold marked points $z_1, \ldots, z_k$, and E an orbifold complex vector bundle, with degree shifting number ι_i at i-th marked point, with a Lagrangian subbundle $\mathcal{L} \to \partial\Sigma$. Then we have

$$index(\overline{\partial}) = \mu(|E|, \mathcal{L}) + n \cdot e(\Sigma) = \mu(E, \mathcal{L}) + n \cdot e(\Sigma) - \sum_{i=1}^{k} 2\iota_i.$$

The second equality follows from Proposition 3.5 (Proposition 6.10 of [**CS**]).

Applying the above index formula to the pull-back orbi-bundle of holomorphic orbi-discs (note that $e(\Sigma) = 1$), we obtain the virtual dimension of the moduli space of bordered stable maps $\mathcal{M}_{k,l}(L, \beta, \boldsymbol{x})$, which proves Lemma 2.5:

$$n + \mu^{de}(\beta, \boldsymbol{x}) + k + 2l - 3 \ = \ n + \mu_{CW}(\beta) + k + 2l - 3 - 2\iota(\boldsymbol{x}).$$

Here we have subtracted the dimension of $Aut(D^2) = 3$ as we consider the moduli space, and k, $2l$ account for the freedom of the boundary, and the interior marked points.

10.2. Construction of Kuranishi structures. We recall a definition of a Kuranishi neighborhood (chart) (V, E, Γ, ψ, s) of a moduli space $\mathcal{M}$: V is a smooth manifold and E is a vector bundle over V, with a group Γ acting on V and E in a compatible way, and $s : V \to E$ is a Γ-equivariant section such that $\psi : s^{-1}(0)/\Gamma \to \mathcal{M}$ is homeomorphic to an open set of the moduli space $\mathcal{M}$. We refer readers to [**FOOO**] for the definition of compatibilities between Kuranishi charts, and for more details.

The general scheme to construct a Kuranishi structure of a moduli space is as follows: First, one constructs a Kuranishi neighborhood of each point in the interior of the moduli space. The proper Fredholm setting for this construction, and the application of the implicit function theorem to it, is by now standard. Then, one also constructs a Kuranishi neighborhood of each point in the boundary of the moduli space or for the stable map. For this, Taubes' type gluing argument is needed, and the gluing construction for interior node [**FO**] (and orbifold interior node [**CR**]), or boundary node [**FOOO**], has been established. Once local Kuranishi neighborhoods are constructed, there is a standard procedure to construct the global Kuranishi charts, for which we refer readers to [**FO**] or [**FOOO**].

We explain the construction of a local Kuranishi neighborhood of

$$((\boldsymbol{\Sigma}, \vec{z}, \vec{z}^{+}), w, \xi) \in \mathcal{M}_{k+1,l}(L(u), \beta, \boldsymbol{x}). \tag{10.1}$$

First, we consider the case where the domain $\boldsymbol{\Sigma} = \mathcal{D}$ is an orbi-disc $\mathcal{D}$, in which case the element (10.1) lies in the interior of the moduli space $\mathcal{M}_{k+1,l}(L(u), \beta, \boldsymbol{x})$. Then, the linearized $\overline{\partial}$-operator at w is given as

$$D_w\overline{\partial} : W^{1,p}(\mathcal{D}, w^*T\mathcal{X}, L) \to L^p(\mathcal{D}, w^*T\mathcal{X} \otimes \Lambda^{0,1}).$$

The obstruction space E can be chosen so that elements of E are smooth, and supported away from marked points and from $\partial\mathcal{D}$, and also that

$$\mathrm{Image}(D_w\overline{\partial}) + E = L^p(\mathcal{D}, w^*T\mathcal{X} \otimes \Lambda^{0,1}).$$

Then the kernel of $D_w\overline{\partial} : W^{1,p}(\mathcal{D}, w^*T\mathcal{X}, L) \to L^p(\mathcal{D}, w^*T\mathcal{X} \otimes \Lambda^{0,1})/E$ is denoted as V^{map}, and the section $s = D_w\overline{\partial}$. One takes $V = V^{map} \times V^{dom}$ where V^{dom} parametrizes the deformation of the domain $(\mathcal{D}, \vec{z}, \vec{z}^{+})$. In this case the automorphism Γ is trivial since the boundary of the disc maps to L, and the disc only intersects toric divisors at finitely many points. Non-trivial Γ appears if $\boldsymbol{\Sigma}$ has a sphere component.

In fact, to consider $D_w\overline{\partial}$ properly, instead of $\mathcal{D}$, one identifies $\mathcal{D}$ with a bordered Riemann surface Σ' of genus 0, with strip-like end (near boundary marked points) and with cylindrical end (near interior marked points). Then, over this domain Σ', we have a Fredholm problem by considering the $D_w\overline{\partial}$ problem with suitable exponential weights as in [**FO**], [**CR**] (for a cylindrical end) and [**FOOO**] (for a strip-like end). In the case of orbifold marked points, we follow Chen-Ruan's construction that such a Riemann surface Σ' still has "orbifold" data near orbifold marked points. Namely, consider an interior marked point $z_1 \in \Sigma$ which has $\mathbb{Z}/m\mathbb{Z}$ singularity. Let ρ be the generator of $\mathbb{Z}/m\mathbb{Z}$. Suppose the equivariancy data ξ of the map w in (10.1) gives a homomorphism $\phi : \mathbb{Z}/m\mathbb{Z} \to G_{w(z_1)}$ where $G_{w(z_1)}$ is the local group of $w(z_1)$. Then a cylindrical end (for z_1) is considered to have a covering cylinder with $\mathbb{Z}/m\mathbb{Z}$ action, and the pull-back bundle over it is considered as an orbifold bundle on it. Hence the change is only for analytical purposes and orbifold data is not lost during the process. Then in setting up the Fredholm problem, one adds the description of the points p_i to which these infinite ends are exponentially converging. For orbifold marked point z_1 as above, Chen-Ruan required that the end of the holomorphic cylinder limit to a point $p_i \in \chi_{\phi(\rho)}$ in the twisted sector. We refer readers to Lemma 3.2.3 of [**CR**] for more details. The construction of $\psi : s^{-1}(0) \to \mathcal{M}_{k+1,l}(L(u), \beta, \boldsymbol{x})$ involves implicit function theorem following [**FOOO**] (and [**CR**] regarding interior orbifold marked points). It is standard and omitted.

Now, we consider the construction when $\kappa := ((\boldsymbol{\Sigma}, \vec{z}, \vec{z}^{+}), w, \xi)$ is in the boundary (or corner) of the moduli space $\mathcal{M}_{k+1,l}(L(u), \beta, \boldsymbol{x})$. We first write the domain $\Sigma = \cup_\eta \pi_\eta(\Sigma_\eta)$ as the union of irreducible components, which are (orbi-)discs and (orbi-)spheres. We recall an important ingredient of Chen-Ruan's construction of Kuranishi structure when the image of some irreducible component of a domain maps entirely into the (orbifold) singular locus of $\mathcal{X}$.

Note that if Σ_η is a disc, it cannot map entirely into the singular locus of $\mathcal{X}$, due to the Lagrangian boundary condition. So, let us suppose that the component Σ_η is an (orbifold) sphere which maps entirely into the singular locus of $\mathcal{X}$ via w. We denote by G_η the group whose elements are stabilizers of all but finitely many points of the image of Σ_η. Namely, after deleting finitely many points $\vec{z}' \supset \vec{z}^+ \cap \Sigma_\eta$ of Σ_η, for any points $p \in w(\Sigma_\eta \setminus \vec{z}')$, local group G_p is isomorphic to a fixed group G_η. Such a G_η exists due to the properties of orbifold J-holomorphic maps. Define

$$G_\kappa = \{(g_\eta) \in \prod_\eta G_\eta \mid g_\eta(z_\eta) = g_\omega(z_\omega) \text{ if } \ \pi_\eta(z_\eta) = \pi_\omega(z_\omega)\}.$$

This G_κ will be added to the Γ of the Kuranishi structure in the following way. The automorphism group $Aut(\kappa)$ of κ acts on G_κ by pull-backs. Hence we get a short exact sequence

$$1 \to G_\kappa \to \Gamma_\kappa \to Aut(\kappa) \to 1.$$

Γ_κ is the finite group Γ of the local Kuranishi neighborhood and the action of Γ_κ on V and E is defined from that of $Aut(\kappa)$ by setting G_κ to act trivially on them. The rest of the construction is carried out in a Γ_κ-equivariant way.

We remark that the general discussion in Chen-Ruan [**CR**] is more complicated as the groups may not be abelian. In the general case of [**CR**], among G_η, one should take the elements which form a global section on $\Sigma_\eta \setminus \vec{z}'$, so that G_η do not change the local group at $w(z_i^+)$'s by conjugation. (Then such elements of G_η commute with local groups at $w(z^+)$.) In our case, the local groups are abelian, and we can take G_η as above.

So, in the case of $\kappa \in \mathcal{M}_{k+1,l}(L(u),\beta,\boldsymbol{x}) \setminus \mathcal{M}^{reg}_{k+1,l}(L(u),\beta,\boldsymbol{x})$, we need Taubes' type gluing construction. Namely, one first replaces Σ_η (equipped with marked points) with the associated Riemann surface with cylindrical and strip-like ends, and apply the construction of the above for each Σ_η. Then, as in section 7.1.3 of [**FOOO**], one can apply gluing construction (of constructing an approximate solution and applying Newton-type iteration arguments to find actual holomorphic curves), where the gluing near a boundary nodal point is carried out in [**FOOO**] and gluing near an interior nodal point is carried out in [**FO**] (and the orbifold nodal points in [**CR**]). We omit the details, and refer readers to the above references. We remark that the construction of Kuranishi structures is a non-trivial task and these have been extensively and very carefully studied recently by McDuff-Wehrheim [**McW**] and Fukaya-Oh-Ohta-Ono [**FOOO6**]. Our construction follows that of [**FOOO2**] and [**FOOO6**]. In particular, we can construct global Kuranishi structure from local Kuranishi charts as explained in Parts 3 and 4 of [**FOOO6**].

10.3. T^n**-equivariant perturbations.** We briefly recall T^n-equivariant Kuranishi structure of the moduli spaces in [**FOOO2**], and show that the moduli space of stable orbi-discs $\mathcal{M}_{k+1,l}(L(u),\beta)$ in this paper also has an analogous structure.

Consider the following family $\mathfrak{A}$ of compatible Kuranishi charts of the moduli space $\mathcal{M}$:

$$\{(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha) | \alpha \in \mathfrak{A}\}.$$

Here $\pi : E_\alpha \to V_\alpha$ is a vector bundle with equivariant Γ_α-action, s_α is a Γ_α equivariant section of E_α, and ψ_α is a homeomorphism $\psi : s^{-1}(0)/\Gamma_\alpha \to \mathcal{M}$.

Recall from [**FOOO2**] Appendix 2, Definition 15.4, that such a Kuranishi structure is said to be T^n-equivariant in the strong sense, if

1) V_α has T^n-action and it commutes with Γ_α-action.
2) E_α is a T^n-equivariant vector bundle.
3) The maps s_α, ψ_α are T^n-equivariant.
4) Coordinate change maps for embeddings of Kuranishi charts are T^n-equivariant.

Recall that a strongly continuous smooth map $ev : \mathcal{M} \to L$ is a family of Γ_α-invariant smooth maps $ev_\alpha : V_\alpha \to L$ (inducing $ev_\alpha : V_\alpha/\Gamma_\alpha \to L$) which are compatible with coordinate changes. The map ev is said to be weakly submersive if each ev_α is a submersion.

Proposition 10.1 (c.f. Prop. 15.7 of [**FOOO2**])**.** *The moduli space $\mathcal{M}_{k+1,l}(L(u),\beta)$ has a T^n-equivariant Kuranishi structure such that $ev_0 : \mathcal{M}_{k+1,l}(L(u),\beta) \to L$ is a T^n-equivariant, strongly continuous, and weakly submersive map.*

Proof. The same line of proof as in that of [**FOOO2**], Proposition 15.7 can be used to prove the existence of T^n-equivariant Kuranishi structure in our case too: The standard complex structure J of X is T^n-invariant, and the Lagrangian submanifold $L(u)$ is a free T^n-orbit. Note that as the torus action on ambient toric variety carries over to the tangent bundles and Cauchy-Riemann equations in a natural way, the main new ingredient is how to choose an obstruction bundle in a T^n-equivariant way.

We have a free T^n action on the Kuranishi neighborhood since the T^n action on the Lagrangian submanifold $L(u)$ is free and the evaluation maps ev are T^n-equivariant as explained in [**FOOO2**]. We can take a multivalued perturbation of the Kuranishi structure that is T^n equivariant. Such a multisection, which is also transversal to 0, is constructed by taking the quotient of the Kuranishi neighborhood, obstruction bundles, and so on by T^n action to obtain a space with Kuranishi structure. Then take a transversal multisection of the quotient Kuranishi structure and lift it to a multisection of the Kuranishi neighborhood. The evaluation map becomes a submersion because of the T^n-equivariance. (The existence of T^n-action simplifies the general construction of [**FOOO**] because the fiber products appearing in the inductive construction are automatically transversal.) q.e.d.

Now, we focus attention on the moduli space of holomorphic discs *without (orbifold) interior marked points*. (The case of orbi-discs will be considered in section 12.) Consider the following map which forgets the $(1, \ldots, k)$-th marked points

$$forget_0 : \mathcal{M}^{main}_{k+1,0}(L(u), \beta) \to \mathcal{M}^{main}_{1,0}(L(u), \beta).$$

As in [**FOOO2**] we can construct our Kuranishi structure so that it is compatible with $forget_0$.

Lemma 10.2 (c.f. Lemma 11.2 in [**FOOO2**]). *For each given $E > 0$, we can take a system of multisections $\mathfrak{s}_{\beta,k+1}$ on $\mathcal{M}^{main}_{k+1,0}(L(u), \beta)$ for $\omega(\beta) < E$ satisfying the following properties:*

1) *They are transversal at zero section, and invariant under T^n-action.*
2) *The multisection $\mathfrak{s}_{\beta,k+1}$ is obtained as the pull-back of the multisection $\mathfrak{s}_{\beta,1}$ by the forgetful map.*
3) *The restriction of the multisection $\mathfrak{s}_{\beta,1}$ to the boundary of $\mathcal{M}^{main}_{1,0}$ $(L(u), \beta)$ is given as the fiber product of the multisection $\mathfrak{s}_{\beta',k'}$ from the following:*

$$\partial \mathcal{M}^{main}_{1,0}(L(u), \beta) = \bigcup_{\beta_1+\beta_2=\beta} \mathcal{M}^{main}_{1,0}(L(u), \beta_1)\, {}_{ev_0}\times_{ev_1} \mathcal{M}^{main}_{2,0}(L(u), \beta_2).$$

4) *$\mathcal{M}^{main}_{1,0}(L(u), \beta_i)$ for $i = 1, \ldots, m$ are not perturbed.*

The proof of the lemma is the same as in [**FOOO**] and is omitted. We obtain the following corollary from dimension arguments:

Corollary 10.3. *The moduli space $\mathcal{M}^{main}_{1,0}(L(u), \beta)^{\mathfrak{s}_\beta}$ is empty if the Maslov index $\mu(\beta) < 0$ or $\beta \neq 0$ and $\mu(\beta) = 0$.*

These T^n-equivariant perturbations define the following open Gromov-Witten invariants for toric orbifolds, as in Lemma 11.7 of [**FOOO2**]. This is because the virtual fundamental chain of $\mathcal{M}^{main}_{1,0}(L(u), \beta)$ is now a cycle due to Corollary 10.3. A homology class $c_\beta[L(u)] \in H_n(L(u); \mathbb{Q})$ can be defined by the pushforward

$$c_\beta[L(u)] = ev_*([\mathcal{M}^{main}_{1,0}(L(u), \beta)^{\mathfrak{s}_\beta}]). \tag{10.2}$$

Lemma 10.4 (Lemma 11.7 of [**FOOO2**]). *The number c_β is well-defined, independent of the choice $s_{\beta,k+1}$ in Lemma* 10.2.

From the classification results (Proposition 9.3), we have $c_{\beta_i} = 1$ for $i = 1, \ldots, m$, where the sign can be computed from [**C1**]. If $\mathcal{X}$ is Fano, then we also have $c_\beta = 0$ for $\beta \neq \beta_i$.

11. Filtered A_∞-algebra and its potential function

11.1. Filtered A_∞-algebra and its deformation theory. We provide a quick summary of the deformation and obstruction theory of [**FOOO**] just to set the notations. We refer readers to [**FOOO**], [**FOOO2**] for details.

For a graded R-module C, its suspension $C[1]$ is defined as $C[1]^k = C^{k+1}$. For $x \in C$, we denote by $deg(x)$ and $deg'(x)$ the original and the shifted degree of x respectively. The bar complex $B(C[1])$, which is a graded coalgebra, is defined as $B(C[1]) = \bigoplus_{k=0}^{\infty} B_k(C[1])$ with

$$B_k(C[1]) = \underbrace{C[1] \otimes \cdots \otimes C[1]}_{k}. \tag{11.1}$$

We have $B_0(C[1]) = R$ by definition.

Definition 11.1. An A_∞-algebra structure on C is given by a sequence of degree one R-module homomorphisms $m_k : B_k(C[1]) \to C[1]$ for $k = 1, 2, \ldots$ such that the equations

$$\sum_{k=1}^{n-1} \sum_{i=1}^{k-i+1} (-1)^{\epsilon} m_{n-k+1}(x_1 \otimes \cdots \otimes m_k(x_i, \ldots, x_{i+k-1}) \otimes \cdots \otimes x_n) = 0, \tag{11.2}$$

which are called the *A_∞-equations*, are satisfied. Here $\epsilon = \deg' x_1 + \cdots + \deg' x_{i-1}$.

This can be written using coderivations as follows. The map m_k can be extended to a coderivation $\widehat{m}_k : B(C[1]) \to B(C[1])$ by

$$\widehat{m}_k(x_1 \otimes \cdots \otimes x_n) = \sum_{i=1}^{k-i+1} (-1)^{\epsilon} x_1 \otimes \cdots \otimes m_k(x_i, \ldots, x_{i+k-1}) \otimes \cdots \otimes x_n. \tag{11.3}$$

If we set $\widehat{d} = \sum_{k=1}^{\infty} \widehat{m}_k$, then the A_∞-equation is equivalent to $\widehat{d} \circ \widehat{d} = 0$.

Since $m_1 \circ m_1 = 0$, the complex (C, m_1) defines the homology of A_∞-algebra. In a filtered case, A_∞-algebra is similarly defined but has an $m_0 : R \to C[1]$ term and we have $m_1 \circ m_1 \neq 0$ in general filtered case.

