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HOMOCLINIC AND HETEROCLINIC ORBITS FOR A SEMILINEAR PARABOLIC EQUATION

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Abstract. We study the existence of connecting orbits for the Fujita equation with a critical or supercritical exponent. For certain ranges of the exponent we prove the existence of heteroclinic connections from positive steady states to zero and a homoclinic orbit with respect to zero.

1. Introduction. In the theory of dynamical systems, heteroclinic connections between equilibria constitute a very important theme, because connecting orbits play a crucial role in the description of the attractor. In particular, for gradient-like systems the attractor consists of equilibria and connecting orbits. For semilinear parabolic equations in one space dimension, the connection problem has been studied extensively since the beginning of 1980's (see, for example, [3, 4, 5, 6, 7, 8, 9, 18, 22, 23, 35]).

In this paper we consider the Fujita equation

$$u_t = \Delta u + |u|^{p-1}u\,, \quad x \in \boldsymbol{R}^N\,, \tag{1}$$

where $u = u(x,t)$ and Δ is the Laplace operator with respect to x. A solution of (1) is called a *global* solution (or an *ancient* solution) if for some $\tau \in \boldsymbol{R}$, the solution exists for all $t > \tau$ (or $t < \tau$). A solution of (1) is called an *entire* solution if the solution exists for all $t \in \boldsymbol{R}$. Our interest is the existence of connecting orbits for (1). By a connecting orbit, we mean an entire solution of (1) that converges to steady states as $t \to \pm\infty$. In particular, if the solution connects two different steady states then it is called a heteroclinic orbit. If the solution converges to the same steady state as $t \to \pm\infty$ then it is called a homoclinic orbit.

Before stating our results on connecting orbits of (1), we introduce several critical exponents. Concerning the existence of positive global solutions of (1), the Fujita exponent

$$p_F := \frac{N+2}{N}$$

is critical. In fact, if $1 < p \le p_F$ then there is no positive global solution of (1). The exponent

$$p_{sg} := \frac{N}{N-2}\,, \quad N > 2\,,$$

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is related to the existence of a singular steady state explicitly given by

$$u = \varphi_\infty(|x|) := L|x|^{-m}, \quad x \in \boldsymbol{R}^N \setminus \{0\}$$

with

$$m := \frac{2}{p-1}, \quad L := \{m\,(N-2-m)\}^{1/(p-1)}.$$

Namely, φ_∞ exists if and only if $p > p_{sg}$. Concerning the existence of positive regular steady states, it is well known (see, e.g., [20, 34]) that the Sobolev exponent

$$p_S := \begin{cases} \dfrac{N+2}{N-2} & \text{for} \quad N > 2, \\ \infty & \text{for} \quad N \le 2, \end{cases}$$

plays a crucial role. Namely, there is a family of positive radial solutions of

$$\Delta\varphi + \varphi^p = 0, \quad x \in \boldsymbol{R}^N,$$

if and only if $p \ge p_S$. We denote the solution by $\varphi = \varphi_\alpha(r)$, $r = |x|$, $\alpha > 0$, where φ_α satisfies

$$\text{(2)} \qquad \begin{cases} (\varphi_\alpha)_{rr} + \dfrac{N-1}{r}(\varphi_\alpha)_r + (\varphi_\alpha)^p = 0, & r > 0, \\ \varphi_\alpha(0) = \alpha, \quad (\varphi_\alpha)_r(0) = 0. \end{cases}$$

For each $\alpha > 0$, the solution φ_α is decreasing in $|x|$ and satisfies $\varphi_\alpha(|x|) \to 0$ as $|x| \to \infty$.

Another important critical exponent is

$$p_c := \begin{cases} \dfrac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & \text{for} \quad N > 10, \\ \infty & \text{for} \quad N \le 10. \end{cases}$$

It is known (see, e.g., [34]) that if $p_S < p < p_c$, every positive radial steady state intersects with other positive radial steady states and the singular steady state infinitely many times. For $p \ge p_c$, Wang [37] showed that the family of steady states $\{\varphi_\alpha\,;\ \alpha > 0\}$ is completely ordered, that is, φ_α is increasing in α for every x. Moreover,

$$\lim_{\alpha \to 0} \varphi_\alpha(|x|) = 0, \qquad \lim_{\alpha \to \infty} \varphi_\alpha(|x|) = \varphi_\infty(|x|).$$

The Lepin exponent

$$p_L := \begin{cases} \dfrac{N-4}{N-10} & \text{for} \quad N > 10, \\ \infty & \text{for} \quad N \le 10, \end{cases}$$

is related to the existence of nonconstant positive radial backward self-similar solutions (see Section 4). Such solutions exist if $p_S < p < p_L$ (see [24, 25]), while for $p > p_L$ nonexistence holds (see [27]).

Before we recall some Liouville-type results on nonexistence of entire or ancient solutions of (1), we introduce one more exponent

$$p_B := \begin{cases} \dfrac{N(N+2)}{(N-1)^2} & \text{for } \ N > 1\,, \\ \infty & \text{for } \ N = 1\,, \end{cases}$$

which first appeared in [2]. For $1 < p < p_B$ there is no positive classical entire solution of (1) (see [2] or in [34, Theorem 21.2]). For $1 < p < p_S$, it was shown in [30, 31] that there is no positive radial classical entire solution of (1). Nonexistence of radial classical entire solution of (1) with finite number of zeros was established in [1] for $1 < p < p_S$. For $p > p_c$ it was shown in [33] that if u is a solution of (1) defined for $t < 0$ and such that

$$\varphi_\alpha \le u(\cdot, t) \le \varphi_\beta, \qquad t < 0\,,$$

for some $0 < \alpha < \beta \le \infty$ then $u(\cdot, t) \equiv \varphi_\gamma$ for some $\gamma > 0$.

Concerning connecting orbits, the studies [11, 14, 16] revealed the possibility of connecting equilibria by non-classical solutions which we call singular connections. By a singular connection we mean a function $u(\cdot, t)$ which is a classical solution on the interval $(-\infty, T)$ for some $T \in \boldsymbol{R}$ and blows up at $t = T$, but continues to exist as a weak solution on $[T, \infty)$. We note here that singular homoclinic orbits which tend to zero as $t \to \pm\infty$ are known to exist for (1) if $p_S < p < p_L$ (see [13, 19]). These singular homoclinics are obtained by continuing a backward self-similar solution which exists for $t < T$ and blows up at $t = T$ by a forward self-similar solution defined for $t > T$ (see Section 6).

The main results of this paper are the following: If $p_S \le p < p_c$ then for every $\alpha > 0$ there is a heteroclinic orbit connecting φ_α to zero. If $p > p_S$ then there is a homoclinic orbit which tends to zero as $t \to \pm\infty$. As far as we know, this is the first example of a homoclinic orbit for a parabolic equation. As is well known, an energy functional is associated with (1), namely

$$E[\phi] := \frac{1}{2}\int_{\boldsymbol{R}^N} |\nabla\phi|^2 dx - \frac{1}{p+1}\int_{\boldsymbol{R}^N} |\phi|^{p+1} dx\,.$$

The homoclinic solution does not belong to the energy space, and this is why the energy does not rule out the existence of a homoclinic orbit (see Corollary 6.3 for more details).

For a discussion of possible complicated dynamics of semilinear parabolic equations we refer to the survey [29, Section 7].

In order to show the existence of connecting orbits, we first analyze the linearization around stationary solutions and a backward self-similar solution. Then we employ the method of Fukao, Morita and Ninomiya [17], which is based on construction of an approximate sequence of solutions by using suitable comparison functions.

This paper is organized as follows. In Section 2 we describe our main results. In Sections 3, 4 and 5, we summarize useful results on stationary solutions, backward self-similar solutions and forward self-similar solutions, respectively. Proofs of the main results and some related results are given in Section 6.

