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Homoclinic and non-wandering points for maps of the circle

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Abstract. For continuous maps f of the circle to itself, we show: (A) the set of nonwandering points of f coincides with that of f^n for every odd n; (B) f has a horseshoe if and only if it has a non-wandering homoclinic point; (C) if the set of periodic points is closed and non-empty, then every non-wandering point is periodic.

1. *Introduction*

In this paper we examine the dynamics of continuous maps of the circle to itself, establishing for such maps versions of three results known to hold for maps of a compact interval.

THEOREM A. *If f is a continuous map of the circle, then the set of non-wandering points of f coincides with that of f^n for every odd n.*

Theorem A is identical to the corresponding result [**3**] for maps of the interval.

THEOREM B. *A continuous map of the circle has a horseshoe if and only if it has a non-wandering homoclinic point.*

Here we have added to the corresponding result [**1**], [**9**], [**10**] for maps of the interval the condition that the homoclinic point is non-wandering. We show by example that on the circle this condition cannot be omitted. Actually we prove a stronger result – see theorem B+ and the remarks following it at the end of § 5.

THEOREM C. *If the set of periodic points of a continuous map of the circle is closed and non-empty, then every non-wandering point is periodic.*

Here we have added to the corresponding result [**4**], [**8**], [**11**] for maps of the interval the obvious requirement that the set of periodic points is non-empty – consider an irrational rotation.

We will prove theorem A by adapting the proof in [**3**] to the circle. In fact we will produce a shorter proof, valid for the interval as well as the circle. We will prove theorems B and C by lifting the map of the circle to a map of the reals, for which these results are known to hold. We will then project back down to the circle.

2. *Preliminaries*

Throughout this paper f will denote a continuous map of the circle to itself. The set of non-wandering points of f will be denoted by $\Omega(f)$ and the set of periodic points by Per (f). $\Omega(f)$ is always closed and non-empty and

$$\text{Per}\,(f) = \text{Per}\,(f^n) \subseteq \Omega(f^n) \subseteq \Omega(f)$$

holds for all n.

Formally we will think of the circle as $\mathbb{R}/\mathbb{Z}$ and use π to denote the canonical projection. Thus every continuous map f of the circle has countably many *lifts*, i.e. continuous maps $F: \mathbb{R} \to \mathbb{R}$ satisfying

$$f \circ \pi = \pi \circ F.$$

Any two such lifts differ by an integer and the unique integer d satisfying

$$F(X+1) = F(X) + d$$

for all lifts F and all X is called the *degree* of f, denoted deg (f).

In addition, we will think of the circle as *oriented* so that π is orientation-preserving. Thus notation such as $[a, b]$ will make sense on the circle as well as on the interval or reals. In all three cases, we will have occasion to refer to points near a given point as being on the positive (+) or negative (−) side of that point. When we wish to speak of a side without specifying which it is we will use the letter S.

We define a (basic) *half-neighbourhood* of a point to be a non-degenerate closed interval having that point as the appropriate endpoint. Thus a positive half-neighbourhood of x is a set of the form $[x, y]$.

A technical concept we will use is that of f-covering. We say of two closed intervals J and K that J *f-covers* K if $f(J') = K$ for some sub-interval J' of J. (For maps of the interval, this is equivalent to $f(J) \supseteq K$; for maps of the circle, it is stronger.) The importance of f-coverings lies in the fact that if J f^n-covers itself for some n, then f has a periodic point in J.

3. *Unstable sets*

The basic tool for proving all three theorems is the analysis of one-sided unstable sets. We review here the basic facts of this theory from [**1**], [**3**] and [**7**]. The definition we adopt is that of [**1**]; the unstable sets in [**3**] and [**7**] are the closures of the ones we consider here.

For a fixed point p of f and a side S, the *one-sided unstable set* of p is

$$W^u(p, f, S) = \bigcap_U \bigcup_{k \geq 0} f^k(U),$$

where the intersection is taken over all S-half-neighbourhoods of p. Each one-sided unstable set is a (possibly degenerate) interval (possibly the whole circle) containing p which is mapped onto itself by f. We remind the reader that there are no universal relations between the two one-sided unstable sets. In particular, a one-sided unstable set need not be a half-neighbourhood.

