5-15-1989

# HOMOCLINIC BIFURCATIONS WITH NONHYPERBOLIC EQUILIBRIA

Bo Deng
*University of Nebraska-Lincoln*, bdeng@math.unl.edu

SIAM J. MATH. ANAL.
Vol. 21, No. 3, pp. 693-720, May 1990

008



# HOMOCLINIC BIFURCATIONS WITH NONHYPERBOLIC EQUILIBRIA*

BO DENG†

**Abstract.** A general geometric approach is given for bifurcation problems with homoclinic orbits to nonhyperbolic equilibrium points of ordinary differential equations. It consists of a special normal form called admissible variables, exponential expansion, strong $\lambda$-lemma, and Lyapunov-Schmidt reduction for the Poincaré maps under Sil'nikov variables. The method is based on the Center Manifold Theory, the contraction mapping principle, and the Implicit Function Theorem.

**Key words.** homoclinic orbit, admissible variables, exponential expansion, strong $\lambda$-lemma, center manifold, saddle-node bifurcation

**AMS(MOS) subject classifications.** 34A34, 34C25, 34C28

**1. Introduction.** In this paper we will study homoclinic bifurcations with nonhyperbolic equilibrium points. The method we will introduce consists of four parts: a special normal form theory, exponential expansions for the Sil'nikov solution, the strong $\lambda$-lemma, and Lyapunov-Schmidt reduction for the Poincaré maps under Sil'nikov variables. Let us begin with a survey on the same method with hyperbolic equilibria. Hopefully, this will help us develop the right intuition to the problems we have in mind.

Consider a system of ordinary differential equations

$$\dot{u} = F(u), \qquad u \in \mathbb{R}^d, \tag{1.1}$$

where $F$ is $C^r$ and $r$ is large enough so that whenever $C^{r-k}$ appears, we have $r-k \geqq 1$. Suppose the origin $u=0$ is a hyperbolic equilibrium point. Let $1 \leqq m \leqq d$ and $1 \leqq n \leqq d$ with $m+n=d$ be the numbers of the eigenvalues with negative and positive real parts, respectively, for the linearization $DF(0)$. Then, up to a linear change of coordinates, we may assume $u=(x, y)$, $DF(0) = \operatorname{diag}(A, B)$, and

$$\dot{x} = Ax + f(x, y), \qquad \dot{y} = By + g(x, y), \tag{1.2}$$

which satisfy that the real parts of the eigenvalues for the $m \times m$ and $n \times n$ matrices $A$ and $B$ are negative and positive, respectively, and $f$, $g$ are vanishing at the origin together with their first derivatives.

Given a triplet $(\tau, x_0, y_1)$, a solution $(x, y)(t)$ is a solution to the Sil'nikov problem if the conditions $x(0)=x_0$ and $y(\tau)=y_1$ are satisfied. Interpreted geometrically in Fig. 1.1, it shows that for a given initial coordinate surface $x=x_0$, an end coordinate surface $y=y_1$, and a time $\tau$, a Sil'nikov solution takes exactly $\tau$ units of time to travel from $x=x_0$ to $y=y_1$. Observe that when $\tau=0$ this problem reduces to the initial value problem. Thus it is not surprising to expect that the Sil'nikov solution is existing, unique, and continuously differentiable in its Sil'nikov data $\tau$, $x_0$, and $y_1$. To be more precise, let $B(\delta) \stackrel{\text{def}}{=} \{(x, y) \mid |x| \leqq \delta, |y| \leqq \delta\}$ be the box neighborhood of the origin; then there exists a small $\delta_0$ such that for every triplet $(\tau, x_0, y) \in \mathbb{R}^+ \times B(\delta_0)$ there exists a unique Sil'nikov solution $(x, y)(t) \stackrel{\text{def}}{=} (x, y)(t, \tau, x_0, y_1)$ in $B(2\delta_0)$ for $0 \leqq t \leqq \tau$. This solution is Lipschitz in the Sil'nikov data $\tau$, $x_0$, and $y_1$ if the nonlinear terms $f$ and $g$ are Lipschitz, or $C^r$ if they are $C^r$. The proof easily follows from the uniform contraction

* Received by the editors December 12, 1988; accepted for publication (in revised form) May 15, 1989.
† Department of Mathematics and Statistics, University of Nebraska, Lincoln, Nebraska 68588-0323.

