ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 26, nº 2 (1998), p. 285-301
<http://www.numdam.org/item?id=ASNSP_1998_4_26_2_285_0>

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXVI (1998), pp. 285-301

Homoclinic and Periodic Orbits for Hamiltonian Systems

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Abstract. This article deals with the existence of homoclinic and periodic solutions for second order Hamiltonian systems. The main purpose is to consider unbounded potentials which do not satisfy the Ambrosetti-Rabinowitz condition. The method is variational and it combines a perturbation argument with Morse index estimates for minimax critical points.

0. – Introduction

During the last twenty years, variational methods have been widely applied to study existence of periodic solutions and connecting orbits for Hamiltonian systems.

An assumption often made on the Hamiltonian is the so-called Ambrosetti-Rabinowitz condition, after their seminal paper [1]. In case of a second order Hamiltonian system this condition states

(S) *There exist* $\mu > 2$ *and* $r > 0$ *such that*

$$V'(q) \cdot q \geq \mu V(q) > 0, \qquad \textit{for all} \quad |q| \geq r. \tag{0.1}$$

Condition (S) implies that the potential grows more than a super quadratic power, and it is crucial in proving that the associated functional satisfies the Palais Smale condition. In this case one finds periodic solutions as critical points of the corresponding functional. Moreover, when studying homoclinic orbits, this condition appears again. It is the main tool in assuring that a family of subharmonics, with growing period, is uniformly bounded. For the relevant literature, we refer the reader to the recent review article by Rabinowitz [7].

In [3], Coti-Zelati, Ekeland and Lions replace the Ambrosetti-Rabinowitz condition by the assumption that the potential is convex and its inverse Hessian approaches zero near infinity. By using Clarke duality and Morse index

* Partially supported by CNPq-CONICYT and Proyecto FONDECYT 1940-505.
** Partially supported by CNPq-CONICYT and CNPq under grant 307014/89-4.
Pervenuto alla Redazione il 13 febbraio 1996 e in forma definitiva il 1 settembre 1997.

estimates, they are able to prove existence theorems for periodic orbits. They consider the case of first and second order Hamiltonian systems.

It is the purpose of this article to extend the results of [3] for the case of second order Hamiltonian systems. Lifting the convexity hypothesis and allowing a much weaker growth at infinity, we show the existence of periodic solutions. Furthermore, by adding a natural extra hypothesis, we construct homoclinic orbits as limit of subharmonic solutions, generalizing the results of Rabinowitz [6].

We now describe the results in a more precise way: in the first part of our work, we consider the autonomous second order Hamiltonian system

$$\ddot{q} + V'(q) = 0, \tag{0.2}$$

with periodic boundary conditions

$$q(0) - q(T) = \dot{q}(0) - \dot{q}(T) = 0. \tag{0.3}$$

Here the potential $V : \mathbb{R}^N \to \mathbb{R}$ is of class C^2, and we assume it satisfies the following general hypotheses:

(V1) $V(0) = 0,\ V'(0) = 0,\ V''(0) = 0,$ *and*

$$V(q) \geq 0, \quad \text{for all} \;\; q \in \mathbb{R}^N. \tag{0.4}$$

(V2) *There exist constants $\alpha > 0$ and $r > 0$, a vector $e \in S^{N-1}$, and a non-negative function $W : \mathbb{R}^N \to \mathbb{R}$, with $\lim_{|q|\to\infty} W(q) = \infty$, such that*

$$V(q) \geq \frac{\alpha}{2}|s|^2 + W(q), \qquad \text{for all} \quad |q| \geq r, \quad \text{and} \tag{0.5}$$

$$V''(q)e \cdot e \geq \alpha, \qquad \text{for all} \quad |q| \geq r, \tag{0.6}$$

where $q = se + \hat{q}$, with $\hat{q}$ orthogonal to e.

Setting $\kappa = N/2 + 1$ or $\kappa = (N+1)/2 + 1$ if N is even or odd, respectively, we can now state our theorem on the existence of T-periodic orbits for (0.2).

THEOREM 0.1. *Assume V is a C^2 potential satisfying* (V1), (V2) *and*

$$\alpha > \kappa^2 \left(\frac{2\pi}{T}\right)^2. \tag{0.7}$$

Then (0.2) *possesses at least one nontrivial T-periodic solution.*

Instead of hypothesis (V2) we may consider the stronger condition

(SV2) *There exist* $\alpha > 0$ *and* $r > 0$ *such that*

$$V''(q)u \cdot u \geq \alpha, \qquad \textit{for all} \quad |q| \geq r, u \in S^{N-1}. \tag{0.8}$$

In this case we may take $\kappa = 1$ in Theorem 0.1 and obtain a similar result. Intermediate conditions can also be considered.

By strengthening hypothesis (V2) in another sense, we prove that system (0.2) possesses infinitely many nontrivial T-periodic solutions. See Theorem 1.1. at the end of Section 1.

Theorem 0.1 can be extended to treat the case of a system like

$$\ddot{q} + Aq + V'(q) = 0, \tag{0.9}$$

where A is a matrix. For the sake of simplicity, we do not treat this problem explicitly, except for the case $A = -aI_N$, where $a > 0$ and I_N is the identity matrix, which is considered in Section 2.

Another extension of Theorem 0.1 is also considered in Section 3, where a nonautonomous system is studied.

Our Theorem 0.1 extends the existence result for second order Hamiltonian systems in [3], allowing much more general potentials. We observe that in particular, the super quadratic growth is greatly relaxed and that, as in [3], the functional associated to the problem does not satisfy the Palais Smale condition in general. In proving our results we work in a direct variational setting and we apply the arguments of [3] with regard to Morse index estimates.

