HOMEOMORPHIC CONTINUOUS CURVES IN 2-SPACE ARE ISOTOPIC IN 3-SPACE

BY
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§I

1. **Introduction.** The following question appears in [12, p. 57]. If K_1 and K_2 are homeomorphic compact continua in E^2, is there an isotopy (or even a homeomorphism) of E^3 onto E^3 which carries K_1 onto K_2? (See also [5, p. 230].)

At present the answer to this question is unknown. If E^3 is replaced by E^4 an affirmative answer has been obtained by Klee [6, Theorem 3.3, p. 36]. Related results for complexes in higher dimensional hyperplanes have been obtained by Bing and Kister [2]. Conditions under which a homeomorphism between planar continuous curves can be extended to E^2 have been obtained by Gehman [4], and Adkisson and MacLane [1]. Conditions under which a continuous curve can be embedded in E^2 have been obtained by Claytor [3]. The following result provides an affirmative answer to the above question when K_1 and K_2 are continuous curves.

EXTENSION THEOREM. *Suppose that S and S' are continuous curves in E^2 and g is a homeomorphism of S onto S'. Then there is a homeomorphism H of E^3 onto E^3 such that*

(a) *$H=g$ on S, and*

(b) *H is realizable by an isotopy.*

Because of space limitations this article omits certain parts of the proof. Complete details will be found in [7].

2. **Outline of proof.** The proof of the extension theorem is divided into two parts. First, the theorem is established for continuous curves without separating points. Next we show that by using an "enlarging process", the general case can be reduced to this special case.

If S and S' contain no separating points the general idea of the proof is as follows: there is a planar disk D containing S' such that the boundary of D is a subset of S'. We construct subdivisions $C_1, C_2, C_3, \ldots$ of D into disks such that, for each i, (a) the interior of every element (disk) in C_i is either a complementary domain of S' or has diameter no greater than $1/2^i$, and (b) the boundary of every disk in C_i

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is a subset of S'. We construct homeomorphisms $H_1, H_2, H_3, \dots$ such that, for each i and each $x \in S$, (a) $H_i(x)$ and $g(x)$ belong to the same disk of C_i, and (b) if $g(x)$ is on the boundary of a disk in C_i, then $H_i(x)=g(x)$. The limit of the H_i's is the required homeomorphism H.

Now assume that S contains separating points. The general idea of the second part of the proof is as follows: we may suppose that S contains a nondegenerate cyclic element C_1. We enlarge C_1 and $g(C_1)$ by attaching arcs to S and S' until we obtain continuous curves $S_2 \supset S$ and $S_2' \supset S'$, and cyclic elements $C_2 \supset C_1$ and $C_2' \supset g(C_1)$ of S_2 and S_2' respectively such that (a) the components of $S_2 - C_2$ and of $S_2' - C_2'$ have small maximum diameter, (b) $g: S \twoheadrightarrow S'$ may be extended to a homeomorphism $g_2: S_2 \twoheadrightarrow S_2'$, and (c) there are homeomorphisms $H_2, H_2': E^3 \twoheadrightarrow E^3$ such that $H_2(S_2)$ and $H_2'(S_2')$ are subsets of E^2.

We continue this process, adding arcs of smaller and smaller diameter to S and S', and thereby constructing larger and larger cyclic elements. Finally, we obtain continuous curves $S_\infty \supset S$ and $S_\infty' \supset S'$ such that (a) S_∞ and S_∞' contain no separating points, (b) there is a homeomorphism $g_\infty: S_\infty \twoheadrightarrow S_\infty'$ which extends $g: S \twoheadrightarrow S'$, and (c) there are homeomorphisms $H, H': E^3 \twoheadrightarrow E^3$ such that $H(S_\infty)$ and $H'(S_\infty')$ are subsets of E^2.

§II

Definitions and Notation.

$E^3 = \{(x_1, x_2, x_3) : x_1, x_2, x_3$ are real numbers$\}$.

$E^2 = \{(x_1, x_2, x_3) \in E^3 : x_3 = 0\}$.

Suppose A and B are point sets. Then: $A+B$ denotes the union (sum) of A and B; Bd (A) denotes the set of boundary points of A; Int (A) denotes the set of interior points of A; $\overline{A}$ denotes the closure of A.

Suppose J is a simple closed curve in E^2. Then In (J) denotes the bounded component (domain) of $E^2 - J$.

Let A and B be metric spaces. Let f_1 and f_2 be continuous functions from A into B. Then: $\|f_1 - f_2\| = \sup \{\text{dist}(f_1(x), f_2(x)) : x \in A\}$; $f_1 = \text{id}$ means that $f_1(x) = x$ for all $x \in A$; $f_1: A \twoheadrightarrow B$ means that f_1 is a continuous function from A onto B.

A *complementary domain* of a planar set M is a component of $E^2 - M$.

A *1-complex* is a finite collection of arcs no two of which intersect in an interior point of either.

A *continuum* is a closed, connected set.

A *continuous curve* is a compact, locally connected, metric continuum.

A *nondegenerate cyclic element* of a continuous curve M is a nondegenerate connected subset of M which contains no separating points and is maximal with respect to the property of being a connected subset without separating points.

The *fence* over a planar set M is the set $\{(x_1, x_2, x_3) \in E^3 : (x_1, x_2, 0) \in M\}$.

An *isotopy* of a space M onto itself is a continuous function H from $M \times I$ onto M (where I is the interval $[0, 1]$) such that for each $t_0 \in I$, the function H_{t_0}, defined by $H_{t_0}(x)=H(x, t_0)$, is a homeomorphism of M onto M. A homeomorphism f from M onto M is *realizable by an isotopy* if there is an isotopy H of M onto M such that $H(x, 0)=x$ and $H(x, 1)=f(x)$ for all $x \in M$.

A finite or infinite sequence is a *null sequence* if the diameters of its elements converge to zero.

If D is a bounded subset of E^2 and C is a circle which encloses both D and its boundary, then the *outer boundary* of D is the boundary of the set of all points x such that x can be joined to some point of C by an arc which contains no point of D or of its boundary. By a theorem of R. L. Moore [8, Theorem 4, p. 259] if D is a bounded complementary domain of a continuous curve, then the outer boundary of D is a simple closed curve.

A *pinched annulus* is any set homeomorphic to the set P defined as follows: Let D_1 and D_2 be planar disks such that $D_2 \subset D_1$ and $\mathrm{Bd}(D_2) \cap \mathrm{Bd}(D_1)$ is a single point. Then $P=\mathrm{Cl}(D_1-D_2)$.

A *separating point* of a set M is a point $p \in M$ such that $M-\{p\}$ is not connected.

An arc B is a *spanning arc* of a disk D if (a) B is contained in D, (b) the endpoints of B are contained in $\mathrm{Bd}(D)$, and (c) except for its endpoints, B misses $\mathrm{Bd}(D)$.

An arc B is a *spanning arc* of an annulus A if (a) B is contained in A, (b) B intersects both boundary components of A, and (c) except for its endpoints, B misses $\mathrm{Bd}(A)$.

§III

1. **Purpose.** In this section we shall prove the extension theorem for the case in which S contains no separating points.

We shall make use of the following:

THEOREM 1 (R. L. MOORE [9, p. 212]). *Suppose X is a nondegenerate continuous curve in E^2. Then the boundary of every domain of E^2-X is a simple closed curve if and only if X contains no separating points.*

2. **Subdividing continuous curves which contain no separating points.** Let S denote a nondegenerate planar continuous curve without separating points. Let J denote the boundary of the unbounded complementary domain of S. By Theorem 1, J is a simple closed curve.

In proving the extension theorem it will be necessary to break up the interior (bounded complementary domain) of J into small pieces so that the boundaries of these pieces lie in S. This is the purpose of Theorem 2 and Theorem 5.

THEOREM 2. *Let* $\mathrm{In}(J)$ *denote the bounded domain of E^2-J. Let D be a bounded complementary domain of S. Then*

$$\mathrm{Cl}(\mathrm{In}(J)) = \bar{D}+R_1+R_2+\cdots,$$

where $R_1, R_2, \ldots$ is a (finite or infinite) null sequence of disks, with disjoint interiors, such that, for each i, $D \cap R_i = \varnothing$ and $\mathrm{Bd}(R_i)$ *is a subset of S.*

Proof. By Theorem 1, the boundary of D is a simple closed curve. We shall consider three cases depending upon how Bd (D) intersects J.

Case 1. Bd $(D) \cap J$ is more than one point.

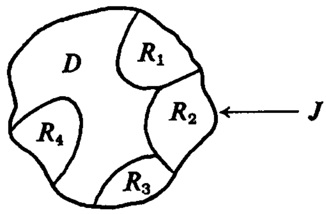

DIAGRAM 1

In this case the theorem is obvious (see Diagram 1). Note that, for each i, Bd $(R_i) \subset J + \text{Bd}(D)$.

Case 2. Bd $(D) \cap J$ is one point.

Let p be the point Bd $(D) \cap J$. S contains no separating points so there is an arc in $S - \{p\}$ from Bd (D) to J. This arc divides the pinched annulus Cl (In $(J)) - D$ into two disks R_1 and R_2. Thus Cl (In $(J)) = \bar{D} + R_1 + R_2$.

Case 3. Bd $(D) \cap J$ is empty. Then, by Lemma 3, there are disjoint arcs A_1 and A_2 in S, both of which run from Bd (D) to J.

Thus Cl (In $(J)) = \bar{D} + R_1 + R_2$.

This completes the proof of Theorem 2.

LEMMA 3 (WHYBURN [10, p. 78]). *If J_1 and J_2 are nondegenerate, closed, and mutually exclusive subsets of S, there exist two mutually exclusive arcs in S joining J_1 and J_2.*

LEMMA 4. *Let J be any simple closed curve in S, ε any positive number, and K any subset of* Cl (In (J)) *such that* diam $(K) < \varepsilon/2$. *Suppose every complementary domain of S contained in* In (J)) *has diameter less than $\varepsilon/4$. Then* Cl (In $(J)) = R_1 + R_2 + \cdots + R_n$, *where $R_1, \ldots, R_n$ are disks, with disjoint interiors, such that*

(1) $\sum_{i=1}^{n}$ Bd (R_i) *is a connected* 1-*complex in S,*

(2) *for each i, $1 \leqq i \leqq n$, if K intersects R_i, then* diam $(R_i) \leqq 4\varepsilon$.