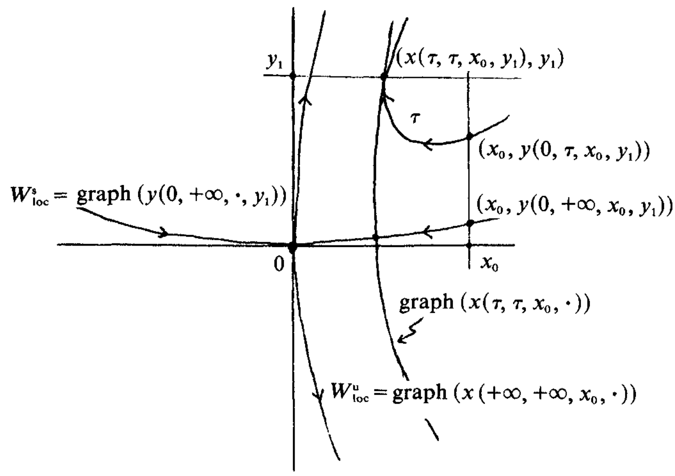


FIG. 1.1. *The hyperbolic structure in terms of the Sil'nikov solutions.*

mapping theorem together with the following equivalent integral equations:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} f(x(s), y(s))\, ds,$$

(1.3)

$$y(t) = e^{B(t-\tau)}y_1 + \int_\tau^t e^{B(t-s)} g(x(s), y(s))\, ds.$$

Also, the hyperbolicity is crucial for the validity of all $\tau \geqq 0$ for it implies the exponential functions are all bounded in the formula above. See Sil'nikov (1967) and Deng (1988a–d) for details. The first natural task is to formulate the Sil'nikov problem when the nonhyperbolicity is taken into consideration.

Before answering this question, let us see first how the Sil'nikov solution can give us a better understanding of the intrinsic local structure near a hyperbolic equilibrium. For instance, the stable and unstable manifolds can be described by the limiting behaviors of the functions $y(0, \tau, x_0, y_1)$ and $x(\tau, \tau, x_0, y_1)$ as $\tau \to +\infty$. Indeed, because the initial point for a given trajectory is given as $(x_0, y(0, \tau, x_0, y_1))$, the uniqueness of the stable manifold will imply that the family of functions $y(0, \tau, \cdot, \cdot)$ converges in the $C^r$-topology as $\tau \to +\infty$ and that the limit, denoted as $y(0, +\infty, \cdot, \cdot)$, does not depend on the $y_1$ variable. Moreover, the local stable manifold $W^s_{\text{loc}}$ is precisely the graph of the $C^r$ function $y(0, +\infty, \cdot, y_1)$ for any fixed $y_1$ since the trajectory through $(x_0, y(0, +\infty, x_0, y_1))$ stays in the box $B(2\delta_0)$ for all the positive time (cf. Fig. 1.1). Similarly, we have $W^u_{\text{loc}} = \text{graph}\,(x(+\infty, +\infty, x_0, \cdot))$, where $x(+\infty, +\infty, \cdot, \cdot)$ is the limit function of $x(\tau, \tau, \cdot, \cdot)$ as $\tau \to +\infty$. As another example, observe that the image of the $n$-dimensional "straight" disc $x = x_0$ under the time $\tau$ mapping of the flow is a curved manifold given as graph $(x(\tau, \tau, x_0, \cdot))$, which converges to the unstable manifold in the $C^r$-topology sense. In fact, this simple observation is just a special case of the so-called $\lambda$-lemma, or inclination lemma, for any $n$-dimensional disc transversely intersecting the stable manifold. This $C^r$ $\lambda$-lemma can be proved by directly using the Implicit Function Theorem and the $C^r$-convergence of the functions $x(\tau, \tau, \cdot, \cdot)$ and $y(0, \tau, \cdot, \cdot)$ as $\tau \to +\infty$. For the complete details, see Deng (1988c).