Definition 11.2. An element $\boldsymbol{e} \in C^0$ is called a *unit* if it satisfies

1) $m_{k+1}(x_1, \ldots, \boldsymbol{e}, \ldots, x_k) = 0$ for $k \geq 2$ or $k = 0$.
2) $m_2(\boldsymbol{e}, x) = (-1)^{\deg x} m_2(x, \boldsymbol{e}) = x$ for all x.

If $m_0(1)$ is a constant multiple of a unit (i.e. $m_0(1) = c\boldsymbol{e}$ for some $c \in R$), then $m_1 \circ m_1 = 0$. Therefore, one can consider the homology of m_1.

To consider filtered A_∞-algebra, let $\bigoplus_{m\in\mathbb{Z}} C^m$ be a free graded $\Lambda_{0,nov}$-module. Let $F^\lambda C^m$ be the submodule of elements with coefficients having T-exponents $\geq \lambda$; then these modules give a natural energy filtration. We define C as the completion with respect to this filtration. Similarly, $B_k C$ and BC are defined as completions. A structure of filtered A_∞-algebra on C is given by a sequence of $\Lambda_{0,nov}$-homomorphisms $\{m_k\}$ satisfying A_∞-equation (11.2) with $k \geq 0$, and additionally satisfying the following properties:

1) $m_0(1) \in F^\lambda C^1$ with $\lambda > 0$,
2) m_k respects the energy filtration,
3) m_k is induced from $\overline{m}_k : B_k\overline{C}[1] \to \overline{C}$ which is an R-module homomorphism, where $\overline{C}$ is the free R-module with the same generating set as C.

In this paper, we follow [**FOOO2**] to work with Λ_0 rather than $\Lambda_{0,nov}$ by forgetting e, and we work with the $\mathbb{Z}_2$-graded complex (see (1.7) for Novikov rings).

For $b \in F^\lambda C^1$ with $\lambda > 0$, consider the following exponential:

$$e^b = 1 + b + b \otimes b + \cdots \in BC.$$

Then, deformed A_∞-algebra $(C, \{m_k^b\})$ is defined by setting m_k^b as

$$m_b^k(x_1, \ldots, x_k) = m(e^b, x_1, e^b, x_2, e^b, x_3, \ldots, x_k, e^b). \tag{11.4}$$

If $m(e^b) = m_0^b$ is a multiple of unit $\boldsymbol{e}$, then m_1^b defines a complex.

Definition 11.3. An element $b \in F^\lambda C^1$ with $\lambda > 0$ is called a *weak bounding cochain* if $m(e^b)$ is a multiple of unit $\boldsymbol{e}$. A filtered A_∞-algebra is called weakly unobstructed if a weak bounding cochain b exists.

We denote by $\widehat{\mathcal{M}}_{weak}(L)$ the set of weak bounding cochains of L. The moduli space $\mathcal{M}_{weak}(L)$ is then defined to be the quotient space of $\widehat{\mathcal{M}}_{weak}(L)$ by suitable gauge equivalence (see section 4.3 of [**FOOO**]). In fact, when $C = H(L, \Lambda_0)$, one can also consider b in $F^0C[1]$ by introducing a non-unitary flat complex line bundle over the Lagrangian submanifold (see [**C3**], [**FOOO2**] for more details).

11.2. Construction of a filtered A_∞-algebra of Lagrangian submanifold. We construct a filtered A_∞-algebra on $H(L(u); \Lambda_0^{\mathbb{R}})$ using the (perturbed) moduli space of holomorphic discs as in [**FOOO2**]. We emphasize that we do *not* use orbi-discs to construst the filtered A_∞-algebra. Orbi-discs will be used for bulk deformations in later sections.

For a T^n-invariant metric on $L(u)$, a differential form x on $L(u)$ becomes harmonic if and only if x is T^n-equivariant, and we identify

$H(L(u), \mathbb{R})$ with the set of T^n-equivariant forms, on which we construct the A_∞-structure.

Consider evaluation maps

$$ev = (ev_1, \dots, ev_k, ev_0) : \mathcal{M}^{main}_{k+1,0}(L(u), \beta)^{\mathfrak{s}_\beta} \to L(u)^{k+1}. \tag{11.5}$$

For $\omega_1, \dots, \omega_k \in H(L(u), \mathbb{R})$, we define

$$m_{k,\beta}(\omega_1, \dots, \omega_k) = (ev_0)_!(ev_1, \dots, ev_k)^*(\omega_1 \wedge \cdots \wedge \omega_k). \tag{11.6}$$

Here $(ev_0)_!$ is an integration along the fiber and it is well-defined as ev_0 is a submersion. (See appendix C of [**FOOO2**] for details on smooth correspondences.)

The resulting differential form is again T^n-equivariant since $\mathfrak{s}_\beta$ and all other maps are T^n-equivariant. As in [**FOOO2**] (and using Lemma 10.2), we obtain the A_∞-formula:

(11.7)
$$\sum_{\beta_1+\beta_2=\beta} \sum_{k_1+k_2=k+1} \sum_{l=1}^{k_1} (-1)^\epsilon m_{k_1,\beta_1}(\omega_1, \dots, m_{k_2,\beta_2}(\omega_l, \dots,), \dots, \omega_k) = 0.$$

Here $\epsilon = \sum_{i=1}^{l-1}(deg'\omega_i)$. We put $m_k = \sum_\beta T^{\omega(\beta)/2\pi} m_{k,\beta}$. We extend the above to ω with coefficients in $\Lambda_0^{\mathbb{R}}$ multi-linearly. This defines an A_∞-structure on $H(L(u), \Lambda_0^{\mathbb{R}})$. The constructed filtered A_∞-algebra is unital with the unit $\boldsymbol{e}$ being the constant $1 \in H^0(L, \mathbb{R})$, which is the Poincaré dual $PD([L(u)])$ of the fundamental class, and this follows from the definition (11.6). Note that the constructed A_∞-algebra is already a canonical model, since we define it on harmonic forms $H(L; \Lambda_0)$ in this case of a toric fiber $L = L(u)$.

As in [**FOOO2**], for $r \in H^1(L, \mathbb{R})$, the A_∞-structure can be explicitly computed:

Lemma 11.1 (c.f. Lemma 11.8 of [**FOOO2**]). *For $r \in H^1(L(u), \Lambda_0^{\mathbb{R}})$ and $\beta \in \pi_2(X, L)$ with $\mu(\beta) = 2$, and for c_β defined in Lemma 10.4, we have*

$$m_{k,\beta}(r, \dots, r) = \frac{c_\beta}{k!}(r(\partial\beta))^k \cdot PD([L(u)]).$$

The proof in Lemma 11.8 of [**FOOO2**] in the manifold case is based on two facts. The first is that the intersection number of r and $\partial\beta$ is determined by the cap product $\partial\beta \cap r = r(\partial\beta)$, and the second is that ev_0 is a diffeomorphism with $\mu(\beta) = 2$ condition. For toric orbifolds, we have shown in Proposition 9.3 and Proposition 9.4 that ev_0 is again a diffeomorphism. Hence the same proof extends to the orbifold case.

From this computation, we have

Proposition 11.2 (c.f. [**FOOO2**], Prop. 4.3). *We have an inclusion*

$$H^1(L(u); \Lambda_+) \hookrightarrow \mathcal{M}_{weak}(L(u)). \tag{11.8}$$

Hence, toric fiber $L(u)$ is weakly unobstructed for any $u \in Int(P)$.

Moreover, one can take $b \in H^1(L(u);\Lambda_0)$, and it is contained in $\mathcal{M}_{weak}(L(u);\Lambda_0)$.

Proof. First, take $b_+ \in H^1(L(u),\Lambda_+)$. We have

$$\sum_{k=0}^{\infty} m_k(b_+,\ldots,b_+) = \sum_{\beta}\sum_{k=0}^{\infty} \frac{c_\beta}{k!}(b_+(\partial\beta))^k T^{\omega(\beta)/2\pi}\cdot PD([L(u)]).$$

By the degree reason, the sum is over β with $\mu(\beta) = 2$. Hence $b_+ \in \widehat{\mathcal{M}}_{weak}(L(u))$ and the gauge equivalence relation is trivial on $H^1(L(u);\Lambda_0)$ and this proves the inclusion.

One can take $b \in H^1(L(u);\Lambda_0)$ in the definition of weak Maurer-Cartan elements as in [**FOOO2**] as follows: For $b = b_0 + b_+$, with $b_0 \in H^1(L(u),\mathbb{C})$ and $b_+ \in H^1(L(u),\Lambda_+)$, we introduce a representation $\rho : \pi_1(L) \to \mathbb{C}^*$ such that $\rho(\gamma) = exp(\int_\gamma b_0)$. We define a non-unitary flat line bundle $\mathcal{L}_\rho$ on L with holonomy given by ρ, and modify the A_∞-structure by

$$m_k^\rho = \sum_{\beta\in\pi_2(M,L)} \rho(\partial\beta) m_{k,\beta}\otimes T^{\omega(\beta)/2\pi}.$$

If the resulting A_∞-structure $\{m_k^\rho\}$ is weakly unobstructed with weak bounding cochain b_+, then the set of such b's are denoted by $\mathcal{M}_{weak}(L(u);\Lambda_0)$, and again called weak bounding cochains. We refer readers to [**C3**], [**FOOO2**] for more details. q.e.d.

For $b \in \widehat{\mathcal{M}}_{weak}(L)$, we have $m_0^b = m(e^b) = c\boldsymbol{e}$ and the A_∞-equation tells us that m_1^b is a differential. Hence, for $b \in \widehat{\mathcal{M}}_{weak}(L)$, we define the Bott-Morse Floer cohomology of L as

$$HF((L;b),(L;b)) = \frac{Ker\ m_1^b}{Im\ m_1^b}, \tag{11.9}$$

We call it *smooth Floer cohomology of L* to emphasize that it does not use the data of bulk deformation by twisted sectors of toric orbifolds.

Recall that for weakly unobstructed L, the potential function PO, as a function from $\widehat{\mathcal{M}}_{weak}(L)$ to Λ_+, is defined by the equation

$$m(e^b) = PO(b)\cdot PD([L]). \tag{11.10}$$

11.3. Smooth potential for toric orbifolds. Given a toric orbifold, the above construction gives filtered A_∞-algebra for $L(u)$, which uses only smooth holomorphic (stable) discs. The potential PO above may be called *smooth potential* for $L(u)$ since it does not use information on orbi-discs. (The bulk deformed potential will be defined using orbi-discs in later sections.)

In this subsection, we discuss the properties of a smooth potential and define leading order smooth potential, which can be explicitly computed.

As in the manifold case, if $\mathcal{X}$ is not Fano, for smooth potential PO, we also need to consider stable disc contributions which are not readily computable.

We choose an integral basis $\mathbf{e}_i \in H^1(L(u);\mathbb{Z})$, which can be done by the identification $L(u) = T^n = (S^1)^n = (\mathbb{R}/\mathbb{Z})^n$. (Here we may use dt_i in de Rham cohomology, where t_i is the coordinate of the i-th factor of $(\mathbb{R}/\mathbb{Z})^n$.)

We choose a weak bounding cochain b as

$$b = \sum x_i \mathbf{e}_i \in H^1(L(u);\Lambda_0).$$

Then, $PO(b)$ depends on $(x_1,\dots,x_n) \in (\Lambda_0)^n$ and $(u_1,\dots,u_n) \in Int(P)$, and hence to emphasize its dependence on u, we may write $PO(b)$ as $PO(x;u) := PO(x_1,\dots,x_n;u_1,\dots,u_n)$. But for simplicity, most of the time we omit u, and write PO and $PO(b)$. (PO^u is used in [**FOOO3**].)

As in [**FOOO2**], it is convenient to introduce $y_1,\dots,y_n$ as follows (also because holonomy is defined up to $2\pi\sqrt{-1}\mathbb{Z}$): We define

$$y_i = e^{x_i} = e^{x_{i,0}} \sum_{k=0}^{\infty} x_{i,+}^k/k!,$$

where we write $x_i = x_{i,0} + x_{i,+}$ with $x_{i,0} \in \mathbb{C}$ and $x_{i,+} \in \Lambda_+$.

Consider a toric orbifold X with moment polytope P and stacky vectors $\vec{\boldsymbol{b}}$. From (1.6) (Lemma 7.1), the following affine function measures the area of smooth discs corresponding to stacky vectors $\boldsymbol{b}_j = (b_{j1},\dots,b_{jn}) \in \mathbb{Z}^n$ for $j = 1,\dots,m$:

$$\ell_j(u) = \langle u, \boldsymbol{b}_j\rangle - p_j.$$

We define the leading order smooth potential function $PO_0(b)$ of toric orbifold:

$$PO_0(b) := \sum_{j=1}^{m} T^{\ell_j(u)}(y_1)^{b_{j1}}\dots(y_n)^{b_{jn}}, \tag{11.11}$$

the j-th term of which corresponds to stacky vector $\boldsymbol{b}_j$ (Corollary 6.4). Remaining terms $PO(b) - PO_0(b)$ correspond to the contributions of stable discs.

We introduce variables z_j as follows (which simplifies $PO_0(b) = z_1 + \dots + z_m$):

$$z_j = T^{\ell_j(u)}(y_1)^{b_{j1}}\dots(y_n)^{b_{jn}}. \tag{11.12}$$

Theorem 11.3 (c.f. Theorem 5.2 of [**FOOO5**]). 1) *$PO(b)$ can be written as*

$$PO(b) = \sum_{i=1}^{m} z_j + \sum_{k=1}^{N} T^{\lambda_k} P_k(z_1,\dots,z_m) \tag{11.13}$$

for $N \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ *and* $\lambda_k \in \mathbb{R}_{>0}$*. If* $N = \infty$*, then* $\lim_{k\to\infty} \lambda_k = \infty$*. Here* $P_k(z_1, \ldots, z_m)$ *are monomials of* $z_1, \ldots, z_m$ *with coefficients in* Λ_0*.*

2) *If* $\mathcal{X}$ *is Fano, then* $P_k = 0$*.*
3) *The above formula* (11.13) *is independent of* u *and depends only on* $\mathcal{X}$*.*

Proof. If $\mathcal{X}$ is Fano, then the usual dimension counting shows that only Maslov index two discs of the classes β_i's for $i = 1, \ldots, m$ contribute to $m_{k,\beta}(b, \ldots, b)$. Hence, to show (2), it is enough to show that the contribution of β_i to the sum $\sum_k m_{k,\beta_i}(b, \ldots, b)$ is given by z_i.

Denote $b = \sum_{i=1}^n x_i \mathbf{e}_i$ with $b = b_0 + b_+$ as before and consider flat line bundle $\mathcal{L}$ on L whose holonomy ρ along e_i^* is $exp(b_{i,0})$. Then, we have

$$
\begin{aligned}
\sum_{k=0}^{\infty} m_k^{\rho}(b_+, \ldots, b_+) &= \sum_{i=1}^{m} \sum_{k=0}^{\infty} T^{\omega(\beta_i)/2\pi} \rho(\partial\beta_i) m_{k,\beta_i}(b_+, \ldots, b_+) \\
&= \sum_{i=1}^{m} \sum_{k=0}^{\infty} e^{\langle \boldsymbol{b}_i, b_0 \rangle} \frac{1}{k!} (b_+(\partial\beta_i))^k T^{\ell_i(u)} \cdot PD([L]) \\
&= \sum_{i=1}^{m} e^{\langle \boldsymbol{b}_i, b \rangle} T^{\ell_i(u)} \cdot PD([L]) \qquad (11.14)
\end{aligned}
$$

where the third inequality follows by writing $b_+(\partial\beta_i) =< \boldsymbol{b}_i, b_+ >$. Since $y_i = e^{x_i}$, we obtain $e^{\langle \boldsymbol{b}_i, b \rangle} = y_1^{b_{i1}} \cdots y_n^{b_{in}}$ and thus, in the Fano case, we have

$$
PO(x; u) = PO_0(b) = \sum_{i=1}^{m} y_1^{b_{i1}} \cdots y_n^{b_{in}} T^{\ell_i(u)}.
$$

Hence, to prove (1), let us assume that $\mathcal{X}$ is not Fano, and find a general expression for stable map contributions. If $\mathcal{X}$ is not Fano, and $\beta \neq \beta_j$, then β is the homotopy class of stable discs (still with $\mu(\beta) = 2$) and from Proposition 9.3 (4), we have that

$$
\partial\beta = \sum k_i \partial\beta_i, \quad \beta = \sum_i k_i \beta_i + \sum_j \alpha_j.
$$

Thus, by computing

$$
\sum_k T^{\omega(\beta)/2\pi} m_{k,\beta}^{\rho}(b_+, \ldots, b_+)
$$

we note that it is a constant multiple of the expression $T^{(\sum_j \omega(\alpha_j)/2\pi)} \prod_i z_i^{k_i}$, which proves the theorem. The proof of (3) is similar to [**FOOO5**] and omitted. q.e.d.

The rest of the procedure to compute smooth Floer cohomology from the smooth potential function is analogous to [**FOOO2**] or [**FOOO5**] of the manifold case. Hence, we only summarize the main results and refer readers to the above references for full details. The following criterion reduces the computation of smooth Floer cohomology to the critical point theory of the potential function.

Theorem 11.4 (c.f. Theorem 5.5 of [**FOOO5**]). *Let $b = \sum x_i \mathbf{e}_i$. The following are equivalent:*

1) *For each of $i = 1, \ldots, n$, we have*
$$\left.\frac{\partial PO}{\partial x_i}\right|_b = 0.$$
2) *We have an isomorphism of modules*
$$HF\big((L(u), b), (L(u), b); \Lambda_0\big) \cong H(T^n; \Lambda_0).$$
3)
$$HF\big((L(u), b), (L(u), b); \Lambda\big) \neq 0.$$

Proof. This is obtained by taking a derivative: note that $\partial b/\partial x_i = \mathbf{e}_i$, and hence
$$\left.\frac{\partial PO}{\partial x_i}\right|_b PD([L]) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} m_{k_1+k_2+1}(\underbrace{b, \ldots, b}_{k_1}, \mathbf{e}_i, \underbrace{b, \ldots, b}_{k_2}) = m_1^b(\mathbf{e}_i).$$
This shows that (1) is equivalent to the condition $m_1^b(\mathbf{e}_i) = 0$ for all $i = 1, \ldots, n$. For the equivalence between the latter condition and (2), we refer readers to section 4.1 of [**C3**] or Lemma 13.1 of [**FOOO**], where one uses the product structure of Floer cohomology classes to show that $\mathbf{e}_i$'s are non-trivial classes. The rest is left as an exercise. q.e.d.