Proof. Let p be a point of K. We consider two cases.

Case 1. dist $(p, J) > \varepsilon$. For any number n, let $D(n)$ be the *circular* disk with center at p and radius equal to n. Then $D(\varepsilon) \cap J = \varnothing$. And

$$(*) \qquad S \cap \overline{[D(7\varepsilon/8) - D(5\varepsilon/8)]} \textit{ separates } p \textit{ and } J \textit{ in } E^2.$$

But then we can find a continuous curve L' such that L' separates p and J, and

$$L' \subset S \cap [E^2 - D(\varepsilon/2)] \cap \text{Int}(D(\varepsilon)).$$

Then the point p is in a bounded domain M of $E^2 - L'$. The outer boundary of M bounds a disk R_1 containing K.

All that remains is to join Bd (R_1) and J by two mutually exclusive arcs in S. The existence of these arcs follows from Lemma 3.

Case 2. dist $(p, J) \leqq \varepsilon$. Let m be a point of J such that dist $(p, m) \leqq \varepsilon$. For any number n, let $D(n)$ be the circular disk, *centered at m*, with radius equal to n. Note that K is a subset of $D(\varepsilon+\varepsilon/2)$.

If Cl (In (J)) has diameter greater than 4ε, there is a point r of J such that r is contained in $E^2 - D(2\varepsilon)$. Then:

(*) $S \cap \overline{[D(\varepsilon+7\varepsilon/8)-D(\varepsilon+5\varepsilon/8)]}$ separates m and r in Cl (In (J)).

(See Diagram 2.)

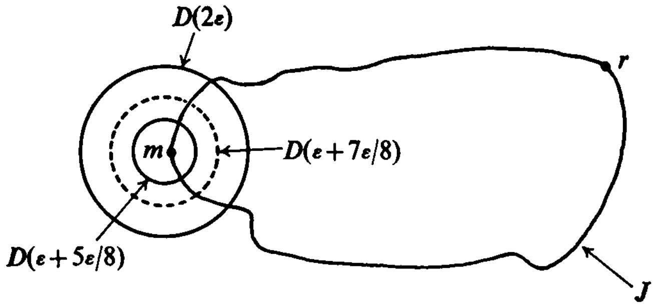

DIAGRAM 2

Hence there is an arc L', in $S \cap (E^2 - D(\varepsilon+9\varepsilon/16)) \cap$ Int $(D(\varepsilon+15\varepsilon/16))$, which runs from A_1 to A_2 (where A_1 and A_2 are the two arcs into which m and r divide J) and such that $A_1 + A_2 + L'$ forms a θ-curve (with $L' \subset$ Cl (In (J))).

Let F_0 and F_1 be the two disks bounded by the θ-curve. Assume $m \in F_0$ and $r \in F_1$. Suppose the diameter of F_0 is greater than 4ε. Then we subdivide F_0, exactly as we subdivided Cl (In (J)), into two disks $F_0^{(1)}$ and $F_1^{(1)}$. If the diameter of $F_0^{(1)}$ is still too large we subdivide again. The subdivision process must terminate at a finite stage because J can contain only a finite number of spanning arcs of

$$\overline{D(2\varepsilon) - D(\varepsilon+15\varepsilon/16)}.$$

Thus, we may write Cl (In $(J)) = F_0 + F_1 + \cdots + F_p$, where diam $(F_0) \leqq 4\varepsilon$.

Suppose the diameter of F_1 is greater than 4ε and K intersects F_1. Then, since K is contained in $D(\varepsilon+\varepsilon/2)$, there is a point m' of J such that $m' \in$ Bd (F_1) and $m' \in D(\varepsilon+\varepsilon/2)$. Hence we may repeat the above process, subdividing F_1 into disks $F_0', F_1', \ldots, F_q'$ with diam $(F_0') \leqq 4\varepsilon$. But this process must terminate at a finite stage because J can contain only a finite number of spanning arcs of

$$\overline{D(\varepsilon\ +\ 9\varepsilon/16) - D(\varepsilon+\varepsilon/2)}.$$

The proof of Lemma 4 is complete.

THEOREM 5. *Let J be any simple closed curve in S, and ε any positive number. Suppose every complementary domain of S contained in* In (J) *has diameter less than $\varepsilon/4$. Then* Cl (In $(J)) = R_1 + R_2 + \cdots + R_n$, *where $R_1, \ldots, R_n$ are disks, with disjoint interiors, such that*

(1) $\sum_{i=1}^{n}$ Bd (R_i) *is a connected* 1-*complex in S,*

(2) *for each i,* $1 \leq i \leq n$, diam $(R_i) \leq 4\varepsilon$.

Proof. Follows from Lemma 4.

3. **The flipping theorem.** In 2 we divided Cl (In(J)) into subdisks with boundaries in S. In 3 we move points of S from one subdisk to another.

THEOREM 6 (CLAYTOR [3, p. 812]). *If X is a continuous curve in E^2, and T is a simple closed curve in X, then there are only a finite number of components of $X-T$ each having diameter greater than a given fixed number.*

Throughout 3 we shall use the following notation unless stated otherwise. S, as usual, denotes a planar continuous curve without separating points. J is a simple closed curve in S. Y is an arc of S which is a spanning arc of Cl (In (J)) (thus $Y+J$ is a θ-curve). D_1 and D_2 are the two domains into which Y divides In (J). Finally, g is a homeomorphism which takes the part of S contained in Cl (In (J)) into Cl (In (J)), and $g=$ id on $J+Y$.

LEMMA 7. *Suppose K is a component of $S \cap (D_1+D_2)$ such that $\bar{K} \cap (J+Y) \subset Y$. Then there is a disk R in* Cl (In (J)) *such that $\bar{K}$ is a subset of R, R intersects $J+Y$ in a subarc of Y, and* Bd $(R) - Y$ *is a subset of K.*

Proof. $J+Y+\bar{K}$ is a planar continuous curve without separating points. Lemma 7 then follows by an application of Theorem 1.

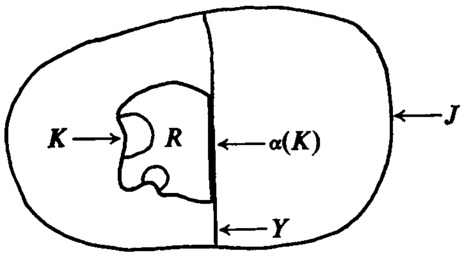

DIAGRAM 3

NOTATION. We shall denote the subarc of Y which is the intersection of R and $J+Y$ by $\alpha(K)$. (See Diagram 3.)

We shall use the following *definitions* based on Claytor [3]. Suppose E and F are distinct components of $S \cap (D_1+D_2)$, and $(\bar{E}+\bar{F}) \cap (J+Y) \subset Y$. Then E and F are on *opposite sides* of Y provided one of the following is true: (a) there are arcs ab in $\bar{E}$, cd in $\bar{F}$, having only their endpoints on Y, such that $(a+b)$ separates

$(c+d)$ on $J+Y$, (b) there are triods T_1 in $\bar{E}$, T_2 in $\bar{F}$ such that $T_1 \cap (J+Y) = T_2 \cap (J+Y) = a+b+c$, where a, b, c are the feet of T_1 and T_2.

If $E_1, E_2, \ldots, E_n$ $(n>1)$ is a finite collection of distinct components of $S \cap (D_1+D_2)$ such that E_i and E_{i+1} are on opposite sides of Y, $1 \leqq i \leqq n-1$, then the set $E_1 + \cdots + E_n$ is called a *chain* joining E_1 to E_n.

If E is a component of $S \cap (D_1+D_2)$ such that $\bar{E} \cap (J+Y) \subset Y$, then the *nest* $n(E)$ determined by E is the set consisting of E together with all components E_x of $S \cap (D_1+D_2)$ for which there exists a chain joining E and E_x.

If E is a component of $S \cap (D_1+D_2)$ such that $\bar{E} \cap (J+Y) \subset Y$, then the *nest arc* $A(E)$ determined by E is the subarc of Y which contains $\mathrm{Cl}\,(n(E)) \cap Y$ and whose endpoints are points of $\mathrm{Cl}\,(n(E)) \cap Y$. (It is possible that $A(E) = Y$.)

LEMMA 8. *Suppose E is a component of $S \cap (D_1+D_2)$ such that E is a subset of D_i and $g(E)$ is a subset of D_j, $i \neq j$. Let the endpoints of the nest arc $A(E)$ be p and m. Then there is a disk M in* $\mathrm{Cl}\,(\mathrm{In}\,(J))$ *such that* $\mathrm{Bd}\,(M) \cap S = \{p, m\}$, *and M contains E.*

LEMMA 9. *The disk M in Lemma* 8 *can be chosen so that if*

$$\varepsilon = \sup \{\mathrm{diam}\,(\alpha(F)) : F \text{ is a component of } n(E)\}$$

and if G is any component of $S \cap (D_1+D_2)$ such that $\bar{G} \cap (J+Y) \subset Y$ and $\mathrm{diam}\,(\alpha(G)) > \varepsilon$, *then* $G \cap \mathrm{Int}\,(M) = \varnothing$. (*For definition of $\alpha(F)$ and $\alpha(G)$, see Notation, following Lemma* 7.)

LEMMA 10. *If A is a subarc of Y, then A is the nest arc for at most a finite number of nests. Moreover, only one nest, having A as its nest arc, can have a component whose closure fails to contain both endpoints of A.*

LEMMA 11. *The collection of nest arcs forms a null sequence.*

LEMMA 12. *Let A be a subarc of Y. Let $n(E_1), \ldots, n(E_p)$ be nests in $S \cap (D_1+D_2)$ having A as a nest arc. Let M be a disk in* $\mathrm{Cl}\,(\mathrm{In}\,(J))$ *such that* $\mathrm{Bd}\,(M) \cap S$ *is the endpoints of A, and M contains the nests $n(E_i)$, $1 \leqq i \leqq p$.*

Then there is a homeomorphism $H: E^3 \twoheadrightarrow E^3$ such that:

(1) $H(S) \subset E^2$,

(2) $H = \mathrm{id}$ *on* $J+Y$,

(3) $H(E_i) \subset D_1$ *if and only if* $g(E_i) \subset D_1$, $1 \leqq i \leqq p$,

(4) $H = \mathrm{id}$ *outside the fence over* M,

(5) $H = \mathrm{id}$ *outside an arbitrarily small neighborhood* (*in* E^3) *of* M.