Of the most importance is to incorporate the Sil'nikov solution into our studies of homoclinic bifurcations. To do this, let us assume that there is a homoclinic orbit

$\Gamma$ to the origin and consider a Poincaré map $\Pi$ around the orbit. Figure 1.2 heuristically illustrates the construction of such a $\Pi$. Here, $\Sigma_0$ and $\Sigma_1$ are two $(d-1)$-dimensional Poincaré cross sections in $B(2\delta_0)$ with the property that they are transverse to $\Gamma \cap W^s_{loc}$ and $\Gamma \cap W^u_{loc}$, respectively. For simplicity, let us assume $\Sigma_0 = \{x^{(1)} = \delta_0\}$ and $\Sigma_1 = \{y^{(1)} = \delta_0\}$ locally. $\sigma_0$ is the set of those points $p = (x_0, y_0)$ of $\Sigma_0$ whose local trajectories hit $\Sigma_1$ at $q = (x_1, y_1)$ at the first time $\tau = \tau(p)$. Thus the local map $\Pi_0$ is defined on $\sigma_0$ with the rule $p \to q \in \Sigma_1$. The global map $\Pi_1$ is defined in the same way by following the trajectories from $\Sigma_1$ back to $\Sigma_0$. Without loss of generality, however, $\Sigma_1$ can be taken as the domain for $\Pi_1$ and all trajectories starting from $\Sigma_1$ take roughly a constant time to reach $\Sigma_0$. In contrast, the domain $\sigma_0$ of the local map is a proper subset of $\Sigma_0$, not containing any point from the local stable manifold, and the time $\tau$ diverges to infinity as the initial point $p$ approaches the stable manifold. The Poincaré map is now defined as $\Pi = \Pi_1 \circ \Pi_0$.

In general, the Poincaré map $\Pi$ is difficult to deal with directly due to the long-time behavior of the local flow. Thus we wish to find a new variable for the Poincaré map such that it becomes tractable in terms of the new variable. The Sil'nikov data $(\tau, x_0, y_1)$ serves us precisely for this purpose. We will see this more clearly later on, but for the moment let us note that $\Delta = \{(\tau, x_0, y_1) \mid \tau \geqq 0, x_0^{(1)} = \delta_0, y_1^{(1)} = \delta_0, |x_0| \leqq \delta_0, \text{ and } |y_1| \leqq \delta_0\}$ is imbedded in $\mathbb{R}^{d-1}$, and that the mapping $\rho_0 : \Delta \to \sigma_0$ with $(\tau, x_0, y_1) \to (x_0, y(0, \tau, x_0, y_1))$ gives rise to a $C^r$ change of variables since its inverse can be easily defined by: $(x_0, y_0) \to (\tau, x_0, y_1)$, where $(x_0, y_0) = p \in \sigma_0$, $(x_1, y_1) = q = \Pi_0(p_0) \in \Sigma_1$ with $\tau = \tau(p)$. $(\tau, x_0, y_1)$ is called the Sil'nikov variable and the change of variables $\rho_0$ transforms these otherwise intractable variables $\tau$, $x_0$, and $y_1$ into independent variables. Moreover, also note that the local map in the new variable is now simply given as $(x(\tau, \tau, x_0, y_1), y_1) = \Pi_0(\rho_0(\tau, x_0, y_1)) \overset{\text{def}}{=} \rho_1(\tau, x_0, y_1)$. Also, the fixed point of $\Pi$, for example, is now equivalent to solving the equation $\rho_0(\tau, x_0, y_1) = \Pi_1(\rho_1(\tau, x_0, y_1))$ for $(\tau, x_0, y_1) \in \Delta$.