2. Main results. Our first result concerns the existence of a homoclinic orbit.

THEOREM 2.1. *Let $p_S < p < p_L$. Then there exists a homoclinic solution u of* (1) *with the following properties*:

(i) *The solution is positive, radially symmetric in space with respect to the origin, decreasing in $|x|$, and satisfies*

$$\lim_{|x|\to\infty} u(x,t) = 0\,, \quad t \in \boldsymbol{R}\,.$$

(ii) *There exists a positive constant C_0 such that*

$$\|u(\cdot,t)\|_{L^\infty(\boldsymbol{R}^N)} = C_0(-t)^{-1/(p-1)} + o((-t)^{-1/(p-1)})$$

as $t \to -\infty$.

(iii) *There exist constants $C_1, C_2 > 0$ such that*

$$C_1 t^{-1/(p-1)} < \|u(\cdot,t)\|_{L^\infty(\boldsymbol{R}^N)} < C_2 t^{-1/(p-1)}$$

for all $t > 1$.

REMARK. If $N > 10$ and $p \geq p_L$, we can find a homoclinic orbit that is positive but not radially symmetric. In fact, we may write $x = (x_1, x_2) \in \boldsymbol{R}^j \times \boldsymbol{R}^{N-j}$ with $3 \leq j \leq 10$ and consider solutions that are constant in the x_2-direction and coincide with the above homoclinic solution in the x_1-direction.

If u is a homoclinic orbit then

$$u^\lambda(x,t) := \lambda^m u(\lambda x, \lambda^2 t)$$

is also a homoclinic orbit for any $\lambda > 0$. In the proof of Theorem 2.1 we construct a homoclinic orbit u with the property that as $\lambda \to \infty$, u^λ approaches a singular homoclinic orbit that consists of a backward self-similar solution and a forward self-similar solution (see Theorem 6.4 in Section 6). Such a singular homoclinic orbit was first found by Galaktionov and Vázquez [19].

The next result concerns the existence of heteroclinic orbits that connect positive radial stationary solutions to the trivial solution.

THEOREM 2.2. *Let $p_S \leq p < p_c$. For every $\alpha > 0$ there exists an entire solution u of* (1) *with the following properties*:

(i) *The solution is positive, radially symmetric in space with respect to the origin, decreasing in $|x|$, and satisfies*

$$\lim_{|x|\to\infty} u(x,t) = 0\,, \quad t \in \boldsymbol{R}\,.$$

Moreover, the solution is decreasing in t.

(ii) *There exist positive constants C_0 and μ_0 such that*

$$\|u(\cdot,t) - \varphi_\alpha(|\cdot|)\|_{L^\infty(\boldsymbol{R}^N)} = C_0 \exp(\mu_0 t) + o(\exp(\mu_0 t))$$

as $t \to -\infty$.

(iii) *There exist constants* $C_1, C_2 > 0$ *such that*

$$C_1 t^{-(N-2)/2} < \|u(\cdot,t)\|_{L^\infty(\boldsymbol{R}^N)} < C_2 t^{-(N-2)/2} \quad \text{if } p = p_S\,,$$
$$C_1 t^{-1/(p-1)} < \|u(\cdot,t)\|_{L^\infty(\boldsymbol{R}^N)} < C_2 t^{-1/(p-1)} \quad \text{if } p_S < p < p_c\,,$$

for all $t > 1$.

The next result can be regarded as the existence of heteroclinic orbits that connect regular steady states to infinity.

THEOREM 2.3. *Let* $p_S \le p < p_c$. *For every* $\alpha > 0$ *there exists an ancient solution* u *of* (1) *with the following properties*:

(i) *The solution is positive*, *radially symmetric in space with respect to the origin*, *decreasing in* $|x|$, *and satisfies*

$$\lim_{|x|\to\infty} u(x,t) = 0\,, \quad t \in (-\infty, T)\,,$$

where $T < \infty$ *is the maximal existence time. Moreover*, *the solution is increasing in* t.

(ii) *There exist positive constants* C_0 *and* μ_0 *such that*

$$\|u(\cdot,t) - \varphi_\alpha(|\cdot|)\|_{L^\infty(\boldsymbol{R}^N)} = C_0 \exp(\mu_0 t) + o(\exp(\mu_0 t))$$

as $t \to -\infty$.

(iii) *The solution blows up at the origin at* $t = T$ *and the blow-up is of Type I* : *There exists* $C > 0$ *such that*

$$\|u(\cdot,t)\|_{L^\infty(\boldsymbol{R}^N)} \le C(T-t)^{1/(p-1)}$$

as $t \uparrow T$.

Finally we establish a result on the nonexistence of ancient solutions.

THEOREM 2.4. *Let* $p_S \le p < p_c$. *Then there is no ancient solution of* (1) *such that*

$$-\varphi_\infty(|x|) \le u(x,t) \le \varphi_\infty(|x|)\,, \quad x \in \boldsymbol{R}^N \setminus \{0\}\,,$$

for all $t < 0$.

This theorem implies that if $p_S \le p < p_c$ then the only entire solution between the singular steady states is the trivial solution.

3. Properties of stationary solutions.

Let $\{\varphi_\alpha(r)\,;\ \alpha > 0\}$ be the family of solutions of (2). The following lemma is due to [37, 26].

LEMMA 3.1. *Let* $p \ge p_S$. *Then the solution of* (2) *has the following properties*:

(i) $\varphi_\alpha > 0$ *for all* $r > 0$ *and* $\varphi_\alpha \to 0$ *monotonically as* $r \to \infty$.

(ii) *If* $p = p_S$, *then* φ_α *intersects exactly once with* φ_β $(\alpha \ne \beta)$.

(iii) *If* $p_S < p < p_c$, *then* φ_α *intersects infinitely many times with* φ_β $(\alpha \ne \beta)$.

(iv) *If* $p \ge p_c$, *then* φ_α *is increasing in* $\alpha \in (0,\infty)$ *for every* $r > 0$, *and* $\varphi_\alpha \to \varphi_\infty$ *as* $\alpha \to \infty$.

Concerning the asymptotic behavior of $\varphi_\alpha(r)$ as $r \to \infty$, the following result was obtained in [37, 26].

LEMMA 3.2. *Let $p \geq p_S$. Then the solution of* (2) *has the following properties*:
(i) *If $p = p_S$, then there exists $C_1 = C_1(\alpha, N) > 0$ such that $\varphi_\alpha(r) = C_1 r^{-(N-2)} + o(r^{-(N-2)})$ as $r \to \infty$.*
(ii) *If $p > p_S$, then there exists $C_2 = C_2(p, N) > 0$ such that $\varphi_\alpha(r) = C_2 r^{-m} + o(r^{-m})$ as $r \to \infty$.*

Let U be the unique solution of the linearized problem

$$\begin{cases} \mu U = U_{rr} + \dfrac{N-1}{r} U_r + p\varphi_\alpha^{p-1} U\,, \quad r > 0\,, \\ U(0) = 1\,, \quad U_r(0) = 0\,. \end{cases} \tag{3}$$

LEMMA 3.3. *Let $p_S \leq p < p_c$. Then there exists a unique $\mu_0 > 0$ such that the solution of* (3) *with $\mu = \mu_0$ satisfies $U(r) > 0$ for all $r > 0$ and $U(r) \to 0$ exponentially as $r \to \infty$.*

PROOF. Take any $\beta \in (0, \alpha)$. Then by Lemma 3.1, $\tilde{U} := \varphi_\alpha - \varphi_\beta$ changes sign. Let $z > 0$ be the first zero of $\tilde{U}$. Then $\tilde{U}$ is positive on $[0, z)$ and satisfies

$$\begin{cases} \tilde{U}_{rr} + \dfrac{N-1}{r} \tilde{U}_r + q(r)\tilde{U} = 0\,, \quad r > 0\,, \\ \tilde{U}(0) = \alpha - \beta > 0\,, \quad \tilde{U}_r(0) = 0\,, \quad \tilde{U}(z) = 0\,, \quad \tilde{U}_r(z) < 0\,, \end{cases} \tag{4}$$

where

$$q(r) := \frac{\varphi_\alpha^p - \varphi_\beta^p}{\varphi_\alpha - \varphi_\beta}\,.$$

Note that $q(r) < p\varphi_\alpha^{p-1}$ for $r \in [0, z)$. First we take $\mu = 0$. Then multiplying (4) by U and (3) by $\tilde{U}$, subtracting, and integrating over $[0, z]$, we obtain

$$\Big[U_r \tilde{U} - \tilde{U}_r U\Big]_0^z = -\int_0^z r^{N-1}\{p\varphi_\alpha^{p-1} - q\} U \tilde{U} dr\,.$$

Here the left-hand side is positive, while the right-hand side is negative if $U > 0$ on $(0, z)$. Hence U must vanish at some $r \in (0, z)$. Since U depends on μ continuously, U must changes sign for every sufficiently small $\mu > 0$.