Proof. We shall induce on the number p of nests $n(E_i)$. If there is only one nest $n(E_1)$, and $E_1 \subset D_i$ and $g(E_1) \subset D_i$, let $H = \mathrm{id}$. If there is one nest $n(E_1)$, and $E_1 \subset D_i$ and $g(E_1) \subset D_j$, $i \neq j$, then we "push in" slightly on $\mathrm{Bd}\,(M)$ to obtain a smaller disk M' whose boundary intersects S in the endpoints of A. We flip M' end-over-end, with A as the axis of rotation.

Now suppose $p>1$. Then, with at most one exception, each E_i has the property that $\bar{E}_i$ contains both endpoints of A (Lemma 10). Then, by Lemma 7, there is a disk R_i corresponding to each E_i, such that $E_i \subset R_i$ and $\mathrm{Bd}\,(R_i) - Y$ is an open arc in E_i with endpoints equal to the endpoints of A.

Let B_1 be the outermost of the open arcs $\mathrm{Bd}\,(R_i) - Y$ which are contained in D_1 (provided some $\mathrm{Bd}\,(R_i) - Y$ lies in D_1; otherwise B_1 does not exist). Let B_2 be the outermost of the open arcs $\mathrm{Bd}\,(R_i) - Y$ which are contained in D_2.

Case 1. B_1 and B_2 both exist and belong to the same nest. But then $p=1$, which is impossible.

Case 2. B_1 and B_2 belong to different nests, or one of B_1, B_2 does not exist. Then one of B_1, B_2, say B_1, has the property that if E_1 is the component of $S \cap (D_1 + D_2)$ containing B_1, then $\bar{E}_1 \cap A$ is exactly the endpoints of A. If $E_1 \subset D_i$ and $g(E_1) \subset D_i$, define $H' = \mathrm{id}$. If $E_1 \subset D_i$ and $g(E_1) \subset D_j$, $i \neq j$, let H' be the end-over-end rotation described above.

Then because $H'(\bar{E}_1) \cap A$ is the endpoints of A, and $H'(B_1)$ is an "outermost" arc, it is clear that there is a disk M_0 in M such that the intersection of $\mathrm{Bd}\,(M_0)$ and $H'(S)$ is the endpoints of A, $H'(E_1)$ does not intersect M_0, and $H'(E_i) \subset M_0$, $2 \leqq i \leqq p$. Then the induction hypothesis is satisfied by M_0 and the nests $H'(n(E_i))$ contained in M_0.

The proof of Lemma 12 is complete.

Lemma 13. *Let ε be a positive number, and suppose that if G is a component of $S \cap (D_1 + D_2)$ and* $\mathrm{diam}\, \alpha(G) > \varepsilon$, *then $G \subset D_1$ if and only if $g(G) \subset D_1$.*

Suppose $E_1, \ldots, E_n$ are components of $S \cap (D_1 + D_2)$ none of which is contained in a nest $n(G)$ where $\mathrm{diam}\, \alpha(G) > \varepsilon$. *Let ε_0 be the diameter of the largest nest arc among the nest arcs determined by $E_1, \ldots, E_n$ and suppose ε_1 is a number such that every component F of $S \cap (D_1 + D_2)$ has diameter less than ε_1, provided $\bar{F} \cap (J+Y) \subset Y$, and F is not a component of a nest $n(G)$ where* $\mathrm{diam}\, \alpha(G) > \varepsilon$.

Then there is a homeomorphism $H\colon E^3 \twoheadrightarrow E^3$ such that:

(1) $H(S) \subset E^2$,

(2) $H = \mathrm{id}$ *on* $J+Y$,

(3) $H(E_i) \subset D_1$ *if and only if* $g(E_i) \subset D_1$, $1 \leqq i \leqq n$,

(4) $H = \mathrm{id}$ *on every component G of $S \cap (D_1 + D_2)$ such that* $\mathrm{diam}\, \alpha(G) > \varepsilon$,

(5) *H moves no point a distance more than* $4(2\varepsilon_1 + \varepsilon_0)$.

Proof. Lemma 13 follows from several applications of Lemmas 8, 9 and 12.

Theorem 14 (The Flipping Theorem). *There is a homeomorphism $H\colon E^3 \twoheadrightarrow E^3$ such that*:

(a) $H(S) \subset E^2$.

(b) $H = \mathrm{id}$ *on* $J+Y$.

(c) $H = \mathrm{id}$ *outside the fence over* $\mathrm{Cl}\,(\mathrm{In}\,(J))$.

(d) $H = \mathrm{id}$ *outside an arbitrary small neighborhood* (*in* E^3) *of* $\mathrm{Cl}\,(\mathrm{In}\,(J))$.

(e) *If x is a point of $S \cap (D_1 + D_2)$ and $g(x) \in D_i$, $i \in \{1, 2\}$, then $H(x) \in D_i$.*

Proof. We shall define a sequence of homeomorphisms $H_1, H_2, H_3, \ldots$ whose limit is the required homeomorphism H.

Suppose that, using Lemma 13, we have defined homeomorphisms $H_1, \ldots, H_k$ such that for each n, $1 \leqq n \leqq k$,

(1) $H_n: E^3 \twoheadrightarrow E^3$ and $H_n(S) \subset E^2$.

(2) $H_n = \mathrm{id}$ on $J+Y$ and $H_n = \mathrm{id}$ outside the fence over Cl (In (J)).

(3) If, for any number η, we define $E(\eta) = \{x \in E^3 : \mathrm{dist}\,(x, J+Y) \geqq \eta\}$, then $H_n = H_{n-1}$ on $E(1/2^{n-1})$.

(4) $\|H_n - H_{n-1}\| < 100/2^{n-1}$.

(5) There are numbers $0 < \beta_n < \beta_{n-1} < \cdots < \beta_1$ such that if F is a component of $S \cap (D_1 + D_2)$, and F intersects $E(1/2^n)$, then diam $\alpha(F) > \beta_n$ (provided $\alpha(F)$ is defined, i.e. provided $\bar{F} \cap (J+Y) \subset Y$).

(6) If F is a component of $S \cap (D_1 + D_2)$ and diam $\alpha(F) > \beta_n$, then $g(F) \subset D_1$ if and only if $H_n(F) \subset D_1$.

(7) If F is a component of $S \cap (D_1 + D_2)$ and F is not in a nest $n(G)$ where diam $\alpha(G) > \beta_n$, then diam $H_n(F) < 1/2^n$, provided $\bar{F} \cap (J+Y) \subset Y$.

(8) If F is a component of $S \cap (D_1 + D_2)$, F is not in a nest $n(G)$ where diam $\alpha(G) > \beta_n$, and $A(F)$ is a nest arc determined by F, then diam $A(F) < 1/2^{n+3}$.

We shall now define H_{k+1}. Choose $\beta_{k+1} < \beta_k$ to be a number such that (a) if G is a component of $S \cap (D_1 + D_2)$, $\bar{G} \cap (J+Y) \subset Y$, and G intersects $E(1/2^{k+1})$, then diam $\alpha(G) > \beta_{k+1}$; and (b) if $n(G)$ is a nest whose nest arc has diameter not less than $1/2^{k+4}$, then diam $\alpha(F) > \beta_{k+1}$ for some component F of $n(G)$.

By an application of Lemma 13, we obtain a homeomorphism $H'_{k+1}: E^3 \twoheadrightarrow E^3$ such that if F is a component of $S \cap (D_1 + D_2)$, with diam $\alpha(F) > \beta_{k+1}$, then $H'_{k+1}(F) \subset D_1$ if and only if $g(F) \subset D_1$ (by Theorem 6 there are only a finite number of components F of $S \cap (D_1 + D_2)$ such that diam $\alpha(F) > \beta_{k+1}$), and H'_{k+1} satisfies conditions (1)–(4) of the inductive hypothesis. Condition (7) will be satisfied if we follow H'_{k+1} by a homeomorphism $h_{k+1}: E^3 \twoheadrightarrow E^3$ which takes the plane onto itself and which "squeezes" the components of $H'_{k+1}(S \cap (D_1 + D_2))$ which are too large in toward their nest arcs. We may choose h_{k+1} carefully enough so that if we let $H_{k+1} = h_{k+1} \circ H'_{k+1}$, then H_{k+1} has all the properties required by the inductive hypothesis.

This defines the sequence $H_1, H_2, \ldots$. Let $H = \lim_{n \to \infty} H_n$.

COROLLARY 15. *Let J be a simple closed curve in S and let $g': S \cap$ Cl (In (J)) $\to$ Cl (In (J)) be a homeomorphism such that $g' = \mathrm{id}$ on J. Suppose* Cl (In (J)) $= R_1 + \cdots + R_n$, *where $R_1, \ldots, R_n$ are disks with disjoint interiors, and $\sum_{i=1}^{n}$* Bd (R_i) *is a connected* 1*-complex in $g'(S)$.*

Then there is a homeomorphism $H: E^3 \twoheadrightarrow E^3$ such that:

(1) $H(S) \subset E^2$,

(2) $H = \mathrm{id}$ *outside the fence over* Cl (In (J)),

(3) $H = \mathrm{id}$ *outside an arbitrarily small neighborhood of* Cl (In (J)),

(4) *If x is a point of $S \cap \mathrm{Cl}(\mathrm{In}(J))$ and $g'(x) \in \mathrm{Bd}(R_i)$, for some i, $1 \leqq i \leqq n$, then* $H(x) = g'(x)$,

(5) *If x is a point of $S \cap \mathrm{Cl}(\mathrm{In}(J))$ and $g'(x) \in R_i$ for some i, $1 \leqq i \leqq n$, then* $H(x) \in R_i$.