For a fixed point p of f^N, the unstable sets (under f^N) of the points in the f-orbit of p are related in the manner stated in the following lemma, which summarizes the relevant portions of lemmas 8.1–8.3 of [**3**].

LEMMA 1. *Let p be a fixed point of f^N and let $S_0 = +$ or $-$. If $W^u(p, f^N, S_0)$ is non-degenerate, then for each $i \geq 0$ there is a side S_i at $f^i(p)$ such that*

(a) *for every S_0-half-neighbourhood U of p, $f^i(U)$ contains an S_i-half-neighbourhood of $f^i(p)$.*

Write W_i in place of $W^u(f^i(p), f^N, S_i)$. Then

(b) $W_i = f^i(W_0)$;

(c) $W_{i+N} = W_i$;

(d) *if $f^i(p) \in \text{int}(W_j)$, then $W_i = W_j$.*

Note that (c) does not assert that $S_{i+N} = S_i$, but only that the one-sided unstable sets are the same. The proof of (d) may require a moment's thought. When W_j is a proper sub-interval of the circle, (d) is just a restatement of lemma 8.3 of [**3**]. On the other hand, if W_j is the whole circle, then (b) implies that f maps the circle onto itself so that W_i is the whole circle as well.

To get some feeling for one-sided unstable sets, the reader is invited to verify the lemma for the examples which appear in § 5.

4. *Non-wandering points: theorem A*

Recall that x is non-wandering under f, denoted $x \in \Omega(f)$, if for every neighbourhood U of x,

$$f^n(U) \cap U \neq \varnothing \qquad \text{for some } n \geq 1.$$

The following two technical lemmas are proved using essentially the same arguments used to prove the corresponding results [**3**] for maps of the interval. For details, see [**6**].

LEMMA 2. *If $x \in \Omega(f)$, then for every neighbourhood U of x, $x \in f^n(U)$ for some $n \geq 1$.*

LEMMA 3. *If $x \in \Omega(f)$ has an infinite orbit, then $x \in \Omega(f^n)$ for every n.*

It is easy to construct, for any pre-assigned even n, a map f with $\Omega(f^n) \neq \Omega(f)$ – just embed the appropriate example from [**3**] in a map of the circle.

It follows from lemma 3 that to prove theorem A it suffices to prove

(*) *If $x \in \Omega(f)$ and the orbit of x contains a fixed point of f^N, then $x \in \Omega(f^n)$ for every odd n.*

We do so by induction on N. Before we begin the induction we state the following technical lemma, which is an immediate consequence of lemmas 1 and 2.

LEMMA 4. *If $x \in \Omega(f)$ and $f^{kN}(x) = p$ is a fixed point of f^N, then there are sides $S_i (i \geq 0)$, consistent with lemma 1, such that*

(a) *for every neighbourhood U of x, $f^{kN+i}(U)$ contains an S_i-half-neighbourhood of $f^{kN+i}(x)$;*

(b) $x \in W_i = W^u(f^i(p), f^N, S_i)$ *for some i.*

The following lemma proves (*) for $N = 1$ and 2. The proof we give is a streamlined version of the proof of Lemma 9.1 of [**3**].

LEMMA 5. *If $x \in \Omega(f)$ and the orbit of x contains a fixed point of f^2, then $x \in \Omega(f^n)$ for every odd n.*

Proof. We may assume that x is not itself periodic. Let $f^{2k}(x)$ be a fixed point of f^2. Using the notation of lemma 4, there exists $i = 0$ or 1 and a side $S = S_i$ of $q = f^{2k+i}(x)$ such that for every neighbourhood U of x, $f^{2k+i}(U)$ contains an S-half-neighbourhood of q and such that $x \in W^u(q, f^2, S)$.

Let U be a neighbourhood of x and let V be an S-half-neighbourhood of q contained in $f^{2k+i}(U)$ with

$$x \notin V \cup f^2(V) \cup f^4(V).$$

Note that $x \in f^{2j}(V)$ for some $j \geq 3$. There are three possibilities for the behaviour of V under f^2:

(1) $f^2(V) \subseteq V$;

(2) $f^2(V) \supseteq V$;

(3) $f^2(V) = V' \cup V''$, where V' is a (possibly degenerate) S-half-neighbourhood of q properly contained in V and V'' is a non-degenerate half-neighbourhood of q on the other side.