In the second part of our work, we consider the problem of finding homoclinic orbits for the Hamiltonian system

$$\ddot{q} - aq + V'(q) = 0, \tag{0.10}$$

that is, solutions of (0.10), satisfying

$$\lim_{t\to\pm\infty} q(t) = 0, \quad \lim_{t\to\pm\infty} \dot{q}(t) = 0. \tag{0.11}$$

Here we consider $a > 0$. As before, we also assume that the potential V is of class C^2 and it satisfies (V1) and a version of (V2) which takes into account the presence of a. This modified (V2) is as follows:

(V2′) *There exist constants* $\alpha > 0$ *and* $r > 0$, *a vector* $e \in S^{N-1}$, *and a non-negative function* $W : \mathbb{R}^N \to \mathbb{R}$, *with* $\lim_{|q|\to\infty} W(q) = \infty$, *such that*

$$V(q) \geq \frac{\alpha}{2}|s|^2 + \frac{a}{2}|\hat{q}|^2 + W(q), \tag{0.12}$$

$$V''(q)e \cdot e \geq \alpha, \qquad \textit{for all} \quad |q| \geq r, \tag{0.13}$$

where $q = se + \hat{q}$, *with* $\hat{q}$ *orthogonal to* e.

In order to prevent linear behavior of V', we assume an extra hypothesis. More specifically, we suppose:

(V3) $\frac{1}{2}V'(q) \cdot q - V(q) > 0$, for all $q \in \mathbb{R}^N \setminus \{0\}$.

Setting $S = -\min\{(V''(q)e \cdot e) \ / \ q \in \mathbb{R}^N\}$, we obtain:

THEOREM 0.2. *Suppose $V \in C^2$ satisfies* (V1), (V2′) *and* (V3). *Assume further that*

$$\alpha\frac{(\pi-2)}{2\pi} - S > a > 0. \tag{0.14}$$

Then (0.10) *possesses a nontrivial homoclinic orbit.*

We prove Theorem 0.2 by an approximation procedure. We start with a family of T-periodic solutions of (0.10), parametrized by T. These solutions are found with arguments like those used in the proof of Theorem 0.1. This family is shown to be uniformly bounded through Morse index estimates inspired by [3]. Then, one finds a homoclinic orbit of (0.10) taking limits.

It is interesting to note that the functional associated to equation (0.10), together with (0.11), does not satisfy the Palais Smale condition, even in the case where condition (S) holds.

Theorem 0.2 extends the existence result of Rabinowitz [6] to more general potentials.

We organize our article in three sections. Section 1 is devoted to the proof of Theorem 0.1. We first modify the potential so that the associated functional satisfies the Palais Smale condition and it keeps the geometric properties of the original one. A solution to the modified problem is found through the Generalized Mountain Pass theorem. Then, by a Morse index estimate, we prove that the solution of the modified problem also solves the original one. Here the arguments make strong use of energy conservation. In the last part of the section we show that under more restricted hypotheses on the potential, the equation has infinitely many T-periodic solutions.

In Section 2, we prove Theorem 0.2. First we find solutions with finite period by using the ideas of Section 1. These solutions are found to be uniformly bounded by an argument that uses Morse index comparison. Here again, we rely on energy conservation arguments. To finish, we let the period go to infinity and carefully study the corresponding sequence. We prove that the sequence has a limit, that is precisely the homoclinic orbit we are looking for.

In Section 3, we briefly extend the results of Section 1 to a nonautonomous system. By adding an hypothesis on the time derivative of the potential V, we are able to show that the energy, even though not conserved, is properly controlled. Then, the argument in Section 2 can be applied.

1. – Existence of T-periodic solutions

In this section we provide a proof of Theorem 0.1. We also show that (0.2) possesses infinitely many T-periodic solutions when α can be taken arbitrarily large. We start by considering the Hilbert space

$$H_T^1 = \{q : [0,T] \to \mathbb{R}^N \ / \ q, \dot{q}, \in L^2[0,T] \text{ and } q(0) = q(T)\}$$

endowed with the usual inner product

$$\langle u, v\rangle = \int_0^T (\dot{u}\cdot\dot{v} + u\cdot v), \qquad \text{for all } u, v \in H_T^1,$$

whose associated norm we denote as $\|\cdot\|$. In H_T^1 we define the functional $I : H_T^1 \to \mathbb{R}$ by

$$I(q) = \frac{1}{2}\int_0^T |\dot{q}|^2 - \int_0^T V(q), \qquad \text{for } q \in H_T^1. \tag{1.1}$$

Since we are assuming that $V \in C^2$, standard arguments imply that the functional I is also of class C^2 and its critical points correspond to the classical T-periodic solutions of (0.2).

The first step in the proof of Theorem 0.1 is to modify the potential V, obtaining an associated functional which preserves the geometric properties of I and satisfies the Palais Smale condition.

Given $k \in \mathbb{N}$, we define the set

$$A_k = \{q \in \mathbb{R}^N \;/\; V(q) \le k\}.$$

The following lemma provides the modified potentials.