Now define

$$\mu_0 := \sup\{\mu > 0\,;\ U(r) \text{ changes sign}\}\,.$$

If $\mu \geq p\alpha^{p-1}$, then $\mu > p\varphi_\alpha^{p-1}$ for all $r > 0$, so that U is increasing. This implies that μ_0 is finite. Again by continuity, $U(r)$ has no zero for $\mu = \mu_0$ and satisfies $U(r) \to 0$ as $r \to \infty$. Thus we have $\mu_0 \in (0, \infty)$, so that the convergence is exponential.

Let U_0 denote the solution of (3) with $\mu = \mu_0$. Then we have

$$U_r U_0 - (U_0)_r U = (\mu - \mu_0) \int_0^r \rho^{N-1} U(\rho) U_0(\rho) d\rho\,.$$

Hence if $\mu > \mu_0$, then it is easy to show that U/U_0 is increasing in $r > 0$ and U can not decay to 0 exponentially for $\mu > \mu_0$. Conversely if $\mu < \mu_0$, then U/U_0 decreases as long as $U > 0$ and U changes its sign. □

PROPOSITION 3.4. *Let $p_S \le p < p_c$. Then the following* (i), (ii) *hold.*

(i) *There exists an entire solution of* (1) *such that u is decreasing in t and $\|u(\cdot, t) - \varphi_\alpha(|\cdot|)\|_{L^\infty(\mathbf{R}^N)} \to 0$ as $t \to -\infty$, and $\|u(\cdot, t)\|_{L^\infty(\mathbf{R}^N)} \to 0$ as $t \to +\infty$.*

(ii) *There exists an ancient solution of* (1) *such that u is increasing in t and $\|u(\cdot, t) - \varphi_\alpha(|\cdot|)\|_{L^\infty(\mathbf{R}^N)} \to 0$ as $t \to -\infty$.*

PROOF. In the case (i) we first construct a subsolution $\underline{u}$ and a supersolution $\overline{u}$ as follows.

Let μ_0 and U be as in Lemma 3.3 and choose $\varepsilon \in (0, \alpha)$ such that $\varphi_\alpha - \varepsilon U > 0$. If

$$\underline{u}(r, t) := \varphi_\alpha(r) - \varepsilon e^{\mu_0 t} U(r)\,, \quad r \ge 0\,, \quad s \le 0\,,$$

then

$$\underline{u}_t - \underline{u}_{rr} - \frac{N-1}{r}\underline{u}_r - \underline{u}^p = \varphi_\alpha^p - \underline{u}^p - p\varepsilon e^{\mu_0 t}\varphi_\alpha^{p-1} U \le 0\,.$$

To construct a supersolution we consider first the case when $1 < p < 2$. In this case we cannot proceed in the same way as in [17] because the nonlinearity $u \mapsto u^p : [0, \infty) \to [0, \infty)$ is not a C^2-function. Let ψ be the solution of the ODE-problem

$$\begin{cases} \psi' = \mu_0 - e^{(p-1)\psi}, & t \le 0\,, \\ \psi(0) = -\dfrac{1}{p-1}\log\left(\dfrac{1+\mu_0}{\mu_0}\right) < 0\,, \end{cases} \tag{5}$$

which can be written explicitly as

$$\psi(t) = \mu_0 t - \frac{1}{p-1}\log\left(1 + \frac{1}{\mu_0}e^{\mu_0(p-1)t}\right)\,.$$

Now we take

$$\overline{u}(r, t) := \varphi_\alpha(r) - \varepsilon e^{\psi(t)} U(r)\,, \quad r \ge 0, \quad t \le 0\,,$$

and then

$$\begin{aligned} &\overline{u}_t - \overline{u}_{rr} - \frac{N-1}{r}\overline{u}_r - (\overline{u})^p \\ &\quad = \varphi_\alpha^p - (\overline{u})^p - p\varepsilon e^{\psi}\varphi_\alpha^{p-1}U - \varepsilon e^{\psi}\psi' U + \mu_0 \varepsilon e^{\psi} U \\ &\quad \ge \varepsilon e^{\psi} U\left[p\big((\overline{u})^{p-1} - \varphi_\alpha^{p-1}\big) - \psi' + \mu_0\right] \\ &\quad \ge \varepsilon e^{\psi} U\left[-p(p-1)\big(\varepsilon e^{\psi} U\big)^{p-1} - \psi' + \mu_0\right] \\ &\quad \ge \varepsilon e^{\psi} U\left[-e^{(p-1)\psi} - \psi' + \mu_0\right] = 0\,, \end{aligned}$$

where we have chosen $\varepsilon > 0$ such that $\varphi_\alpha > 2\varepsilon U$ and $p(p-1)(\varepsilon U)^{p-1} \le 1$.

If $p \geq 2$ then we take ψ satisfying (5) with $p = 2$, choose $\varepsilon > 0$ such that $p(p-1)\varepsilon U \varphi_\alpha^{p-2} \leq 1$, and use the inequalities

$$\varphi_\alpha^{p-1} - (\overline{u})^{p-1} \leq (p-1)\varepsilon U \varphi_\alpha^{p-2} e^{\psi} \leq \frac{1}{p} e^{\psi} .$$

Using the supersolution $\overline{u}$ and a sequence of solutions $\{u_i\}$ of (1) with the initial data $u_i(x, -i) = \overline{u}(|x|, -i)$, we can show as in [17] that the sequence $\{u_i\}$ converges to a solution u of (1) defined for $t < 0$ and decreasing in t. Therefore the solution can be continued for $t \geq 0$, and since there is no steady state between φ_α and 0, we have $u \to 0$ as $t \to \infty$.

In the case (ii) we construct a subsolution $\underline{u}$ and a supersolution $\overline{u}$ in the following way.