If (1) holds, then $x \notin W^u(q, f^2, S)$. If (2) holds, then $x \in f^{2m+2j}(V)$ for every m. If (3) holds, then $f^2(V \cup V'')$ is not contained in $V \cup V''$, so either

(3a) $f^2(V'') \supseteq V''$, or

(3b) $f^2(V'') \supseteq V$.

If (3a) holds, then

$$x \in f^{2(m-1)+2j}(V'') \subseteq f^{2m+2j}(V) \qquad \text{for every } m.$$

If (3b) holds, then $x \in f^{4m+2j}(V)$ for every m. In all three possible situations – (2), (3a), (3b) – we have $x \in f^{4m+2j+2k+i}(U)$ for every m. Thus $\{r \mid x \in f^r(U)\}$ contains all sufficiently large integers in some residue class modulo 4. But every such class contains arbitrarily large multiples of every odd number. Thus $x \in \Omega(f^n)$ for every odd n. □

LEMMA 6. *Suppose $x \in \Omega(f)$ has a finite orbit. If, with notation as in lemma 4, $x \in \operatorname{int}(W_i)$ for some i, then $x \in \overline{\operatorname{Per}(f)}$ and hence $x \in \Omega(f^n)$ for every n.*

Proof. We again assume that x is not itself periodic. Let $f^{kN}(x) = p$ be a fixed point of f^N. There are sides S at x and S_i at $f^{kN+i}(x) = q$ such that for every S-half-neighbourhood U of x, $f^{kN+i}(U)$ contains an S_i-half-neighbourhood of q.

Let G be a lift of f^N with a fixed point Q satisfying $\pi(Q) = q$. Let X be such that $\pi(X) = x$ and $Q \in (X-1, X)$. Since $x \in \operatorname{int}(W^u(q, f^N, S_i))$ and x is not periodic, at least one of the following must hold:

(1) $X \in \operatorname{int}(W^u(Q, G, S_i))$;

(2) $X - 1 \in \operatorname{int}(W^u(Q, G, S_i))$;

(3) both X and $X-1$ are in $W^u(Q, G, S_i)$.

In each case it follows that if U is a small enough S-half-neighbourhood of x contained in W_i, then U $f^{jN+kN+i}$-covers itself for some j. Thus every such U contains a periodic point and hence $x \in \overline{\operatorname{Per}(f)}$. □

Now comes the inductive step.

LEMMA 7. *Let $N \geq 3$. If $x \in \Omega(f) - \overline{\text{Per}(f)}$ and the orbit of x contains a fixed point of f^N, then there exist $m \geq 2$ and $M < N$ such that for $g = f^m$, $x \in \Omega(g) - \overline{\text{Per}(g)}$ and the g-orbit of x contains a fixed point of g^M.*

Proof. Let $f^{kN}(x) = p$ be a fixed point of f^N. Without loss of generality, N is the least period of p, for otherwise the conclusion holds with $M =$ the least period of p, and $m = N/M$.

Let W_i, $i \geq 0$, be given by lemma 4. Note that if $x \in W_0$, then the conclusion holds with $m = N$ and $M = 1$. Suppose then that $x \notin W_0$ and hence that $x \in W_i$ for some i, $1 \leq i \leq N-1$.

Claim 1. The endpoints of W_i are x and p.
By lemma 6, x must be an endpoint of W_i. Since W_i is f^N-invariant, $p \in W_i$. If p were not an endpoint of W_i, then by lemma 1, $W_0 = W_i$ which contains x. This proves claim 1.

Claim 2. W_i contains only p and $f^i(p)$ from the orbit of p.
We use lemma 1 repeatedly. Suppose $f^j(p) \in W_i$ where $f^j(p) \neq p$ and $f^j(p) \neq f^i(p)$. Then $f^j(p) \in \text{int}(W_i)$ and hence $W_i = W_j$. Let $t = |i-j|$. Then W_i is f^t-invariant and hence contains the entire f^t-orbit of p. In particular, for some r,

$$f^t(p), f^{2t}(p), \ldots, f^{rt}(p) \in \text{int}(W_i) \quad \text{and} \quad f^{(r+1)t}(p) = p.$$

Then $W_i = W_{rt}$ and hence $W_0 = f^t(W_{rt}) = W_i$ which contains x. This proves claim 2.