However, the property of the uniform convergences of $x(\tau, \tau, x_0, y_1)$ and $y(0, \tau, x_0, y_1)$ as $\tau \to +\infty$ alone is not enough to make full use of the nice representations above for the local map of the flow. This is because the intersection of the stable and unstable manifolds along a homoclinic orbit must not be transverse. But, on the other

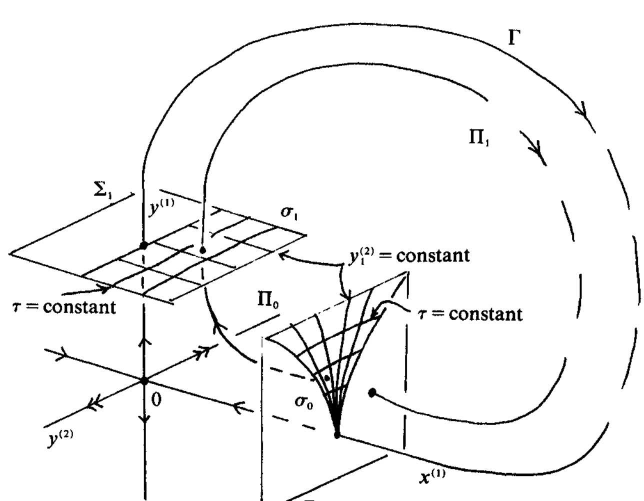


FIG. 1.2. *The Poincaré map for flows and the Sil'nikov variables.*

hand, it is quite sufficient for studying the dynamics of a transverse homoclinic point for diffeomorphisms. To see this, we refer our readers to Sil'nikov (1967) and Moser (1973). It turns out that to compensate for this loss of transversality in vector fields, we need to know a finer and subtler structure of the local flow on an exponentially small scale. Since we will also encounter the same difficulty in our nonhyperbolic case, let us explore this idea a little further.

To be more precise, let the coordinates $x$ and $y$ be chosen so that the matrices $A$ and $B$ are in their real Jordan canonical form:

$$A = \operatorname{diag}(A_0, A_1) \quad \text{and} \quad B = \operatorname{diag}(B_0, B_1)$$

with the property that the real parts for the eigenvalues of the $p \times p$ ($q \times q$, respectively) matrix $A_0$ ($B_0$, respectively) are a single number $\lambda_0 < 0$ ($\mu_0 > 0$, respectively), and that those of the $(m-p) \times (m-p)$ ($(n-q) \times (n-q)$, respectively) matrix $A_1$ ($B_1$, respectively) are strictly less (greater, respectively) than $\lambda_0$ ($\mu_0$, respectively). $A_0$ ($B_0$, (respectively) is called the principal stable (unstable, respectively) block and its eigenvalues principal stable (unstable, respectively) eigenvalues. Define

$$A_* = \operatorname{diag}(A_0, \lambda_0 I_{(m-p)}) \quad \text{and} \quad B_* = \operatorname{diag}(B_0, \mu_0 I_{(n-q)}),$$

where $I_i$ is the $i \times i$ identity matrix. Then, the Sil'nikov solution is said to admit a $C^l$ exponential expansion if it can be expressed as

$$\text{(1.4a)} \qquad \begin{aligned} x(t) &= e^{A_* t}[\psi(t-\tau, x_0, y_1) + R_1(t, \tau, x_0, y_1)], \\ y(t) &= e^{B_*(t-\tau)}[\varphi(t, x_0, y_1) + R_2(t, \tau, x_0, y_1)] \end{aligned}$$

for all $0 \leqq t \leqq \tau$ and all sufficiently small $|x_0|$ and $|y_1|$ with the properties that the coefficient functions $\psi$ and $\varphi$ are $C^l$ satisfying