COROLLARY 16. *Let J be a simple closed curve in S and let g': $S \cap \mathrm{Cl}(\mathrm{In}(J)) \to \mathrm{Cl}(\mathrm{In}(J))$ be a homeomorphism such that $g' = \mathrm{id}$ on $J + \mathrm{Bd}(D)$, where D is a complementary domain of $g'(S)$ such that $\mathrm{Bd}(D) \cap J$ is more than one point. Then there is a homeomorphism H: $E^3 \to E^3$ such that*:

(1) $H(S) \subset E^2$,

(2) $H = \mathrm{id}$ *outside the fence over* $\mathrm{Cl}(\mathrm{In}(J))$,

(3) $H = \mathrm{id}$ *outside an arbitrarily small neighborhood of* $\mathrm{Cl}(\mathrm{In}(J))$,

(4) $H = \mathrm{id}$ *on* $\mathrm{Bd}(D)$,

(5) *D is a complementary domain of $H(S)$.*

Proof. By Theorem 2, Case 1, $\mathrm{Cl}(\mathrm{In}(J)) = \bar{D} + R_1 + R_2 + \cdots$, where $R_1, R_2, \ldots$ is a null sequence of disks with disjoint interiors such that $D \cap R_i = \varnothing$, $\mathrm{Bd}(R_i) \cap J$ is an arc, $i = 1, 2, 3, \ldots$, and $\sum_{i=1}^{\infty} \mathrm{Bd}(R_i) \subset \mathrm{Bd}(D) + J$.

If there is only a finite number of disks R_i, then we apply Corollary 15.

If there is an infinite number, then $\mathrm{Bd}(R_i) \cap J$ and $\mathrm{Bd}(R_j) \cap J$, $i \neq j$, can have at most one point in common. By Theorem 6, the components of $S \cap [\sum_{i=1}^{\infty} \mathrm{Int}(R_i) + D]$ form a null sequence. We can apply Theorem 14 a countable number of times to obtain a sequence of homeomorphisms whose limit is H.

LEMMA 17. *Let H: $E^3 \twoheadrightarrow E^3$ be the homeomorphism obtained in Theorem* 14. *Let N be a complementary domain of $H(S)$ such that N is a subset of* $\mathrm{In}(J)$. *Suppose N contains a point $H(x)$ such that the distance between x and S is greater than or equal to $1/2^m$, for some integer m. Then at last one of the following holds*:

(a) *$H^{-1}(N)$ is a complementary domain of S, and* $\mathrm{diam}\, H^{-1}(N) \geqq 1/2^m$, *or*

(b) $\mathrm{Bd}(N)$ *intersects the θ-curve $J + Y$ in at least two points.*

Proof. H is the limit of a sequence of homeomorphisms $H_1, H_2, H_3, \ldots$, where H_n differs from H_{n-1} by a finite number of homeomorphisms $h_1, \ldots, h_m$ each of which is a rotation of a disk end-over-end with a subarc of Y as the axis of rotation.

But (a) or (b) holds for each rotation h_i. Hence (a) or (b) holds for each H_n, $n = 1, 2, 3, \ldots$. Suppose $\mathrm{Bd}(N) \cap (J + Y)$ is at most one point. Then $\mathrm{Bd}(N)$ is contained in the closure of a component F of $H(S) \cap (D_1 + D_2)$. By the way in which H was defined, there is an integer n such that $H^{-1}(F) = H_n^{-1}(F)$.

4. The extension theorem for continuous curves without separating points.

THEOREM 18. *Let S and S' be continuous curves in E^2, and suppose S contains no separating points. Let g be a homeomorphism of S onto S'. Then there is a homeomorphism H: $E^3 \twoheadrightarrow E^3$ such that $H = g$ on S and H is realizable by an isotopy.*

Proof. All homeomorphisms defined below can be chosen to be the identity outside some preassigned cube containing S and S', and hence the final homeomorphism H may be realized by an isotopy.

By Corollary 15, we may assume that the boundary J of the unbounded complementary domain of S' is the same as the boundary of the unbounded complementary domain of S, and that $g=\text{id}$ on J.

We shall define a sequence of homeomorphisms whose limit is the required homeomorphism H.

Suppose we have defined homeomorphisms $H_1, H_2, \ldots, H_m$ such that for any n, $1 \leqq n \leqq m$, we have:

(1) $H_n\colon E^3 \twoheadrightarrow E^3$, $H_n(S) \subset E^2$, and $H_n=\text{id}$ outside the fence over Cl (In (J)).

(2) Cl (In $(J)) = \bar{D}_1 + \cdots + \bar{D}_{p_n} + R_1^n + R_2^n + \cdots$, where

(a) D_i, $1 \leqq i \leqq p_n$, is a complementary domain of S' and of $H_n(S)$.

(b) $R_1^n, R_2^n, \ldots$ is a null sequence of disks with disjoint interiors, each having diameter no more than $1/2^n$.

(c) $\sum_{i=1}^{\infty} \text{Bd}(R_i^n) \subset S'$.

(d) $D_i \cap R_j^n = \varnothing$, for all i, j, $1 \leqq i \leqq p_n$, $j=1, 2, \ldots$.

(3) If x is a point of S, then one of (a), (b), (c) holds:

(a) $g(x) \in \text{Bd}(D_i)$ for some i, $1 \leqq i \leqq p_n$, and $H_n(x)=g(x)$.

(b) $g(x) \in \text{Bd}(R_i^n)$ for some i, $i=1, 2, \ldots$, and $H_n(x)=g(x)$.

(c) $g(x) \in \text{Int}(R_i^n)$ for some i, $i=1, 2, \ldots$, and $H_n(x) \in \text{Int}(R_i^n)$.

(4) If for any number η, we let $E(\eta)=\{x \in E^3 : \text{dist}(x, S) \geqq \eta\}$, and if N is a complementary domain of $H_n(S)$ such that N is contained in some disk R_i^n and N intersects $H_n(E(1/2^k))$ for some integer k, then one of (a), (b) holds:

(a) $H_n^{-1}(N)$ is a complementary domain of S and diam $H_n^{-1}(N) \geqq 1/2^k$.

(b) Bd (N) intersects Bd (R_i^n) in at least two points.

(5) R_i^n misses $H_n(E(1/2^n))$ for all i, $i=1, 2, 3, \ldots$.

(6) $H_n=H_{n-1}$ on $E(1/2^{n-1})$.

(7) $\|H_n - H_{n-1}\| \leqq 100/2^{n-1}$.

We shall now define H_{m+1}.

Let $N_1, \ldots, N_l$ be the domains of $E^2 - S$ such that diam $(N_i) \geqq 1/2^{m+1}$, $1 \leqq i \leqq l$.

Let $\delta = \min\{1/2^{m+1}, \text{diam}\, g(\text{Bd}(N_1)), \ldots, \text{diam}\, g(\text{Bd}(N_l))\}$.

Since the disks $\{R_i^m\}$ form a null sequence there are only a finite number which have diameter greater than $\delta/2$. We may assume that R_1^m is the only disk with diameter greater than $\delta/2$.

Let $D_{m_1}, \ldots, D_{m_q}$ be the complementary domains of S' such that

(a) $D_{m_i} \subset R_1^m$, and

(b) diam $D_{m_i} \geqq 4^{-2}(\delta/2)$, $1 \leqq i \leqq q$.

By Theorem 2, we may subdivide R_1^m as follows: $R_1^m = \bar{D}_{m_1} + R_1 + R_2 + \cdots$ where $R_1, R_2, \ldots$ is a (finite or infinite) null sequence of disks, with disjoint interiors, such that $D_{m_1} \cap R_i = \varnothing$ and Bd (R_i) is a subset of S', $i=1, 2, \ldots$.

There is a homeomorphism $h_1: E^3 \twoheadrightarrow E^3$ such that: (a) $h_1 = \text{id}$ outside the fence over R_1^m, (b) h_1 moves no point outside a small neighborhood (in E^3) of R_1^m, (c) $h_1 = \text{id}$ on $H_n(E(1/2^m))$, (d) if x is a point of $H_m(S) \cap R_1^m$ and $g \circ H_m^{-1}(x) \in \text{Bd}(R_i)$ for some i, $i = 1, 2, \ldots$, then $g \circ H_m^{-1}(x) = h_1(x)$, and (e) if x is a point of $H_m(S) \cap R_1^m$, and $g \circ H_m^{-1}(x) \in R_i$ for some i, $i = 1, 2, \ldots$, then $h_1(x) \in R_i$.

The existence of h_1 follows from Corollary 15 in the case where $R_1, R_2, \ldots$ is a finite sequence, or from Corollary 16 in the case where $R_1, R_2, \ldots$, is an infinite sequence.

We may assume that D_{m_2} is contained in R_1. We then repeat the above procedure, with R_1 in place of R_1^m, D_{m_2} in place of D_{m_1}, and $g \circ H_m^{-1} \circ h_1^{-1}$ in place of $g \circ H_m^{-1}$.

We repeat the procedure a finite number of times, until we obtain a homeomorphism $H': E^3 \twoheadrightarrow E^3$ and a subdivision of R_1^m, so that: $R_1^m = \bar{D}_{m_1} + \cdots + \bar{D}_{m_q} + R_1' + R_2' + \cdots$, where $R_1', R_2', \ldots$ is a null sequence of disks with disjoint interiors such that $D_{m_i} \cap R_j' = \varnothing$ for all i and j, $1 \leqq i \leqq q$, $j = 1, 2, \ldots$, and $\text{Bd}(R_j') \subset S'$ for $j = 1, 2, \ldots$. If x is a point of $H_m(S) \cap R_1^m$, and $g \circ H_m^{-1}(x) \in \text{Bd}(R_i')$ for some i, $i = 1, 2, \ldots$, or $g \circ H_m^{-1}(x) \in \text{Bd}(D_{m_i})$ for some i, $1 \leqq i \leqq q$, then $g \circ H_m^{-1}(x) = H'(x)$. If x is a point of $H_m(S) \cap R_1^m$ and $g \circ H_m^{-1}(x) \in R_i'$ for some i, $i = 1, 2, \ldots$, then $H'(x) \in R_i'$. $H' = \text{id}$ outside the fence over R_1^m, $H' = \text{id}$ outside a small neighborhood (in E^3) of R_1^m, and H' moves no point of $H_m(E(1/2^m))$.