Claim 3. Each W_j contains exactly two members of the orbit of p.
This follows immediately from claim 2.
We now complete the proof of the lemma. If $N = 2i$, the conclusion follows with $M = 2$ and $n = N/2$. Suppose that $N \neq 2i$. Then the points $p, f^i(p)$, and $f^{2i}(p)$ are distinct. Now W_{2i} contains both $f^i(p) \in \text{int}(W_i)$ and $f^{2i}(p) \notin W_i$. Thus $p \notin W_{2i}$ and hence $x \in \text{int}(W_{2i})$. Then by lemma 6, $x \in \text{Per}(f)$. □

Assertion (*) and hence theorem A now follow easily from lemmas 5 and 7. Note that our proof of theorem A works for maps of the interval as well as for maps of the circle.

5. *Homoclinic points: theorem B*

Recall that x is a *homoclinic point* if for some N there is a fixed point p of f^N such that $x \neq p$, p is in the f^N-orbit of x, and $x \in W^u(p, f^N, S)$ for some side S. In this case we say that x is homoclinic to p. We sometimes call such a point a 'strong' homoclinic point, to distinguish it from a 'weak' homoclinic point, which we define below.

A point x is a *weak homoclinic point* if for some N there is a fixed point p of f^N such that $x \neq p$, p is in the f^N-orbit of x, and $x \in W^u(q, f^N, S)$ for some q in the f-orbit of p and some side S. It follows from lemma 4 that any point in $\Omega(f) - \text{Per}(f)$ with a finite orbit is a weak homoclinic point. Conversely, a non-wandering weak homoclinic point has a finite orbit but is not periodic. The distinction between strong and weak homoclinic points is illustrated by the following example.

Example 1. Let f be a map of the circle with the following properties: f has a periodic orbit $\{p, q\}$ of period 2 with p and q diametrically opposite; f maps $[p, q]$ isometrically onto $[q, p]$ preserving orientation; and for some $x \in (q, p)$, f collapses $[q, x]$ to p and uniformly stretches $[x, p]$ onto $[p, q]$ preserving orientation. Then x is a weak homoclinic point but there are no strong homoclinic points.

We note in passing that this example shows that on the circle, unlike the interval [**6**], [**8**], [**11**], the existence of a weak homoclinic point does not imply the existence of a strong homoclinic point.

We say that *f has a horseshoe* if for some N there are disjoint closed intervals J and K such that each of J and K f^N-covers both J and K. When f has a horseshoe as above, the f^N-invariant set

$$H = \bigcap_{i=0}^{\infty} f^{-iN}(J \cup K),$$

the set of points whose f^N-orbit lies in $J \cup K$, has the full one-sided shift on two symbols as a continuous factor, via the map that assigns to each point in H its itinerary under f^N. This factor map takes the periodic points of $f^N | H$ onto the periodic points of the shift, as a consequence of the fact that any interval which f^i-covers itself contains a fixed point of f^i.

The Homoclinic Point Theorem [**1**], [**9**], [**10**] states (in part) that a map of the interval (or the reals) which has a strong homoclinic point also has a horseshoe.

It is easy to construct a map of the circle with a strong homoclinic point but no horseshoes.

Example 2. Let f be a map of the circle with the following properties: f has a fixed point p, $x \neq p$, f collapses $[x, p]$ to p and homeomorphically stretches (p, x) onto the complement of p, with every point moving a positive distance forward. Then x is homoclinic to p but f cannot have a horseshoe, since for any closed interval J which does not contain p, and for any n, $f^n(J)$ does not contain J.

We will prove the two implications of theorem B separately. The easier implication is:

Proposition 1. *If f has a horseshoe, then it has a non-wandering homoclinic point.*

A preliminary technical observation will streamline our proof. If J f-covers K, it is clear that we can choose the subinterval J' for which $f(J') = K$ so that its endpoints map onto the endpoints of K. We then refer to J' as a *precise pre-image* of K in J. If $J' = [a, b]$ is a precise pre-image of $K = [c, d]$ then f either preserves the endpoint order $(f(a) = c, f(b) = d)$ or reverses it $(f(a) = d, f(b) = c)$.