Let μ_0 and U be as in Lemma 3.3 and let $\varepsilon_1 > 0$ be specified later. If

$$\underline{u}(r,t) := \varphi_\alpha(r) + \varepsilon_1 e^{\mu_0 t} U(r) , \quad r \geq 0 , \quad s \leq 0 ,$$

then

$$\underline{u}_t - \underline{u}_{rr} - \frac{N-1}{r} \underline{u}_r - \underline{u}^p = \varphi_\alpha^p - \underline{u}^p + p\varepsilon_1 e^{\mu_0 t} \varphi_\alpha^{p-1} U \leq 0 .$$

To construct a supersolution we again consider first the case when $1 < p < 2$. Let Ψ be the solution of

$$\begin{cases} \Psi' = \mu_0 + e^{(p-1)\Psi}, & t \leq 0, \\ \Psi(0) = 0 , \end{cases} \tag{6}$$

which can be written explicitly as

$$\Psi(t) = \mu_0 t - \frac{1}{p-1} \log\left[1 + \frac{1}{\mu_0}\big(1 - e^{\mu_0(p-1)t}\big)\right] .$$

If we set

$$\overline{u}(r,t) := \varphi_\alpha(r) + \varepsilon_2 e^{\Psi(t)} U(r) , \quad r \geq 0, \quad t \leq 0, \quad \varepsilon_2 > 0 ,$$

then

$$\begin{aligned} &\overline{u}_t - \overline{u}_{rr} - \frac{N-1}{r}\overline{u}_r - (\overline{u})^p \\ &\quad = \varphi_\alpha^p - (\overline{u})^p + p\varepsilon_2 e^{\Psi} \varphi_\alpha^{p-1} U + \varepsilon_2 e^{\Psi} \Psi' U - \mu_0 \varepsilon_2 e^{\Psi} U \\ &\quad \geq \varepsilon_2 e^{\Psi} U \left[-p\big((\overline{u})^{p-1} - \varphi_\alpha^{p-1}\big) + \Psi' - \mu_0\right] \\ &\quad \geq \varepsilon_2 e^{\Psi} U \left[-p(p-1)\big(\varepsilon_2 e^{\Psi} U\big)^{p-1} + \Psi' - \mu_0\right] \\ &\quad \geq \varepsilon_2 e^{\Psi} U \left[-e^{(p-1)\Psi} + \Psi' - \mu_0\right] = 0 , \end{aligned}$$

where we have chosen $\varepsilon_2 > 0$ such that $\varphi_\alpha > \varepsilon_2 U$ and $p(p-1)(\varepsilon_2 U)^{p-1} \leq 1$.

If $p \geq 2$ then we take Ψ satisfying (6) with $p = 2$, choose $\varepsilon_2 > 0$ such that $\varphi_\alpha > \varepsilon_2 U$, $p(p-1)\varepsilon_2 U (2\varphi_\alpha)^{p-2} \leq 1$, and use the inequalities

$$(\overline{u})^{p-1} - \varphi_\alpha^{p-1} \leq (p-1)\varepsilon_2 U (2\varphi_\alpha)^{p-2} e^{\Psi} \leq \frac{1}{p} e^{\Psi} .$$

If we now take $\varepsilon_1 > 0$ such that

$$\varepsilon_1 < \varepsilon_2 \left(1 + \frac{1}{\mu_0}\right)^{-1/(p-1)}$$

then

$$0 < \overline{u}(r,t) - \underline{u}(r,t) < \varepsilon_2 e^{\mu_0 t}\,, \qquad r \geq 0\,, \quad t \leq 0\,.$$

Since the subsolution $\underline{u}$ is increasing in t, by using a sequence of solutions $\{u_i\}$ of (1) with the initial data $u_i(x, -i) = \underline{u}(|x|, -i)$, we can show as in [17] that the sequence $\{u_i\}$ converges to a solution u of (1) defined for $t < 0$ and increasing in t. □

4. Properties of backward self-similar solutions. For a solution u of (1) defined for $t \in (-\infty, 0)$, we set

(7) $$w(y,s) = (-t)^{1/(p-1)} u(x,t)\,, \qquad y = \frac{x}{\sqrt{-t}}\,, \quad s = -\log(-t)\,.$$

Then we obtain the following equation for w:

(8) $$w_s = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{1}{p-1} w + |w|^{p-1} w\,, \quad y \in \boldsymbol{R}^N\,,\ s \in \boldsymbol{R}\,.$$

Let $w = b(r)$, $r = |y|$, be any radially symmetric steady state of this equation. Then b must satisfy

(9) $$\begin{cases} b_{rr} + \dfrac{N-1}{r} b_r - \dfrac{r}{2} b_r - \dfrac{1}{p-1} b + |b|^{p-1} b = 0\,, & r > 0\,, \\ b(0) = \alpha\,, \quad b_r(0) = 0\,, \end{cases}$$

for some $\alpha > 0$. If $b(r) > 0$ for all $r > 0$, then

(10) $$u = B(x,t) := (-t)^{-1/(p-1)} b((-t)^{-1/2}|x|)\,, \quad x \in \boldsymbol{R}^N,\ t < 0\,,$$

is called a (positive radial) backward self-similar solution of (1). We note that $b = \varphi_\infty(r)$ satisfies the equation in (9).

For a proof of the following lemma we refer to [24, 25] or [19, Theorem 12.1].

LEMMA 4.1. *Let $p_S < p < p_L$. Then there exists $\alpha_* > 0$ such that the solution b of* (9) *has the following properties*:

(i) *If $\alpha = \alpha_*$, then $b(r)$ is positive for all $r > 0$, decreasing in $r > 0$, and intersects exactly twice with φ_∞. Moreover, there exists a constant $l \in (0, L)$ such that $b(r) = l r^{-m} + o(r^{-m})$ as $r \to \infty$.*

(ii) *If $\alpha < \alpha_*$, then $b(r)$ vanishes at some finite r.*

Let b be the solution of (9) with $b(0) = \alpha_*$, and W be the unique solution of the linearized problem

(11) $$\begin{cases} \mu W = W_{rr} + \dfrac{N-1}{r} W_r - \dfrac{r}{2} W_r - \dfrac{1}{p-1} W + p b^{p-1} W\,, & r > 0\,, \\ W(0) = 1\,, \quad W_r(0) = 0\,. \end{cases}$$

LEMMA 4.2. *Let $p_S < p < p_L$. Then there exists a unique $\mu_0 > 0$ such that the solution of* (11) *with $\mu = \mu_0$ satisfies $W(r) > 0$ for all $r > 0$ and $W(s) = o(e^{-r^2/4})$ as $r \to \infty$.*

PROOF. Set

$$\tilde{W} := b - \varphi_\infty .$$

Then $\tilde{W}$ satisfies

$$\tilde{W}_{rr} + \frac{N-1}{r}\tilde{W}_r - \frac{r}{2}\tilde{W}_r - \frac{1}{p-1}\tilde{W} + \tilde{q}(r)\tilde{W} = 0\,, \quad r > 0\,,$$

where

$$\tilde{q}(r) := \frac{b^p - \varphi_\infty^p}{b - \varphi_\infty} .$$

By Lemma 4.1, $\tilde{W}$ has exactly two zeros $0 < z_1 < z_2 < \infty$ so that $\tilde{W}(z_1) = \tilde{W}(z_2) = 0$ and $\tilde{q}(r) < pb^{p-1}$ for $r \in (z_1, z_2)$. By the Sturm comparison theorem, W must vanish at some $r \in (z_1, z_2)$.

Next, define

$$\mu_0 := \sup\{\mu > 0\,;\ W(r) \text{ changes sign}\} .$$

If $\mu \geq -1/(p-1) + p\alpha^{p-1}$, then $\mu > -1/(p-1) + p\varphi_\alpha^{p-1}$ for all $r > 0$. Then by the equation

$$r^{N-1}e^{-r^2/4}W_r = \int_0^r \rho^{N-1}e^{-\rho^2/4}\left(\mu + \frac{1}{p-1} - pb(\rho)^{p-1}\right)W(\rho)\,d\rho\,,$$

W and $r^{N-1}e^{-r^2/4}W_r$ are increasing in $r > 0$. Hence μ_0 is finite. Since W depends on μ continuously and every zero of W is nondegenerate, W does not change sign for $\mu = \mu_0$. If W changes sign for some $\mu_1 > 0$, then by the Sturm comparison theorem, W changes sign for all $\mu < \mu_1$. Thus W changes sign for all $\mu < \mu_0$.