By Corollary 15, Theorem 5, and the fact that no R_i', $i = 1, 2, \ldots$, contains a domain of $E^2 - S'$ which has diameter as large as $4^{-2}(\delta/2)$, we may assume, in addition, that each R_i' has diameter no more than $\delta/2$.

By induction hypothesis (4), and Lemma 17, it is clear that if N is a domain of $E^2 - H' \circ H_m(S)$, if N intersects $H' \circ H_m(E(1/2^k))$ for some integer k, and if N is contained in a disk in the sequence $\{R_1', R_2', \ldots, R_2^m, R_3^m, \ldots\}$ then either $\text{Bd}(N)$ intersects the boundary of the disk in at least two points, or $(H' \circ H_m)^{-1}(N)$ is a domain of $E^2 - S$ with diameter no smaller than $1/2^k$.

A finite number of disks in the null sequence $\{R_1', R_2', \ldots, R_2^m, R_3^m, \ldots\}$ may intersect $H' \circ H(E(1/2^{m+1}))$. We must subdivide further to eliminate these intersections. We may assume that R_2^m is the only disk in the sequence which intersects $H' \circ H_m(E(1/2^{m+1}))$.

Let $M_1, \ldots, M_r$ be the domains of $E^2 - H' \circ H_m(S)$ which are contained in R_2^m and which intersect $H' \circ H_m(E(1/2^{m+1}))$. By our choice of δ, the boundary of each M_i, $1 \leqq i \leqq r$, intersects $\text{Bd}(R_2^m)$ in at least two points. Hence, for each M_i, $1 \leqq i \leqq r$, we may choose a pair of points p_i, q_i such that $\{p_i, q_i\}$ is contained in $\text{Bd}(M_i) \cap \text{Bd}(R_2^m)$.

Let $\varepsilon = \min\{\text{dist}(p_i, q_i) : 1 \leqq i \leqq r\}$.

Let $D_{n_1}, \ldots, D_{n_y}$ be the domains of $E^2 - S'$ such that (a) $D_{n_i} \subset R_2^m$, and (b) $\text{Bd}(D_{n_i})$ contains a pair of points of $\text{Bd}(R_2^m)$ which are a distance at least ε apart, $1 \leqq i \leqq y$.

There is a homeomorphism $G: E^3 \twoheadrightarrow E^3$ and a subdivision of R_2^m so that: $R_2^m = \bar{D}_{n_1} + \cdots + \bar{D}_{n_y} + R_1'' + R_2'' + \cdots$ where the subdivision and the homeomorphism G have properties analogous to the subdivision of R_1^m and the homeomorphism H' above.

Further, we may choose G so that if D is a complementary domain of $G \circ H' \circ H_m(S)$, and D intersects $G \circ H' \circ H_m(E(1/2^{m+1}))$, then Bd (D) contains two points p, q of Bd (R_2^m) such that dist $(p, q) \geqq \varepsilon$. But then $D = D_{n_i}$ for some i, $1 \leqq i \leqq y$. Thus D misses R_i'' for all i, $i = 1, 2, \ldots$.

Hence, if we let $H_{m+1} = G \circ H' \circ H_m$, relabel $\{D_{m_1}, \ldots, D_{m_q}, D_{n_1}, \ldots, D_{n_y}\}$ as $\{D_{p_m+1}, \ldots, D_{p_{m+1}}\}$, and relabel $\{R_1', R_2', \ldots, R_1'', R_2'', \ldots, R_3^m, R_4^m, \ldots\}$ as $\{R_1^{m+1}, R_2^{m+2}, \ldots\}$, then conditions (1)–(7) of the inductive hypothesis are fulfilled. And $H = \lim_{n\to\infty} H_n$ is the required homeomorphism.

§IV

1. **Introduction.** In this section we shall prove the extension theorem. Henceforth S will denote any planar continuous curve.

We shall make use of the following theorems:

THEOREM 19 [10, Theorem 3.1, p. 108]. *If A and B are compact subsets of E^2 such that $A \cap B = T$ is totally disconnected and a, b are points of $A - (A \cap B)$ and $B - (A \cap B)$, respectively, and ε is any positive number, then there exists a simple closed curve J in E^2 which separates a and b and is such that $J \cap (A + B) \subset A \cap B$, and every point of J is at a distance less than ε from some point of A.*

THEOREM 20 [10, Theorem 2.1, p. 66]. *If C is a nondegenerate cyclic element of S, and T is a component of $S - C$, then there exists a point x of C such that T is a component of $S - \{x\}$.*

2. **Enlarging cyclic elements.** Suppose C is a nondegenerate cyclic element of S, and suppose T is a component of $S - C$. In 2 we shall demonstrate a way of moving S with a homeomorphism H, and then adding planar arcs $A_1, \ldots, A_n$ to the planar curve $H(S)$ so that $H(C)$ and a certain part of $H(T)$ belong to the same cyclic element of $H(S) + A_1 + \cdots + A_n$. We shall do this in such a way that if S' is a given planar homeomorphic image of S, then we can move S' with a homeomorphism H', and add planar arcs $A_1', \ldots, A_n'$ to the planar curve $H'(S')$ so that there is an extension of the natural homeomorphism of $H(S)$ onto $H'(S')$ to a homeomorphism of $H(S) + A_1 + \cdots + A_n$ onto $H'(S') + A_1' + \cdots + A_n'$.

LEMMA 21. *Let g be a homeomorphism of S into E^2, and let J be a simple closed curve in the boundary of a complementary domain of S. Let A be an arc in J, and let p be an endpoint of $g(A)$. Then there is a point x of $g(A) - \{p\}$ such that x and p lie on the boundary of a complementary domain of $g(S)$.*

THEOREM 22. *Let e be a point of S, and let T be a component of $S - \{e\}$. Let J be a simple closed curve in $\bar{T}$. Suppose f_1 and f_2 are homeomorphisms of S into E^2. Suppose D_1 and D_2 are disks in E^2 such that, for each i, $i = 1, 2$,* Bd $(D_i) \cap f_i(S)$

$=\{f_i(e)\}$, and $f_i(T)\subset D_i$. Then for each i, $i=1, 2$, there is a homeomorphism H_i: $E^3 \twoheadrightarrow E^3$, and a finite collection of arcs $A_1^i, \ldots, A_n^i$ such that:

(1) $H=\mathrm{id}$ *outside the fence over D_i, and outside an arbitrarily small neighborhood of D_i, and $H_i \circ f_i(S)\subset E^2$,*

(2) *for each j, $1\leqq j\leqq n$, A_j^i lies in D_i and has an endpoint on $H_i\circ f_i(T)$,*

(3) Bd (D_i) *and $H_i\circ f_i(J)$ belong to the same cyclic element of $H_i\circ f_i(S)+$* Bd (D_i) $+A_1^i+\cdots+A_n^i$,

(4) *there is a homeomorphism g of $H_1\circ f_1(S)+$* Bd $(D_1)+A_1^1+\cdots+A_n^1$ *onto* $H_2\circ f_2(S)+$ Bd $(D_2)+A_1^2+\cdots+A_n^2$ *such that $g=H_2\circ f_2\circ f_1^{-1}\circ H_1^{-1}$ on $H_1\circ f_1(S)$, and* $g($Bd $(D_1))=$ Bd (D_2).

Proof. Let M_1 be the simple closed curve in $f_1(S)$ such that (a) M_1 lies on the boundary of the complementary domain of $f_1(S)+$ Bd (D_1) whose outer boundary is Bd (D_1), (b) $f_1(J)\subset$ Cl (In (M_1)). The existence of M_1 follows from [11, Theorem 17, p. 369].

LEMMA 23. *Suppose $f_1(J)$ and M_1 belong to different cyclic elements of $f_1(S)$* $+$Bd (D_1). *Let C' be the cyclic element of $f_1(S)+$* Bd (D_1) *which contains M_1 and let T' be the component of $f_1(S)-C'$ whose closure contains $f_1(J)$. Then we may assume without loss of generality that $\overline{T}'$ does not intersect M_1.*

Proof. Suppose $\overline{T}'$ intersects M_1. Then, by Theorem 20, this intersection is a single point p. By an application of Theorem 19, there is a disk E such that Bd$(E)\cap f_1(S)=\{p\}$, and Bd(E) separates T' and $M_1-\{p\}$ (and thus $E\subset$Cl(In(M_1))). Let A be an arc in D_1 from Bd $(D_1)-\{f_1(e)\}$ to p such that A lies, except for its endpoints, in $E^2-[f_1(S)+$Bd $(D_1)]$. Let d be a small arc with one endpoint on A and the other equal to p so that the resulting disk E_1, bounded by d and a subarc of A, intersects $f_1(S)+$Bd (D_1) only at p (see Diagram 4).

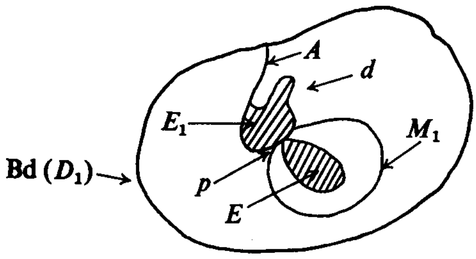

DIAGRAM 4

Let $G_1: E^3 \twoheadrightarrow E^3$ be a homeomorphism which interchanges E and E_1 so that $G_1(f_1(S)+$Bd $(D_1))\subset E^2$, $G_1=\mathrm{id}$ on $[f_1(S)+$Bd $(D_1)]-E$, and $G_1(p)=p$.

It is clear that if Q is a simple closed curve in $G_1\circ f_1(S)$, and $G_1\circ f_1(J)$ is contained in Cl (In (Q)), then $f_1(J)$ is contained in Cl (In $(G_1^{-1}(Q))$).