In practice, we use derivatives with respect to y_i, and $\frac{\partial}{\partial x_i}$ is the same as $y_i \frac{\partial}{\partial y_i}$. In fact, the variable y depends on u and written as $y^{\mathbf{u}}$ in [**FOOO5**], but potential function given as (11.13) is independent of u. Thus, we may take $u = 0$, and write y for $y^{\mathbf{0}}$ as in [**FOOO5**] and write $z_j = T^{\ell_j(0)}(y_1)^{b_{j1}} \ldots (y_n)^{b_{jn}}$. In [**FOOO5**], they introduce $(\eta_1, \cdots \eta_n) \in (\Lambda \setminus \{0\})^n$ as a possible domain for $(y_1, \ldots, y_n)$ and consider
$$A(Int(P)) = \{(\eta_1, \cdots \eta_n) \in (\Lambda \setminus \{0\})^n \mid (\mathfrak{v}_T(\eta_1), \ldots, \mathfrak{v}_T(\eta_n)) \in Int(P)\}.$$
Then, the relevant information of u from a y variable can be read off from the valuation $\mathfrak{v}_T$ of y variables, and PO can be considered as a function on $A(Int(P))$.

Theorem 11.5 (c.f. Theorem 5.9 of [**FOOO5**]). *For $u \in Int(P)$, the following two conditions are equivalent.*

1) *There exists $b \in H^1(L(u); \Lambda_0)$ such that we have an isomorphism as modules*
$$HF\big((L(u), b), (L(u), b); \Lambda_0\big) \cong H(T^n; \Lambda_0).$$

2) *There exists* $\eta = (\eta_1, \cdots \eta_n) \in A(Int(P))$ *such that*

$$\eta_i \frac{\partial PO}{\partial y_i}(\eta) = 0$$

for $i = 1, \ldots, n$ *and that*

$$(\mathfrak{v}_T(\eta_1), \ldots, \mathfrak{v}_T(\eta_n)) = u.$$

Once the potential PO is defined, the proof of Theorem 5.9 of [**FOOO5**] is rather algebraic (or combinatorial), and the proof easily extends to the orbifold case. We discuss examples of the smooth Floer cohomology of Lagrangian torus fibers for teardrop orbifolds and weighted projective spaces in section 15.

12. Bulk deformations of Floer cohomology and bulk orbi-potential

Bulk deformations were introduced in [**FOOO**] as a way to deform A_∞-algebra of a Lagrangian submanifold by an ambient cycle of the symplectic manifold. It gives further ways to deform Floer theory, which was found to be a very effective way of locating non-displaceable torus fibers in toric manifolds ([**FOOO3**]).

For an orbifold $\mathcal{X}$ and a smooth Lagrangian submanifold L, bulk deformations from inertia components of $\mathcal{X}$ play a much more important role, because J-holomorphic orbi-discs come into Floer theory only via bulk deformations. This is because the domain of holomorphic orbi-discs has an interior orbifold singularity, and we have used an interior orbifold marked point to record the orbifold structure of such a domain.

We will see in examples in section 15 that these bulk deformations are very important to understand symplectic geometry of orbifolds, because the very rigid features of Hamiltonian dynamics of orbifolds are detected by bulk deformations via twisted sectors.

In this section, we first explain our setting of bulk deformations for toric orbifolds, set up bulk deformed A_∞-algebras, and analyze their bulk potentials.

12.1. Bulk deformation.

We follow [**FOOO**] and [**FOOO3**] to set up bulk deformations of A_∞-algebras as follows. The new feature is that for toric orbifold $\mathcal{X}$, we consider bulk deformation via the fundamental class of twisted sectors.

Definition 12.1. For each $\nu \in Box'$, consider fundamental cycles $1_{\mathcal{X}_\nu} \in H^0(\mathcal{X}_\nu; R)$ of inertia component $\mathcal{X}_\nu$, and regard it as an element with degree $2\iota_{(\nu)}$ (i.e. $deg(1_{\mathcal{X}_\nu}) = 2\iota_{(\nu)}$) as in [**CR**]. Also, consider the toric divisor D_i of $\mathcal{X}$. We take a finite dimensional graded R-vector space H generated by these $1_{\mathcal{X}_\nu}$'s and D_i's:

$$H = \oplus_{\nu \in Box'} R < 1_{\mathcal{X}_\nu} > \oplus_{i=1}^m R < D_i > . \tag{12.1}$$

Note that we do *not* consider more general bulk deformations by $H^*_{orb}(\mathcal{X})$ in this paper. To simplify notation, we label elements of Box' as

$$Box' = \{\nu_{m+1}, \dots, \nu_B\}. \tag{12.2}$$

We define

$$H_a = \begin{cases} D_a & \text{for } 1 \le a \le m \\ 1_{\mathcal{X}_{\nu_a}} & \text{for } m+1 \le a \le B. \end{cases} \tag{12.3}$$

These H_a's for $a = 1, \dots, B$ form a basis of H.

For $\mathfrak{b}_a \in \Lambda_+$ for each a, we consider an element

$$\mathfrak{b} = \sum_a \mathfrak{b}_a H_a \in H \otimes \Lambda_+. \tag{12.4}$$

Bulk deformations use the following family of operators:

$$\mathfrak{q}_{\beta;\ell,k} : E_\ell(H[2]) \otimes B_k(H^*(L;R)[1]) \to H^*(L;R)[1]. \tag{12.5}$$

Here, degree shiftings $H[2]$ and $H^*(L;R)[1]$ are introduced so that the degree of the map $\mathfrak{q}_{\beta;\ell,k}$ is $1 - \mu(\beta)$, where 2 and 1 correspond to the degrees of freedom of interior and boundary marked points in D^2 respectively.

The symmetrization $E_\ell C$ of $B_\ell C$ can be defined as invariant elements of $B_\ell C$ under symmetric group action. Consider the standard coproduct $\Delta : BC \to BC$ and $\Delta^{n-1} : BC \to (BC)^{\otimes n}$ or $EC \to (EC)^{\otimes n}$, which is defined by

$$\Delta^{n-1} = (\Delta \otimes \underbrace{id \otimes \cdots \otimes id}_{n-2}) \circ (\Delta \otimes \underbrace{id \otimes \cdots \otimes id}_{n-3}) \circ \cdots \circ \Delta. \tag{12.6}$$

The image of an element $\mathbf{x} \in BC$ under Δ^{n-1} can be written as

$$\Delta^{n-1}(\mathbf{x}) = \sum_c \mathbf{x}_c^{n;1} \otimes \cdots \otimes \mathbf{x}_c^{n;n}, \tag{12.7}$$

for c running over some index set for each $\mathbf{x}$. Shifted degree of the element $\mathbf{x} = x_1 \otimes \cdots \otimes x_k$ is given by $\deg' \mathbf{x} = \sum \deg' x_i$.

Theorem 12.1 (c.f. Theorem 3.8.32 of [**FOOO**])**.** *For toric orbifold $\mathcal{X}$ and Lagrangian torus fiber L, the operators $\mathfrak{q}_{\beta;l,k}$ can be constructed to have the following properties.*

1) *For β and $\mathbf{x} \in B_k(H(L;R)[1])$, $\mathbf{y} \in E_l(H[2])$, we have*

$$0 = \sum_{\beta_1+\beta_2=\beta} \sum_{c_1,c_2} (-1)^\epsilon \mathfrak{q}_{\beta_1}(\mathbf{y}_{c_1}^{2;1}; \mathbf{x}_{c_2}^{3;1} \otimes \mathfrak{q}_{\beta_2}(\mathbf{y}_{c_1}^{2;2}; \mathbf{x}_{c_2}^{3;2}) \otimes \mathbf{x}_{c_2}^{3;3}) \tag{12.8}$$

where

$$\epsilon = \deg' \mathbf{x}_{c_2}^{3;1}(1 + \deg \mathbf{y}_{c_1}^{2;2}) + \deg \mathbf{y}_{c_1}^{2;1}. \tag{12.9}$$

Here, we write $\mathfrak{q}_\beta(\mathbf{y};\mathbf{x})$ for $\mathfrak{q}_{\beta;l,k}(\mathbf{y};\mathbf{x})$.

2) *For* $1 \in E_0(H[2])$ *and* $\mathbf{x} \in B_k(H(L;R)[1])$, *we have*

$$\mathfrak{q}_{\beta;0,k}(1;\mathbf{x}) = m_{k,\beta}(\mathbf{x}), \tag{12.10}$$

where $m_{k,\beta}$ *is the filtered* A_∞ *structure on* $H(L;R)$ *constructed in* (11.6).

3) *Consider* $\mathbf{x} = \mathbf{x}_1 \otimes \mathbf{e} \otimes \mathbf{x}_2 \in B(H(L;R)[1])$. *Then*

$$\mathfrak{q}_\beta(\mathbf{y};\mathbf{x}) = 0 \tag{12.11}$$

except

$$\mathfrak{q}_{\beta_0}(1;\mathbf{e}\otimes x) = (-1)^{\deg x}\mathfrak{q}_{\beta_0}(1;x\otimes \mathbf{e}) = x, \tag{12.12}$$

where we have $\beta_0 = 0 \in H_2(X,L;\mathbb{Z})$ *and* $x \in H(L;R)[1]$.

We explain the construction of $\mathfrak{q}$ in the next subsection 12.2 using the moduli space of J-holomorphic orbi-discs. After the construction of moduli spaces and their T^n-equivariant Kuranishi perturbations satisfying suitable compatibility conditions, the rest of the proof of the above theorem is analogous to the manifold case given in Theorem 3.8.32 of [**FOOO**] and Theorem 2.1 of [**FOOO3**], and we omit further details.

Using the notation in (12.4), we define

$$m_k^{\mathfrak{b}}(x_1,\ldots,x_k) = \sum_\beta \sum_{l=0}^\infty T^{\omega(\beta)/2\pi}\mathfrak{q}_{\beta;l,k}(\mathfrak{b}^{\otimes l};x_1,\ldots,x_k). \tag{12.13}$$

The above theorem implies that

Lemma 12.2 (Lemma 2.2 of [**FOOO3**]). *The operations* $\{m_k^{\mathfrak{b}}\}_{k=0}^\infty$ *define a structure of filtered* A_∞*-algebra on* $H(L;\Lambda_0)$.

The element $b \in H^1(L;\Lambda_+)$ is called a *weak bounding cochain* of the filtered A_∞ algebra $(H(L;\Lambda_0),\{m_k^{\mathfrak{b}}\})$ if

$$m_k^{\mathfrak{b}}(e^b) = \sum_{k=0}^\infty m_k^{\mathfrak{b}}(b,\ldots,b) = cPD([L]), \tag{12.14}$$

for some constant $c \in \Lambda_+$. In fact, one can extend it for $b \in H^1(L;\Lambda_0)$ exactly the same way as in Proposition 11.2, and we omit the details. We define the potential $PO(\mathfrak{b},b)$ by the equation (12.14):

$$PO(\mathfrak{b},b) = c \in \Lambda_+. \tag{12.15}$$

Definition 12.2. The set of the pairs $(\mathfrak{b},b)$ such that b is a weak bounding cochain of $(H(L;\Lambda_0),\{\mathfrak{m}_k^{\mathfrak{b}}\})$ is denoted as $\widehat{\mathcal{M}}_{weak,def}(L;\Lambda_0)$. $PO(\mathfrak{b},b)$ defines *the potential function* $PO : \widehat{\mathcal{M}}_{weak,def}(L;\Lambda_0) \to \Lambda_+$.

We also use the notation $PO^{\mathfrak{b}}(b)$, and $PO^{\mathfrak{b}}(b,u)$ sometimes for $PO(\mathfrak{b},b)$.

For $(\mathfrak{b}, b) \in \widehat{\mathcal{M}}_{weak,def}(L;\Lambda_0)$, we have the differential satisfying $m_1^{\mathfrak{b},b} \circ m_1^{\mathfrak{b},b} = 0$:

$$m_1^{\mathfrak{b},b} = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} m^{\mathfrak{b}}_{k+\ell+1}(b^{\otimes k}, x, b^{\otimes \ell}). \tag{12.16}$$

Definition 12.3 ([**FOOO**], Definition 3.8.61)**.** For $(\mathfrak{b}, b) \in \widehat{\mathcal{M}}_{weak,def}$, we define Floer cohomology with deformation $(\mathfrak{b}, b)$ by

$$HF((L,\mathfrak{b},b),(L,\mathfrak{b},b);\Lambda_0) = \frac{\mathrm{Ker}(m_1^{\mathfrak{b},b})}{\mathrm{Im}(m_1^{\mathfrak{b},b})}. \tag{12.17}$$

12.2. Construction of $\mathfrak{q}$ for toric orbifolds. In this section, we construct the operator $\mathfrak{q}$ using the moduli space of holomorphic (orbi-)discs to prove Theorem 12.1. Recall from Definition 12.1 that we consider bulk deformation by elements of H, where H is generated by the fundamental classes $[1_{\mathcal{X}_\nu}]$'s for $\nu \in Box'$ and by the divisors D_i's (for $i = 1, \ldots, m$).

First, we consider the relevant moduli spaces. Recall that we write $\underline{l} = \{1, \ldots, l\}$ and consider the map $\boldsymbol{x} : \underline{l} \to Box$, where a stable map $\big((\Sigma, \vec{z}, \vec{z}^+), w, \xi\big)$ is said to be of type $\boldsymbol{x}$ if for $i = 1, \ldots, l$,

$$ev_i^+\big((\Sigma, \vec{z}, \vec{z}^+), w, \xi\big) \in \mathcal{X}_{\boldsymbol{x}(i)}.$$

To include the interior intersection condition with toric divisors, we introduce the following notations. We use a function

$$\mathbf{p} : \underline{l} \to \{1, \ldots, B\}$$

to describe bulk intersection, and write $|\mathbf{p}| = l$. The set of all such $\mathbf{p}$ are denoted as $Map(\underline{l}, \underline{B})$. From $\mathbf{p}$, we define $\boldsymbol{x} : \underline{l} \to Box$ as follows.

$$\boldsymbol{x}(j) = \begin{cases} \nu_j & \text{if } \ m+1 \le j \le B \\ 0 & \text{if } \ \mathbf{p}(j) \in \{1, \ldots, m\} \end{cases}$$

We enumerate the set of all $j \in \underline{l}$ with $\boldsymbol{x}(j) = 0$ as $\{j_1, \ldots, j_{l_1}\}$.

We define a fiber product

$$\mathcal{M}^{main}_{k+1,l}(L(u), \beta; \mathbf{p}) = \mathcal{M}^{main}_{k+1,l}(L(u), \beta, \boldsymbol{x})_{(ev^+_{j_1}, \ldots, ev^+_{j_{l_1}})} \times_{X^{l_1}} \prod_{i=1}^{l_1} D_{\mathbf{p}(j_i)}. \tag{12.18}$$

The virtual dimension of the above fiber product is (see section 10.1)

$$n + \mu(\beta) + k + 2l + 1 - 3 - \sum_{j=1}^{l} 2\iota_{(\boldsymbol{x}(j))}. \tag{12.19}$$

We also remark that we take the fiber product only at smooth interior marked points, and hence the above fiber product is the usual fiber product, not the orbifold one.

The following lemma follows from Proposition 10.1, as in Lemma 6.3 of [**FOOO3**].

Lemma 12.3. *Moduli space $\mathcal{M}^{main}_{k+1,l}(L(u),\beta;\mathbf{p})$ has a T^n-equivariant Kuranishi structure, and the evaluation map*

$$ev = (ev_0, ev_1, \ldots, ev_k) : \mathcal{M}^{main}_{k+1,l}(L(u),\beta;\mathbf{p}) \to L(u)^{k+1} \tag{12.20}$$

is weakly submersive and T^n-equivariant. It is oriented and has a tangent bundle.

The consideration of the boundary of a moduli space is by now standard, and can be done as in [**FOOO3**] of Lemma 6.4. (We skip the details and refer readers to [**FOOO3**].)

Lemma 6.5 of [**FOOO3**] also generalizes to our situation. Let

$$\mathfrak{forget}_0 : \mathcal{M}^{\mathrm{main}}_{k+1,l}(L(u),\beta;\mathbf{p}) \to \mathcal{M}^{main}_{1,l}(L(u),\beta;\mathbf{p}) \tag{12.21}$$

be the forgetful map which forgets all the boundary marked points except the 0-th one. We may choose our Kuranishi structures so that (12.21) is compatible with $forget_0$ of Lemma 10.2.

Lemma 12.4 (c.f. Lemma 6.5 of [**FOOO3**])**.** *Fix $E > 0$. Then there exists a system of multisections $\mathfrak{s}_{\beta,k+1,l,\mathbf{p}}$ on $\mathcal{M}^{main}_{k+1,l}(L(u),\beta;\mathbf{p})$ for $\omega(\beta) < E$, $\mathbf{p} \in Map(\underline{l},\underline{B})$, satisfying the following properties.*

1) *They are transversal to zero section and invariant under T^n-action.*
2) *The multisection $\mathfrak{s}_{\beta,k+1,l,\mathbf{p}}$ is given by the pull-back of the multisection $\mathfrak{s}_{\beta,1,l,\mathbf{p}}$ via the forgetful map* (12.21)*.*
3) *The multisection at the boundary is compatible with those from its fiber product structures as in Lemma 6.5 of* [**FOOO3**]*.*
4) *For $l = 0$, the multisection $\mathfrak{s}_{\beta,k+1,0,\emptyset}$ is the same as the one defined in Lemma* 10.2*.*
5) *The multisection $\mathfrak{s}_{\beta,k+1,l,\mathbf{p}}$ is invariant under permutation of the interior marked points.*

Proof. This is proved by induction over the symplectic area $\omega(\beta)$. As we have T^n-equivariant perturbations, the transversalities are much easier to achieve, which was used in the proof of Lemma 6.5 in [**FOOO3**], regarding the manifold cases, and this continues to hold in the orbifold cases. q.e.d.

We use the above moduli spaces to define the operators $\mathfrak{q}_{\beta;k,l}$ as follows. We put

$$H(\mathbf{p}) = H_{\mathbf{p}(1)} \otimes \cdots \otimes H_{\mathbf{p}(l)} \in H^{\otimes l}.$$

Then, $\mathfrak{q}$ is defined as in (11.6) by pulling back differential forms and pushing forward:

$$\mathfrak{q}_{\beta;l,k}(H(\mathbf{p}); h_1, \ldots, h_k) = \frac{1}{l!}(ev_0)_!(ev_1, \ldots, ev_k)^*(h_1 \wedge \cdots \wedge h_k). \tag{12.22}$$

We define $\mathfrak{q}_{\beta;l,k}$ for $(\beta, l, k) \neq (0,0,0), (0,0,1)$ by the above and put (12.23)

$$\mathfrak{q}_{0;0,1}(h) = (-1)^{n+\deg h+1} dh, \quad \mathfrak{q}_{0;0,2}(h_1,h_2) = (-1)^{\deg h_1(\deg h_2+1)} h_1 \wedge h_2.$$

Remark 12.4. One needs to fix E_0 and construct $\mathfrak{q}_{\beta;k,l}$ for $\beta\cap\omega < E_0$ and take the inductive limit, due to Kuranishi perturbation (see sections 7.2 and 7.4 of [**FOOO**]). As in [**FOOO3**], we can use $A_{n,K}$ structure in place of A_∞ structure, and we omit the details.