Lemma 8. *Suppose J f-covers K and J′ is a precise pre-image of K in J. If f preserves (respectively reverses) the endpoint order on J′, then every sub-interval L of K has a precise pre-image in J′ on which f preserves (respectively reverses) the endpoint order.*

We omit the straightforward proof.

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Proof of proposition 1. Suppose f has a horseshoe, exhibited by $J=[a_0, b_0]$, K, and f^N. Since a non-wandering point for f^N is non-wandering for f, we may assume that $N=1$.

In particular, J f-covers itself, and hence it f^2-covers itself as well. A precise pre-image J' of J which reverses endpoint order must, by lemma 8, contain a precise pre-image J'' of J' on which f also reverses endpoint order. But then f^2 maps J'' onto J preserving endpoint order. Thus (replacing f by f^2 if necessary) we can assume that $J_1=[a_1, b_1]$ is a precise pre-image of J in J on which f preserves endpoint order. Invoking lemma 8 inductively, we find a nested sequence of intervals $J_m=[a_m, b_m]$ such that J_m is a precise pre-image of J_{m-1} in J_{m-1} on which f preserves endpoint order. In particular, we have

$$a_{m-1}=f(a_m)\leq a_m < b_m \leq f(b_m)=b_{m-1},$$

so that the sequences a_m and b_m converge monotonically to fixed points, say a and b, of f. Furthermore, given a negative half-neighbourhood U of a and a positive half-neighbourhood V of b, $a_m \in U$ and $b_m \in V$ for all sufficiently large m. This implies that $a_i \in W^u(a, f, -)$ and $b_i \in W^u(b, f, +)$ for $i=0, 1, 2, \ldots$. Now, since $[a_0, b_0]$ f-covers K and

$$f[a_1, b_1]\cap K=\varnothing,$$

either $[a_0, a_1]$ or $[b_1, b_0]$ f-covers K. Without loss of generality, we assume the latter. Then, since the f-invariant set $W^u(b, f, +)$ contains $[b, b_0]$, it contains K as well.

On the other hand, K f-covers J, so there is a point $x \in K$ such that $f(x)=b$ and the image of every neighbourhood of x contains a positive half-neighbourhood of b. Then x is non-wandering and homoclinic to b. □

An examination of the proof of the Homoclinic Point Theorem in [**1**] reveals that, for maps of the interval or the reals, the intervals exhibiting the horseshoe can be chosen to lie inside any pre-assigned neighbourhood of the periodic point involved. In particular, if some lift F of f has a homoclinic point, then F has a horseshoe exhibited by intervals which are contained in an interval of length less than one, and hence which project under π to disjoint intervals on the circle. These latter intervals exhibit a horseshoe for f. Thus we have

LEMMA 9. *If some lift of f has a homoclinic point, then f has a horseshoe.*

We will make use of the following fact, which (for maps of the interval) is implicit in [**1**] and explicit in [**8**] and [**11**].

LEMMA 10. *Let F be a map of the interval or reals. If there is a fixed point P of F and a point $X>P$ with $X \in W^u(P, F, -)-W^u(P, F, +)$, then F has a homoclinic point.*

We formulate a strengthened converse of proposition 1 as

PROPOSITION 2. *If a map of the circle has a non-periodic non-wandering point with a finite orbit, then it has a horseshoe.*

We first prove a special case.

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LEMMA 11. *If $x \in \Omega(f)$ is not a fixed point, but has a fixed point in its orbit, then f has a horseshoe.*

Proof. We may assume that $f(x) = \pi(0)$ is a fixed point and choose a lift F of f such that $F(0) = 0$. By lemma 9, we may assume that F has no homoclinic points. Let X be the unique point between 0 and 1 such that $\pi(X) = x$. Then either X or $X - 1$ belongs to

$$W^u(0, F, +) \cup W^u(0, F, -).$$

We assume that it is X, noting that the proof in the other case is similar. Then by lemma 10, $X \in W^u(0, F, +)$, for otherwise F has a homoclinic point.