LEMMA 1.1. *Given $k \in \mathbb{N}$, there exists $V_k : \mathbb{R}^N \to \mathbb{R}$ of class C^2 satisfying:*

(a) *$V_k(q) = V(q)$ in A_k,*
(b) *$V_k(q) \ge 0$, for all $q \in \mathbb{R}^N$, and $\lim_{|q|\to\infty} V_k(q) = \infty$, uniformly in $k \in \mathbb{N}$,*
(c) *$V_k''(q)e\cdot e \ge \alpha$, for all $|q| \ge r$,*
(d) *There exist $R_k > r+1$ and $\alpha_k > 0$ such that*

$$V_k(q) = \alpha_k\big(|q|^2 - (R_k-1)^2\big)^3, \qquad \textit{for all } |q| \ge R_k + 1.$$

PROOF. For every $k \in \mathbb{N}$, hypothesis (V2) guarantees the existence of $R_k > r+1$ so that

$$A_k \subset B(0, R_k - 1)$$

Let $\chi : \mathbb{R} \to [0,1]$ be a nonincreasing function of class C^∞ such that $\chi(s) = 1$, for $s \le 0$, and $\chi(s) = 0$, for $s \ge 1$. Taking $\chi_k(s) = \chi(s - R_k)$, for $s \in \mathbb{R}$, we define

$$V_k(q) = \chi_k(|q|)V(q) + \alpha_k\big(|q|^2 - (R_k-1)^2\big)_+^3, \tag{1.2}$$

where we use the notation $a_+ = \max\{a, 0\}$. That (a) and (d) hold is obvious from the definition. Moreover, by choosing α_k large enough, we obtain that (c) also holds and V_k satisfies

$$V_k(q) \ge \chi_k(|q|)V(q) + \big(1 - \chi_k(|q|)\big)|q|^6, \quad \text{for all } \; q \in \mathbb{R}^N.$$

Condition (b) is a consequence of (V1), (V2) and the above inequality. □

Now, we consider the functional associated to the modified potential $I_k : H_T^1 \to \mathbb{R}$ given by

$$I_k(q) = \frac{1}{2}\int_0^T |\dot{q}|^2 - \int_0^T V_k(q), \qquad \text{for } q \in H_T^1. \tag{1.3}$$

We will find critical points of I_k by using the Generalized Mountain Pass theorem in the version given in Rabinowitz [5], Theorem 5.9. In order to do that, we split the space H_T^1 in the sum of two subspaces, $X_1 = \{q \ / \ q(t) = q_0 \in \mathbb{R}^N\}$ and $X_2 = X_1^{\perp}$, obtaining:

LEMMA 1.2. *The functional I_k is of class C^2 and satisfies the Palais Smale condition. Moreover, there are constants $\rho > \sigma > 0$, $\gamma \geq \beta > 0$, a function $\phi \in \partial B(0,1) \cap X_2$, and $k_0 \in \mathbb{N}$ so that, for all $k \geq k_0$, the following holds*

(I1) $I_k(q) \geq \beta > 0$, *for all* $q \in \partial B(0,\sigma) \cap X_2$,
(I2) $I_k(q) \leq \gamma$, *for all* $q \in Q_\rho$,
(I3) $I_k(q) \leq 0$, *for all* $q \in \partial Q_\rho$,
where $Q_\rho = \{q = \lambda\phi + q_0 \ / \ 0 \leq \lambda, q_0 \in X_1, \| \lambda\phi + q_0 \| \leq \rho\}$.

PROOF. Condition (d) implies that V_k satisfies condition (S). Hence I_k satisfies the Palais Smale condition for every $k \in \mathbb{N}$.

Now we show that the geometric conditions (I1)-(I3) are satisfied by the functional I. From (V1), we have that $V''(0) = 0$. Hence, there exists $\sigma > 0$ and $\beta > 0$ such that (I1) is satisfied by I.

For proving (I2) and (I3), we first observe that $I(q_0) \leq 0$, for every $q_0 \in X_1$, since V is non-negative. Moreover, if we choose the function ϕ as

$$\phi(t) = \bar{r}\sin(2\pi t/T)e,$$

where e is given in (V2) and $\bar{r}$ is chosen so that $||\phi|| = 1$. Then, for $q = \lambda\phi + q_0$, with $\lambda > 0$ and $q_0 \in X_1$, it follows from (V2) that

$$I(q) \leq \frac{T}{4}(\bar{r}\lambda)^2\left(\left(\frac{2\pi}{T}\right)^2 - \alpha\right) - \int_0^T W(\lambda\phi + q_0) + bT,$$

with $b \geq \max\{V(q) \ / \ |q| \leq r\}$. As $\alpha > (2\pi/T)^2$ and $W(q) \to \infty$, when $|q| \to \infty$, we easily conclude that I satisfies (I2) and (I3) when $\rho > \sigma > 0$ is sufficiently large.

Finally, by using condition (a) in Lemma 1.1, and choosing k_0 large enough, we obtain that (I1), (I2) and (I3) are also satisfied by I_k, for all $k \geq k_0$. □

Now, we are in a position to apply the Generalized Mountain Pass theorem to find a critical point q_k of I_k. By the Minimax characterization of the corresponding critical value, we have

$$0 < \beta \leq \gamma_k \equiv I_k(q_k) \leq \gamma < \infty. \tag{1.4}$$

Moreover, we can use Theorem 1 in Solimini [8] in order to find a critical point of I_k at level γ_k, which we still call q_k, so that

$$M(I_k, q_k) \leq N + 1, \tag{1.5}$$

where $M(I, q)$ denotes the Morse index of the functional I at the critical point q. Observe that I_k'' is a Fredholm operator, as a standard argument shows.

In our next step, we will use the Morse index estimate (1.5) in order to prove that the critical points q_k are uniformly bounded on $[0, T]$. Thus, in view of Lemma 1.1, for k large enough, we will have that q_k is a critical point of I and, consequently, a T-periodic solution of equation (0.2). This concludes the proof of Theorem 0.1.

Considering r given in (V2), we define

$$B_k = \{t \in [0, T] \ / \ |q_k(t)| < r\},$$

We have the following lemma

LEMMA 1.3. *If* $\lim_{k\to\infty} \| q_k \|_\infty = \infty$, *then*

$$\lim_{k\to\infty} \mu(B_k) = 0,$$

where μ *denotes the Lebesgue measure in* $\mathbb{R}$.