We put $\mathfrak{q}_{l,k} = \sum_\beta T^{\omega\cap\beta/2\pi}\mathfrak{q}_{\beta;l,k}$, and by extending linearly to $H\otimes\Lambda_+^{\mathbb{R}}$, we obtain an operator $\mathfrak{q}_{l,k}$ for Theorem 12.1. The proof of (12.8) is the same as that of Theorem 2.1 of [**FOOO3**] and is omitted. By taking T^n-invariant differential forms on L, we in fact obtain a canonical model $(H(L(u);\Lambda_0(\mathbb{R})), \{m_k^{\mathfrak{b},\mathrm{can}}\}_{k=0}^\infty)$ as before.

12.3. Bulk orbi-potential of toric orbifolds. Recall that in section 11.3, we have discussed smooth potential PO for toric orbifolds. In this subsection, we discuss the bulk (orbi-)potential $PO^{\mathfrak{b}}$ (Definition 12.2) of toric orbifolds, which should be considered as a bulk deformation of the smooth potential PO.

Even for Fano orbifolds, it is very difficult to compute the bulk potential when we take $\mathfrak{b}$ from inertia components. The reason is related to the fact that constant orbi-spheres with several orbifold marked points are in general obstructed, and Chen and Ruan [**CR2**] introduced the Chen-Ruan cohomology ring of an orbifold from it. We have found holomorphic orbi-discs with one orbifold marked point, and proved its Fredholm regularity. But to consider bulk deformations $\mathfrak{b}$, we need to consider several insertions of $\mathfrak{b}$'s, and in general, even the constant orbi-spheres will make the relevant compactified moduli spaces obstructed. Hence, it is hard to compute them directly. We remark that in [**CCLT**], some of these bulk orbi-potentials are computed, which are then used to give geometric understanding of the (open) Crepant resolution conjecture and change of variable formulas.

We will define a notion of leading order potential for toric orbifold, which we can compute explicitly using the classification of basic orbi-discs. This will be enough to determine Floer cohomology deformed by $(b,\mathfrak{b})$.

First we consider the dimension restrictions. From (12.19), the moduli space $\mathcal{M}_{1,l}^{main}(L(u),\beta;\mathbf{p})$ contributes to the bulk potential if the following equality holds:
(12.24)
$$n+\mu(\beta)+1+2l-3-\sum_{j=1}^{l}2\iota_{(\boldsymbol{x}(j))}=n, \quad\text{or}\quad \mu(\beta)=2+\sum_{j=1}^{l}(2\iota_{(\boldsymbol{x}(j))}-2)$$
and $\beta\neq 0$. In such a case, note that the moduli space defined in Lemma 12.4,
$$\mathcal{M}_{1,l}^{main}(L(u),\beta;\mathbf{p})^{\mathfrak{s}_{\beta,1,l,\mathbf{p}}},$$
has a virtual fundamental *cycle*, because boundary strata involve moduli spaces of lower dimension, but due to T^n-equivariant condition, such

boundary contribution vanishes as the expected dimension is less than n as in Lemma 10.4. Hence we can define the following orbifold open Gromov-Witten invariants.

Definition 12.5. The number $c(\beta;\mathbf{p}) \in \mathbb{Q}$ is defined by

$$c(\beta;\mathbf{p})[L(u)] = ev_{0*}([\mathcal{M}^{main}_{1,l}(L(u),\beta;\mathbf{p})^{\mathfrak{s}_{\beta,1,l,\mathbf{p}}}]).$$

Lemma 12.5. *The number $c(\beta;\mathbf{p})$ is well-defined and independent of the choice of $\mathfrak{s}_{\beta,k+1}$ in Lemma* 12.4.

The lemma essentially follows from T^n-equivariance and a dimension formula as in the proof of Lemma 11.7 of [**FOOO2**]. Namely, the codimension one stratum should have an evaluation image of dimension $n-1$, but this should be zero because T^n-equivariance implies that any evaluation image is of dimension n.

From the classification results, Proposition 9.4 (5), we know the one-point orbifold disc invariants.

Lemma 12.6. *For $|\mathbf{p}| = 1$, we have*

$$c(\beta_a;\mathbf{p}) = \begin{cases} 0 & \text{if } \mathbf{p}(1) \neq a, \\ 1 & \text{if } \mathbf{p}(1) = a. \end{cases}$$

Lemma 12.7. *For $r \in H^1(L(u);\Lambda_+)$, $\beta \in H_2(X,L;\mathbb{Z})$, and $\mathbf{p} \in Map(\underline{l},\underline{B})$ satisfying the dimension condition* (12.24)*, we have*

$$\mathfrak{q}_{\beta;l,k}(H(\mathbf{p}); r,\ldots,r) = \frac{c(\beta;\mathbf{p})}{l!k!}(r(\partial\beta))^k \cdot PD([L(u)]).$$

Note that (12.24) is needed to have non-zero value by dimension counting. Once we have defined $c(\beta_a;\mathbf{p})$ in Definition 12.5, this lemma can be proved as Lemma 11.1, and we omit its proof.

From this calculation, as in Proposition 11.2, we have the following.

Proposition 12.8. *There is a canonical inclusion*

$$(H \otimes \Lambda_+) \times H^1(L(u);\Lambda_0) \hookrightarrow \widehat{\mathcal{M}}_{weak,def}(L(u)).$$

Remark 12.6. We do not know how to extend the above to $H \otimes \Lambda_0$. Namely, it is desirable in several cases to have a bulk insertion with energy zero, but it is hard to make it rigorously defined in the orbifold case. On the contrary, for toric manifolds, bulk deformation can be extended over Λ_0 for degree 2 classes of ambient symplectic manifold, because the related open Gromov-Witten invariants can be readily computed using a divisor equation.

We choose $\mathfrak{b} \in H \otimes \Lambda_+$ and $b \in H^1(L(u);\Lambda_0)$. Thus, we have a weak bounding cochain $(\mathfrak{b}, b)$, and Definition 12.2 defines the bulk potential $PO(\mathfrak{b}, b)$. If we set $\mathfrak{b} = 0$, we get $PO(0,b) = PO(b)$, the smooth potential discussed in section 11.3.

Next, we describe the leading order bulk potential for toric orbifolds. Leading order bulk potential is a part of the full potential, and can be explicitly computed by the classification of basic (orbi-)discs. Furthermore, we show in the next section that non-displaceability of a Lagrangian torus fiber can be obtained by studying the leading term equation, which will be derived from leading order bulk potential.

Let us write

$$\mathfrak{b} = \mathfrak{b}_{sm} + \mathfrak{b}_{orb}$$

where

$$\begin{cases} \mathfrak{b}_{sm} = \sum_{i=1}^{m} \mathfrak{b}_i D_i & \mathfrak{b}_i \in \Lambda_+ \\ \mathfrak{b}_{orb} = \sum_{\nu \in Box'} \mathfrak{b}_\nu 1_{\mathcal{X}_\nu} & \mathfrak{b}_\nu \in \Lambda_+ \end{cases}$$

Recall that for each $\nu_a \in Box'$, we denoted the corresponding lattice vector as $\boldsymbol{b}_a$.

Definition 12.7. We define the leading order potential $PO^{\mathfrak{b}}_{orb,0}(b)$ as

$$\begin{aligned} PO^{\mathfrak{b}}_{orb,0}(b) = &\sum_{j=1}^{m} T^{\ell_j(u)}(y_1)^{b_{j1}} \dots (y_n)^{b_{jn}} \\ &+ \sum_{\nu_a \in Box'} \mathfrak{b}_{\nu_a} T^{\ell_a(u)} (y_1)^{b_{a1}} \cdots (y_n)^{b_{an}}. \end{aligned} \tag{12.25}$$

Note that the first summations are the leading order terms $PO_0(b)$ of the smooth potential $PO(b)$ and the second summations are contributions from Box'. More precisely, in the classification of holomorphic orbi-discs (Corollary 6.3), we have found one-to-one correspondence between the basic holomorphic orbi-disc (modulo T^n-action) and twisted sectors Box' of the toric orbifold. These basic orbi-disc contributions are the new terms in $PO^{\mathfrak{b}}_{orb,0}(b) - PO_0(b)$, since $PO_0(b)$ are contributions from basic smooth holomorphic discs.

We remark that for the case of toric manifolds in [**FOOO3**], leading order bulk potential $PO^{\mathfrak{b}}_0(b)$ is the same as the leading order potential for the potential $PO_0(b)$ (without bulk), since all bulk contributions come from holomorphic discs (by adding interior marked points). But in our case of toric orbifolds, addition of $\mathfrak{b}_\nu$ allows holomorphic orbi-discs into the theory, and provides new terms in the leading order potential as well. Hence it is quite different from the case of manifolds.

It is important to note that the smooth potential PO_0 is independent of $\mathfrak{b}$, but $PO^{\mathfrak{b}}_{orb,0}(b)$ *depends* on the choice of $\mathfrak{b}_\nu$. In particular, we will see that the freedom to choose this coefficient $\mathfrak{b}_\nu$ allows us to find much more non-displaceable Lagrangian torus fibers in toric orbifolds.

In our applications (in the next section and in examples), we will choose simpler types of bulk deformations such as $\mathfrak{b}_\nu = c_\nu T^{\lambda_\nu}$ for some $c_\nu \in \mathbb{C}$ and $\lambda_\nu > 0$, but in general, one can work with more general cases. In fact, we may define leading order potential by just taking the term of $\mathfrak{b}_\nu$ with the smallest T-exponent for each ν, as it will give rise to the same leading term equation later on.

To discuss the general form of the bulk potential, we need a notion of G-gappedness for a discrete monoid G, for which we refer readers to Definition 3.3 of [**FOOO3**]. The discrete monoid G in this setting is defined as in [**FOOO3**].

(12.26)
$$G(X) = \langle \{\omega(\beta)/2\pi \mid \beta \in \pi_2(X) \text{ is realized by a holomorphic sphere}\}\rangle.$$

The actual discrete monoid to be used, G_{bulk}, will be defined in Definition 13.2, and $G(X)$ is a subset of G_{bulk}.

We discuss the general form of the bulk potential for toric orbifolds, roughly given by the leading order bulk potential with additional higher order terms.

Theorem 12.9 (c.f. [**FOOO3**], Theorem 3.5)**.** *Let X be a compact symplectic toric orbifold and let $\mathfrak{b} \in H(\Lambda_+)$ be a G_{bulk}-gapped element. Then the difference of the bulk orbi-potential and its leading order potential can be written as follows:*

$$PO(\mathfrak{b}; b) - PO^{\mathfrak{b}}_{orb,0}(b) = \sum_{\zeta=1}^{\infty} c_\zeta y_1^{v'_{\zeta,1}} \cdots y_n^{v'_{\zeta,n}} T^{\ell'_\zeta + \rho_\zeta}, \tag{12.27}$$

for some $c_\zeta \in \mathbb{Q}$, $e^i_\zeta \in \mathbb{Z}_{\geq 0}$, $\rho_\zeta \in G_{bulk}$, and $\rho_\zeta > 0$, such that $\sum_{i=1}^B e^i_\zeta > 0$. Here

$$v'_{\zeta,k} = \sum_{a=1}^B e^a_\zeta b_{a,k}, \quad \ell'_\zeta = \sum_{a=1}^B e^a_\zeta \ell_a. \tag{12.28}$$

If we have infinitely many non-zero c_ζ's, we have

$$\lim_{\zeta\to\infty} \rho_\zeta = \infty. \tag{12.29}$$

Proof. The proof is along the same lines as that of [**FOOO3**], Theorem 3.5. Let $\mathfrak{b} = \sum_{a=1}^B \mathfrak{b}_a H_a$ with $\mathfrak{b}_a \in \Lambda_+$, where $\mathfrak{b}_a$ is G_{bulk}-gapped. Note that $c(\beta : \mathbf{p})$ determines $\mathfrak{q}_{\beta;|\mathbf{p}|,k}(H(\mathbf{p}); b, \ldots, b)$ from Lemma (12.7). Hence, proceeding as in (11.14), we obtain that

$$\begin{aligned} PO(\mathfrak{b}; b) &= \sum_{\beta,\mathbf{p},k} \mathfrak{b}^{\mathbf{p}} T^{\omega(\beta)/2\pi} \frac{c(\beta;\mathbf{p})}{k!|\mathbf{p}|!} (b(\partial\beta))^k \\ &= \sum_{\beta,\mathbf{p}} \frac{1}{|\mathbf{p}|!} \mathfrak{b}^{\mathbf{p}} T^{\omega(\beta)/2\pi} c(\beta;\mathbf{p}) \exp(b(\partial\beta)). \end{aligned} \tag{12.30}$$

Now, we consider the cases of $|\mathbf{p}| = 0, 1$ or $|\mathbf{p}| \geq 2$. If $|\mathbf{p}| = 0$, there is no interior marked point, and hence there is no orbifold disc contributions. The statement in this case follows from (11.13).

When $|\mathbf{p}| = 1$, the case where the interior marked point is smooth is similar to the case of a smooth manifold, and it is enough to consider the case where the interior marked point is an orbifold marked point. In this case, additional orbi-disc contributions for basic orbi-disc classes are computed from Lemma 12.6. For other homology classes, the statement follows from Proposition 9.4.

We next study terms for $|\mathbf{p}| \geq 2$. We first consider the case $\beta = \beta_a$ for $a = 1, \dots, B$. In this case we obtain the following term:

$$cT^{\ell_a(u)+\rho} y^{\boldsymbol{b}_a}, \tag{12.31}$$

where $c \in \mathbb{Q}$ and ρ is obtained by summing over the exponents of $\mathfrak{b}_{\mathbf{p}(j)}$ for various j. As $l \neq 0$ and $\mathfrak{b}_{\mathbf{p}(j)} \in \Lambda_+$, this is non-zero. Hence $\rho \in G_{bulk} \setminus \{0\}$. Therefore the form of (12.31) equals the right hand side of (12.27).

Now, we consider $\beta \neq \beta_a$ $(a = 1, \dots, B)$. We may assume that $c(\beta; \mathbf{p}) \neq 0$. Then by Proposition 9.4 (4), we have e^i and ρ' satisfying

$$\frac{\omega(\beta)}{2\pi} = \sum_{a=1}^{B} e^a \ell_a(u) + \rho'.$$

Here $e^a \in \mathbb{Z}_{\geq 0}$, $\sum e^a > 0$, and ρ' corresponds to a sum of symplectic areas of holomorphic spheres (divided by 2π). Hence these give rise to an expression

$$cT^{\sum_a e^a \ell_a(u)+\rho+\rho'} y^{\sum e^a \boldsymbol{b}_a},$$

where $c \in \mathbb{Q}$ and ρ is obtained by summing over the exponents of $\mathfrak{b}_{\mathbf{p}(j)}$ for various j. This form agrees with the right hand side of (12.27).

The proof of (12.29) is based on the idea that to have infinitely many terms, either infinitely many bulk insertions contribute to the potential, or the contribution of energy from the sphere component should go to infinity. The proof is similar to that of [**FOOO3**], and omitted. q.e.d.

Theorem 11.4 and Theorem 11.5 can be easily generalized to the bulk setting.

Theorem 12.10. *Let $b = \sum x_i \mathbf{e}_i$, and $\mathfrak{b} \in H(\Lambda_+)$. Theorem 11.4 and Theorem 11.5 hold in the bulk case too by replacing PO, $(L(u), b)$ with $PO^{\mathfrak{b}}$, $(L(u), (\mathfrak{b}, b))$ respectively.*

Once the bulk orbi-potential is set up, the proofs of the theorems in [**FOOO3**] are rather algebraic, and hence can be easily adapted to the case of orbifolds. This applies to the proof of Theorem 12.10 and we omit the details. We also remark that the above can be extended to the following general form as in [**FOOO3**]: If $(\mathfrak{b}, b)$ satisfies

$$\mathfrak{y}_i \frac{\partial PO^u}{\partial y_i}(\mathfrak{b}, \mathfrak{y}) \equiv 0 \mod T^{\mathcal{N}}, \tag{12.32}$$

then we have

$$HF((L(u_0),\mathfrak{b},b),(L(u_0),\mathfrak{b},b);\Lambda_0/T^{\mathcal{N}}) \cong H(T^n;\Lambda_0/T^{\mathcal{N}}). \tag{12.33}$$

13. Leading term equation and bulk deformation

From Theorem 12.10, if the (bulk) potential function is known, then Floer cohomology is determined by considering the critical points of the potential function. But for toric orbifolds, the full bulk potential is very difficult to compute even for Fano orbifolds. For toric manifolds, the notion of leading term equations was introduced in section 4 of [**FOOO3**], so that one can determine the Floer cohomology only from the knowledge of leading order potential, which is explicitly calculable. Namely, given the solutions of leading term equations, they show that there exists a bulk deformation $\mathfrak{b}$ such that potential $PO(\mathfrak{b},b)$ has an actual critical point. The bulk-balanced Lagrangian fibers can be located by this method.

In this section, we define a leading term equation for toric orbifolds. The construction is similar to that of section 4 of [**FOOO3**]. Instead of repeating their construction, we make our construction in the same form as that of [**FOOO3**], so that once we prove Proposition 13.1 in this paper, which plays the role of Proposition 4.14 of [**FOOO3**], the rest of the construction, which is rather long, becomes the same and can be omitted.

Note that leading term equations are determined from the leading order equation (defined in Definition 12.7), and for toric orbifolds, they depend on $\mathfrak{b}_\nu$. Thus, the crucial difference from [**FOOO3**] is that [**FOOO3**] deals with the leading term equation of $PO_0(b)$ (which is independent of $\mathfrak{b}$), whereas we deal with that of $PO^{\mathfrak{b}}_{orb,0}$ which depends on the choice of $\mathfrak{b}_{orb}$.

We first set up some notations. Recall from (1.4) that bulk deformation terms corresponding to twisted sectors are

$$\mathfrak{b}_{orb} = \sum_{\nu\in Box'} \mathfrak{b}_\nu 1_{\mathcal{X}_\nu},$$

and we denote

$$\mathfrak{b}_\nu = \mathfrak{b}_{\nu,0} + \mathfrak{b}_{\nu,+} \tag{13.1}$$

where $\mathfrak{b}_{\nu,0} = c_\nu T^{\lambda_\nu}$ for some $c_\nu \in \mathbb{C}$, $\lambda_\nu > 0$, and $\mathfrak{b}_{\nu,+}$ satisfies $\mathfrak{v}_T(\mathfrak{b}_{\nu,+}) > \lambda_\nu$.