Set

$$W = e^{r^2/4}\hat{W} .$$

Then we have the following relations:

$$W_r = \frac{1}{2}re^{r^2/4}\hat{W} + e^{r^2/4}\hat{W}_r\,,$$
$$W_{rr} = \left(\frac{1}{2} + \frac{1}{4}r^2\right)e^{r^2/4}\hat{W} + re^{r^2/4}\hat{W}_r + e^{r^2/4}\hat{W}_{rr} .$$

Substituting these in (11), we find that $\hat{W}$ satisfies the initial value problem

$$\begin{cases} \hat{W}_{rr} + \left(\dfrac{N-1}{r} + \dfrac{r}{2}\right)\hat{W}_r + \left(-\mu - \dfrac{1}{p-1} + \dfrac{N}{2} + pb^{p-1}\right)\hat{W} = 0\,, \\ \hat{W}(0) = 1\,, \quad \hat{W}_r(0) = 0\,. \end{cases}$$

Hence $\hat{W}$ tends to ∞ or 0 exponentially as $r \to \infty$. Since $b(r)$ is decreasing in $r > 0$, the function

$$\hat{q}(r) := -\mu - \frac{1}{p-1} + \frac{N}{2} + pb^{p-1}(r)$$

changes sign at most once:

$$\hat{q}(r) > 0 \text{ for } r \in [0, r_0), \quad \hat{q}(r_0) = 0, \quad \hat{q}(r) < 0 \text{ for } r \in (r_0, \infty)$$

for some r_0. Note that $\hat{W}_r < 0$ for $r \in (0, r_0)$.

Let $\mu < \mu_0$. Then $\hat{W}$ is decreasing in $r \in (0, z)$. Indeed, if $\hat{W}_r(r_1) = 0$ and $W(r_1) > 0$, then $r_1 > r_0$ and $W_r(r) > 0$ for all $r > r_1$ so that $\hat{W}$ has no zero for $r > r_1$, a contradiction. Taking the limit $r_1 \to r_0$, we find that $\hat{W}$ is positive and decreasing for all $r > 0$. Moreover, it is easy to see from the equation that $\hat{W} \to 0$ as $r \to \infty$. This implies $W(r) = o(e^{-r^2/4})$ as $r \to \infty$. Finally, by the same argument as in the proof of Lemma 3.3, $W > 0$ for all $r > 0$ but can not decay to 0 as $r \to \infty$ for $\mu > \mu_0$. □

PROPOSITION 4.3. *Let $p_S < p < p_L$, and let b be the solution of* (9) *with* $b(0) = \alpha_*$. *Then there exists an ancient solution of* (8) *such that w is decreasing in s and $\|w(\cdot, s) - b(|\cdot|)\|_{L^\infty(\mathbf{R}^N)} \to 0$ as $s \to -\infty$.*

PROOF. To show the existence of an ancient solution of (8) that converges to b exponentially as $s \to -\infty$, we employ an idea of Fukao, Morita and Ninomiya [17] to (8). We first construct a subsolution $\underline{w}$ and a supersolution $\overline{w}$ as follows.

Let μ_0 and W be as in Lemma 4.2 and choose $\varepsilon \in (0, \alpha_*)$ such that $b - \varepsilon W > 0$. If

$$\underline{w}(r, s) := b(r) - \varepsilon e^{\mu_0 s} W(r), \quad r \geq 0, \quad s \leq 0,$$

then

$$\begin{aligned} \underline{w}_s - \underline{w}_{rr} - \frac{N-1}{r}\underline{w}_r + \frac{r}{2}\underline{w}_r - \underline{w}^p + \frac{1}{p-1}\underline{w} \\ = b^p - \underline{w}^p - p\varepsilon e^{\mu_0 s} b^{p-1} W \leq 0. \end{aligned}$$

To construct a supersolution we distinguish two cases as before. Consider first the case when $1 < p < 2$. Let ψ be the solution of the ODE-problem (5) and set

$$\overline{w}(r, s) := b(r) - \varepsilon e^{\psi(s)} W(r), \quad r \geq 0, \quad s \leq 0.$$

Then

$$\begin{aligned} \overline{w}_s - \overline{w}_{rr} - \frac{N-1}{r}\overline{w}_r + \frac{r}{2}\overline{w}_r - (\overline{w})^p + \frac{1}{p-1}\overline{w} \\ = b^p - (\overline{w})^p - p\varepsilon e^{\psi} b^{p-1} W - \varepsilon e^{\psi}\psi' W + \mu_0 \varepsilon e^{\psi} W \\ \geq \varepsilon e^{\psi} W \left[p\big((\overline{w})^{p-1} - b^{p-1}\big) - \psi' + \mu_0 \right] \\ \geq \varepsilon e^{\psi} W \left[-p(p-1)\big(\varepsilon e^{\psi} W\big)^{p-1} - \psi' + \mu_0 \right] \\ \geq \varepsilon e^{\psi} W \left[-e^{(p-1)\psi} - \psi' + \mu_0 \right] = 0, \end{aligned}$$

where we have chosen $\varepsilon > 0$ such that $b > 2\varepsilon W$ and $p(p-1)(\varepsilon W)^{p-1} \leq 1$.

If $p \geq 2$ then we take ψ satisfying (5) with $p = 2$, choose $\varepsilon > 0$ such that $p(p-1)\varepsilon W b^{p-2} \leq 1$, and use the inequalities

$$b^{p-1} - (\overline{w})^{p-1} \leq (p-1)\varepsilon W b^{p-2} e^{\psi} \leq \frac{1}{p} e^{\psi} .$$

Using the supersolution $\overline{w}$, we define a sequence of solutions $\{w_i\}$ of (8) with the initial data $w_i(y, -i) = \overline{w}(|y|, -i)$. Although the right-hand side of (8) includes a gradient term, we can show as in [17] that the sequence $\{w_i\}$ converges to a solution w of (8) defined for $s < 0$ and decreasing in s. □

5. Properties of forward self-similar solutions. For a solution u of (1) defined for $t \in (0, \infty)$, we set

$$v(y, s) = t^{1/(p-1)} u(x, t) , \quad y = \frac{x}{\sqrt{t}} , \quad s = \log t . \tag{12}$$

Then we obtain the following equation for v:

$$v_s = \Delta v + \frac{1}{2} y \cdot \nabla v + \frac{1}{p-1} v + |v|^{p-1} v , \quad s \in \boldsymbol{R} , \ y \in \boldsymbol{R}^N . \tag{13}$$

Let $v = f(r)$, $r = |y|$, be any radially symmetric steady state of this equation. Then f must satisfy the initial value problem

$$\begin{cases} f_{rr} + \dfrac{N-1}{r} f_r + \dfrac{r}{2} f_r + \dfrac{m}{2} f + |f|^{p-1} f = 0 , \quad r > 0 , \\ f(0) = \alpha , \quad f_r(0) = 0 , \end{cases} \tag{14}$$

for some $\alpha > 0$. We denote the solution of the initial value problem by $f_\alpha(r)$. If $f_\alpha(r) > 0$ for all $r > 0$, then

$$u = F_\alpha(x, t) := t^{-1/(p-1)} f_\alpha(t^{-1/2}|x|) , \qquad x \in \boldsymbol{R}^N , \ t > 0 , \tag{15}$$

is called a (positive radial) forward self-similar solution of (1). We note that $f = \varphi_\infty(r)$ satisfies the equation in (14).

It was shown in [21] that (14) has a unique global solution f_α, and it satisfies the condition

$$f_\alpha(r) = l_\alpha r^{-m} + o(r^{-m}) \quad \text{as } r \to \infty$$

with some constant l_α depending continuously on α.

The next result is due to [21, 28, 36].