PROOF. Given $\varepsilon > 0$, we choose $R > 2r$ so that

$$\frac{2r}{(R - r)} < \varepsilon, \tag{1.6}$$

and define

$$\hat{B}_k = \{t \in [0, T] \ / \ |q_k(t)| < R\}.$$

Since $\hat{B}_k$ is open, it can be written as a countable union of mutually disjoint open intervals $\hat{B}_k = \cup_{n=1}^{\infty}(s_n, t_n)$. We will show that if k is large enough, then

$$\mu\big((s_n, t_n) \cap B_k\big) \leq \varepsilon(t_n - s_n), \tag{1.7}$$

from where we obtain that $\mu(B_k) \leq T\varepsilon$, since $B_k \subset \hat{B}_k$. That finishes the proof.

Now, we prove (1.7). We start with some preliminary facts. Since q_k is a solution of (0.2), by conservation of energy there is a constant E_k such that

$$E_k = \frac{1}{2}|\dot{q}_k(t)|^2 + V_k\big(q_k(t)\big), \qquad \text{for all } t \in [0, T], \tag{1.8}$$

and, if k is large enough,

$$E_k = \frac{1}{2}|\dot{q}_k(t)|^2 + V\big(q_k(t)\big), \qquad \text{for all } t \in \hat{B}_k. \tag{1.9}$$

Then, setting $\widehat{M} = \sup\{V(q) \ / \ |q| \leq R\}$, the following estimates hold

$$(1.10) \qquad |\dot{q}_k(t)|^2 \leq 2E_k \quad \text{and} \quad |\dot{q}_k(t)|^2 \geq 2(E_k - \widehat{M}), \qquad \text{for all } t \in \hat{B}_k.$$

We observe that from the hypothesis and condition (b) in Lemma 1.1, we have that

$$(1.11) \qquad \lim_{k\to\infty} E_k = \infty \quad \text{and} \quad \lim_{k\to\infty} |\dot{q}_k(t)| = \infty, \quad \text{uniformly in } t \in \hat{B}_k.$$

Now, let us consider one of the intervals that build up $\hat{B}_k$, say $(s_n, t_n) = (s, t)$. Then, $|q_k(s)| = |q_k(t)| = R$ and $|q_k(\tau)| < R$ for all $\tau \in (s,t)$. First, we provide an upper estimate for $(t-s)$. Invoking the Mean Value theorem and using that q_k solves (0.2) on $\hat{B}_k$ for k large , we obtain:

$$(1.12) \qquad | \dot{q}_k(\tau) - \dot{q}_k(s) | \leq \hat{m}T \qquad \text{for all } \tau \in (s,t),$$

where $\hat{m} = \sup\{V'(q) \ / \ |q| \leq R\}$. Thus, from (1.12), we find

$$2R \geq (t-s)(|\dot{q}(s)| - \hat{m}T),$$

and then, using (1.10),

$$(1.13) \qquad (t-s) \leq \frac{2R}{\sqrt{2(E_k - \widehat{M})} - \hat{m}T}.$$

We claim that, for k large enough, $q_k(\tau)$ belongs to the cylinder with axis $q(s) + \lambda\dot{q}_k(s)$, $\lambda \in \mathbb{R}$, and radius $r/2$ and it moves in the direction $\dot{q}_k(s)$, that is

(i) $| q_k(\tau)\cdot v - q_k(s)\cdot v | \leq \frac{r}{2}$, for all $\tau \in (s,t)$ and $v \perp \dot{q}_k(s)$, $| v |= 1$, and
(ii) $\dot{q}_k(\tau)\cdot\dot{q}_k(s) > 0$, for all $\tau \in (s,t)$.

In fact, taking into account that q_k solves (0.2) on $\hat{B}_k$, by the Mean Value theorem and (1.13), we have

$$| \dot{q}_k(\tau)\cdot u - \dot{q}_k(s)\cdot u | \leq \hat{m}(t-s), \qquad \text{for all } u \in \mathbb{R}^N, | u |= 1.$$

Choosing $u = v$ and $u = \dot{q}_k(s)/|\dot{q}_k(s)|$, (i) and (ii) follow directly from (1.11), (1.13) and the above inequality, if k is large enough.

Now, we can estimate $\mu((s,t) \cap B_k)$. When $q_k(\tau)$ hits the ball $B(0,r)$, there exist s^* and t^* so that $| q_k(s^*) |=| q_k(t^*) |= r$ and $r <| q_k(\tau) |< R$ for all $\tau \in (s,s^*) \cup (t^*,t)$. Using i) and ii), we get that $|q(\tau)| \leq 2r$, for all $\tau \in (s^*,t^*)$, for k sufficiently large. Proceeding as in (1.13), we find

$$(1.14) \qquad \mu\big((s,t) \cap B_k\big) \leq (t^* - s^*) \leq \frac{4r}{\sqrt{2(E_k - M)} - mT}.$$

where $m = \sup\{|V'(q)| \,/\, |q| \leq 2r\}$ and $M = \sup\{V(q) \,/\, |q| \leq 2r\}$. Finally, we estimate $(t-s)$ from below. From (1.10), we have that

$$R - r \leq | q_k(s^*) - q_k(s) | \leq \sqrt{2E_k}(s^* - s)$$

and

$$R - r \leq | q_k(t) - q_k(t^*) | \leq \sqrt{2E_k}(t - t^*),$$

from where

$$(t-s) \geq \frac{2(R-r)}{\sqrt{2E_k}}. \tag{1.15}$$

Thus, from (1.6), (1.14) and (1.15), we get

$$\mu\big((s,t) \cap B_k\big) \leq \frac{\sqrt{2E_k}}{(R-r)} \frac{2r}{\sqrt{2(E_k - M)} - mT}(t-s) \leq \varepsilon(t-s),$$

for k large enough. Observe that the size of k does not depend on the particular interval (s,t) we are considering. □