$$\text{(1.4b)} \qquad \begin{aligned} &\frac{\partial \psi}{\partial x_0}(t-\tau, 0, 0) = \operatorname{diag}(I_p, 0), \qquad \frac{\partial \psi}{\partial y_1}(t-\tau, 0, 0) = 0 \quad \text{for all } 0 \leqq t \leqq \tau, \\ &\frac{\partial \varphi}{\partial y_1}(t, 0, 0) = \operatorname{diag}(I_q, 0), \qquad \frac{\partial \varphi}{\partial x_0}(t, 0, 0) = 0 \quad \text{for all } t \geqq 0, \end{aligned}$$

and that the remainder terms $R_1$ and $R_2$ are also $C^l$ satisfying

$$\text{(1.4c)} \qquad |D^i R_1(t, \tau, x_0, y_1)| \leqq K e^{-\sigma t}, \qquad |D^i R_2(t, \tau, x_0, y_1)| \leqq K e^{\sigma(t-\tau)},$$

for all $0 \leqq t \leqq \tau$ and all sufficiently small $|x_0|$ and $|y_1|$, where $K$ and $\sigma$ are some constants independent of $t$, $\tau$, $x_0$, and $y_1$, and $D^i$ is the $i$th differentiation operator up to the order $0 \leqq i \leqq l$.

It turns out that a sufficient condition for the exponential expansion requires that the coordinate $(x, y)$ be admissible in the following sense that, besides being of higher order, $f$ has the order $\sum_{k=1}^{p} |x^{(k)}|^2 + \sum_{k=p+1}^{m} |x^{(k)}|$ while $g$ has the order $\sum_{k=1}^{q} |y^{(k)}|^2 + \sum_{k=q+1}^{n} |y^{(k)}|$ as $(x, y) \to (0, 0)$. Note that this necessarily implies that $W^s_{loc} = \{y = 0\}$ and $W^u_{loc} = \{x = 0\}$ locally. Fortunately, an admissible coordinate can be obtained by a $C^{r-2}$ change of variables for (1.1), and the exponential expansion is $C^{r-4}$. For the complete but nontrivial details we refer to Deng (1988a, b, d). A counterexample against the exponential expansion when the coordinate is not admissible is also given in Deng (1988b).

Bearing in mind the questions of what are the admissible variables and what are the corresponding exponential expansions for nonhyperbolic equilibria, let us see what kinds of additional information we can draw from the expansion. First, the local strong unstable manifold $W^{uu}_{loc}$ is given by the level set $\varphi(0, 0, y) = 0$ of the expansion coefficient function. By (1.4b) it can be expressed as the graph of a $C^{r-4}$ function over the last $n-q$ variables $y^{(q+1)}, \cdots, y^{(n)}$. Second, when the system does not have the nonprincipal blocks, the exponential expansion implies that $C^1$-linearization theorem, constructively (cf. Deng (1988a)). Third, when the principal unstable block is only one-dimensional, we have

$$\lim_{\tau\to+\infty} y^T(0,\tau,x_0,y_1)\cdot\frac{\partial y^T}{\partial\tau}(0,\tau,x_0,y_1)[y^T(0,\tau,x_0,y_1)\cdot y(0,\tau,x_0,y_1)]^{-1}=B_0,$$

which is precisely the theoretical scheme for the convergence of the Feigenbaum number, where $a^T$ means the transpose of $a$ (see, e.g., Collet and Eckmann (1980)). It is my personal belief that this formula also holds true for all finite-dimensional principal blocks with $y(0,\tau,x_0,y_1)$ above being replaced by an $n\times(n-q)$ matrix