To show that f has a horseshoe, it suffices to show

$$W^u(0, F, +) \text{ contains some } Y > 1. \qquad (*)$$

To see this, note first that we may assume that $Y < 2$. There exist $m > 0$ and points

$$0 < X_0 < X_1 < X_2 < Y - 1$$

such that

$$F^m(X_2) = Y, \quad F^m(X_1) = 1 \quad \text{and} \quad F^m(X_0) = X_2.$$

Then the intervals $\pi[0, X_0]$ and $\pi[X_1, X_2]$ are disjoint and each f^m-covers both.

To prove $(*)$, note that since $f(x) = \pi(0)$ is a fixed point, $F(X)$ must be an integer, which is non-zero since otherwise X is homoclinic to 0. Furthermore, if $F(X) > 1$, then $(*)$ holds with $Y = F(X)$. Thus we have two cases: $F(X) = 1$ and $F(X) < 0$.

Case 1. $F(X) = 1$.

We may assume that $F(1) = 1$; for if $F(1) < 0$, then some point in $(X, 1)$ is homoclinic to 0; if $F(1) = 0$, then 1 is homoclinic to 0; and if $F(1) > 1$, then $(*)$ holds with $Y = F(1)$. Thus we have $F(X) = F(1) = 1$. For every neighbourhood U of X, $F(U)$ contains at least a half-neighbourhood of 1, otherwise $x \notin \Omega(f)$. In fact, it contains a positive half-neighbourhood, since if $F(U)$ is a negative half-neighbourhood of 1, then by lemma 2, $X \in W^u(1, F, -)$, making X homoclinic to 1. But since $F(U)$ contains a positive half-neighbourhood of 1, $(*)$ holds for some $Y \in F(U)$.

Case 2. $F(X) < 0$.

If $F^2(X) < 0$ as well, then $\deg(f) > 0$ and hence $F^m(X) < 0$ for all $m \geq 1$. Now $F[0, X]$ contains no point to the right of 0, since otherwise some point in $[0, X]$ is homoclinic to 0. Similarly, for all $m \geq 1$, $F^m[0, X]$ contains no point to the right of 0 and hence $X \notin W^u(0, F, +)$.

Suppose then that $F^2(X) \geq 0$. If $F^2(X) = 0$, then X is homoclinic to 0. If $F^2(X) = 1$, then $F(X) = -1$, so $\deg(f) = -1$ and hence $F(X - 1) = 0$, making $X - 1$ homoclinic to 0. This leaves only $F^2(X) > 1$, and so $(*)$ holds with $Y = F^2(X)$. This proves $(*)$ and hence the lemma. □

Proof of proposition 2. Let $x \in \Omega(f)$ be non-periodic and suppose $f^{kN}(x) = p$ is a fixed point of f^N. If $x \in \text{Per}(f)$, then $x \in \Omega(f^N)$. By lemma 11, f^N has a horseshoe and hence so does f.

Suppose then that $x \notin \text{Per}(f)$. Using the notation of lemma 4, we have $x \in W_i$ for some i, $0 \leq i \leq N - 1$. If $x \in W_0$, then $x \in \Omega(f^N)$ and f has a horseshoe as in the preceding paragraph. Suppose then that $x \notin W_0$ and $x \in W_i$ where $1 \leq i \leq N-1$. As in the proof of lemma 7, the endpoints of W_i are x and p. Thus W_i is a compact, f^N-invariant, proper sub-interval of the circle. Let $q = f^i(p)$ and suppose without loss of generality that $S_i = +$ and hence that $W_i = W^u(q, f^N, +)$.

If $W_i = [p, x]$, then $f^{jN}(y) = x$ for some $y \in (q, x)$ and some $j \geqslant k$. Since $f^{jN}(x) = p$, there exists $z \in (y, x)$ such that $f^{jN}(z) = q$. But then z is a homoclinic point for $f^N \mid W_i$. By the Homoclinic Point Theorem, $f^N \mid W_i$ has a horseshoe, and hence f has one as well.

If $W_i = [x, p]$, then by lemma 10, either $f^N \mid W_i$ has a homoclinic point or $x \in W^u(q, f^N \mid W_i, -)$. In either case, as in the preceding paragraph, f has a horseshoe. □

Note that we have proved a stronger version of theorem B, analogous to the proposition in [**8**].