End of the Proof of Theorem 0.1. To finish the proof of Theorem 0.1 we will make a Morse index estimate under the assumption that $\| q_k \|_\infty$ goes to infinity. We define the 2κ dimensional subspace

$$W = \operatorname{span}\left\{ \sin\left(\frac{2\pi}{T}\ell t\right) e, \cos\left(\frac{2\pi}{T}\ell t\right) e \,/\, 1 \leq \ell \leq \kappa \right\},$$

on which the following inequalities hold

$$\int_0^T |\dot{w}|^2 \leq \kappa^2 \left(\frac{2\pi}{T}\right)^2 \int_0^T w^2 \quad \text{and} \quad \| w \|_\infty \leq b \int_0^T w^2, \tag{1.16}$$

for alll $w \in W$ and for some $b > 0$. Taking $w \in W$ such that $\int_0^T w^2 = 1$, we have

$$\int_{B_k} V_k''(q_k) w \cdot w \leq \widetilde{M} b^2 \mu(B_k),$$

where $\widetilde{M} = \sup\{|V''(q)| \,/\, |q| \leq r\}$. Then, from Lemma 1.3, we obtain $\lim_{k\to\infty} \int_{B_k} V_k''(q_k) w \cdot w = 0$. On the other hand, from (V2),

$$\int_{[0,T]\setminus B_k} V_k''(q_k) w \cdot w \geq \alpha \int_{[0,T]\setminus B_k} w^2.$$

Thus, applying Lemma 1.3 once more, $\liminf_{k\to\infty} \int_{[0,T]\setminus B_k} V_k''(q_k) w \cdot w \geq \alpha$, uniformly on w.

Consequently, given $\varepsilon > 0$, there exists k large enough so that

$$\langle I_k''(q_k) w, w \rangle = \int_0^T |\dot{w}|^2 - \int_0^T V_k''(q_k) w \cdot w \leq \kappa^2 \left(\frac{2\pi}{T}\right) - \alpha + \varepsilon.$$

Since we are assuming (0.7), we conclude that $M(I_k, q_k) \geq 2\kappa \geq N + 2$. But this contradicts (1.5). Thus, the sequence $\{q_k\}$ is uniformly bounded, as desired. Finally, since $I(q_k) \geq \beta > 0$ we concludes that q_k is nontrivial. The proof is complete. □

As a consequence of Theorem 0.1, we have

THEOREM 1.1. *Assume V is a C^2 potential satisfying* (V1). *Moreover, suppose that, for every $\alpha > 0$, there is $r = r(\alpha) > 0$ such that* (V2) *holds. Then,* (0.2) *possesses infinitely many nontrivial T-periodic solutions for every $T > 0$.*

PROOF. The existence of infinitely many T-periodic solutions is based on a well known argument. Let $\{q_1, \dots q_{m-1}\}$ be a finite set of nontrivial T-periodic solutions of (0.2) and consider $T_1, \dots, T_{m-1}$, the associated minimal periods. Taking $l_m \in \mathbb{N}$ such that $T_m = T/l_m < \min\{T_1, \dots, T_{m-1}\}$, we choose $\alpha_m > 0$ so that

$$\alpha_m > \kappa^2 \left(\frac{2\pi}{T_m}\right)^2 .$$

Hence, by Theorem 0.1, equation (0.2) possesses a nontrivial T_m-periodic solution $q_m \notin \{q_1, \dots, q_{m-1}\}$. This method and a simple induction argument provide the proof of Theorem 1.1. □

REMARK 1.1. Note that Theorem 1.1 shows that (0.2) possesses infinitely many T-periodic solutions when V satisfies (V1) and the condition assumed in [3], that is:

(SV2∞) $\lim_{|q|\to\infty} V''(q)u \cdot u = \infty$, uniformly for $u \in S^{N-1}$.

2. – Existence of homoclinic orbits

In this section we will prove Theorem 0.2, following a standard approach. We will find a homoclinic orbit for (0.10) as the limit of subharmonic solutions.

Given $T > 0$, we consider the functional $I^T : H_T^1 \to \mathbb{R}$ defined as

$$I^T(q) = \frac{1}{2}\int_0^T |\, \dot{q} \,|^2 + a \,|\, q \,|^2 - \int_0^T V(q), \qquad \text{for all } q \in H_T^1.$$

The critical points of I^T are the classical T-periodic solutions of (0.10).

We proceed as in Section 1 modifying the potential V to obtain a sequence $\{V_k\}$ satisfying the conditions in Lemma 1.1. Then, we define a corresponding functional I_k^T, which is of class C^2, and satisfies the Palais Smale condition, and, due to (0.14), it has a mountain pass geometry if T is large enough. We use the Mountain Pass theorem to obtain a critical point q_k^T of I_k^T such that

$$\gamma_k^T = I_k^T(q_k^T) = \inf_{g\in\Gamma_k} \sup_{0\le t\le 1} I_k^T(g(t)),$$

where $\Gamma_k = \{g \in C([0, T], H_T^1) \;/\; g(0) = 0, g(1) = u_k\}$, and u_k is such that $I_k^T(u_k) \le 0$. Moreover, using a result of Hofer [4], see also Solimini [8], we may assume that