To define the leading term equation in our case, we fix $u \in P$ and bulk deformation $\mathfrak{b}$ as above. We start with relabeling the indices $a = 1,\ldots,B$. Recall that for $1 \leq a \leq m$, $\boldsymbol{b}_a \in N$ is the stacky vector corresponding to the a-th facet of the polytope, and ℓ_a is the symplectic area of the corresponding basic disc (intersecting that facet). For $m+1 \leq a \leq B$, a labels elements of Box', corresponding to the lattice vector $\boldsymbol{b}_a$

in N, and the area of the corresponding basic orbi-disc is ℓ_a (see (7.1) for ℓ_a in this case).

We compare the areas ℓ_a for $a = 1, \dots, m$ and $\ell_a + \lambda_{\nu_a}$ for $a = m + 1, \dots, B$, because the orbi-disc for ν_j has an additional energy coming from $\mathfrak{b}_{\nu_j,0} = c_\nu T^{\lambda_{\nu_j}}$. We enumerate energy levels as

$$(13.2)\qquad \begin{aligned}&\{S_l \mid l = 1, 2, \dots, \mathcal{L}\}\\ &\quad = \{\ell_i(u), \ell_j(u) + \lambda_{\nu_j} \mid i = 1, 2, \dots, m, j = m + 1, \dots, B\},\end{aligned}$$

so that $S_i < S_{i+1}$ and $S_i \in \mathbb{R}_+$. Note that these are the exponents of T in the terms in $PO^{\mathfrak{b}}_{orb,0}$. The indices of $\boldsymbol{b}_k$'s can be re-enumerated: We write $\{\boldsymbol{b}_{l,1}, \dots, \boldsymbol{b}_{l,a(l)}\}$ for all $\boldsymbol{b}_k$'s satisfying

$$\ell_k(u) = S_l, k \le m \ \text{ or } \ \ell_k(u) + \lambda_{\nu_k} = S_l, k \ge m.$$

By the following procedure, we determine an optimal energy level, so that $\boldsymbol{b}_a$'s with smaller or equal T-exponent span $N_{\mathbb{R}}$. Let $A_l^\perp \subset N_{\mathbb{R}}$ be the $\mathbb{R}$-vector space generated by $\boldsymbol{b}_{l',r}$ for $l' \le l$, $r = 1, \dots, a(l')$. The smallest integer l such that $A_l^\perp = N_{\mathbb{R}}$ is denoted by K. Let $d(l)$ be the difference in the dimension of $A_l^\perp$ and $A_{l-1}^\perp$, and $d(1)$ the dimension of $A_1^\perp$. Then, $\sum_{l=1}^K d(l) = n$. In the notation $\boldsymbol{b}_{l,r}$, we always have $l \le K$.

To relate to the original indices, integer $i(l,r) \in \{1, \dots, B\}$ is defined by $\boldsymbol{b}_{l,r} = \boldsymbol{b}_{i(l,r)}$. We renumber the set of $\boldsymbol{b}_i$'s for $i = 1, \dots, B$ as

$$\{\boldsymbol{b}_{l,r} \mid l = 1, \dots, K, r = 1, \dots, a(l)\} \cup \{\boldsymbol{b}_i \mid i = \mathcal{K} + 1, \dots, B\}$$

where $\mathcal{K}$ is given by $\mathcal{K} = \sum_{l=1}^K a(l)$.

The rest of the procedure to define the leading term equation is similar to that of [**FOOO2**], section 4, and hence we only briefly sketch the construction. Now, at each energy level S_l, the vectors $\boldsymbol{b}_{l,1}, \dots, \boldsymbol{b}_{l,a(l)}$ may not be linearly dependent (if $a(l) \ne d(l)$), and the next procedure chooses a suitable basis of the subspace spanned by these vectors. We denote the dual basis of $\{\mathbf{e}_i\}$ of $H^1(L(u);\mathbb{Z}) \cong M$ by $\{\mathbf{e}_i^*\}$, which becomes the basis of $N_{\mathbb{R}}$. Then, basis $\mathbf{e}^*_{l,s}$ of $N_{\mathbb{R}}$ can be chosen so that $\mathbf{e}^*_{1,1}, \dots, \mathbf{e}^*_{l,d(l)}$ becomes a $\mathbb{Q}$-basis of $A_l^\perp$ and also that any lattice vector of $A_l^\perp$ is given as an integer linear combination of $\mathbf{e}^*_{l,s}$'s. In particular $\mathbf{e}_i^*$ can be written as an integer linear combination of $\mathbf{e}^*_{l,s}$'s.

By considering $\mathbf{e}_i^*$ and $\mathbf{e}^*_{l,s}$ as functions on $M_{\mathbb{R}}$, we may write them as x_i and $x_{l,s}$, and define $y_{l,s}$ as $e^{x_{l,s}}$. Then, y_i is given as a monomial of $y_{l,s}$'s and hence so is $y^{\boldsymbol{b}_{l,r}}$ (see Lemma 13.1 of [**FOOO3**]).

In this way, we can define the T^{S_l} part of the potential $PO^{\mathfrak{b}}_{orb,0}$:

$$(13.3)\qquad (PO^{\mathfrak{b}}_{orb,0})_l = \sum_{r=1, i(l,r)\le m}^{a(l)} y^{\boldsymbol{b}_{l,r}} + \sum_{r=1, i(l,r)>m}^{a(l)} c_{\nu_{i(l,r)}} y^{\boldsymbol{b}_{l,r}}, \ \text{ for } \ l = 1, \dots, K,$$

where $c_{\nu_{i(l,r)}}$ is given in (13.1). Then, $(PO^{\mathfrak{b}}_{orb,0})_l$ can be written as a Laurent polynomial of $y_{l',s}$ with $s \le d(l')$ and $l' \le l$.

Definition 13.1. The *leading term equation* of $PO^{\mathfrak{b}}_{orb,0}$ (or that of $PO^{\mathfrak{b}}(b)$) is the system of equations

$$\frac{\partial (PO^{\mathfrak{b}}_{orb,0})_l}{\partial y_{l,s}} = 0, \quad \text{for } \; l = 1, \ldots, K;\; s = 1, \ldots, d(l) \tag{13.4}$$

with $y_{l,s} \in R \setminus \{0\}$.

Note that we only take derivatives of $(PO^{\mathfrak{b}}_{orb,0})_l$ with respect to the variables $y_{l,s}$ for $s = 1, \ldots, d(l)$, but not with respect to the variables $y_{l',s'}$ for $l' < l$.

In view of Theorem 12.10, we need critical points of the actual bulk potential $PO^{\mathfrak{b}}(b)$, but the solutions of the leading term equation (13.4) equal those of $PO^{\mathfrak{b}}(b)$ from Lemma 4.4 of [**FOOO3**]. (The solutions of the equation $y_k \frac{\partial PO^{\mathfrak{b}}(b)}{\partial y_k} = 0$ correspond to the solutions of the equation $y_{l,s}\frac{\partial PO^{\mathfrak{b}}(b)}{\partial y_{l,s}} = 0$ by Lemma 4.2 of [**FOOO**].)

The following Proposition 13.1 on the shape of bulk orbi-potential, is analogous to Proposition 4.14 of [**FOOO3**]. We first define the monoid G_{bulk}. Recall the definition of $G(X)$ from (12.26). We define

$$\begin{aligned} G(L(u)) = \langle \{ \omega(\beta)/2\pi \mid \beta \in H_2(X, L(u)) \\ \text{corresponds to a holomorphic orbi-disc}\} \rangle. \end{aligned} \tag{13.5}$$

Definition 13.2. G_{bulk} is a discrete submonoid of $\mathbb{R}$ which is generated by $G(X)$ and the subset

$$\{\lambda - S_l \mid \lambda > S_l, \lambda \in G(L(u)), \quad l = 1, \ldots, K, \} \subset \mathbb{R}_+.$$

Note that $G(L(u)) \subset G_{bulk}$.

Condition 13.3. Bulk deformation $\mathfrak{b}$ is of the form

$$\mathfrak{b} = \sum_{l=1}^{K} \sum_{r=1}^{a(l)} \mathfrak{b}_{i(l,r)} H_{i(l,r)} \in H(\Lambda_+) \tag{13.6}$$

such that each $\mathfrak{b}_{i(l,r)}$ is G_{bulk}-gapped. Here $\mathfrak{b}_{i(l,r)}$ means $\mathfrak{b}_{\nu_{i(l,r)}}$ in case $i(l,r) > m$.

The main proposition to prove in our orbifold case is the following.

Proposition 13.1 (c.f. Proposition 4.14 [**FOOO3**]). *Assume that* $\mathfrak{b}$ *satisfies Condition* (13.3) *and consider*

$$\mathfrak{b}' = \mathfrak{b} + cT^{\lambda} H_{i(l,r)}, \tag{13.7}$$

for $c \in R$, $\lambda \in G_{bulk} + \mathfrak{b}_{\nu_{i(l,r)},0}$, $l \le K$.

Then the difference of the corresponding bulk orbipotentials is given by

$$
\begin{aligned}
PO^{\mathfrak{b}'}(b) - PO^{\mathfrak{b}}(b) &= cT^{\lambda+\ell_{i(l,r)}(u)} y^{\boldsymbol{b}_{i(l,r)}} + \sum_{h=2}^{\infty} c_h T^{h\lambda+\ell_{i(l,r)}(u)} y^{\boldsymbol{b}_{i(l,r)}} \\
&\quad + \sum_{h=1}^{\infty} \sum_{\zeta} c_{h,\zeta} T^{h\lambda+\ell'_{\zeta}(u)+\rho_\zeta} y^{\boldsymbol{b}_\zeta}. \qquad (13.8)
\end{aligned}
$$

Here $c_h, c_{h,\zeta} \in R, \rho_\zeta \in G_{bulk}$, *and there exist* $e^i_\zeta \in \mathbb{Z}_{\geq 0}$ *such that* $\boldsymbol{b}_\zeta = \sum e^i_\zeta \boldsymbol{b}_i$, $\ell'_\zeta = \sum e^i_\zeta \ell_i$, *and* $\sum_i e^i_\zeta > 0$. *Also, in the third summand of the right hand side of* (13.8)*, in the case that* $h = 1$ *and* $\rho_\zeta = 0$*, we have* $c_{h,\zeta} = 0$.

Remark 13.4. In Proposition 4.14 of [**FOOO**], the last assertion is not written, but is shown in their proof and it is needed in the induction for the theorem.

Proof. We first remark that the proof in the toric orbifold case is somewhat different than that of toric manifolds. For toric manifolds, the bulk deformation contribution when $\mathfrak{b}$ is from toric divisors is explicitly computable for the basic disc classes by homology arguments similar to the divisor equation (see for example Proposition 4.7 of [**FOOO3**]). But, for toric orbifolds, such arguments do not work for basic disc classes, as there is no divisor type equation for orbifold marked points. As we will see, the proposition does not require this explicit computation.

Note that the dimension condition (12.24) needs to be satisfied for a possible contribution to the potential. We will divide the contribution of

$$PO^{\mathfrak{b}'}(b) - PO^{\mathfrak{b}}(b) \qquad (13.9)$$

into several cases and subcases. First, consider the terms corresponding to the case with no interior marked points. As they do not have insertions from $\mathfrak{b}$, they give no contribution to (13.9).

Next, consider the case of one interior marked point. Recall that a one-point open orbifold Gromov-Witten invariant is computed in Lemma 12.6, and it is non-zero only if the disc class is of $\beta_{i(l,r)}$, in which case we get the first term of (13.8).

Thus, from now on, we consider the case with two or more interior marked points. We remark that if the bulk insertion $\mathfrak{b}$ is used in all of the interior insertions, then obviously such terms contribute 0 to (13.9). So we assume that at least one of the interior marked points is used for the insertion of $T^\lambda H_{i(l,r)}$. We divide it into three cases as follows:

1) $\beta = \beta_{i(l,r)}$.

We consider the following two subcases:

a) All bulk inputs are $T^\lambda H_{i(l,r)}$: In this case, it is easy to see that the contribution is equal to the second term of the RHS of (13.8).
b) Both bulk inputs $T^\lambda H_{i(l,r)}$ and $\mathfrak{b}$ are used at least once: In this case, it contributes to the 3rd term of the RHS of (13.8), with $h \geq 1$, $\ell'_\zeta(u) = \ell_{i(l,r)}$, and $\rho_\zeta > 0$ since it receives non-trivial contribution from $\mathfrak{b}$.

2) β equals the basic disc class β_a for $a = 1, \ldots, B$, and $a \neq i(l,r)$.
a) All bulk inputs are $T^\lambda H_{i(l,r)}$: The possible contribution is to the third term of RHS of (13.8), with $h \geq 2$, $\ell'_\zeta(u) = \ell_a$, and $\rho_\zeta = 0$.
b) Both bulk inputs $T^\lambda H_{i(l,r)}$, and $\mathfrak{b}$ are used at least once: In this case, it contributes to the 3rd term of RHS of (13.8), with $h \geq 1$, $\ell'_\zeta(u) = \ell_a$, and $\rho_\zeta > 0$ since it receives non-trivial contribution from $\mathfrak{b}$.

3) $\beta \neq \beta_a$ for $a = 1, \ldots, B$.
We may write

$$\beta = \sum_{i=1}^{B} e^i_\beta \beta_i + \sum_j \alpha_{\beta,j}.$$

Then we have

$$\begin{aligned} \frac{\omega(\beta)}{2\pi} &= \sum_{i=1}^{B} e^i_\beta \ell_i(u) + \sum_j \frac{\omega(\alpha_{\beta,j})}{2\pi} \\ exp(b(\partial\beta)) &= y^{\sum_{i=1}^{B} e^i_\beta \boldsymbol{b}_i}. \end{aligned}$$

We have $e^i \geq 0$ and $\sum_i e^i_\beta > 0$. Thus the contributions belong to the third term of (13.8) with $\ell'_\zeta(u) = \sum_i e^i_\beta \ell_i(u)$ and ρ_ζ is the sum of the contribution from the sphere class $\omega(\alpha_{\beta,j})$ together with contributions from $\mathfrak{b}$. Now we consider the following subcases:
a) All bulk inputs are $T^\lambda H_{i(l,r)}$: The possible contribution is to the third term of the RHS of (13.8), with $h \geq 2$, $\ell'_\zeta(u)$, and ρ_ζ as described above.
b) Both bulk inputs $T^\lambda H_{i(l,r)}$, and $\mathfrak{b}$ are used at least once: in this case, it contributes to the 3rd term of RHS of (13.8), with $h \geq 1$, $\ell'_\zeta(u)$ and ρ_ζ as described above, and we have $\rho_\zeta > 0$ since it receives non-trivial contribution from $\mathfrak{b}$.

This proves the proposition. q.e.d.

For convenience, given $\mathfrak{b}$ as in (1.4), (13.1), we denote (by taking the least exponent terms)

$$\mathfrak{b}_{orb,0} = \sum_{\nu \in Box'} \mathfrak{b}_{\nu,0} 1_{\mathcal{X}_\nu}.$$

The leading order equation and the leading term equation of $PO^{\mathfrak b}_{orb,0}$ (see Definition 13.1) only depend on $\mathfrak b_{orb,0}$, not the entire $\mathfrak b_{orb}$.

Theorem 13.2 (c.f. Theorem 4.5 of [**FOOO3**]). *The following conditions on u are equivalent:*

1) *The leading term equation of $PO^{\mathfrak b}_{orb,0}(u)$ has a solution $y_{l,s} \in R \setminus \{0\}$ $(l = 1, \dots, K, s = 1, \dots, d(l))$.*
2) *There exists $\widetilde{\mathfrak b} \in H(\Lambda_+)$ such that $\widetilde{\mathfrak b}_{orb,0} = \mathfrak b_{orb,0}$ and $PO^{\widetilde{\mathfrak b}}(u)$ has a critical point on $(\Lambda_0 \setminus \Lambda_+)^n$.*
3) *There exists $\widetilde{\mathfrak b} \in H(\Lambda_+)$ such that $\widetilde{\mathfrak b}_{orb,0} = \mathfrak b_{orb,0}$ and $y_{l,s} \in R \setminus \{0\}$ $(l = 1, \dots, K, s = 1, \dots, d(l))$ in the item (1) above is a critical point of $PO^{\widetilde{\mathfrak b}}(u)$.*

Proof. We have set up our case in a form similar to that of [**FOOO3**] so that the same proof of Theorem 4.5 in [**FOOO3**] works in our case too, as we have proved Proposition 13.1, which plays the role of Proposition 4.14 of [**FOOO3**]. We refer readers to [**FOOO3**] for the full proof and briefly explain the rest of the procedure to prove Theorem 13.2. The argument is exactly the same, except the point regarding $\mathfrak b_{orb,0}$.

Given a solution $\eta_{l,s}$ of the leading term equation, we need to find $\mathfrak b$ such that $\eta_{l,s}$ satisfies the actual critical point equation:

$$\eta_{l,s} \frac{\partial PO^{\widetilde{\mathfrak b}}}{\partial y_{l,s}}(\eta) = 0. \tag{13.10}$$

We first enumerate elements of G_{bulk} so that

$$G_{bulk} = \{\lambda^b_j \mid j = 0, 1, 2 \dots\}$$

where $0 = \lambda^b_0 < \lambda^b_1 < \cdots$.

Then we define $\widetilde{\mathfrak b}$ inductively by choosing $\widetilde{\mathfrak b}(k)$ for each k for the terms with energy $S_l + \lambda^b_k$, and also for $1 \le l \le K$ (see Definition 4.15 of [**FOOO3**]).

First, we take

$$\widetilde{\mathfrak b}(0) = \mathfrak b_{orb,0}.$$

If the critical point equation (13.10) is satisfied up to the level k, then we introduce the bulk deformation $\widetilde{\mathfrak b}(k)$ to make the equation (13.10) satisfied up to level $k+1$ (see [**FOOO**, Proposition 4.18]).

In this process, the equation (13.10) is satisfied up to $S_l + \lambda^b_k$ hence we need to kill the error terms with T-exponent $S_l + \lambda^b_{k+1}$. This can be done by choosing appropriate $\widetilde{\mathfrak b}(k+1)$, and using Proposition 13.1 to cancel the error term with the first term of the RHS of (13.8). As two other terms of the RHS of (13.8) have higher T-exponent, it does not introduce any other error terms on the level $k+1$. Note that we need to choose λ of Proposition 13.1 so that $\lambda + \ell_{i(l,r)}(u)$ equals $S_l + \lambda^b_{k+1}$. As

S_l equals $\ell_{i(l,r)}(u) + \lambda_{\nu_{i(l,r)}}$, we choose λ here to be $\lambda^b_{k+1} + \lambda_{\nu_{i(l,r)}}$. Thus, the leading term $\widetilde{\mathfrak{b}}_{orb,0}$ is not changed and equals $\mathfrak{b}_{orb,0}$.

Then one takes the limit as $k \to \infty$ to define $\widetilde{\mathfrak{b}}$ such that the equation (13.10) is satisfied. We refer readers to section 4 of [**FOOO3**] for details.

q.e.d.