Let $G_1(M_1^{(1)})$ be the simple closed curve in $G_1\circ f_1(S)$ such that (a) $G_1(M_1^{(1)})$ lies on the boundary of the complementary domain of $G_1(f_1(S)+$Bd $(D_1))$ whose outer boundary is Bd (D_1), and (b) $G_1 \circ f_1(J)$ is a subset of Cl (In $(G_1(M_1^{(1)}))$). Then $M_1^{(1)}$ and M_1 belong to different cyclic elements of $f_1(S)+\text{Bd}(D_1)$, and $f_1(J) \subset \text{Cl}(\text{In}(M_1^{(1)}))$.

Suppose $M_1^{(1)}$ and $f_1(J)$ belong to different cyclic elements of $f_1(S)+\text{Bd}(D_1)$. Let C'' be the cyclic element containing $M_1^{(1)}$, and let T'' be the component of $f_1(S)-C''$ whose closure contains $f_1(J)$. If $\overline{T}''$ intersects $M_1^{(1)}$, then, as above, we obtain a homeomorphism $G_2: E^3 \twoheadrightarrow E^3$ which moves $G_1 \circ f_1(J)$ out of In $(G_1(M_1^{(1)}))$.

We continue this process. It must terminate after a finite number of steps, however, because the simple closed curves $M_1, M_1^{(1)}, M_1^{(2)}, \ldots$ belong to different cyclic elements of $f_1(S)+\text{Bd}(D_1)$, and each $M_1^{(i)}$ contains $f_1(J)$ in the closure of its interior. But the nondegenerate cyclic elements of $f_1(S)+\text{Bd}(D_1)$ form a null sequence [10, Theorem 4.2, p. 71]. The proof of Lemma 23 is complete.

Resuming the proof of Theorem 22, let M_2 be the simple closed curve in $f_2(S)$ such that (a) M_2 lies on the boundary of the complementary domain of $f_2(S)+\text{Bd}(D_2)$ whose outer boundary is Bd (D_2), and (b) $f_2(J)$ is contained in Cl (In (M_2)). Again, we may assume that either $f_2(J)$ and M_2 belong to the same cyclic element of $f_2(S)+\text{Bd}(D_2)$, or if C' is the cyclic element containing M_2, then the closure of the component of $f_2(S)-C'$ containing $f_2(J)$ does not intersect M_2.

For each i, $i=1, 2$, let p_i be the point of M_i such that $M_i-\{p_i\}$ and Bd $(D_i)-\{p_i\}$ belong to different components of $[f_i(S)+\text{Bd}(D_i)]-\{p_i\}$. For each i, $i=1, 2$, let N_i be the complementary domain of $f_i(S)+\text{Bd}(D_i)$ whose outer boundary is Bd (D_i). One may show that either $f_2 \circ f_1^{-1}(p_1)$ is contained in Bd (N_2), or $f_1 \circ f_2^{-1}(p_2)$ is contained in Bd (N_1). Assume that $f_2 \circ f_1^{-1}(p_1)$ is contained in Bd (N_2).

Let $f=f_2 \circ f_1^{-1}$. There is a disk E such that Bd $(E) \cap [f_2(S)+\text{Bd}(D_2)]=\{f(p_1)\}$, Bd $(E)-\{f(p_1)\}$ is contained in the domain N_2, and Bd (E) separates $f(M_1)-\{f(p_1)\}$ and Bd $(D_2)-\{f(p_1)\}$.

By an application of Lemma 21, there is an arc B such that one endpoint of B is $f(p_1)$, the other endpoint m is in $f(M_1)$, and except for its endpoints, B misses $f_2(S)+\text{Bd}(D_2)+\text{Bd}(E)$. Then the simple closed curve C, formed by B and an arc in $f(M_1)$ from m to $f(p_1)$, lies, except for $f(p_1)$, in the interior of E.

Let E_1 be a disk which intersects $f_2(S)+\text{Bd}(D_2)$ only at $f(p_1)$, and which lies, except for $f(p_1)$, in In (C), and such that E_1 and C intersect the same complementary domain of $f_2(S)+\text{Bd}(D_2)$.

There is a homeomorphism $G: E^3 \twoheadrightarrow E^3$ such that (a) G interchanges the pinched annulus bounded by Bd $(E_1)+C$ and the one bounded by Bd $(E)+C$, (b) $G=\text{id}$ on $[f_2(S)+\text{Bd}(D_2)]-E$, (c) $G=\text{id}$ on C, and (d) $G(f_2(S)+\text{Bd}(D_2)) \subset E^2$. Thus the image, under G, of Bd (E_1) is the set Bd (E). Hence there is an arc A from a point of $G \circ f(M_1-\{p_1\})$, m for example, to a point of $G(\text{Bd}(E_1)-\{f(p_1)\}) = \text{Bd}(E)-\{f(p_1)\}$, so that, except for its endpoints, A lies in a complementary domain of $G(f_2(S)+\text{Bd}(D_2))$. There is also an arc A' from the endpoint of A on Bd (E) to a point of Bd $(D_2)-\{f_2(e)\}$ such that A' lies, except for its endpoints, in a complementary domain of $G(f_2(S)+\text{Bd}(D_2))+A$. Let A_1^2 be the arc $A+A'$.

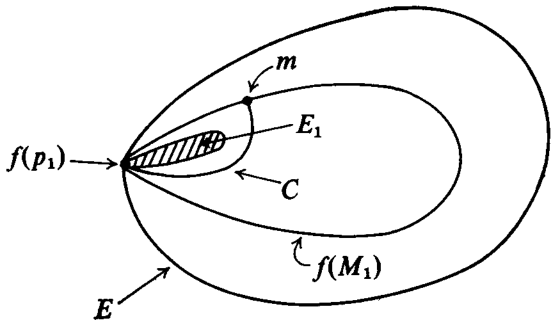

DIAGRAM 5

We may add a corresponding arc A_1^1 to $f_1(S)+\mathrm{Bd}\,(D_1)$ so that one endpoint of A_1^1 is the image, under $(G\circ f)^{-1}$, of the endpoint of A_1^2 on $G\circ f(M_1)$, and the other endpoint of A_1^1 lies on $\mathrm{Bd}\,(D_1)-\{f_1(e)\}$. Then there is a homeomorphism of $f_1(S)+\mathrm{Bd}\,(D_1)+A_1^1$ onto $G\circ f_2(S)+\mathrm{Bd}\,(D_2)+A_1^2$ which extends the homeomorphism $G\circ f_2\circ f_1^{-1}$ of $f_1(S)$ onto $G\circ f_2(S)$. Note that $\mathrm{Bd}\,(D_1)$ and M_1 belong to the same cyclic element of $f_1(S)+\mathrm{Bd}\,(D_1)+A_1^1$.

LEMMA 24. *If Q is a simple closed curve in $G\circ f_2(S)+\mathrm{Bd}\,(D_2)+A_1^2$ such that $G\circ f_2(J)$ is contained in* $\mathrm{Cl}\,(\mathrm{In}\,(Q))$, *and Q belongs to a cyclic element of $G\circ f_2(S)+\mathrm{Bd}\,(D_2)+A_1^2$ different from the one containing* $\mathrm{Bd}\,(D_2)$, *then $f_2(J)$ is contained in* $\mathrm{Cl}\,(\mathrm{In}\,(G^{-1}(Q)))$.

Now suppose $f_1(J)$ and M_1 do not belong to the same cyclic element of $f_1(S)+\mathrm{Bd}\,(D_1)+A_1^1$. Then we shall further enlarge the cyclic element containing M_1 (and $\mathrm{Bd}\,(D_1)$).

First, we describe disks D_1' and D_2' which play roles similar to those played above by D_1 and D_2.

Let C_1 be the cyclic element of $f_1(S)+\mathrm{Bd}\,(D_1)+A_1^1$ which contains M_1. Let T_1 be the component of $[f_1(S)+\mathrm{Bd}\,(D_1)+A_1^1]-C_1$ which contains $f_1(J)$. Then $\overline{T}_1\cap C_1$ is a point e'. Clearly, the boundary of the domain of E^2-C_1 containing T_1 is also the outer boundary of a domain N of $E^2-[f_1(S)+\mathrm{Bd}\,(D_1)+A_1^1]$. Hence there is a disk D_1' such that

$$\mathrm{Bd}\,(D_1')\cap[f_1(S)+\mathrm{Bd}\,(D_1)+A_1^1]=\{e'\},$$

$\mathrm{Bd}\,(D_1')-\{e'\}$ lies in the domain N, and $\mathrm{Bd}\,(D_1')$ separates T_1 and $C_1-\{e'\}$ (and thus $T_1\subset D_1'$).

By Theorem 19, there is a disk D_2' such that

$$\mathrm{Bd}\,(D_2')\cap[G\circ f_2(S)+\mathrm{Bd}\,(D_2)+A_1^2]=\{G\circ f(e')\},$$

and $\mathrm{Bd}\,(D_2')$ separates $G\circ f(T_1)$ and $G\circ f(C_1-\{e'\})$ (and thus, $G\circ f(T_1)\subset D_2'$).

LEMMA 25. *We may assume without loss of generality that the image, under $G\circ f$, of the outer boundary of the domain N contains at least two points which* *belong to the boundary of* N', *where* N' *is the complementary domain of* $G \circ f_2(S) + \text{Bd}(D_2) + A_1^2$ *containing* $\text{Bd}(D_2') - \{G \circ f(e')\}$.

By Lemma 25, there are arcs B_1 in D_1, B_2 in D_2 such that (a) one endpoint of B_1 is on $C_1 - \{e'\}$, the other endpoint is on $\text{Bd}(D_1') - \{e'\}$, and except for its endpoints, B_1 misses $f_1(S) + \text{Bd}(D_1) + A_1^1 + \text{Bd}(D_1')$; (b) there is a homeomorphism of $f_1(S) + \text{Bd}(D_1) + A_1^1 + B_1 + \text{Bd}(D_1')$ onto $G \circ f_2(S) + \text{Bd}(D_2) + A_1^2 + B_2 + \text{Bd}(D_2')$ which extends the previous homeomorphism of $f_1(S) + \text{Bd}(D_1) + A_1'$ onto $G \circ f_2(S) + \text{Bd}(D_2) + A_1^2$.

Then $\text{Bd}(D_1)$ and $\text{Bd}(D_1')$ belong to the same cyclic element of $f_1(S) + \text{Bd}(D_1) + A_1^1 + B_1 + \text{Bd}(D_1')$.