$$M(I_k^T, q_k^T) \le 1. \tag{2.1}$$

Now we show that the point u_k can be chosen independently of k and T. From (0.14) we can choose $T_0 > 0$ so that $\alpha - a > (2\pi/T_0)^2$. For $T > T_0$, we can consider the function $\phi \in H_T^1$ defined as $\phi(t) = \sin\left(\frac{2\pi}{T_0}t\right)e$, if $t \in [0, T_0]$, and $\phi(t) = 0$, for $t \in [T_0, T]$. Then, using hypothesis (V2), we find that

$$I^T(\lambda\phi) = \frac{1}{2}\lambda^2 \int_0^{T_0} |\dot{\phi}|^2 + a|\phi|^2 - \int_0^{T_0} V(\lambda\phi) \leq \lambda^2 \frac{T_0}{4}\left((2\pi/T_0)^2 + a - \alpha\right) + bT_0,$$

for $\lambda \in \mathbb{R}$ and an appropriate b. We see then that there exists $\bar{\lambda} > 0$, independent of T, so that $u = \bar{\lambda}\phi$ satisfies $I^T(u) \leq 0$. From the properties of V_k in Lemma 1.1, we obtain k_0 sufficiently large so that

$$I_k^T(\lambda u) = I_{k_0}^T(\lambda u), \qquad \text{for all } \lambda \in [0, \bar{\lambda}].$$

From this discussion and the the minimax characterization of γ_k^T we see that there exists $\gamma > 0$ so that

$$0 < \gamma_k^T \leq \gamma, \qquad \text{for all} \quad k \geq k_0 \text{ and } T \geq T_0. \tag{2.2}$$

Now we can continue with the arguments, as in Section 1, comparing the Morse indices, to show that for every $T \geq T_0$, there is $k \geq k_0$ so that q_k^T is a T-periodic solution of (0.10). We use here $\kappa = 1$.

In order to complete the proof of Theorem 0.2, we will prove several lemmas. Considering the two dimensional subspace

$$W_T = \operatorname{span}\left\{\sin\left(\frac{2\pi}{T}t\right)e, \cos\left(\frac{2\pi}{T}t\right)e\right\},$$

we have

LEMMA 2.1. *Let $A \subset [0, 1]$ be a measurable set. Then*

$$\int_A w^2 \geq \eta(\mu(A)) \int_0^1 w^2 \qquad \textit{for all } w \in W_1, \tag{2.3}$$

where η is the increasing function $\eta(s) = s - \frac{1}{\pi}\sin(\pi s)$.

PROOF. If $w \in W_1$, $w(t) = a\sin(2\pi t + \delta)$, for $a > 0$ and $\delta \in \mathbb{R}$. It is easy to see that

$$\int_A w^2 \geq 4a \int_0^{\mu(A)/4} \sin^2(2\pi t),$$

from which the result follows. □

REMARK 2.1. If we consider the space $\widetilde{W_T} = \operatorname{span}\{e, \sin\left(\frac{2\pi}{T}t\right)e\}$ instead of W_T we do not improve the constant in (2.3). However, for the subspace $\widehat{W_T} = \operatorname{span}\{e_1, e_2\}$, where $e_1, e_2 \in S^{N-1}$ are orthogonal, we find that the appropriate function in (2.3) is $\hat{\eta}(s) = s$.

LEMMA 2.2. *The families $\{q^T\}$ and $\{\dot{q}^T\}$ are uniformly bounded.*

PROOF. We prove first that $\{q^T\}$ is uniformly bounded. Let us assume the contrary, then there is a sequence $\{T_n\}$, such that $T_n \to \infty$ and $\| q^{T_n} \|_\infty \to \infty$ as $n \to \infty$. From now on in the proof we write $q^n = q^{T_n}$. We have, for every $T \geq T_0$,

$$\gamma^T = \frac{1}{2}\int_0^T | \dot{q}^T |^2 + a | q^T |^2 - \int_0^T V(q^T), \tag{2.4}$$

and, by conservation of energy,

$$E^T = \frac{1}{2} | \dot{q}^T |^2 - \frac{1}{2} a | q^T |^2 + V(q^T), \quad \text{for all } t \in [0, T]. \tag{2.5}$$

It follows from this last relation and from (V2′), that for large n the energy $E^n = E^{T_n}$ is positive, actually it converges to infinity.

We define the set $B^n = \{t \in [0, T_n] \ / \ |q(t)| \leq r\}$. Following [3], from the above relations, we obtain:

$$\gamma + T_n E^n \geq \gamma^{T_n} + T_n E^n \geq 2\int_0^T E^n + a|q^n|^2 - V(q^n) \geq 2(E^n - M)\mu(B^n), \tag{2.6}$$

from where we find that

$$\limsup_{n\to\infty} \left(\mu(B^n) - \frac{T_n}{2} \right) \leq 0. \tag{2.7}$$

Let $w \in W_T$ with $\int_0^T w^2 = 1$. In view of (V2′), we have

$$\langle I''(q^n)w, w\rangle \leq \left(\frac{2\pi}{T_n}\right)^2 + a\int_{B^n} V''(q^n)w \cdot w - \alpha \int_{[0,T_n]\setminus B^n} w^2. \tag{2.8}$$

Applying Lemma 2.1 and (2.7), for $\varepsilon > 0$ and n large enough, we have

$$\int_{[0,T_n]\setminus B^n} w^2 \geq \eta \left(\frac{1}{2} - \varepsilon \right). \tag{2.9}$$

Then, from (2.8), (2.9) and the hypothesis (0.14), we find that

$$\langle I''(q^n)w, w\rangle \leq 0, \quad \text{for all } w \in W.$$

But this is impossible since the Morse index of q^n is less than or equal to 1. We conclude then, that there is a constant $C > 0$ such that

$$|q^T(t)| \leq C, \qquad \text{for all } t \in \mathbb{R}, T \geq T_0. \tag{2.10}$$