14. Floer homology of Lagrangian intersections in toric orbifolds

So far, we have discussed the Bott-Morse version of Floer cohomology of Lagrangian submanifolds. In this section, we discuss the Lagrangian Floer cohomology between two Lagrangian submanifolds L and $\psi_1(L)$, for a Hamiltonian isotopy ψ_1, and its relationship to A_∞-algebra and bulk deformed A_∞-algebra, which are constructed in the previous sections. We note that the Lagrangian submanifold L lies in the smooth part of the orbifold $\mathcal{X}$.

There are two versions of Floer cohomology of Lagrangian intersections, as we have two versions of A_∞-algebras. Namely, there is a smooth Lagrangian Floer cohomology where we consider J-holomorphic strips and discs from a smooth domain. By a *smooth domain*, we mean that the domain does not have orbifold singularities (but could have nodal singularity). We emphasize that the maps from the smooth domain can intersect orbifold points, as we have seen in the case of smooth holomorphic discs corresponding to stacky vectors $\boldsymbol{b}_i$ for $i = 1, \ldots, m$.

And there is a version thatincludes orbifold J-holomorphic strips and discs, which are maps from an orbifold domain. To denote the orbifold structure of the domain, we have introduced orbifold marked points in the interior of the Riemann surface, and their deformation theory is entirely analogous to that of a Riemann surface with interior marked points. Namely, orbifold marked points cannot disappear, be created, or be combined when we consider sequences of orbifold J-holomorphic maps of a given type.

Thus, when we consider only maps from smooth domains (into an orbifold), a degeneration which appears in the compactification of the moduli space of such maps is still from a smooth domain. Hence, we have a smooth Floer theory for orbifolds. Such theory still is non-trivial. Namely, we show in subsection 15.2 that smooth Floer theory finds a central fiber to be non-displaceable in weighted projective spaces.

But Lagrangian Floer theory involving orbifold strips and discs provides much more information, as we will see in several examples (actually the example of a teardrop already shows such phenomena). Yet to have an orbifold structure in the domain strip or disc, we need an orbifold marked point to record the orbifold structure of the domain. Hence this always requires interior marked points, and hence they appear as a *bulk deformation theory* of the smooth theory.

14.1. Smooth Lagrangian Floer homology. First, we consider Hamiltonian vector fields in an effective orbifold $\mathcal{X}$. By definition, a smooth function $H : \mathcal{X} \to \mathbb{R}$ is a function $H : X \to \mathbb{R}$, which locally has its lifting $\tilde{H}_V := H \circ \pi$ in any uniformizing chart (V, G, π) such that $\tilde{H}_V$ is smooth. Note that $\tilde{H}_V$ is invariant under G-action: $\tilde{H}_V(g \cdot x) = \tilde{H}_V(x)$. Hamiltonian vector field X_H can be defined by $i_{X_{\tilde{H}}}\omega = d\tilde{H}$, and $X_{\tilde{H}}$ is preserved by G-action because the symplectic form ω (on the chart V) is also invariant.

Hamiltonian isotopy ψ_t^H of the flow X_H is well-defined without much difficulty as we consider effective orbifolds: It is well-known that effective orbifolds can be always considered as a global quotient of a manifold, say M, by a compact Lie group action, and one can use this presentation to define the flow of a vector field, by integrating the flow after pull-back to the manifold M.

One can also consider time-dependent Hamiltonian functions (we still denote it by H for simplicity), and define time-dependent Hamiltonian isotopy. The resulting Hamiltonian isotopies are regular (in the sense of Definition 2.2), and hence, are good maps and the related group homomorphisms are isomorphisms, as the inverses are also good. This implies the following simple lemma.

Lemma 14.1. *For any Hamiltonian isotopy ψ_t^H, the isotropy group of the point x and $\phi_t^H(x)$ are isomorphic. This in particular implies that by a Hamiltonian isotopy, smooth points always move to smooth points of an orbifold.*

In particular, for our Lagrangian torus fiber, which lies away from the singular set $\Sigma\mathcal{X}$ of toric orbifold $\mathcal{X}$, $\psi_t^H(L)$ also does not intersect $\Sigma\mathcal{X}$.

For smooth Lagrangian Floer theory, we only consider J-holomorphic strips and discs (not orbifold ones). We may additionally consider smooth interior marked points to consider smooth bulk deformations, but we will not consider interior marked points here.

Lagrangian intersection Floer cohomology between L and $\psi_1^H(L)$, is constructed from the A_∞-bimodule.

Theorem 14.2 (c.f. Theorem 3.7.21 of [**FOOO**], Theorem 15.1 of [**FOOO2**]). *Let (L, L') be an arbitrary relatively spin pair of compact Lagrangian smooth submanifolds. Then the family $\{\mathfrak{n}_{k_1,k_2}\}$ of operators*

$$B_{k_1}(C(L)[1]) \mathbin{\widehat{\otimes}_{\Lambda_0}} C(L, L') \mathbin{\widehat{\otimes}_{\Lambda_0}} B_{k_1}(C(L')[1]) \to C(L, L')$$

for $k_1, k_2 \geq 0$ define a left $(C(L), m)$ and right $(C(L'), m')$ filtered A_∞-bimodule structure on $C(L, L')$.

The construction of such an A_∞-bimodule is standard, by considering the compactified moduli space of J-holomorphic strips. Note that here we are only considering J-holomorphic maps from stable strips *without* any orbifold marked point insertions, and also the Lagrangian submanifolds are away from the orbifold loci. Hence the construction for the theorem would be a direct adaptation of the proofs of [**FOOO**] and [**FOOO2**].

The above includes the case of clean intersection also, and in the case of $L = L'$ the maps $\mathfrak{n}_{k_1,k_2}$ are defined as $\mathfrak{n}_{k_1,k_2} = m_{k_1+k_2+1}$.

Now, let L, L' be weakly unobstructed. We define $\delta_{b,b'} : C(L, L') \to C(L, L')$ by

$$\delta_{b,b'}(x) = \sum_{k_1,k_2} \mathfrak{n}_{k_1,k_2}(b^{\otimes k_1} \otimes x \otimes b'^{\otimes k_2}) = \widehat{\mathfrak{n}}(e^b, x, e^{b'}).$$

One can check that the equation $\delta_{b,b'} \circ \delta_{b,b'} = 0$ holds if the potential functions $PO(b)$ and $PO(b')$ agree. In such a case, Floer cohomology is defined by

$$HF((L, b), (L', b'); \Lambda_0) = \operatorname{Ker} \delta_{b,b'} / \operatorname{Im} \delta_{b,b'}.$$

Proposition 14.3 (c.f. Lemma 12.9 of [**FOOO2**], section 5.3 of [**FOOO**])**.** *For the case of $L' = \psi(L)$ and $b' = \psi_* b$ we have*

$$HF((L, b), (L', b'); \Lambda) \cong HF((L, b), (L, b); \Lambda)$$

In particular, we have

$$\#(\psi(L) \cap L) \geq \operatorname{rank}_\Lambda HF((L, b), (L, b); \Lambda).$$

This proposition explains the Hamiltonian invariance of Lagrangian Floer homology, by considering an isomorphism to the Bott-Morse model. Construction of such an isomorphism is by now standard, and our case is also analogous since we are not considering orbifold marked points. We can use this proposition to obtain non-displaceability results, but we will see that the consideration of bulk deformation by twisted sectors as in the next section provides much stronger non-displaceability results.

14.2. Bulk deformed Lagrangian Floer cohomology. Now, we consider J-holomorphic orbifold discs and strips, whose information gives rise to the bulk deformation of the smooth Lagrangian Floer theory for toric orbifolds. This is similar to the construction in section 12, where we constructed bulk deformed A_∞-algebra from smooth A_∞-algebra by considering holomorphic orbi-discs. As explained before, the bulk deformation here is a bit different from that of [**FOOO3**] in that we considered bulk deformation by fundamental classes of twisted sectors. But the general formalism and algebraic structures are the same.

In fact, the construction of the A_∞-bimodule for the bulk deformed Floer theory in our case is entirely analogous to that of [**FOOO3**] except the following issue of time dependent $\{J_t\}$-holomorphic maps, which we first explain.

Consider a transversal pair of Lagrangian submanifolds L and $\psi_1(L)$, for a Hamiltonian isotopy ψ_t with $\psi_0 = id$. To define Lagrangian Floer cohomology between them, and to show invariance under other Hamiltonian isotopies, one considers J-holomorphic strips of several kinds with Lagrangian boundary conditions. In general, one takes a family of J's parametrized by the domain of the strip. For example, to define the differential of the Floer complex $C(L, \psi_1(L))$, one takes a one-parameter family of compatible almost complex structures $\mathcal{J} := \{J_t\}_{t\in[0,1]}$ such that J_0 is the (almost) complex structure J of $\mathcal{X}$, and $J_1 = \psi_*(J)$, and consider $\{J_t\}$-holomorphic strips

$$\frac{\partial u}{\partial \tau} + J_t(\frac{\partial u}{\partial t}) = 0.$$

Now, if the domain is an orbifold strip, namely it is $\mathbb{R} \times [0,1]$ with interior orbifold marked points $z_1^+, \ldots, z_l^+$, then it is not obvious what it means to have a $\mathcal{J}$-holomorphic strip. Namely, for an orbifold J-holomorphic strip, by definition, local lifts near orbifold marked points are J-holomorphic. For orbifold discs, we use a fixed almost complex structure J which is invariant under local group action, and hence this does not cause any problem. But when we consider a family of almost complex structures which are t-dependent, the coordinate t of the domain strip becomes complicated when we consider the branch covering near a given marked point.

We find that this issue actually does not cause much trouble since the lift satisfies the $\mathcal{J}'$-holomorphic equation where $\mathcal{J}'$ is a family of compatible almost complex structures of $\mathcal{X}$ parametrized by a domain. We explain it in more detail as follows. Consider an orbifold point $z^+ = (\tau_0, t_0) \in \mathbb{R} \times I$ with $\mathbb{Z}/k$ orbifold structure. Holomorphic structure near z^+ is given by the coordinate $\tau + it$ (normalized so that at z^+, $\tau = t = 0$), and we consider a local neighborhood U of z^+, and a branch covering $br : \widetilde{U} \to U$. Denote the coordinate of $\widetilde{U}$ as $\tilde{\tau} + i\tilde{t}$ and the branch covering map is given by

$$br(\tilde{\tau} + i\tilde{t}) = (\tilde{\tau} + i\tilde{t})^k.$$

Then, the t-coordinate of $br(\tilde{\tau} + i\tilde{t})$ is its imaginary part, $Im(br(\tilde{\tau} + i\tilde{t}))$, which is a polynomial function of $\tilde{\tau}$ and $\tilde{t}$. We define $u : (\mathbb{R} \times [0,1], \mathbb{R}\times\{0\}, \mathbb{R}\times\{1\}) \to (\mathcal{X}, L, L')$ with interior orbifold marked points $\vec{z}^+$ to be an orbifold $\mathcal{J}$-holomorphic strip if it is $\mathcal{J}$-holomorphic away from orbifold marked points and at each orbifold marked point, with coordinate parametrized as above, the local lift $\widetilde{u}$ satisfies

$$\frac{\partial \widetilde{u}}{\partial \tilde{\tau}} + J_{Im(br(\tilde{\tau}+i\tilde{t}))}\frac{\partial \widetilde{u}}{\partial \tilde{t}} = 0.$$

We denote by $\mathcal{J}' = \{J_{Im(br(\tilde{\tau}+i\tilde{t}))}\}$ a family of compatible almost complex structures, parametrized by the domain $\widetilde{U}$. The way to deal with domain dependent almost complex structure $\mathcal{J}'$ is also standard in Floer theory, and adds no additional difficulty in the construction of Kuranishi structures and moduli spaces. For example, already in [**FO**], authors used such a domain dependent case to prove the Arnold conjecture. Note that the dependence is smooth since $Im(br(\tilde{\tau}+i\tilde{t}))$ is a polynomial function.

The rest of the details to construct the bulk deformed A_∞-bimodule are a direct adaptation of the manifold case, following section 8 of [**FOOO3**], where they describe the de Rham version of bulk deformed Lagrangian Floer cohomology of a pair (L, L') of Lagrangian submanifolds.

From this construction, we obtain the following proposition for toric orbifolds. Let $L(u)$ be a Lagrangian torus fiber, and let $L' = \psi(L(u))$. Consider the bounding cochain $(\mathfrak{b}, b)$, and $(\mathfrak{b}, \psi_* b)$.

Proposition 14.4 (c.f. [**FOOO3**], Proposition 8.24)**.** *Lagrangian Floer cohomology between $(L(u), (\mathfrak{b}, b))$ and $(\psi(L(u)), (\mathfrak{b}, \psi_* b))$ can be defined as in* [**FOOO3**]*, and satisfies*

$$HF((L(u), \mathfrak{b}, b), (\psi(L(u)), \mathfrak{b}, \psi_* b); \Lambda) \cong HF((L(u), \mathfrak{b}, b), (L(u), \mathfrak{b}, b); \Lambda).$$

Here the latter has been defined in Definition 12.3.

The notion of balanced and bulk-balanced fibers can be defined in exactly the same way as in Definition 4.11 of [**FOOO2**] and Definition 3.17 of [**FOOO3**], and we omit the details.

The above proposition implies the following non-displaceability results for torus fibers with non-vanishing Lagrangian Floer homology.

Corollary 14.5 (Proposition 3.19 of [**FOOO3**])**.** *If $L(u) \subset X$ is bulk-balanced, then $L(u)$ is non-displaceable. Given a Hamiltonian diffeomorphism $\psi : X \to X$ such that $\psi(L(u))$ is transversal to $L(u)$, then, we have*

$$\#(L(u) \cap \psi(L(u))) \geq 2^n. \tag{14.1}$$

15. Examples

15.1. Teardrop orbifold. We first consider a teardrop orbifold $\mathbb{P}(1, a)$ for some positive integer $a \geq 2$ (see Figure 1). The labelled polytope, corresponding to $\mathbb{P}(1, a)$, is given by the interval

$$P = [-1, \frac{1}{a}]$$

with label a on the vertex $\frac{1}{a}$.

To find an associated fan and stacky vectors, recall that the polytope P is defined by $\langle x, \boldsymbol{b}_j \rangle \geq p_j$ for $j = 1, 2$. In this case we have the lattice

$N = \mathbb{Z}$, and

$$\boldsymbol{b}_0 = -a, \boldsymbol{b}_1 = 1, p_0 = p_1 = -1.$$

The stacky vectors $\boldsymbol{b}_0$ and $\boldsymbol{b}_1$ generate two 1-dimensional cones $\sigma_0 = R_{\leq 0}$, $\sigma_1 = R_{\geq 0}$ of the fan Σ.

$\mathbb{P}(1,a)$ is given as the quotient orbifold $(\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^*$ where $\mathbb{C}^*$ acts by $t \cdot (z_1, z_2) = (tz_1, t^a z_2)$. The unique orbifold point is $[0,1]$ with isotropy group $\mathbb{Z}_a$. Thus inertia components are labelled by $\mathbb{Z}_a$.

$$Box' = \{\nu_i \mid \nu_i = i \in \mathbb{Z}/a \ \text{ for } i = 1, \ldots, a-1\}.$$

We take $u \in (-1, \frac{1}{a})$, and consider the Lagrangian circle fiber $L(u)$.

The classification theorem (Corollary 6.4) tells us that there are two smooth holomorphic discs with Maslov index two of class β_0 and β_1, corresponding to the stacky vectors $\boldsymbol{b}_0, \boldsymbol{b}_1$. Explicitly, the holomorphic disc $w_0 : (D^2, \partial D^2) \to (\mathbb{P}(1,a), L(u))$ of class β_0 is given by $w_0(z) = [cz, 1]$ for some constant c to make $w_0(\partial D^2) \subset L(u)$ (up to $Aut(D^2)$). The image of w_0 is an a-fold uniformizing cover of a neighborhood of the orbifold point $[0,1]$. Holomorphic discs of β_1 classes are $w_1(z) = [c, z]$.

The smooth potential function of $\mathbb{P}(1,a)$ thus has two terms corresponding to these two smooth discs:

$$PO(b) = PO(b)_0 = T^{1-au}y^{-a} + T^{1+u}y.$$

To find a fiber $L(u)$ with holonomy whose smooth Floer cohomology is non-vanishing, we find critical points of $PO(b)$. If the T-exponents of the two terms of $PO(b)$ are not equal, then $PO(b)$ does not have non-trivial critical points. (Since $y = e^x$, the exponent cannot be zero.)

The areas of two smooth discs are the same, or, $1 - au = 1 + u$, exactly when $u = 0$. Notice that $u = 0$ is not at the center of the polytope P, since the smooth disc of class β_1 wraps around the orbifold point a times.

In this case, the critical point equation becomes $y^{a+1} = 1/a$, which has solutions

$$y = \frac{1}{\sqrt[a+1]{a}} exp(\frac{2\pi k i}{a+1}) \ \text{ for } \ k = 0, \ldots, a.$$

Thus the fiber $L(0)$ with flat line bundle of (non-unitary) holonomy, as one of the above, has non-trivial Floer cohomology (see Figure 1).

Now, we consider bulk deformations by orbi-discs. From the classification theorem, we have $a-1$ holomorphic orbi-discs corresponding to the elements of Box' (see Corollary 6.3). These correspond to holomorphic orbi-discs, wrapping around the orbifold points $1, \ldots, a-1$ times.

The leading order bulk potential $PO^{\mathfrak{b}}_{orb,0}$ can be explicitly written as

$$PO^{\mathfrak{b}}_{orb,0} = T^{1-au}y^{-a} + T^{1+u}y + \sum_k T^{k(1/a-u)}\mathfrak{b}_{\nu_k} y^{-k}.$$

In this example, we can set $\mathfrak{b}_{\nu_k} = 0$ for $k = 2, 3, \ldots, a-1$ as $\mathfrak{b}_{\nu_1}$ is enough in this case.

For $-\frac{1}{2}(1-\frac{1}{a}) < u < 0$, we have $\frac{1}{a} - u < 1+u < 1-au$. Here, we will use $\mathfrak{b}_{\nu_1}$ to make $\mathfrak{b}_{\nu_1}T^{\frac{1}{a}-u}$ and T^{1+u} of the same energy level. Namely, we take

$$\mathfrak{b}_{\nu_1} = T^{(1+u)-(\frac{1}{a}-u)}.$$

Then, the leading term equation (with $S_1 = 1+u$) is

$$\frac{\partial}{\partial y}(y + y^{-1}) = 0.$$

This equation has solutions $y = \pm 1 \in \mathbb{C}^*$.