$$(y(0,\tau,x_0,y_{1,1}),\cdots,y(0,\tau,x_0,y_{1,n-q})),$$

with the property that the matrix

$$(\varphi(0,x_0,y_{1,1}),\cdots,\varphi(0,x_0,y_{1,n-q}))$$

has the maximal rank $q$. Last, but not finally, by using the exponential expansion and the Implicit Function Theorem, we can prove the strong $\lambda$-lemma, which states that for every point $u_0$ on the stable manifold there is associated a $(d-n+q)$-dimensional linear space $W(u_0)$, which contains the stable tangent space at $u_0$ as a subset such that, for every $(n-q)$-dimensional $C^{r-7}$ disc $D_0$ transverse to this critical affine plane $W(u_0)$, the image $D_\tau$ under the time $\tau$ map of the flow approaches the strong unstable manifold $W^{uu}_{loc}$ in the $C^{r-7}$-topology as $\tau\to+\infty$. See Deng (1988d) for a proof. Figure 1.3 illustrates the use of the strong inclination property in classifying some homoclinic

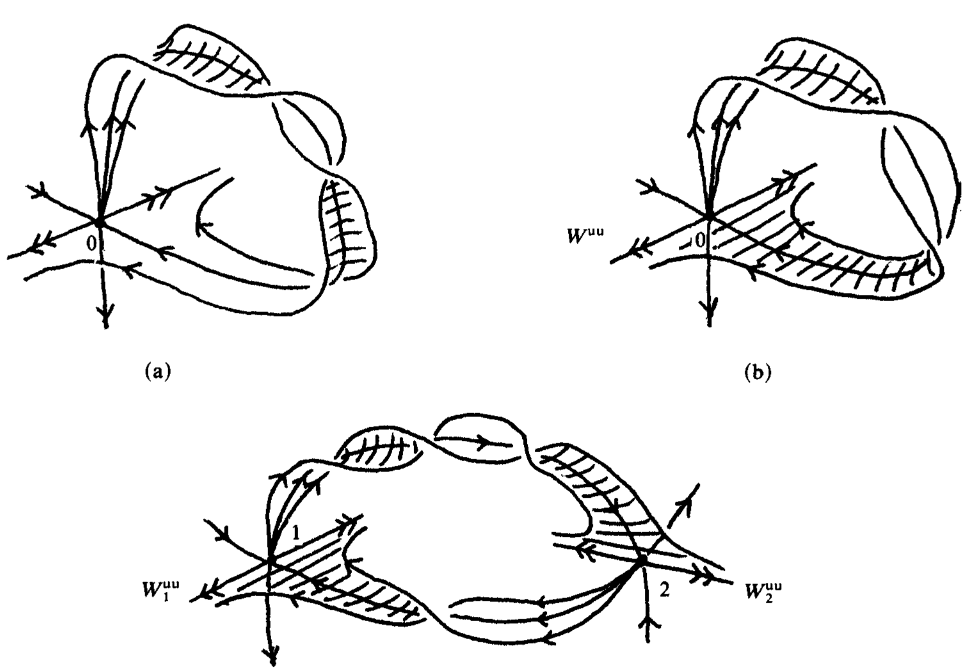


FIG. 1.3. *The phase portaits of some nondegenerate orbits.* (a) *Nontwisted homoclinic orbit.* (b) *Twisted homoclinic orbit.* (c) *Double twisted heteroclinic loop.*

and heteroclinic bifurcations for the flow. What is the strong $\lambda$-lemma for nonhyperbolic equilibria, and how can we use it, if at all, to classify homoclinic and heteroclinic orbits? Most important, how can we solve a given homoclinic bifurction problem by combining all these ideas?