THEOREM B+. *For a continuous map f of the circle, the following are equivalent*:
(1) *f has a horseshoe*;
(2) *f has a non-wandering (strong) homoclinic point*;
(3a) *f has a non-wandering weak homoclinic point*;
(3b) *f has a non-periodic non-wandering point with a finite orbit.*

We remark that these conditions are also equivalent to each of the following:
(4) *f has positive topological entropy*;
(5) *f has periodic points with least periods $n < m$ where m/n is not a power of* 2.

(1) implies (5) follows from the fact that the factor map from the horseshoe preserves periods, (5) implies (4) from [**2**], and (4) implies (1) from [**5**].

6. *Maps with closed periodic set: theorem C*

Our proof of theorem C will follow from an analysis of the non-wandering set for maps with no horseshoes. That this is the right situation to look at follows from

LEMMA 12. *If f has a horseshoe, then* Per (f) *is not closed.*

Proof. Recall from the earlier discussion of horseshoes that for some N there is a compact f^N-invariant set H such that $f^N|H$ has the full one-sided shift on two symbols as a continuous factor. Furthermore, Per $(f^N|H)$ maps onto the set of periodic points of the shift. If Per (f) is closed, then so is Per $(f^N|H)$ and hence also the set of periodic points of the shift. But this last set is not closed. □

LEMMA 13. *If for some lift F of f, there is an interval $J \subseteq [0, 1]$ of length less than one such that for some $m \geq 1$, $F^m(J)$ contains three consecutive integers, then f has a horseshoe.*

The proof of lemma 13 is straightforward (see lemma 5.10 of [**6**]).
Lemmas 12 and 13 allow us to concentrate on maps of degree 0 or ±1. We handle these cases separately.

PROPOSITION 3. *Suppose f has degree zero and F is a lift of f. We have*

(1) $\pi[\Omega(F)]=\Omega(f)$;

(2) *if* Per (f) *is closed, then so is* Per (F).

Proof. Since the range of F is compact, F has a fixed point, and so we can assume without loss of generality that $F(0)=0$. Let I be a compact interval which contains the range of F and has (distinct) integer endpoints. Then $\Omega(F)=\Omega(F|I)$ and $\pi|I$ is a finite-to-one factor map of $F|I$ onto I. We will abuse notation slightly by using F in place of $F|I$ and π in place of $\pi|I$.

To show (1), we need only show that $\Omega(f)\subseteq \pi[\Omega(F)]$. Given $x\in\Omega(f)$, we may assume that $0\notin\pi^{-1}(x)$, since otherwise $x\in\pi[\Omega(F)]$. Let

$$\pi^{-1}(x)=\{X^{(1)},\ldots,X^{(m)}\};$$

observe that $X^{(j)}\in\text{int}(I)$ for all j. By lemma 2, there exist $x_i\to x$ and $n_i\geq 1$ such that $f^{n_i}(x_i)=x$. We may assume that $0\notin\pi^{-1}(x_i)$ and hence that each $\pi^{-1}(x_i)$ consists of exactly m points in I,

$$\pi^{-1}(x_i)=\{X_i^{(1)},\ldots,X_i^{(m)}\}.$$

We can label these points so that $X_i^{(j)}\to X^{(j)}$ for each j. Note that $F(X_i^{(j)})$ depends on i but not on j, since f has degree zero.

Now $\pi[F^{n_i}(X_i^{(j)})]=x$ for all i and all j, and $\pi^{-1}(x)$ is finite while $F^{n_i}(X_i^{(j)})$ is independent of j. It follows that for some k,

$$F^{n_i}(X_i^{(j)})=X^{(k)}$$

for infinitely many i and all j, and hence that

$$X^{(k)}\in\Omega(F)\cap\pi^{-1}(x).$$

This proves (1).

To show (2), suppose $X\in\overline{\text{Per}(F)}-\text{Per}(F)$. Then $\pi(X)\in\overline{\text{Per}(f)}=\text{Per}(f)$. Thus $\pi(X)$ has a finite orbit and so X does too. This makes X a weak homoclinic point. Then F must have a strong homoclinic point as well (see [**6**], [**8**] or [**11**]). Hence by lemma 9, f has a horseshoe and then by lemma 12, Per (f) is not closed. □

To obtain the analogue of proposition 3 for maps of degree one we need to assume more.