Now we can prove that the derivative is also bounded. This follows easily from the uniform bound for q^T, and for $\ddot{q}^T$, as is obtained from the equation. We can assume that $|\dot{q}^T(t)| \leq C$, for all $t \in \mathbb{R}$ and $T \geq T_0$, by enlarging C if necessary. $\square$

With the estimates given in Lemma 2.2, we can use the Arzelà-Ascoli theorem with a diagonal procedure in order to find a sequence $T_n \to \infty$, as $n \to \infty$, and a function $q : \mathbb{R} \to \mathbb{R}^N$ such that the functions $q^n = q^{T_n}$, $\dot{q}^n$ and $\ddot{q}^n$ converge uniformly over bounded intervals to q, $\dot{q}$ and $\ddot{q}$, respectively, and

$$\ddot{q}(t) - aq + V'(q(t)) = 0, \qquad \text{for all } t \in \mathbb{R}.$$

Our next lemma studies the properties of the limiting function q.

LEMMA 2.3. *The limiting function defined above is nonzero, and it satisfies*

$$\lim_{t\to\pm\infty} q(t) = \lim_{t\to\pm\infty} \dot{q}(t) = 0.$$

PROOF. First we show that q is nontrivial. From hypothesis (V1), there exists $\delta > 0$ so that

$$V'(q)\cdot q \leq \frac{a}{2}|q|^2, \qquad \text{for all } |q| \leq \delta.$$

If we assume that $\| q^n \|_\infty \leq \delta$, then it follows, from the equation that q^n satisfies, that

$$a\int_0^{T_n} |q^n|^2 \leq \int_0^{T_n} |\dot{q}^n|^2 + a|q^n|^2 = \int_0^{T_n} V'(q^n)q^n \leq \frac{a}{2}\int_0^{T_n} |q^n|^2.$$

But this is impossible, since $q^n \not\equiv 0$, as follows from (2.2). We conclude that $\| q^n \|_\infty \geq \delta$. Consequently, for each $n \in \mathbb{N}$, there exists $t_n \in [0, T_n]$ so that

$$|q^n(t_n)| = \sup\{|q^n(t)| \;/\; t \in [0, T_n]\} \geq \delta.$$

Since the system is autonomous, we can redefine t so that $q^n : \left[-\frac{T_n}{2}, \frac{T_n}{2}\right] \to \mathbb{R}^N$ and $t_n = 0$, for all n. Now we can assure that the limit q of the sequence $\{q^n\}$ is nontrivial. Actually, $|q(0)| \geq \delta$.

The next step is to study the limit of q as t approaches infinity. For $0 < \sigma < \frac{\delta}{2}$, we define the sets $D_\sigma^n = \left\{t \in \left[-\frac{T_n}{2}, \frac{T_n}{2}\right] \;/\; |q^n(t)| \geq \sigma\right\}$. We will estimate the size of D_σ^n. Since q^n is a critical point of I^{T_n}, we have

$$\gamma^{T_n} = I^{T_n}(q^n) - \frac{1}{2}\langle (I^{T_n})'(q^n), q^n\rangle = \int_0^{T_n} \frac{1}{2}V'(q^n)\cdot q^n - V(q^n).$$

From (V3) there is a constant $\beta > 0$ so that

$$\frac{1}{2}V'(q)\cdot q - V(q) \geq \beta, \qquad \text{for all } \sigma < |q| < C,$$

where C is given in (2.10). Putting together the above inequalities and using (2.2) we obtain:

$$\mu(D_\sigma^n) \leq \frac{\gamma}{\beta}. \tag{2.11}$$

Let us consider t_0 such that $|q^n(t_0)| \geq 2\sigma$. Since $|\dot{q}^n(t)| \leq C$, for all t, there is $\varepsilon > 0$ so that $[t_0 - \varepsilon, t_0 + \varepsilon] \subset D^n_\sigma$, and this ε can be chosen independently of t_0 and n. Then, it follows from (2.11) that $D^n_{2\sigma}$ is contained in a finite number of L intervals of the form $[t^n_m - \varepsilon, t^n_m + \varepsilon]$, $1 \leq m \leq L$. By taking a subsequence and reordering the intervals if necessary, we can assume that for $1 \leq m \leq \ell$, $t^n_m \to t_m$, and $t^n_m \to \pm\infty$, for $\ell < m \leq L$. We note that $L > 0$ since $|q^n(0)| \geq \delta > 2\sigma$. Now choose $M = M(\sigma)$ large, so that

$$\cup_{m=1}^{\ell}[t_m - 2\varepsilon, t_m + 2\varepsilon] \subset [-M, M].$$

Thus, given $K > M$, for n large enough, we have

$$|q^n(t)| \leq 2\sigma, \qquad \text{if } t \in [-K, K] \setminus [-M, M],$$

so that

$$|q(t)| \leq 2\sigma, \qquad \text{for all } |t| \geq M.$$

Consequently $\lim_{t\to\pm\infty} q(t) = 0$. Finally, as $\ddot{q}$ is uniformly bounded, the last conclusion implies that $\lim_{t\to\pm\infty} \dot{q}(t) = 0$. □

The discussion given in this section together with Lemmas 2.2 and 2.3 completes the proof of Theorem 0.2.

REMARK 2.2. Hypothesis (V2′) expresses, somehow, the weakest growth assumption we need in order to apply our methods. However it may happen that condition (SV2) hold, or even a condition like (V2), where we have super-α growth in two directions only. In these cases, in view of Remark 2.1, we may replace (0.14) by

$$\frac{\alpha}{2} - S > a > 0.$$

We do not know if this condition is optimal.