LEMMA 5.1. *Let $p \geq p_S$. Then for any $\alpha > 0$, the solution $f_\alpha(r)$ of* (14) *is positive for all $r > 0$, and l_α is positive and continuous in $\alpha > 0$. Moreover, there exists $\alpha^* = \alpha^*(p, N) \in (0, \infty]$ with the following properties*:

(i) *$\alpha^* < \infty$ if $p_S \leq p < p_c$ and $\alpha^* = \infty$ if $p \geq p_c$.*
(ii) *The solution $f_\alpha(r)$ of* (14) *is increasing in $\alpha \in (0, \alpha^*)$ for every $r > 0$.*
(iii) *l_α is increasing in $\alpha \in (0, \alpha^*)$.*
(iv) *If $p_S \leq p < p_c$, then $l_\alpha \to 0$ as $\alpha \to 0$ and $l_{\alpha^*} > L$.*

LEMMA 5.2. *Let $p \geq p_S$. For any $l \in (0, L)$, there exists $\alpha \in (0, \alpha^*)$ such that $f_\alpha(r) = lr^{-m} + o(r^{-m})$ as $r \to \infty$ and $0 < f_\alpha(r) < \varphi_\infty(r)$ for all $r > 0$.*

PROOF. By Lemma 5.1, for any $l \in (0, L)$, there exists $\alpha \in (0, \alpha^*)$ such that $l_\alpha = l$. Suppose that f_α intersects with φ_∞ and let $0 < z_1 < z_2$ be two zeros of $Y(r) := f_\alpha - \varphi_\infty$ such that $Y(z_1) = Y(z_2) = 0$ and $Y(r) > 0$ for $r \in (z_1, z_2)$. We take $\beta > 0$ such that $\beta \in (\alpha, \alpha^*)$ and set $Z(r) := f_\beta - \varphi_\infty$.

The functions $Y(r)$ and $Z(r)$ satisfy the equations

$$Y_{rr} + \frac{N-1}{r} Y_r + \frac{r}{2} Y_r + \frac{1}{p-1} Y + \eta(r) Y = 0 \,, \quad r > 0 \,,$$
$$Z_{rr} + \frac{N-1}{r} Z_r + \frac{r}{2} Z_r + \frac{1}{p-1} Z + \zeta(r) Z = 0 \,, \quad r > 0 \,,$$

respectively, where

$$\eta(r) := \frac{f_\alpha^p - \varphi_\infty^p}{f_\alpha - \varphi_\infty} \,, \qquad \zeta(r) := \frac{f_\beta^p - \varphi_\infty^p}{f_\beta - \varphi_\infty} \,.$$

Here since $f_\alpha < f_\beta$, we have $\eta(r) < \zeta(r)$ for $r \in [z_1, z_2]$. Then by the Sturm comparison theorem, Z must vanish at some $r \in (z_1, z_2)$, a contradiction. □

6. Proofs of the main results. In this section we complete the proofs of Theorems 2.1 through 2.4. We also give some related results.

We begin with the following lemma.

LEMMA 6.1. *Assume that $u(\cdot, 0) \in L^\infty(\boldsymbol{R}^N)$.*

(i) *Let $p_{sg} < p < p_c$. If $0 \leq u(x, 0) \leq \varphi_\infty(|x|)$ for all $x \in \boldsymbol{R}^N \setminus \{0\}$, then the solution of* (1) *satisfies $\|u(\cdot, t)\|_{L^\infty(\boldsymbol{R}^N)} \to 0$ as $t \to \infty$.*

(ii) *Let $p \geq p_c$. If $0 \leq u(x, 0) \leq \varphi_\infty(|x|)$ for all $x \in \boldsymbol{R}^N \setminus \{0\}$ and*

$$\limsup_{|x| \to \infty} |x|^m u(x, 0) < L \,,$$

then the solution of (1) *satisfies $\|u(\cdot, t)\|_{L^\infty(\boldsymbol{R}^N)} \to 0$ as $t \to \infty$.*

PROOF. For (i), see [19, 36]. The assertion (ii) is proved in [32, Theorem 4.5]. □

By Lemma 5.2, for any $l \in (0, L)$, there exists a unique $\alpha_l > 0$ such that the solution of (14) satisfies the following two conditions:

$$f_{\alpha_l}(r) = lr^{-m} + o(r^{-m}) \quad \text{as} \quad r \to \infty \,,$$
$$0 < f_{\alpha_l}(r) < Lr^{-m} \quad \text{for all} \quad r > 0 \,.$$

We give a sufficient condition for the convergence of solutions of (13) to $f_{\alpha_l}(|y|)$.

LEMMA 6.2. *Let $p_S \leq p < p_c$. Suppose that v satisfies the following two conditions:*

$$0 \leq v(y, 0) \leq \varphi_\infty(|y|) \ \text{for} \ \ y \in \boldsymbol{R}^N \setminus \{0\} \,,$$
$$v(y, 0) = l|y|^{-m} + o(|y|^{-m}) \ \text{as} \ \ |y| \to \infty \,,$$

with some $l \in (0, L)$. *Then the solution of* (13) *exists for all* $s > 0$ *and* $\|v(\cdot, s) - f_{\alpha_l}(|\cdot|)\|_{L^\infty(\boldsymbol{R}^N)} \to 0$ *as* $s \to \infty$.

PROOF. Let u be the corresponding solution of (1). Then by Lemma 6.1, u exists for all $t > 0$ and becomes smaller than f_{α^*} in finite time. Then we can apply [15, Lemma 3.1] to show that for any $\varepsilon > 0$, there exists $s_\varepsilon > 0$ such that

$$f_{\alpha_l - \varepsilon}(|y|) < v(y, s) < f_{\alpha_l + \varepsilon}(|y|) \text{ for all } y \in \boldsymbol{R}^N, \ s > s_\varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, this implies the uniform convergence to f_{α_l}. □

Now we are in a position to complete the proof of Theorem 2.1.

PROOF OF THEOREM 2.1. Let w be the ancient solution of (8) given in Proposition 4.3. Then the corresponding solution u of (1) given by

$$u = (-t)^{-1/(p-1)} w(y, s), \quad y = \frac{x}{\sqrt{-t}}, \quad s = -\log(-t), \ t < 0,$$

is an ancient solution of (1) satisfying (i) and (ii). Since there is no positive steady state of (9) below b, the solution w becomes smaller than φ_∞ in finite time. This implies that u also satisfies $u < \varphi_\infty$ for some finite t. Then from Lemma 6.2, we see that (iii) holds. □

COROLLARY 6.3. *Let* $N = 3, 4$ *and* $p_S < p < p_L$. *If* u *is the homoclinic orbit constructed in the proof of Theorem* 2.1, *then* $E[u(\cdot, t)] = +\infty$ *for all* $t \in \boldsymbol{R}$.

PROOF. Since

$$E[u(\cdot, t)] = \int_0^R r^{N-1}\left(\frac{1}{2}u_r^2 - \frac{1}{p+1}u^{p+1}\right)dr + \int_R^\infty r^{N-1}\left(\frac{1}{2}u_r^2 - \frac{1}{p+1}u^{p+1}\right)dr =: I_1 + I_2 ,$$

where I_1 is finite and

$$I_2 \sim h(l, p) \int_R^\infty r^{N-3-2m} dr, \quad h(l, p) := \frac{l^2}{2} - \frac{l^{p+1}}{p+1}, \tag{16}$$

for R large enough, we obtain $I_2 = \infty$ if $N - 3 - 2m > -1$ and $h(l, p) > 0$, which is equivalent to $p > p_S$ and

$$l < \left(\frac{p+1}{2}\right)^{1/(p-1)} .$$

The last inequality holds for $N = 3, 4$ because $l < L$ and $p > p_S$. □

We note that in the case $N > 4$, the right-hand side of (16) is infinite if $h(l, p) \neq 0$, but it is not easy to determine the sign of $h(l, p)$.