We are now in a natural position to outline our paper, giving hints as to the answers. In § 2, specifically in Lemma 2.2, we will use the Center Manifold Theory only to obtain a $C^{r-2}$ admissible coordinate for the system (1.1), with $F$ having additional $l$ eigenvalues of the linearization $DF(0)$ that lie on the imaginary axis of the complex plane. A coordinate $u=(x, y, z)$ is called admissible in this case if in terms of the new variable (1.1) takes the following form:

$$\dot{x}=Ax+f(x, y, z), \quad \dot{y}=By+g(x, y, z), \quad \dot{z}=\theta(z)+h(x, y, z),$$

where $f$, $g$, and $h$ are higher-order terms satisfying $f=O(|x|+|y|+|z|)|x|$, $g=O(|x|+|y|+|z|)|y|$, and $h=O(|x|\,|y|)$, and $\dot{z}=\theta(z)$ describes the flow on the local center manifold $\{x=0, y=0\}$ with $\theta$ being $C^{r-1}$. Note that the admissible coordinate directly implies the $C^{r-1}$ "straight" invariant foliations on the center-stable and center-unstable manifolds as $W^{cs}=\bigcup_{|z_0|\ll 1}\{z=z_0, y=0\}$ and $W^{cu}=\bigcup_{|z_0|\ll 1}\{z=z_0, x=0\}$, respectively. In particular, when $z_0=0$, which is $W^{u}_{loc}$, it is analogous to $W^{uu}_{loc}=\{\varphi(0,0,y)=0\}$ for the hyperbolic exponential expansion (see Fig. 1.4). The foliations will be very useful in § 5 in establishing the bifurcation equations and the homoclinic and heteroclinic connections between bifurcated equilibria.

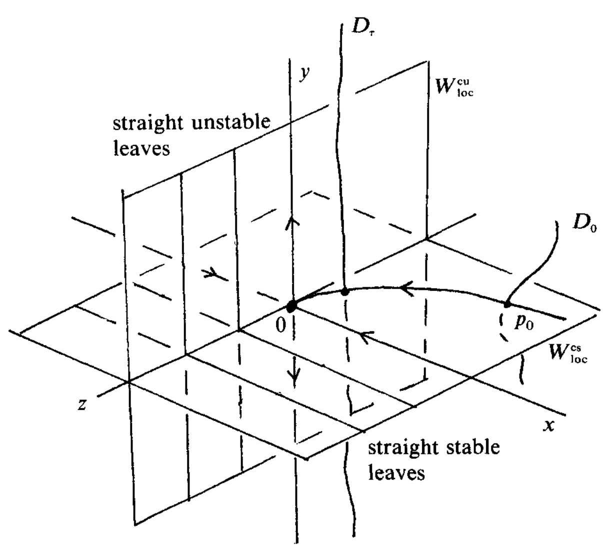


FIG. 1.4. *The straight foliations and the strong $\lambda$-lemma.*

In § 3, we will formulate the Sil'nikov solution according to its center flow. Roughly speaking, for every local center flow $z^c(t)$ (i.e., $\dot{z}^c(t)=\theta(z^c(t))$) defined on the positive maximum interval $0\leqq t<\tau^c$ with respect to a fixed small neighborhood of the equilibrium point, there exists a unique local flow $(x, y, z)(t)$ satisfying $x(0)=x_0$, $y(\tau)=y_1$, and $z(0)=z_0=z^c(0)$ (or $z(\tau)=z_1=z^c(\tau)$) for a given triplet $(\tau, x_0, y_1)$ with $0\leqq\tau<\tau^c$ and small $|x_0|$ and $|y_1|$. Moreover, this solution can be expanded according to its center flow in the sense that $z(t)=z^c(t)+R(t)$ and the exponential bounds $|D^ix(t)|\leqq K e^{\lambda t}$, $|D^iy(t)|\leqq K e^{\mu(t-\tau)}$, and $|D^iR(t)|\leqq K e^{\lambda t+\mu(t-\tau)}$ are valid for all $0\leqq t\leqq\tau<\tau^c$, and all sufficiently small $|x_0|$ and $|y_1|$, where $\lambda_0 < \lambda < 0 < \mu < \mu_0$ and $K$ are constants independent of $t$, $\tau$, $\tau^c$, $x_0$, $y_1$, $z_0$ (or $z_1$), and the derivatives are taken in $t$, $\tau$, $x_0$, $y_1$, and $z_0$ (or $z_1$) up to the orders $i \leqq r-4$. However, the regularity $r$ here must be finite if we want to find those constants. For the precise statement, see Lemma 3.1. The proof is directly based on the uniform contraction mapping principle and has much in common with the existence, uniqueness, and continuous dependence of the Sil'nikov problem (or the initial value problem) for the hyperbolic case. To obtain the exponential bounds, certain weighted Banach spaces are used for functions over $0 \leqq t \leqq \tau$ that are bounded up to some weighted exponential scales—for instance, $e^{-\lambda t}$, $e^{-\mu(t-\tau)}$, and $e^{-\lambda t-\mu(t-\tau)}$ are used for $x(t)$, $y(t)$, and $R(t)$, respectively.