As above we enlarge the cyclic element containing $\text{Bd}(D_1')$ to include a new simple closed curve M', where, by Lemma 24, we may assume that $G^{-1}(M')$ contains $f_1(J)$ in the closure of its interior.

We continue the process. We must stop after a finite number of steps, however, because the simple closed curves $M_1, G^{-1}(M'), \ldots$ belong to different cyclic elements of $f_1(S) + \text{Bd}(D_1)$, and by Lemma 24, each element in the sequence $\{M_1, G^{-1}(M'), \ldots\}$ either (a) contains $f_1(J)$ in the closure of its interior, or (b) the closure of the interior of its image, under f, contains $f_2(J)$. But the collection of nondegenerate cyclic elements of a continuous curve forms a null sequence [10, Theorem 4.2, p. 71].

The proof of Theorem 22 is complete.

LEMMA 26. NOTATION SAME AS IN THEOREM 22. *For each* i, $i = 1, 2$, *suppose* K_i *is a closed subset of* $E^3 - f_i(S)$ *and* ε_i *is a number such that if* Q *is the outer boundary of a complementary domain of* $f_i(S)$ *and* Q *has diameter less than* ε_i, *then* K_i *does not intersect* $\text{In}(Q)$.

Then the homeomorphisms H_1, H_2 *of Theorem* 22 *may be chosen so that for each* i, $i=1, 2$, *if* $H_i(Q)$ *is the outer boundary of a complementary domain of* $H_i \circ f_i(S) + A_1^i + \cdots + A_n^i + \text{Bd}(D_i)$ *and* $H_i(Q)$ *does not belong to the cyclic element of* $H_i \circ f_i(S) + A_1^i + \cdots + A_n^i + \text{Bd}(D_i)$ *containing* $\text{Bd}(D_i)$ *then* (a) Q *is the outer boundary of a complementary domain of* $f_i(S)$, *and* (b) *if the diameter of* Q *is less than* ε_i, *then* $H_i(K_i)$ *does not intersect* $\text{In}(H_i(Q))$.

Proof. It suffices to prove that H_1 may be chosen to satisfy conditions (a) and (b). H_1 is a finite composition of homeomorphisms of two types: the first type is the interchange of two disks which meet at a point; the second type is the interchange of two pinched annuli which meet along a simple closed curve.

It suffices to assume that H_1 is a single homeomorphism of the first or second type. If H_1 is a homeomorphism of the first type the proof is easy.

Suppose H_1 is a homeomorphism of the second type. Let E and E_1 be the disks such that the sum of the two pinched annuli which H_1 interchanges is the pinched annulus $\text{Cl}(E - E_1)$. Let p be the point $\text{Bd}(E) \cap \text{Bd}(E_1)$. Then $\text{Bd}(E) - \{p\}$ misses $f_1(S)$, $E_1 - \{p\}$ misses $f_1(S)$, and $H_1(p)$ is a point of the cyclic element of

$H_1 \circ f_1(S)+A_1^1+\cdots+A_n^1+\text{Bd}(D_1)$, containing Bd (D_1). There is a simple closed curve $f_1(J)$ in E (this is the J of Theorem 22), and an arc Z in $f_1(S)$ from a point of $f_1(J)$ to p such that (1) $Z-\{p\}$ intersects the boundary of the complementary domain of $f_1(S)$ containing $E_1-\{p\}$, and (2) a subarc of $H_1(Z)$ belongs to the same cyclic element of $H_1 \circ f_1(S)+A_1^1+\cdots+A_n^1+\text{Bd}(D_1)$ as Bd (D_1). The existence of Z follows from Lemma 23.

If E contains no point of K_1, choose H_1 to be the identity on K_1.

Suppose $H_1(Q)$ is the outer boundary of a complementary domain of $H_1 \circ f_1(S)$ $+A_1^1+\cdots+A_n^1+\text{Bd}(D_1)$; suppose $H_1(Q)$ does not belong to the same cyclic element as Bd (D_1); suppose the diameter of Q is less than ε_1.

Suppose Q is contained in E. If Cl (In (Q)) contains E_1, then the half-open arc $Z-\{p\}$ must be contained in In (Q), hence $f_1(J)$ must be contained in In (Q). One may show that this contradicts Lemma 23. Hence Cl (In (Q)) is contained in Cl $(E-E_1)$. Hence H_1 (Cl (In (Q)))$=$Cl (In $(H_1(Q))$) and the lemma is obvious.

If Q is not contained in E, then $H_1(Q)=Q$, and again the lemma is obvious.

The proof of Lemma 26 is complete.

Remark. The conclusion of Lemma 26 is true even if H_i, $i=1, 2$, is a finite composition of homeomorphisms each obtained from Theorem 22.

3. **The extension theorem.** We shall prove the extension theorem by means of Theorem 18 and the following:

Theorem 27. *Let S and S' be planar continuous curves, and let g be a homeomorphism of S onto S'. Then there are continuous curves S_∞ and S'_∞ in E^3, and homeomorphisms H and H' of E^3 onto E^3 such that*:

(a) *S is a subset of S_∞, S' is a subset of S'_∞, and there is a homeomorphism G of S_∞ onto S'_∞ such that $G=g$ on S*;

(b) *$H(S_\infty)$ and $H'(S'_\infty)$ are subsets of E^2*;

(c) *S_∞ contains no separating points*;

(d) *H and H' may be realized by isotopies.*

Proof. All homeomorphisms defined below can be chosen to be the identity outside some preassigned cube containing S and S', and hence the final homeomorphisms may be realized by isotopies.

We may suppose that S is nondegenerate.

We may also assume that there is a nondegenerate cyclic element C_1 in S such that C_1 contains at least two points on the boundary of the unbounded complementary domain of S, and $g(C_1)=C_1'$ contains at least two points on the boundary of the unbounded complementary domain of S'.

We shall define sequences of homeomorphisms $H_1, H_2, \ldots$ and $H_1', H_2', \ldots$ whose limits are the required homeomorphisms H and H' respectively. We shall define nested sequences of continuous curves $S_1, S_2, \ldots$ and $S_1', S_2', \ldots$ whose sums are the continuous curves S_∞ and S'_∞ respectively. S_{i+1} will be obtained from S_i by the addition of a finite number of arcs. These arcs will enlarge a certain cyclic element C_i of S_i so that the components of S_{i+1} minus the enlarged cyclic element C_{i+1} will have smaller maximum diameter than the components of $S_i - C_i$. In addition, we shall need some technical δ and ε conditions on the homeomorphisms $\{H_i\}$ and $\{H_i'\}$ so that the limits H and H' will be one-one functions.

Let $S_1 = S$, $S_1' = S'$, $H_1 = H_1' = \text{id}$.

Suppose we have defined homeomorphisms $H_1, H_2, \ldots, H_m$; $H_1', \ldots, H_m'$, and continuous curves $S_1, \ldots, S_m$; $S_1', \ldots, S_m'$ such that for each k, $2 \leqq k \leqq m$, we have:

(1) H_k and H_k' map E^3 onto E^3, and $H_k(S) + H_k'(S')$ is contained in E^2.

(2) $H_k(S_k) = H_k(S) + A_1 + A_2 + \cdots + A_{n_k}$, where $A_1, \ldots, A_{n_k}$ are planar arcs, each having an endpoint in $H_k(S)$, such that the diameter of A_i, for $n_{(k-1)} + 1 \leqq i \leqq n_k$, is at most $1/2^{k-1}$; $H_k'(S_k') = H_k'(S') + A_1' + \cdots + A_{n_k}'$, where $A_1', \ldots, A_{n_k}'$ are planar arcs, each having an endpoint in $H_k'(S')$, such that the diameter of A_i', for $n_{(k-1)} + 1 \leqq i \leqq n_k$, is at most $1/2^{k-1}$.

(3) There is a homeomorphism g_k of S_k onto S_k' which extends $g_{k-1}: S_{k-1} \twoheadrightarrow S_{k-1}'$, and $g = g_1$.

(4) There is a cyclic element C_k of S_k such that (a) $g_k(C_k) = C_k'$, (b) C_{k-1} is contained in C_k, and (c) the sum of the arcs $A_1, \ldots, A_{n_k}$ is contained in $H_k(C_k)$.

(5) Let $\varepsilon_k = \inf \{\text{dist}(H_k(x_1), H_k(x_2)) : (x_1, x_2) \in S \times S \text{ and dist}(x_1, x_2) \geqq 1/k\}$,

let $\delta_1 = \min \{\varepsilon_1, 1/2\}$,

let $\delta_k = \min \{\varepsilon_k, 1/2^k, \delta_{k-1}/2(1/1000)\}$.

Then, for $k < m$, $\|H_{k+1} - H_k\| \leqq \delta_{k-1}/100$.

(5′) A similar condition holds for ε_k', δ_1', δ_k' and H_k'.

(6) For $k < m$, H_{k+1} is the composition of three homeomorphisms $F_k \circ G_k \circ H_k$, where G_k takes the plane onto itself, and $\|H_{k+1} - G_k \circ H_k\| \leqq \delta_k/100$.

(6′) A similar condition holds for H_k', G_k', δ_k'.

(7) If, for any number η, we define $E(\eta) = \{x \in E^3 : \text{dist}(x, S) \geqq \eta\}$, and $E'(\eta) = \{x \in E^3 : \text{dist}(x, S') \geqq \eta\}$, then, for $k < m$:

$H_{k+1} = G_k \circ H_k = H_k$ on $E(1/2^k)$ and on C_k,

$H_{k+1}' = G_k' \circ H_k' = H_k'$ on $E'(1/2^k)$ and on C_k',

$\text{H}_2 = \text{id}$ on C_1, $H_2' = \text{id}$ on C_1'.