If B is the backward self-similar solution given by (10) with b as in Lemma 4.1 (i), and F_α is the forward self-similar solution given by (15) with $\alpha \in (0, \alpha^*)$ and $l_\alpha = l$, then

$$\lim_{t \uparrow 0} B(x, t) = \lim_{t \downarrow 0} F_\alpha(x, t) = l|x|^{-m}, \quad x \in \boldsymbol{R}^N \setminus \{0\},$$

and

$$u^\infty(x,t) := \begin{cases} B(x,t) & (x,t) \in \boldsymbol{R}^N \times (-\infty, 0)\,, \\ l|x|^{-m} & (x,t) \in (\boldsymbol{R}^N \setminus \{0\}) \times \{0\}\,, \\ F_\alpha(x,t) & (x,t) \in \boldsymbol{R}^N \times (0,\infty)\,, \end{cases}$$

is an entire weak solution of (1) (see [19]), which converges uniformly to zero as $t \to \pm\infty$.

If u is a homoclinic orbit, then

$$u^\lambda(x,t) := \lambda^m u(\lambda x, \lambda^2 t)\,, \quad \lambda > 0\,,$$

is also a homoclinic orbit. In the next theorem we show the convergence of u^λ to u^∞ as $\lambda \to \infty$.

THEOREM 6.4. *Let u be the homoclinic orbit constructed in the proof of Theorem* 2.1. *Then $u^\lambda(x,t)$ approaches the singular homoclinic orbit u^∞ in the following sense*:

(i) *For any $\tau < 0$, $u^\lambda(x,t)/B(x,t) \to 1$ as $\lambda \to \infty$ uniformly in $(x,t) \in \boldsymbol{R}^N \times (-\infty,\tau)$, where B is the backward self-similar solution given by* (10) *with b as in Lemma* 4.1 (i).

(ii) *For any $\tau > 0$, $u^\lambda(x,t) \to F_\alpha(x,t)$ as $\lambda \to \infty$ uniformly in $(x,t) \in \boldsymbol{R}^N \times (\tau,\infty)$, where F_α is the forward self-similar solution given by* (15) *with $\alpha \in (0,\alpha^*)$ and $l_\alpha = l$.*

PROOF. Let $w(y,s)$ be the ancient solution of (8) given as above. Then we have

$$\frac{w(y,s)}{b(|y|)} \to 1 \text{ as } s \to -\infty$$

uniformly in $y \in \boldsymbol{R}^N$. In the original variables x and $t < 0$, this implies

$$\frac{(-t)^{1/(p-1)}u(x,t)}{b((-t)^{-1/2}|x|)} \to 1 \text{ as } t \to -\infty\,.$$

Transforming $t \to \lambda^2 t$, $x \to \lambda x$ with $\lambda > 0$, this is rewritten as

$$\frac{(-\lambda^2 t)^{1/(p-1)}u(\lambda x\,,\lambda^2 t)}{b((-\lambda^2 t)^{-1/2}|\lambda x|)} \to 1 \text{ as } \lambda^2 t \to -\infty\,.$$

Hence for any $\tau < 0$, we have

$$\frac{\lambda^m u(\lambda x, \lambda^2 t)}{(-t)^{-1/(p-1)}b((-t)^{-1/2}|x|)} = \frac{u^\lambda(x,t)}{B(x,t)} \to 1 \text{ as } \lambda \to \infty$$

uniformly in $(x,t) \in \boldsymbol{R}^N \times (-\infty,\tau)$. Thus (i) is proved.

Next, let us consider the properties of the entire solution for $t > 0$. Let v be the corresponding solution of (13). By Lemma 6.2, we have

$$v(y,s) \to f_{\alpha_l}(|y|) \text{ as } s \to \infty$$

uniformly in $y \in \boldsymbol{R}^N$. In the original variables x and $t > 0$, this implies

$$t^{1/(p-1)}u(x,t) \to f_{\alpha_l}(t^{-1/2}|x|) \text{ as } t \to +\infty\,.$$

Transforming $t \to \lambda^2 t$, $x \to \lambda x$ with $\lambda > 0$, this is rewritten as

$$(\lambda^2 t)^{1/(p-1)} u(\lambda x, \lambda^2 t) \to f_{\alpha_l}((\lambda^2 t)^{-1/2}|\lambda x|) \ \text{ as } \ \lambda^2 t \to +\infty .$$

Hence for any $\tau > 0$, we have

$$\lambda^m u(\lambda x, \lambda^2 t) \to t^{-1/(p-1)} f_{\alpha_l}(t^{-1/2}|x|) \ \text{ as } \ \lambda \to \infty$$

uniformly in $\boldsymbol{R}^N \times (\tau, \infty)$. This completes the proof of (ii). □

The following result is a consequence of [10, Theorems 1.2 and 1.3].

LEMMA 6.5. *Let $p > p_F$. Suppose that*

$$0 \le u(x, 0) \le \varphi_\infty(|x|) \ \textit{for} \ \ x \in \boldsymbol{R}^N \setminus \{0\}$$

and

$$k_1|x|^{-d} \le u(x, 0) \le k_2|x|^{-d} \ \textit{for} \ \ |x| > R$$

with some positive constants $d > m$, k_1, k_2 and R. Then the solution of (1) *exists globally in time and there are constants $C_1, C_2 > 0$ and $t_1 > 1$ such that*

$$C_1 g_d(t) \le \|u(\cdot, t)\|_{L^\infty(\boldsymbol{R}^N)} \le C_2 g_d(t) \ \textit{for} \ \ t \ge t_1 ,$$

where

$$g_d(t) := \begin{cases} t^{-d/2} & \textit{if } m < d < N , \\ t^{-N/2} \ln t & \textit{if } d = N , \\ t^{-N/2} & \textit{if } d > N . \end{cases}$$

PROOF OF THEOREM 2.2. Let u be the ancient solution of (1) given in Proposition 3.4 (i). Then the assertion (i) is clear from the construction of the ancient solution. Since u is below the steady state φ_α and decreasing in t, u exists globally in time and becomes smaller than φ_∞ in finite time. Then the conclusion immediately follows from Lemmas 3.2 and 6.5. □

PROOF OF THEOREM 2.3. Let u be the ancient solution of (1) given in Proposition 3.4 (ii). Then the assertion (i) is clear from the construction of the ancient solution defined for $t < T$, where $T > 0$ is the maximal existence time of the solution.

Let B_1 be the unit ball centered at the origin. Since the solution is radial and decreasing in $r = |x|$, we can show in a similar manner to [16] that the solution is bounded on ∂B_1 for $t \in (-\infty, T)$. On the other hand, since $U = u_t$ satisfies the inequality

$$U_t \ge \Delta U ,$$

it follows from the maximum principle that $u_t = U > \delta$ for all $t \in (0, T)$ and $x \in B_1$.

Following an argument in [16], we set

$$J := u_t - \varepsilon u^p ,$$

and take $\varepsilon > 0$ so small that $J(x, 0) > 0$ for $x \in B_1$ and $J(x, t) > 0$ for $(x, t) \in \partial B_1 \times (0, T)$. Since

$$\begin{aligned} J_t - \Delta J &= u_{tt} - \varepsilon p u^{p-1} u_t - \{\Delta u_t - \varepsilon p(p-1) u^{p-2} |\nabla u|^2 - \varepsilon p u^{p-1} \Delta u\} \\ &= (u_t - \Delta u)_t - \varepsilon p u^{p-1}(u_t - \Delta u) + \varepsilon p(p-1) u^{p-2} |\nabla u|^2 \\ &\geq p u^{p-1} u_t - \varepsilon p u^{2p-1} \\ &= p u^{p-1} J, \end{aligned}$$

we have $J(x, t) > 0$ for all (x, t) by the maximum principle. Thus we obtain the inequality

$$u_t \geq \varepsilon u^p$$

for all $(x, t) \in B_1 \times (0, T)$.