For $0 < u < \frac{1}{a}$, we have $\frac{1}{a} - u < 1 - au < 1+u$. Here, we will use $\mathfrak{b}_{\nu_1}$ to make $\mathfrak{b}_{\nu_1}T^{\frac{1}{a}-u}$ and T^{1-au} of the same energy level. Namely, we take

$$\mathfrak{b}_{\nu_1} = T^{(1-au)-(\frac{1}{a}-u)}.$$

Then, the leading term equation (with $S_1 = 1-au$) is

$$\frac{\partial}{\partial y}(y^{-a} + y^{-1}) = 0.$$

This equation becomes $y^{a-1} = -a$, which has non-trivial solutions in $\mathbb{C}^*$.

Therefore, the fiber $L(u)$ with $-\frac{1}{2}(1-\frac{1}{a}) < u < \frac{1}{a}$ is non-displaceable by any Hamiltonian isotopy of $\mathbb{P}(1,a)$ from Theorem 13.2. The result holds even for $u = \frac{1}{2}(-1+\frac{1}{a})$ by the standard limit argument, and thus, exactly half of the interval $[-1, \frac{1}{a}]$ containing the image of the orbifold point corresponds to non-displaceable circles in $\mathbb{P}(1,a)$ (see Figure 1 for the region of non-displaceability).

15.2. Weighted projective spaces. We consider the smooth Floer homology of weighted projective space $\mathbb{P}(1, a_1, \ldots, a_n)$ for positive integers $a_i \in \mathbb{N}$, $i = 1, \ldots, n$. The bulk deformed theories are much more complicated, and we will discuss several examples in more detail in later subsections.

The polytope P for $\mathbb{P}(1, a_1, \ldots, a_n)$ is defined by
(15.1)

$$P = \{(x_1, \ldots, x_n) \mid x_j + 1 \geq 0,\ j = 1, \ldots, n, \quad -(\sum_{j=1}^{n} a_j x_j) + 1 \geq 0\}.$$

Here $N = \mathbb{Z}^n$ and we take $\boldsymbol{b}_0 = (-a_1, \ldots, -a_n)$ and $\boldsymbol{b}_i = e_i$ for $i = 1, \ldots, n$. Then $\mathbb{P}(1, a_1, \ldots, a_n)$ is obtained as a quotient orbifold of $\mathbb{C}^{n+1} \setminus \{0\}/S^1$ where the circle acts with weight $(1, a_1, \ldots, a_n)$.

There are $n+1$ smooth holomorphic discs corresponding to stacky vectors $\boldsymbol{b}_0, \ldots, \boldsymbol{b}_n$ whose homology classes are denoted as $\beta_0, \ldots, \beta_n$.

Thus the smooth potential $PO(b) = PO(b)_0$ is

$$PO(b) = T^{1-\langle \vec{a}, u\rangle}\frac{1}{y_1^{a_1}\cdots y_n^{a_n}} + T^{u_1+1}y_1 + \cdots + T^{u_n+1}y_n.$$

The only non-trivial critical points can occur only when all the exponents of T are the same, and hence $u = 0$, i.e. the central fiber $u = 0$ admits $n + 1$ smooth holomorphic discs of Maslov index two with the same area

$$\ell_0(u) = \cdots = \ell_n(u) = 1.$$

This is an analog of the Clifford torus in projective spaces.

In this case the critical point equations, $\frac{\partial}{\partial y_i} PO(b)_{u=0} = 0$ for all i, have a solution: $\frac{1}{y_1^{a_1} \cdots y_n^{a_n}} = \frac{y_i}{a_i}$ or $y_i = a_i \lambda$ with λ given as $\lambda = (a_1^{a_1} \cdots a_n^{a_n})^{1/(1-a_1-\cdots-a_n)}$.

Proposition 15.1. *The central fiber $L(0)$ in the above weighted projective space $\mathbb{P}(1, a_1, \ldots, a_n)$ with holonomy $a_i\lambda$ as above, has non-trivial smooth Floer cohomology. Thus $L(0)$ is non-displaceable by any Hamiltonian isotopy.*

15.3. Bulk Floer homology for $\mathbb{P}(1, a, a)$. Consider the space $\mathbb{P}(1, a, a)$ for a positive integer $a \geq 2$. We explain how to use bulk deformation to detect non-displaceable torus fibers.

The labelled polytope is the same as that given in (15.1), where the facet corresponding to $\boldsymbol{b}_0$ (to which $\boldsymbol{b}_0$ is normal) has a label a on it. The whole divisor corresponding to $\boldsymbol{b}_0$ has an isotropy group $\mathbb{Z}_a$. It is not hard to check that

$$Box' = \{\nu_k := \frac{k}{a}\boldsymbol{b}_0 \mid k = 1, \ldots, a-1\}.$$

The smooth potential $PO(b)$ is given by

$$PO(b) = T^{1-au_1-au_2} \frac{1}{y_1^a y_2^a} + T^{u_1+1} y_1 + T^{u_2+1} y_2.$$

We may take

$$\mathfrak{b}_{\nu_1} = T^{\alpha},\ \mathfrak{b}_{\nu_k} = 0 \text{ for } k \neq 1,$$

for some $\alpha > 0$. Then, the leading order bulk potential (with the above choice of bulk deformation) becomes

$$PO^{\mathfrak{b}}_{orb,0} = PO(b) + T^{\alpha} T^{\frac{1}{a}-u_1-u_2} \frac{1}{y_1 y_2}.$$

Now, we try to find α such that the leading term equation of $PO^{\mathfrak{b}}_{orb,0}$ (which depends on α) has a non-trivial solution. The idea is that on a given energy level, say S_l, if the vectors $\boldsymbol{b}$ corresponding to energy S_l span $d(l)$ dimensional space, then we need at least $d(l) + 1$ vectors $\boldsymbol{b}$ to have a non-trivial solution of the leading term equation of level S_l. In our case, we need at least three $\boldsymbol{b}$ vectors to correspond to the minimal energy level S_1.

As the area of the basic disc corresponding to $\boldsymbol{b}_0$ is a times bigger than that of ν_1, and as $PO^{\mathfrak{b}}_{orb,0}$ has only four terms, the three terms excluding that of $\boldsymbol{b}_0$ should have the same T-exponent in order to have a non-trivial solution of the leading term equation.

Thus basic discs corresponding to $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ should have the same area, which then is equal to the sum of α and the area of the ν_1 orbi-disc:

$$\ell_1 = \ell_2 = \alpha + \ell_{\nu_1}.$$

This implies that we have

$$1 + u_1 = 1 + u_2 = \frac{1}{a} - u_1 - u_2 + \alpha,$$

which gives

$$u_1 = u_2, u_1 = \frac{1}{3}(\alpha + \frac{1}{a} - 1). \tag{15.2}$$

Also, we need to require that the area of $\boldsymbol{b}_0$ is bigger than that of $\boldsymbol{b}_1$ or $\boldsymbol{b}_2$. Thus, we have

$$1 - au_1 - au_2 > 1 + u_1.$$

With the condition that $u_1 = u_2$, we have $u_1 < 0, u_2 < 0$. Since $\alpha > 0$,

$$u_1 = u_2 > \frac{1-a}{3a}.$$

Thus, (u_1, u_2) lies on the line segment connecting $(0,0)$ and $(\frac{1-a}{3a}, \frac{1-a}{3a})$.

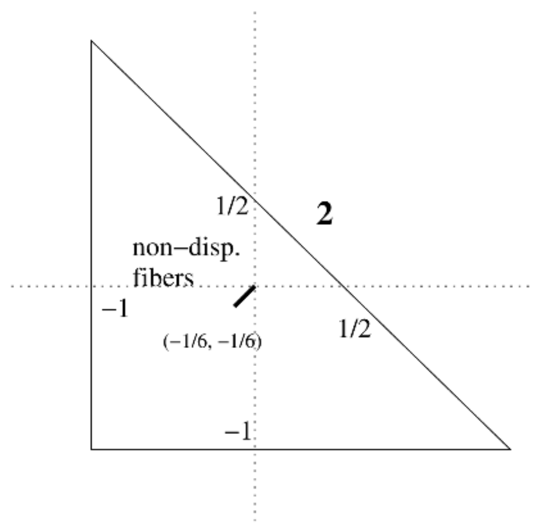

Figure 2. Orbifold $\mathbb{P}(1,2,2)$

Indeed for a fixed (u_1, u_2) in the line segment above, we choose α to satisfy (15.2), then the leading term equation (with the minimal energy $S_1 = 1 + u_1$) is nothing but

$$\frac{1}{y_1 y_2} + y_1 + y_2 = 0.$$

It is easy to check that this equation has a non-trivial critical point, which describes the (non-unitary) holonomies to be put on the Lagrangian torus fiber at (u_1, u_2) so that the resulting Floer cohomology is non-trivial and isomorphic to the singular cohomology of the torus from Theorem 13.2 and Theorem 12.10.

15.4. Bulk Floer homology for $\mathbb{P}(1,1,a)$. Now, we discuss the case of $\mathbb{P}(1,1,a)$ for a positive integer $a \geq 3$. The corresponding moment polytope is shown in Figure 3. The elements of Box' are

$$Box' = \{\nu_k := \frac{k}{a}\boldsymbol{b}_0 + \frac{k}{a}\boldsymbol{b}_1 = (0,-k) \mid k = 1, \ldots, a-1\}.$$

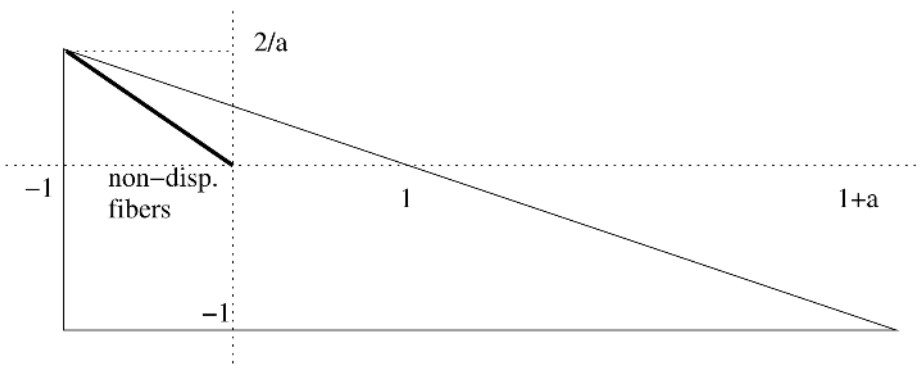

Figure 3. Orbifold $\mathbb{P}(1,1,a)$

The smooth potential $PO(b)$ is given by

$$PO(b) = T^{1-u_1-au_2}\frac{1}{y_1 y_2^a} + T^{u_1+1}y_1 + T^{u_2+1}y_2.$$

We take

$$\mathfrak{b}_{\nu_1} = T^{\alpha}, \ \mathfrak{b}_{\nu_k} = 0 \text{ for } k \neq 1,$$

for some $\alpha > 0$. Since

$$\ell_{\nu_1} = \frac{1}{a}\ell_0 + \frac{1}{a}\ell_1 = \frac{2}{a} - u_2,$$

the leading order potential with the above choice of bulk deformations becomes:

$$PO^{\mathfrak{b}}_{orb,0} = PO(b) + T^{\alpha}T^{\frac{2}{a}-u_2}\frac{1}{y_2}.$$

We try to find α, which makes $PO^{\mathfrak{b}}_{orb,0}$ have a solution in its leading term equation. Note that ν_1 and $\boldsymbol{b}_2$ are in the opposite direction.

As in the previous example, we need three terms of $PO^{\mathfrak{b}}_{orb,0}$ to have the same T-exponent. In the case $\ell_0 = \ell_1 = \ell_2$, we get a solution by Proposition 15.1. One can check that the remaining case with non-trivial solution of a leading term equation is $\ell_0 = \ell_1 = \ell_{\nu_1} + \alpha$. (Other cases contain both ν_1 and $\boldsymbol{b}_2$, and the corresponding leading term equations do not have a solution, as ν_1 and $\boldsymbol{b}_2$ are linearly dependent.)

This implies that

$$1 + u_1 = 1 - u_1 - au_2 = \frac{2}{a} - u_2 + \alpha,$$

which gives

$$u_2 = -\frac{2}{a}u_1, u_1 = -1 + \alpha\frac{a}{a-2}. \tag{15.3}$$

Also, we need to require that $\ell_1 \leq \ell_2$. Thus, $u_1 \leq u_2$. This implies that $u_1 \leq 0$ and $u_2 \geq 0$. Thus, (u_1, u_2) lies in the interior of the line segment connecting $(-1, 2/a)$ and $(0, 0)$, as drawn in Figure 3. It is not hard to check that the corresponding leading term equation has a solution in such a case.

Remark 15.1. It is shown in [**WW**], Example 4.9, that in the case of $a = 2$, the analogous line segment is also the location of non-displaceable torus fibers. But unfortunately, we do not know how to prove it using our methods. In these computations, we need $\mathfrak{b}_\nu$, which lies in $\Lambda_0 \setminus \Lambda_+$ for the case $a = 2$, which is not possible since we cannot make sense of infinite sums in the definition of bulk deformation with such $\mathfrak{b}_\nu$. We leave it for future research.

15.5. Bulk Floer homology for $\mathbb{P}(1,3,5)$. The example $\mathbb{P}(1,3,5)$ has been found to be very interesting recently; see [**M**], and also [**WW**] and [**ABM**]. We show that the torus fibers which are inverse images of points in the shaded region in the polytope (in Figure 4) are non-displaceable by Hamiltonian isotopy using our methods.

As the shape indicates, we need a little detailed analysis on comparing the sizes of the areas of holomorphic discs and orbi-discs.

First we identify elements of Box'. We denote the sectors involving $\boldsymbol{b}_0$ and $\boldsymbol{b}_1$ by

$$\begin{aligned}
\nu_1 &= \tfrac{1}{5}\boldsymbol{b}_0 + \tfrac{3}{5}\boldsymbol{b}_1 = && (0, -1), \\
\nu_2 &= \tfrac{2}{5}\boldsymbol{b}_0 + \tfrac{1}{5}\boldsymbol{b}_1 = && (-1, -2), \\
\nu_3 &= \tfrac{3}{5}\boldsymbol{b}_0 + \tfrac{4}{5}\boldsymbol{b}_1 = && (-1, -3), \\
\nu_4 &= \tfrac{4}{5}\boldsymbol{b}_0 + \tfrac{2}{5}\boldsymbol{b}_1 = && (-2, -4).
\end{aligned}$$

The sectors involving $\boldsymbol{b}_0$ and $\boldsymbol{b}_2$ are

$$\begin{aligned}
\nu_5 &= \tfrac{1}{3}\boldsymbol{b}_0 + \tfrac{2}{3}\boldsymbol{b}_2 = && (-1, -1), \\
\nu_6 &= \tfrac{2}{3}\boldsymbol{b}_0 + \tfrac{1}{3}\boldsymbol{b}_2 = && (-2, -3).
\end{aligned}$$

The areas of holomorphic discs and orbi-discs are

$$\ell_0 = 1 - 3u_1 - 5u_2,\ \ell_1 = 1 + u_1,\ \ell_2 = 1 + u_2$$

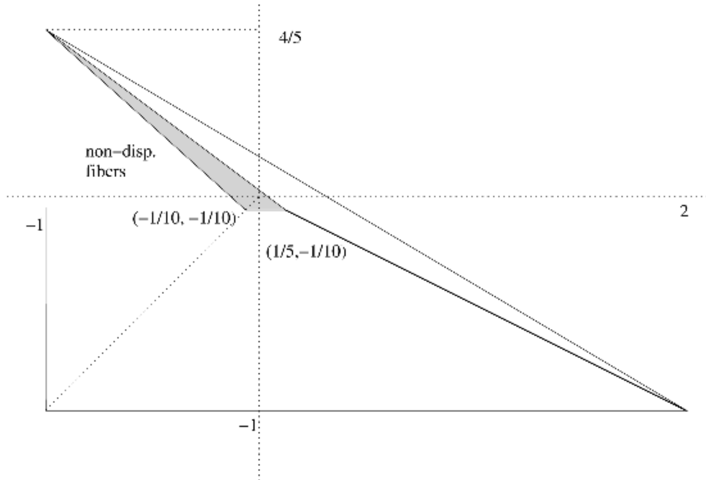

Figure 4. Orbifold $\mathbb{P}(1,3,5)$

$$\ell_{\nu_1} = \frac{1}{5}\ell_0 + \frac{3}{5}\ell_1 = \frac{4}{5} - u_2,\ \ell_{\nu_2} = \frac{3}{5} - u_1 - 2u_2$$

The areas $\ell_{\nu_3}, \dots, \ell_{\nu_6}$ can be computed similarly.

Near the vertex $(-1, 4/5)$ of the moment triangle in Figure 4, the areas of the following (orbi-)discs (depending on the position $u \in P$),

$$\ell_0, \ell_1, \ell_{\nu_1}, \ell_{\nu_2}, \ell_{\nu_3}, \ell_{\nu_4},$$

are smaller than others, and could give relevant terms in the leading term equation.

As it is two-dimensional, we would like to have a triple of them to have the same energy S_1. Although the symplectic areas ℓ_0, ℓ_1 are already fixed, we could add the bulk deformation term $\mathfrak{b}_{\nu_i}$ to ℓ_{ν_i} suitably to increase the energy level. Thus if ℓ_0 is not equal to ℓ_1, we need two other orbi-discs to make the triple, and for this we need them to have smaller symplectic areas.

More precisely, we proceed as follows. First, we consider the region where ℓ_0 and ℓ_1 are smaller than ℓ_2. This implies that

$$\ell_0 < \ell_2 \Rightarrow u_2 > -u_1/2, \quad \ell_1 < \ell_2 \Rightarrow u_2 > u_1.$$

We divide it further into three cases:

1) $\ell_0 < \ell_1$, or $u_1 > -4u_2/5$: To have at least three terms of least energy, we need

$$\ell_{\nu_1} < \ell_0 \text{ and } \ell_{\nu_2} < \ell_0. \tag{15.4}$$

The first inequality gives $3u_1 + 4u_2 < 1/5$ and the second inequality gives $2u_1 + 3u_2 < 2/5$. If this happens, we can add bulk deformation

$$\mathfrak{b}_{\nu_1} = T^{\ell_0 - \ell_{\nu_1}}, \mathfrak{b}_{\nu_2} = T^{\ell_0 - \ell_{\nu_2}},$$

which will make $\boldsymbol{b}_0, \boldsymbol{b}_{\nu_1}, \boldsymbol{b}_{\nu_2}$ to contribute to the leading term equation of $PO^{\mathfrak{b}}_{orb,0}$ of the same energy S_1. We note that the first inequality implies the second inequality for the points in P; hence, for the region bounded by

$$u_1 > -\frac{4u_2}{5}, \;\; 3u_1 + 4u_2 < \frac{1}{5}, \;\; u_2 > -\frac{u_1}{2},$$

we can choose bulk deformation as above, so that the corresponding leading term equation has a solution.