(8) If T is a component of $S_k - C_k$ (of $S_k' - C_k'$), then:

(a) diam $(T) \leqq 1/2^k$

(b) diam $(H_k(T)) \leqq \delta_{k-1}/1000$ (diam $H_k'(T) \leqq \delta_{k-1}'/1000$)

(c) diam $(G_k \circ H_k(T)) \leqq \delta_k/1000$ (diam $G_k' \circ H_k'(T) \leqq \delta_k'/1000$)

(d) If D is a bounded complementary domain of $H_k(\overline{T})$ (of $H_k'(\overline{T})$), then D does not intersect $H_k(E(1/2^k))$, (does not intersect $H_k'(E'(1/2^k))$).

(9) $H_k(C_k)(H_k'(C_k'))$ contains at least two points on the boundary of the unbounded complementary domain of $H_k(S_k)$ (of $H_k'(S_k')$).

We shall now define H_{m+1}, H_{m+1}', S_{m+1}, S_{m+1}'.

There is a homeomorphism $G_m: E^3 \twoheadrightarrow E^3$ such that (a) $G_m(E^2) = E^2$, (b) $G_m = \text{id}$ on $H_m(E(1/2^m))$, (c) $\|G_m \circ H_m - H_m\| \leqq (3/1000)\delta_{m-1}$, (d) no component of $G_m \circ H_m(S_m - C_m)$ has diameter greater than $(1/1000)\delta_m$.

Similarly there is a homeomorphism $G'_m: E^3 \twoheadrightarrow E^3$ which reduces the size of components of $H'_m(S'_m - C'_m)$.

Let $D_1, \ldots, D_p$ be the complementary domains of $G_m \circ H_m(S_m)$ which intersect $G_m \circ H_m(E(1/2^{m+1}))$.

Let $\beta_m = \min \{\text{diam}(\text{Bd}(D_i)) : 1 \leqq i \leqq p\}$.

Let α_m be a number less than $1/2^{m+1}$ such that if A is a subset of S_m and A has diameter no greater than α_m, then $G_m \circ H_m(A)$ has diameter less than β_m.

Define β'_m and α'_m similarly.

Let $V_1, \ldots, V_r$ be disjoint open sets in E^3 such that if T is a component of $S_m - C_m$ and either the diameter of T is greater than α_m or the diameter of $g_m(T)$ is greater than α'_m, then $G_m \circ H_m(\overline{T})$ is contained in V_i for some i, $1 \leqq i \leqq r$. We may assume that the diameter of each V_i, $1 \leqq i \leqq r$, is no greater than $(4/1000)\delta_m$. We also may assume that each V_i, $1 \leqq i \leqq r$, misses $G_m \circ H_m(E(1/2^m))$.

Define $V'_1, \ldots, V'_r$ similarly.

All the remaining homeomorphisms making up H_{m+1} will be the identity outside $\sum_{i=1}^{r} V_i$, so that we shall have $\|H_{m+1} - G_m \circ H_m\| \leqq (4/1000)\delta_m$, and similarly for the remaining homeomorphisms making up H'_{m+1}.

Let T be a component of $S_m - C_m$ such that either T has diameter greater than α_m, or $g_m(T)$ has diameter greater than α'_m. Assume that T has diameter greater than α_m. Then there is an arc B_1 in $\overline{T}$ such that (a) B_1 has diameter greater than $(\alpha_m)/3$, (b) B_1 lies on the boundary of a complementary domain of S (it is clear that T is a subset of S, since the part of S_m not in S belongs to C_m), (c) one endpoint of B_1 is the point $\overline{T} \cap C_m$, and (d) the other endpoint of B_1 is either contained in a simple closed curve in $\overline{T}$, or is not a limit point of simple closed curves of $\overline{T}$.

By an application of Theorem 22, there are homeomorphisms L and L' of E^3 onto E^3, and planar arcs $A_1^{(1)}, \ldots, A_x^{(1)}, A_1^{(2)}, \ldots, A_x^{(2)}$ such that (a) $L = \text{id}$ outside $\sum_{i=1}^{r} V_i$, $L' = \text{id}$ outside $\sum_{i=1}^{r} V'_i$, $L = \text{id}$ on $G_m \circ H_m(C_m)$, $L' = \text{id}$ on $G'_m \circ H'_m(C'_m)$, (b) $\sum_{i=1}^{x} A_i^{(1)}$ is contained in $\sum_{i=1}^{r} V_i$, $\sum_{i=1}^{x} A_i^{(2)}$ is contained in $\sum_{i=1}^{r} V'_i$, (c) $L \circ G_m \circ H_m(B_1)$, $A_1^{(1)}, \ldots, A_x^{(1)}$, and $L \circ G_m \circ H_m(C_m)$ belong to the same cyclic element of $L \circ G_m \circ H_m(S_m) + A_1^{(1)} + \cdots + A_x^{(1)}$, and (d) there is an extension of g_m to a homeomorphism of

$$S_m + (L \circ G_m \circ H_m)^{-1}(A_1^{(1)} + \cdots + A_x^{(1)})$$

onto

$$S'_m + (L' \circ G'_m \circ H'_m)^{-1}(A_1^{(2)} + \cdots + A_x^{(2)}).$$

Let the cyclic element of $S_m + (L \circ G_m \circ H_m)^{-1}(A_1^{(1)} + \cdots + A_x^{(1)})$ containing B_1 and C_m be denoted by $\tilde{C}_m$.

If $[S_m + (L \circ G_m \circ H_m)^{-1}(A_1^{(1)} + \cdots + A_x^{(1)})] - \tilde{C}_m$ contains a component T_1 such that either T_1 has diameter greater than α_m or $g_m(T_1)$ has diameter greater than α'_m, we repeat the above process, obtaining an arc B_2 and adding additional arcs to obtain a new cyclic element containing B_2.

This process must terminate at a finite stage, however, because S contains only a finite number of arcs $B_1, B_2, \ldots, B_q$ such that (a) each B_i is contained in the boundary of a complementary domain of S, (b) each B_i has diameter greater than $(\alpha_m)/3$, and (c) B_i and B_j, $i \neq j$, do not intersect in an interior point of both; and similarly for S' (see [11, Theorem 6, p. 359]).

Assume this process stops with the addition of the arcs $A_1^{(1)}, \ldots, A_x^{(1)}$. Let $\tilde{C}_m = C_{m+1}$; relabel $A_1^{(1)}, \ldots, A_x^{(1)}$ as $A_{n_m+1}, \ldots, A_{n_{m+1}}$; let

$$S_{m+1} = S_m + (L \circ G_m \circ H_m)^{-1}(A_1^{(1)} + \cdots + A_x^{(1)}).$$

Then, by Lemma 26 and the remark following Lemma 26, if T is a component of $S_{m+1} - C_{m+1}$, and D is a bounded domain of $E^2 - L \circ G_m \circ H_m(\overline{T})$, then D misses $L \circ H_m \circ G_m(E(1/2^{m+1}))$. This is because if Q is the outer boundary of D, then $L^{-1}(Q)$ is the outer boundary of a complementary domain of $G_m \circ H_m(S_m)$, and the diameter of $L^{-1}(Q)$ is less than β_m (because the diameter of T is at most α_m).

Finally, we let $K_m \colon E^3 \twoheadrightarrow E^3$ be a homeomorphism defined similarly to G_m, so that if $H_{m+1} = K_m \circ L \circ G_m \circ H_m$, then every component of $H_{m+1}(S_{m+1} - C_{m+1})$ has diameter no greater than $(1/1000)\delta_m$.

Define K'_m and H'_{m+1} similarly.

This completes the definition of the sequences $\{H_1, H_2, \ldots\}$, $\{H'_1, H'_2, \ldots\}$, $\{S_1, S_2, \ldots\}$ and $\{S'_1, S'_2, \ldots\}$.

Let $H = \lim_{n\to\infty} H_n$, let $H' = \lim_{n\to\infty} H'_n$, let $S_\infty = \sum_{i=1}^\infty S_i$, let $S'_\infty = \sum_{i=1}^\infty S'_i$, let G be the common extension of $g_1, g_2, \ldots$. H is one-one on $E^3 - S$ by the fact that $H_{k+1} = H_k$ on $E(1/2^k)$ for $k = 2, 3, \ldots$. By the same fact, we see that $H(S) \cap H(E^3 - S) = \varnothing$. It remains to show that H is one-one on S. Let x and y be distinct points of S. If x and y are contained in $\sum_{i=1}^\infty C_i$ then there is an integer k such that x and y are contained in C_k, and hence $H(x) = H_k(x) \neq H_k(y) = H(y)$. If x is contained in C_k for some k, and y is not contained in $\sum_{i=1}^\infty C_i$, then there is an integer m and a point p such that (a) if T is the component of $S_m - C_m$ containing y, then p is the point $\overline{T} \cap C_m$, (b) $C_k \subset C_m$, (c) dist $(p, x) \geqq 1/m$. The point p exists because the diameters of the components of $S_n - C_n$ become arbitrarily small as n becomes large. Then:

$$\operatorname{dist}(G_m \circ H_m(x), G_m \circ H_m(y)) \geqq (49/50)\delta_m.$$

This is because:

$$\begin{aligned} \delta_m &\leqq \operatorname{dist}(G_m \circ H_m(x), G_m \circ H_m(p)), \quad \text{since } G_m = \text{id on } H_m(C_m), \\ &\leqq \operatorname{dist}(G_m \circ H_m(x), G_m \circ H_m(y)) + (1/1000)\delta_m. \end{aligned}$$

Hence, by definition of δ_m, $H(x) \neq H(y)$.

If neither x nor y is contained in $\sum_{i=1}^\infty C_i$, the proof that $H(x) \neq H(y)$ is similar to the above.

Thus H and H' are homeomorphisms.

S_∞ is a continuous curve because $S_\infty = S + \sum_{i=1}^{\infty} H^{-1}(A_i)$ where $\{H^{-1}(A_1), H^{-1}(A_2), \ldots\}$ is a null sequence of arcs each having an endpoint on S; G is a homeomorphism for the same reason. S_∞ has no separating points because the components of $S_n - C_n$ become arbitrarily small as n becomes large.

The proof of Theorem 27 is complete.

THEOREM 28 (THE EXTENSION THEOREM). *Let S and S' be continuous curves in E^2, and let g be a homeomorphism of S onto S'. Then there is a homeomorphism H of E^3 onto E^3 such that* (a) *$H = g$ on S, and* (b) *H is realizable by an isotopy.*

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