Therefore $m(t) := \max u = u(0, t)$ satisfies the inequality

$$m_t \geq \varepsilon m^p$$

for some $\varepsilon > 0$. This shows that $T < \infty$ and the blow-up is of Type I. □

PROOF OF THEOREM 2.4. Suppose that there exists such an ancient solution u. By Lemma 5.1 (iv), there is a forward self-similar solution $F_\alpha(x, t)$ with $l_\alpha > L$. Since

$$F_\alpha(x, t) \to l|x|^{-m} \quad \text{as} \quad t \downarrow 0\,,$$

and u is bounded by the singular steady state, we have

$$-F_\alpha(x, t+\tau) < u(x, t) < F_\alpha(x, t+\tau)\,, \quad x \in \boldsymbol{R}^N,\ t > \tau\,,$$

where $\tau > 0$ is an arbitrary constant. Setting $t = 0$, we obtain

$$-F_\alpha(x, \tau) < u(x, 0) < F_\alpha(x, \tau)\,, \quad x \in \boldsymbol{R}^N\,.$$

Letting $\tau \to \infty$, we have $F_\alpha(x, \tau) \to 0$ so that $u(x, 0) \equiv 0$, a contradiction. □

REFERENCES

[1] T. BARTSCH, P. POLÁČIK AND P. QUITTNER, Liouville-type theorems and asymptotic behavior of nodal radial solutions of semilinear heat equations, J. Eur. Math. Soc. 13 (2011), 219–247.

[2] M.-F. BIDAUT-VÉRON, Initial blow-up for the solutions of a semilinear parabolic equation with source term, Equations aux dérivées partielles et applications, articles dédiés à Jacques-Louis Lions, 189–198, Gauthier-Villars, Éd. Sci. Méd. Elsevier, Paris, 1998.

[3] P. BRUNOVSKÝ AND B. FIEDLER, Connecting orbits in scalar reaction diffusion equations, Dynam. Report. Ser. Dynam. Systems Appl. 1 (1988), 57–89.

[4] P. BRUNOVSKÝ AND B. FIEDLER, Connecting orbits in scalar reaction diffusion equations II: The complete solution, J. Differential Equations 81 (1989), 106–135.

[5] N. CHAFEE AND E. INFANTE, A bifurcation problem for a nonlinear partial differential equation of parabolic type, Applicable Anal. 4 (1974/75), 17–37.

[6] B. FIEDLER, Global attractors of one-dimensional parabolic equations: sixteen examples, Tatra Mt. Math. Publ. 4 (1994), 67–92.

[7] B. FIEDLER AND C. ROCHA, Heteroclinic orbits of scalar semilinear parabolic equations, J. Differential Equations 125 (1996), 239–281.

[8] B. FIEDLER AND C. ROCHA, Realization of meander permutations by boundary value problems, J. Differential Equations 156 (1999), 282–308.

[9] B. FIEDLER AND C. ROCHA, Orbit equivalence of global attractors of semilinear parabolic differential equations, Trans. Amer. Math. Soc. 352 (2000), 257–284.

[10] M. FILA, J. R. KING, M. WINKLER AND E. YANAGIDA, Linear behaviour of solutions of a superlinear heat equation, J. Math. Anal. Appl. 340 (2008), 401–409.

[11] M. FILA AND H. MATANO, Connecting equilibria by blow-up solutions, Discrete Contin. Dynam. Systems 6 (2000), 155–164.

[12] M. FILA, H. MATANO AND P. POLÁČIK, Existence of L^1-connections between equilibria of a semilinear parabolic equation, J. Dynam. Differential Equations 14 (2002), 463–491.

[13] M. FILA AND N. MIZOGUCHI, Multiple continuation beyond blow-up, Differential Integral Equations 20 (2007), 671–680.

[14] M. FILA AND P. POLÁČIK, Global solutions of a semilinear parabolic equation, Adv. Differential Equations 4 (1999), 163–196.

[15] M. FILA, M. WINKLER AND E. YANAGIDA, Convergence to selfsimilar solutions for a semilinear parabolic equation, Discrete Contin. Dyn. Syst. 21 (2008), 703–716.

[16] A. FRIEDMAN AND B. MCLEOD, Blow-up of positive solutions of semilinear heat equations, Indiana Univ. Math. J. 34 (1985), 425–447.

[17] Y. FUKAO, Y. MORITA AND H. NINOMIYA, Some entire solutions of the Allen-Cahn equation, Taiwanese J. Math. 8 (2004), 15–32.

[18] G. FUSCO AND C. ROCHA, A permutation related to the dynamics of a scalar parabolic PDE, J. Differential Equations 91 (1991), 111–137.

[19] V. GALAKTIONOV AND J. L. VÁZQUEZ, Continuation of blow-up solutions of nonlinear heat equations in several space dimensions, Comm. Pure Appl. Math. 50 (1997), 1–67.

[20] B. GIDAS AND J. SPRUCK, Global and local behavior of positive solutions of nonlinear elliptic equations, Comm. Pure Appl. Math. 34 (1981), 525–598.

[21] A. HARAUX AND F. B. WEISSLER, Nonuniqueness for a semilinear initial value problem, Indiana Univ. Math. J. 31 (1982), 167–189.

[22] D. HENRY, Geometric theory of semilinear parabolic equations, Springer-Verlag, New York, 1981.

[23] D. HENRY, Some infinite dimensional Morse–Smale systems defined by parabolic differential equations, J. Differential Equations 59 (1985), 165–205.

[24] L. A. LEPIN, Countable spectrum of eigenfunctions of a nonlinear heat-conduction equation with distributed parameters, Differentsial'nye Uravneniya 24 (1988), 1226–1234 (English translation: Differential Equations 24 (1988), 799–805).

[25] L. A. LEPIN, Self-similar solutions of a semilinear heat equation, (in Russian), Mat. Model. 2 (1990), 63–74.

[26] Y. LI, Asymptotic behavior of positive solutions of equation $\Delta u + K(x)u^p = 0$ in $\boldsymbol{R}^n$, J. Differential Equations 95 (1992), 304–330.

[27] N. MIZOGUCHI, Nonexistence of backward self-similar blowup solutions to a supercritical semilinear heat equation, J. Funct. Anal. 257 (2009), 2911–2937.

[28] Y. NAITO, An ODE approach to the multiplicity of self-similar solutions for semilinear heat equations, Proc. Roy. Soc. Edinburgh Sect. A 136 (2006), 807–835.

[29] P. POLÁČIK, Parabolic equations: Asymptotic behavior and dynamics on invariant manifolds, Handbook of dynamical systems (B. Fiedler ed.), Vol. 2, Chapter 16, Elsevier, Amsterdam, 2002.

[30] P. POLÁČIK AND P. QUITTNER, A Liouville-type theorem and the decay of radial solutions of a semilinear heat equation, Nonlinear Anal. 64 (2006), 1679–1689.

[31] P. POLÁČIK, P. QUITTNER AND PH. SOUPLET, Singularity and decay estimates in superlinear problems via Liouville-type theorems. Part II: Parabolic equations, Indiana Univ. Math. J. 56 (2007), 879–908.

[32] P. POLÁČIK AND E. YANAGIDA, On bounded and unbounded global solutions of a supercritical semilinear

heat equation, Math. Ann. 327 (2003), 745–771.

[33] P. POLÁČIK AND E. YANAGIDA, A Liouville property and quasiconvergence for a semilinear heat equation, J. Differential Equations 208 (2005), 194–214.

[34] P. QUITTNER AND PH. SOUPLET, Superlinear parabolic problems. Blow-up, global existence and steady states, Birkhäuser Advanced Texts, Birkhäuser, Basel, 2007.

[35] J. SMOLLER, Shock waves and reaction-diffusion equations, Grundlehren Math. Wiss. 258, Springer-Verlag, New York, 1983.

[36] PH. SOUPLET AND F. B. WEISSLER, Regular self-similar solutions of the nonlinear heat equation with initial data above the singular steady state, Ann. Inst. H. Poincaré Anal. Non Linéaire 20 (2003), 213–235.

[37] X. WANG, On the Cauchy problem for reaction-diffusion equations, Trans. Amer. Math. Soc. 337 (1993), 549–590.

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