3. – Periodic orbits in a nonautonomous case

In this section, we study the existence of T-periodic solutions for a class of nonautonomous Hamiltonian system using arguments similar to those employed in Section 1.

We consider the system

$$\ddot{q}(t) + V'\big(t, q(t)\big) = 0, \quad \text{for all } t \in (0, T) \tag{3.1}$$

with periodic boundary conditions

$$q(0) - q(T) = \dot{q}(0) - \dot{q}(T) = 0, \tag{3.2}$$

where the potential V is of class C^2 and T-periodic in the variable t. As in the autonomous case, we assume that V satisfies the following hypotheses

(W1) *$V(t,0)=0$, $V'(t,0)=0$ and $V''(t,0)=0$, for all $t \in [0,T]$, and*

$$V(t,q) \geq 0, \quad \textit{for all} \quad t \in [0,T], q \in \mathbb{R}^N. \tag{3.3}$$

(W2) *There exist constants $\alpha > 0$ and $r > 0$, a vector $e \in S^{N-1}$, and a non-negative function $W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, with $\lim_{|q|\to\infty} W(t,q) = \infty$, such that*

$$V(t,q) \geq \frac{\alpha}{2}|s|^2 + W(t,q), \qquad \textit{for all} \quad t \in [0,T], \ |q| \geq r \tag{3.4}$$

$$V''(t,q)e \cdot e \geq \alpha, \qquad \textit{for all} \quad t \in [0,T], \ |q| \geq r, \tag{3.5}$$

where $q = se + \hat{q}$, with $\hat{q}$ orthogonal to e.

Furthermore, we assume a condition which controls the time derivative of V.

(W3) *There exists $\sigma > 0$ such that*

$$|\, \partial_t V(t,q) \,| \leq \sigma V(t,q), \quad \textit{for all} \quad t \in [0,T], q \in \mathbb{R}^N. \tag{3.6}$$

Here and in what follows we denote by $\partial_t V$, the partial derivative of V with respect to t. Considering κ, as given in the Introduction, we state our theorem.

THEOREM 3.1. *Assume $V \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ is T-periodic in the time variable, and it satisfies* (W1)-(W3) *and*

$$\alpha > \kappa^2 \left(\frac{2\pi}{T}\right)^2. \tag{3.7}$$

Then (3.1) *possesses at least one nontrivial T-periodic solution.*

PROOF. As in the proof of Theorem 0.1, we define $A_k = \{q \in \mathbb{R}^N \ / \ V(t,q) \leq k, t \in [0,T]\}$, for every $k \in \mathbb{N}$, and construct, in the same way, a sequence of C^2 potentials satisfying

(a) $V_k(t,q) = V(t,q)$ in A_k.
(b) $V_k(t,q) \geq 0$, for all $t \in [0,T]$ and $q \in \mathbb{R}^N$, and $\lim_{|q|\to\infty} V_k(t,q) = \infty$, uniformly on $t \in [0,T]$ and $k \in \mathbb{N}$.
(c) $V_k''(t,q)e \cdot e \geq \alpha$, for all $t \in [0,T]$, $|q| \geq r$
(d) There exist $R_k > r+1$ and α_k such that

$$V_k(t,q) = \alpha_k \left(|q|^2 - (R_k - 1)^2\right)^3, \quad \text{for all } t \in [0,T], |q| \geq R_k + 1.$$

Furthermore, by construction, we also have

(e) $\partial_t V_k(t,q) = \chi_k(|q|)\partial_t V(t,q)$, for all $t \in [0,T]$, $q \in \mathbb{R}^N$.

Using these properties, we proceed as in Section 1 and prove the existence of a critical point q_k of the functional I_k, defined as in (1.3), such that

$$0 < \beta \leq \gamma_k \equiv I_k(q_k) \leq \gamma < \infty, \tag{3.8}$$

for certain constants β and γ, and

$$M(I_k, q_k) \leq N + 1. \tag{3.9}$$

In view of (a), to prove Theorem 3.1 it suffices to show that $\{q_k\}$ possesses a uniformly bounded subsequence on $[0, T]$. For obtaining such result, we consider the energy-like function

$$E_k(t) = \frac{1}{2}|\dot{q}_k(t)|^2 + V_k\big(t, q_k(t)\big), \quad 0 \leq t \leq T. \tag{3.10}$$

Differentiating (3.10), and using (3.1) and (e), we find

$$\dot{E}_k(t) = \chi_k(|\ q\ |)\partial_t V\big(t, q_k(t)\big).$$

Hence, (W1), (W3) and (3.10) imply

$$|\dot{E}_k(t)| \leq \sigma V_k\big(t, q_k(t)\big) \leq \sigma E_k(t), \quad 0 \leq t \leq T. \tag{3.11}$$

Supposing that $\|\ q_k\ \|_\infty \to \infty$, as $k \to \infty$, from (3.10), (3.11) and (b), we also get $\|\ E_k\ \|_\infty \to \infty$, as $k \to \infty$, and

$$E_k(t) \geq \exp(\sigma T)\ \|\ E_k\ \|_\infty, \quad 0 \leq t \leq T,$$

if k is large enough.

Considering r given in (W2), and defining $B_k = \{t \in [0, T]/|q_k(t)| < r\}$, we can use the above results and argue as in Lemma 1.3 to show that $\mu(B_k) \to 0$, as $k \to \infty$. This fact and (3.7) implies $M(I_k, q_k) \geq N+2$, for k sufficiently large, and contradicts (3.9). Consequently, the sequence $\{q_k\}$ is uniformly bounded. The proof is now complete. □

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