2) $\ell_0 > \ell_1$ or $u_1 < -4u_2/5$: To have at least three terms of least energy, we need

$$\ell_{\nu_1} < \ell_1 \text{ and } \ell_{\nu_2} < \ell_1. \tag{15.5}$$

Both equations translate to the inequality $u_1 + u_2 > -1/5$. Thus, in the region

$$u_1 < -\frac{4u_2}{5}, \quad u_1 + u_2 > -\frac{1}{5}, \quad u_2 > u_1,$$

we can choose bulk deformation as

$$\mathfrak{b}_{\nu_1} = T^{\ell_1 - \ell_{\nu_1}}, \mathfrak{b}_{\nu_2} = T^{\ell_1 - \ell_{\nu_2}},$$

so that the corresponding leading term equation has a solution.

3) $\ell_0 = \ell_1$: We similarly obtain that the line segment $u_1 = -\frac{4u_2}{5}, u_1 < 0$ supports bulk deformation whose leading term equation has a solution. We leave this as an exercise.

The above do not cover the whole shaded region of Figure 4. The rest of the region that is not covered is the triangle Δ formed by three points

$$(-1/10, -1/10), (0,0), (1/5, -1/10).$$

For this region, the leading term equation involves two equations of energy levels S_1 and S_2. Note that the vectors $\boldsymbol{b}_2 = (0,1)$ and $\boldsymbol{b}_{\nu_1} = (0,-1)$ are opposite to each other. So in Δ, ℓ_2 is smaller than ℓ_1 and ℓ_0, and hence we take

$$\mathfrak{b}_{\nu_1} = T^{\ell_2 - \ell_{\nu_1}} = T^{2u_2 + \frac{1}{5}}.$$

This makes the terms corresponding to $\boldsymbol{b}_2$ and $\boldsymbol{b}_{\nu_1}$ contribute to the leading term equation of energy level $S_1 = \ell_2$. Now, for the next level S_2, we have terms from $\boldsymbol{b}_1$ and $\boldsymbol{b}_{\nu_2}$. We again have a solution for the S_2 energy level leading term equation. We leave the details to readers.

We remark that the line segment from $(1/5, -1/10)$ to $(2, -1)$ is known to be non-displaceable by [**WW**], but our methods cannot prove it yet (cf. Remark 15.1).

15.6. Polytope with nontrivial integer labels.

Proposition 15.2. *If P is a compact rational simple polytope with m facets, and if the integer labels $c_1, \ldots, c_m$ for facets satisfy*

$$c_i \geq 2 \;\; \text{for all} \;\; i = 1, \ldots, m,$$

then for any $u \in Int(P)$, $L(u)$ is non-displaceable.

Proof. The first proof is due to Kaoru Ono, who provided this alternative proof after the first author gave a talk on this paper and this proposition.

Here is the first proof. As all facets have non-trivial integer labels, points in any toric divisors are not smooth points. Take a torus fiber $L(u)$ for any interior point $u \in Int(P)$. If $L(u)$ is displaceable by Hamiltonian isotopy ψ_1^H, i.e. $L(u) \cap \psi_1^H(L(u)) = \emptyset$, then we can modify H so that its support lies in $Int(P)$, and still satisfy the above displacing property as Hamiltonian isotopy sends smooth points to smooth points. But this is a contradiction since Lagrangian torus fibers in $(\mathbb{C}^*)^n$ are not displaceable by any compactly supported Hamiltonian isotopy. As a symplectic manifold, the inverse image L of $Int(P)$ is symplectically embedded in T^*T^n. Then a T^n-orbit is considered as the graph of a T^n-invariant 1-form η on T^n, which is closed, embedded in T^*T^n. By a symplectomorphism, which comes from the fiberwise addition of $-\eta$, we may assume that the T^n-orbit is the zero section of T^*T^n. The non-displaceability of the zero section in the cotangent bundle is well-known.

The second proof is by using bulk deformation. Let $\boldsymbol{b}_i$ be a stacky vector corresponding to the i-th facet. We take ν_i to be the minimal integral vector in the direction of $\boldsymbol{b}_i$ such that $\boldsymbol{b}_i = c_i\nu_i$. We consider bulk deformations $\mathfrak{b}_{\nu_i} = -c_i T^{\ell_i - \ell_{\nu_i}}$ for each $i = 1, \ldots, m$. Then corresponding leading term potential $PO^{\mathfrak{b}}_{orb,0}$ becomes

$$\sum_{i=1}^{m} T^{\ell_i}(y^{\boldsymbol{b}_i} - c_i y^{\boldsymbol{b}_{\nu_i}}), \tag{15.6}$$

since each contribution of $\mathfrak{b}_{\nu_i}$ is chosen to match with the term of the potential corresponding to $\boldsymbol{b}_i$. For generic u, we may assume that all the areas ℓ_i are distinct. Then, the leading term equation is the critical point equation of each summand of (15.6) up to the dimension of P. By denoting $y_l = y^{\boldsymbol{b}_{\nu_i}}$, the summand equals $y_l^{c_i} - c_i y_l$, and clearly has a non-trivial critical point $y_l = 1$. This shows that generic $u \in Int(P)$ is non-displaceable. But by the standard limit argument, this implies that $L(u)$ is non-displaceable for all $u \in Int(P)$. q.e.d.

16. Appendix: Preliminaries on orbifold maps

In this appendix, we recall definitions regarding maps between orbifolds following [**CR**], [**CR2**], with the added condition that the domain orbifold may have a nontrivial smooth boundary.

Let $\mathcal{X}$ be a differentiable (C^∞) orbifold with boundary and let X be its underlying topological space. In applications we will often deal with the case when X is an orbifold Riemann surface with smooth boundary (i.e. orbifold singularity lies in the interior).

An uniformizing system for an open connected set $U \subset X$ is a triple (V, G, π) where V is a smooth connected manifold with boundary ∂V (which may be empty), G is a finite group acting smoothly on V (preserving ∂V), and $\pi : V \to U$ is a continuous map that induces a homeomorphism between V/G and U. The orbifold analogue of inclusion of open sets in manifolds is the notion of injection of uniformizing charts.

Definition 16.1. Let $i : U \hookrightarrow U'$ be an inclusion of open sets uniformized by (V, G, π) and (V', G', π') respectively. An injection $(\phi, \rho) : (V, G, \pi) \to (V', G', \pi')$ consists of an injective group homomorphism $\rho : G \to G'$ and a ρ-equivariant open embedding $\phi : V \to V'$ such that

1) $i \circ \pi = \pi' \circ \phi$,
2) ρ induces an isomorphism $\ker G \to \ker G'$ where $\ker G := \{g \in G : g \cdot x = x \text{ for all } x \in V\}$.

If $\ker G$ is trivial for every uniformizing system, the orbifold is called effective or reduced.

An injection (ϕ, ρ) is an isomorphism of uniformizing systems if an inverse injection exists. An important fact is that, given an automorphism (ϕ, ρ) of an uniformizing system (V, G, π) of an open set $U \subset X$ in a C^∞ orbifold $\mathcal{X}$ (with boundary), there exists an element $g \in G$ such that $\phi(x) = gx$ and $\rho(h) = ghg^{-1}$ (see Lemma 2.11 of [**MM**]). This correspondence is one-to-one if $\ker G$ is trivial.

Definition 16.2. A compatible cover of an open set Y in an orbifold $\mathcal{X}$ is an open cover $\mathcal{U}$ of Y together with a uniformizing system (V, G, π) for each $U \in \mathcal{U}$ and a collection of injections such that:

1) If $U \subset U'$, then there exists an injection $(V, G, \pi) \to (V', G', \pi')$.
2) For every $p \in U_1 \bigcap U_2$, where $U_1, U_2 \in \mathcal{U}$, there exists a $U \in \mathcal{U}$ such that $p \in U \subset U_1 \bigcap U_2$.

Definition 16.3. Let $\mathcal{X}$ and $\mathcal{X}'$ be orbifolds, possibly with boundary. Suppose $U \subset X$, $U' \subset X'$ are open sets uniformized by (V, G, π) and (V', G', π') respectively. Given a continuous map $f : U \to U'$, a C^l lift of f is a C^l map $\tilde{f} : V \to V'$ satisfying

1) $\pi' \circ \tilde{f} = f \circ \pi$.
2) Given any $g \in G$, there exists a $g' \in G'$ such that $\tilde{f}(gx) = g'\tilde{f}(x)$ for all $x \in V$.

Note that the correspondence $g \to g'$ is not required to be a group homomorphism.

Definition 16.4. Two lifts $\tilde{f}_i : (V_i, G_i, \pi_i) \to (V'_i, G'_i, \pi'_i)$, $i = 1, 2$, are isomorphic if there are isomorphisms $(\phi, \rho) : (V_1, G_1, \pi_1) \to (V_2, G_2, \pi_2)$ and $(\phi', \rho') : (V'_1, G'_1, \pi'_1) \to (V'_2, G'_2, \pi'_2)$ such that $\phi' \circ \tilde{f}_1 = \tilde{f}_2 \circ \phi$.

Let $\tilde{f} : (V, G, \pi) \to (V', G', \pi')$ be a C^l lift of $f : U \to U'$. Let W, W' be open sets such that $W \subset U$ and $f(W) \subset W' \subset U'$. Then $\tilde{f}$ naturally induces a unique isomorphism class of lift for $f : W \to W'$.

Definition 16.5. Two lifts $\tilde{f}_i : (V_i, G_i, \pi_i) \to (V'_i, G'_i, \pi'_i)$, $i = 1, 2$, of $f : X \to X'$ over open sets U_1 and U_2, are said to be equivalent at $p \in U_1 \bigcap U_2$ if they induce isomorphic lifts of $f : U \to U'$ for some open sets U containing p and U' containing $f(p)$.

Definition 16.6. A local C^l lift of $f : X \to X'$ at a point $p \in X$ is a C^l lift $\tilde{f}_p : V_p \to V'_{f(p)}$, for some uniformizing systems (V_p, G_p, π_p) and $(V'_{f(p)}, G'_{f(p)}, \pi'_{f(p)})$ on open sets containing p and $f(p)$ respectively.

Definition 16.7. Let $\mathcal{X}$ and $\mathcal{X}'$ be orbifolds, possibly with boundary. Given a continuous map $f : X \to X'$, a C^l lift $\tilde{f} : \mathcal{X} \to \mathcal{X}'$ of f is a choice of a local C^l lift $\tilde{f}_p : V_p \to V'_{f(p)}$ for each point p, such that $\tilde{f}_p$ is equivalent to $\tilde{f}_q$ for each $q \in U_p$.

Example 16.8. Consider the orbifold $\mathbb{C}/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts by reflection about the origin. Consider the holomorphic coordinates z on $\mathbb{C}$ and $w = z^2$ on $\mathbb{C}/\mathbb{Z}_2$. Regard S^1 as $\mathbb{R}/2\pi\mathbb{Z}$. Take the map $f : S^1 \to \mathbb{C}/\mathbb{Z}_2$ defined by $w \circ f(\theta) = e^{i\theta}$. Consider the covering of S^1 by the open sets $U_1 = (0, 2\pi)$ and $U_2 = (-\pi, \pi)$. The lifts $\tilde{f}_j : U_j \to \mathbb{C}$ given by

$$z \circ \tilde{f}_j(\theta) = e^{i\theta/2} \text{ for } j = 1, 2, \tag{16.1}$$

define a C^∞ lift of f.

Note that not every continuous map of underlying spaces admits a lift. As an example, the map $h : \mathbb{C} \to \mathbb{C}/\mathbb{Z}_2$ defined by $w \circ h(t) = t$ does not admit even a C^0 lift near the origin.

Definition 16.9. Two lifts $\{\tilde{f}_{p,i} : (V_{p,i}, G_{p,i}, \pi_{p,i}) \to (V'_{f(p),i}, G'_{p,i}, \pi'_{f(p),i})\}$, $i = 1, 2$, of f are said to be equivalent if for each $p \in X$, $\tilde{f}_{p,1}$ and $\tilde{f}_{p,2}$ are equivalent at p.

Definition 16.10. A C^l map of orbifolds $\mathbf{f} : \mathcal{X} \to \mathcal{X}'$ is a continuous map $f : X \to X'$ of underlying spaces together with the equivalence class of a C^l lift $\tilde{f}$ of f.

Now we recall the crucial notion of a good map in [**CR**]. Chen and Ruan used the notion of compatible system to describe a good map. A compatible system roughly consists of compatible covers of the domain and range of the map by uniformizing charts, choice of lifts on each chart, and some algebraic data for injections of charts that encode how the lifts fit together. This enables one to define the pull-back of an orbifold vector bundle with respect to a good map. The notion of a good map is very closely related to the notions of a strong map [**MP**] and an orbifold morphism [**ALR**].

Definition 16.11. Let $\mathbf{f} : \mathcal{X} \to \mathcal{X}'$ be a C^l map between orbifolds with boundary whose underlying continuous map is denoted by f. Suppose $\mathcal{U}$ and $\mathcal{U}'$ are compatible covers of X and an open set containing $f(X)$ respectively, satisfying the following conditions:

1) There is a bijection between $\mathcal{U}$ and $\mathcal{U}'$, given by $U \leftrightarrow U'$, such that $f(U) \subset U'$ and $U_2 \subset U_1$ implies $U_2' \subset U_1'$.
2) There exists a collection of local C^l lifts $\{\tilde{f}_{UU'} : (V, G, \pi) \to (V', G', \pi')\}$ of f, and an assignment of an injection $\lambda(i) : (V_2', G_2', \pi_2') \to (V_1', G_1', \pi_1')$ to every injection $i : (V_2, G_2, \pi_2) \to (V_1, G_1, \pi_1)$, such that
   a) $\tilde{f}_{U_1U_1'} \circ i = \lambda(i) \circ \tilde{f}_{U_2U_2'}$, and
   b) $\lambda(j \circ i) = \lambda(j) \circ \lambda(i)$ for each composition $j \circ i$ of injections.
3) The C^l lift of f defined by the collection $\{\tilde{f}_{UU'}\}$ is in the equivalence class corresponding to $\mathbf{f}$.

Then we say that $\{\tilde{f}_{UU'}, \lambda\}$ is a compatible system of $\mathbf{f}$.

Note that if $\mathcal{X}'$ is reduced, each automorphism $g \in G$ of (V, G, π) is assigned an automorphism $\lambda(g) \in G'$ giving rise, by condition (2)(b), to a group homomorphism $\lambda_{UU'} : G \to G'$ with respect to which $\tilde{f}_{UU'}$ is equivariant.

Definition 16.12. A C^l map $\mathbf{f} : \mathcal{X} \to \mathcal{X}'$ is called a good C^l map if it admits a compatible system.

When an orbifold is reduced, it may be represented as the quotient of a manifold by the effective action of a compact Lie group by the so-called frame bundle trick. However, a good map between $\mathcal{X} = M/G$ and $\mathcal{X}' = N/H$ may not be represented by an equivariant map from M to N. This has to do with the fineness of the compatible cover of $\mathcal{X}$ used to define the good map. Indeed, a similar problem occurs for a good map from a manifold to an orbifold. For instance, consider the C^∞ map $S^1 \to \mathbb{C}/\mathbb{Z}_p$ given by the lift $t \mapsto t^{1/p}$. We need to use a suitable cover of S^1 to make sense of continuity of the lift.

A good C^∞ map is what Chen and Ruan [**CR**] call a good map. Not all orbifold maps admit a compatible system. See example 4.4.2a of [**CR**].

Chen and Ruan prove (cf. Lemma 4.4.6 and Remark 4.4.7 of [**CR**]) that, given two compatible systems $\xi_1 = \{\tilde{f}_{1,UU'}, \alpha_1 : U \in \mathcal{U}, U' \in \mathcal{U}'\}$

and $\xi_2 = \{\tilde{f}_{2,RR'}, \alpha_2 : R \in \mathcal{R}, R' \in \mathcal{R}'\}$ for a C^∞ map $\mathbf{f} : \mathcal{X} \to \mathcal{X}'$, there exist

1) common refinements $\mathcal{W}$ of $\mathcal{U}$ and $\mathcal{R}$, and $\mathcal{W}'$ of $\mathcal{U}'$ and $\mathcal{R}'$, that satisfy condition (1) of Definition 16.11;
2) compatible systems $\{\tilde{f}_{1,WW'}, \lambda_1\}$ and $\{\tilde{f}_{2,WW'}, \lambda_2\}$, where $W \in \mathcal{W}$ and $W' \in \mathcal{W}'$, for $\mathbf{f}$ induced by ξ_1 and ξ_2 respectively.

Chen-Ruan's proof actually works for any C^l map where $l \geq 0$. An important consequence of Lemma 4.4.6 of [**CR**] is that the compatible systems $\{\tilde{f}_{i,WW'}, \lambda_i\}$ can be assumed to be geodesic compatible systems (see Definition 4.4.5 of [**CR**]). In particular, the open sets of $\mathcal{W}$ and $\mathcal{W}'$ are images of the exponential map from some subset of the tangent space at some point in their interiors. This property is crucial to relate compatible systems with pull-back of vector bundles, especially the tangent bundle. This will continue to hold if $\mathcal{X}$ is an orbifold Riemann surface with smooth boundary, for an appropriate choice of Riemannian metric on $\mathcal{X}$. (The idea is to choose a Riemannian metric on the double $\mathcal{Y}$ of $\mathcal{X}$ that agrees with a metric on the manifold Y away from a small neighborhood of the singular set. Then use the positivity of the injective radius of the metric on Y.)

The following definition is equivalent to, but different from, the one in [**CR**].

Definition 16.13. Two compatible systems ξ_1 and ξ_2 of a good C^l map $\mathbf{f}$ are said to be isomorphic if there exist induced compatible systems $\{\tilde{f}_{1,WW'}, \lambda_1\}$ and $\{\tilde{f}_{2,WW'}, \lambda_2\}$ corresponding to ξ_1 and ξ_2 respectively, and an automorphism $\delta_{V'}$ of the uniformizing system (V', G', π') for each $W' \in \mathcal{W}'$, such that

1) $\delta_{V'} \circ \tilde{f}_{1,WW'} = \tilde{f}_{2,WW'}$ and
2) for each injection $i : (W_2, G_2, \pi_2) \to (W_1, G_1, \pi_1)$, the relation $\lambda_2(i) = \delta_{V_1'} \circ \lambda_1(i) \circ (\delta_{V_2'})^{-1}$ holds.

The proof of the following lemma is similar to Proposition 4.4.8 of [**CR**].

Lemma 16.1. *Suppose $\mathbf{f} : \mathcal{X} \to \mathcal{X}'$ is a good C^∞ map where $\mathcal{X}$ is an orbifold with smooth boundary. Then two compatible systems ξ_1 and ξ_2 are isomorphic if and only if the pull-backs of any orbifold vector bundle on $\mathcal{X}'$ by ξ_1 and ξ_2 are isomorphic.*

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