PROPOSITION 4. *Suppose f is a map of degree one which has no horseshoes. If F is a lift of f which has a fixed point, then*

(1) Per $(F)=\pi^{-1}[\text{Per}(f)]$;

(2) $\Omega(F)=\pi^{-1}[\Omega(f)]$.

Proof. We may assume that 0 is a fixed point of F. We first establish

$$X-1<F^n(X)<X+1 \qquad \text{for all } X \text{ and all } n. \tag{*}$$

Since deg $(f)=1$, $F^n(X+k)=F^n(X)+k$, so we need prove (*) only for $X\in(0,1)$. If $F^n(X)\geq X+1$, then

$$F^{2n}[0,X]\supseteq F^n[0,X+1]\supseteq[0,X+2],$$

and so by lemma 13 f has a horseshoe. Similarly, if $F^n(X) \le X-1$, then

$$F^{2n}[X, 1] \supseteq [X-2, 1],$$

and again f has a horseshoe. This proves (*).

To prove (1), it suffices to show that if $\pi(X) \in \text{Per}(f)$, then $X \in \text{Per}(F)$. If $\pi(X) \in \text{Per}(f)$, say $f^n(\pi(X)) = \pi(X)$, then $F^n(X) = X + k$ for some integer k. By (*), $k = 0$ and hence $X \in \text{Per}(F)$.

To prove (2), it suffices to show that if $x = \pi(X) \in \Omega(f)$, then $X \in \Omega(F)$. Since $x \in \Omega(f)$, lemma 2 implies that there exist $x_i \to x$ and $n_i \ge 1$ such that $f^{n_i}(x_i) = x$. Let X_i be the point closest to X in $\pi^{-1}(x_i)$. Then $X_i \to X$ and $F^{n_i}(X_i) = X + k_i$ for some integer k_i. The convergence of X_i to X together with (*) imply that for i sufficiently large, $k_i = -1$, 0, or 1. Thus, for a subsequence of X_i (which we still denote X_i) $F^{n_i}(X_i)$ is constant and equals $X-1$, X, or $X+1$.

Suppose $F^{n_i}(X_i) = X+1$. If $F(X) < X$, then for some $\delta > 0$ and all sufficiently large i, $F(X_i) < X - \delta$. Using (*) again, for these i we have

$$F^{n_i}(X_i) = F^{n_i-1}(F(X_i)) < F(X_i) + 1 < X + 1 - \delta,$$

contradicting the assumption that $F^{n_i}(X_i) = X+1$. Using the fact that

$$F^{n_i+1}(X_i) = F(X) + 1,$$

similar arguments lead to a contradiction when $F(X) > X$. Hence $F(X) = X$ and $X \in \Omega(F)$.

In the same way, if $F^{n_i}(X_i) = X - 1$, then $F(X) = X$ and' $X \in \Omega(F)$. Finally, if $F^{n_i}(X_i) = X$, then $X \in \Omega(F)$ by definition. □

We now assemble a proof of theorem C. Suppose Per (f) is closed and non-empty. By lemma 12, f has no horseshoes. It follows from lemma 13 that deg $(f) = 0$ or ± 1. If deg $(f) = 0$, then by proposition 3, for a lift F of f, Per (F) is closed and non-empty. Then by [**8**], $\Omega(F) = \text{Per}(F)$, and by proposition 3 again,

$$\Omega(f) = \pi[\text{Per}(F)] = \text{Per}(f).$$

If deg $(f) = \pm 1$ and $x \in \Omega(f) - \text{Per}(f)$, then x must have an infinite orbit, since otherwise by theorem B f has a horseshoe. But then by lemma 3, $x \in \Omega(f^n)$ for all n. Choose n even (so that f^n has degree one) and such that f^n has a fixed point. Let G be a lift of f^n which has a fixed point. Since f^n has no horseshoes, it follows from proposition 4 that Per (G) is closed and non-empty, and hence by [**8**] again that $\Omega(G) = \text{Per}(G)$. But then

$$x \in \Omega(f^n) = \pi[\Omega(G)] = \pi[\text{Per}(G)] = \text{Per}(f^n) = \text{Per}(f).$$

The proof is complete. □

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