In § 4, we will prove the strong $\lambda$-lemma, Lemma 4.1, which is heuristically illustrated in Fig. 1.4. Roughly speaking, it states that if a trajectory on the center-stable manifold approaches the equilibrium point $u=0$, then for every $C^{r-3}$ $n$-dimensional disc $D_0$ that transversely intersects the center-stable manifold through a point of the trajectory, the image $D_\tau$ under the time $\tau$ mapping of the flow converges to the unstable manifold as $\tau \to +\infty$ in the $C^{r-3}$-topology. Moreover, the convergence rate is the same as that of the center trajectory, but the tangent space, being normal to the center-stable manifold, "stretches" exponentially rapidly. Note that when the disc $D_0$ happens to be one of the straight leaves $\{z=z_0, x=0\}$ on the center unstable manifold, the preceding description makes perfect sense, since in terms of the straight foliation mentioned above, $D_\tau = \{z^c(t, z_0), x=0\}$ locally.

In § 5, we will first classify nondegenerate homoclinic orbits in general by the strong $\lambda$-lemma and just consider three types of nonhyperbolic equilibria in particular: saddle-node, transcritical, and pitchfork. The generic codimension-2 bifurcation unfoldings are obtained through a modified Lyapunov-Schmidt reduction for the Poincaré map in the Sil'nikov variable (see Theorems 5.1-5.3 for the precise statements). One parameter here governs the bifurcations of the equilibria and the other the breaking of the homoclinic orbits. Due to the lack of oscillatory structures for the center flows, all the dynamics considered are nonchaotic. The chaotic bifurcations of a homoclinic orbit to a Hopf equilibrium, or of a transverse homoclinic point to a nonhyperbolic fixed point of a map, are not studied here mainly because many difficulties in analyzing chaos are still under investigation.

As we have seen, the next three sections consist of the foundation of our nonlinear and nonhyperbolic analysis. It allows us to reduce a complex problem simply to an individual case study on the local center manifold. Then the dominant role of the center flow in the bifurcations theory should prevail as usual. Unexpectedly, however, the exponential expansion and the strong $\lambda$-lemma for the nonhyperbolic case are much more easily and directly obtained than their hyperbolic counterparts. Of course, to see this we need to compare the proofs with Deng (1988a, d). Also, for the answers that cannot be included in this Introduction, we will refer our readers to Chow, Deng, and Terman (1987), Deng (1988e), and Chow, Deng, and Fiedler (1988) for homoclinic and heteroclinic bifurcation problems with hyperbolic equilibria, which have much in common with the spirit of § 5. For another important topic that is not treated in this paper, we will refer the reader to Schecter (1987) for an example of the saddle-node homoclinic bifurcation in $\mathbb{R}^2$, and to Dangelmayr, Armbruster, and Neveling (1985) and Ju (1988) for an example of the pitchfork homoclinic bifurcation in $\mathbb{R}^2$ as well. The former models the dynamics of the forced Josephson junction, and the latter, the laser with a saturable absorber.

Let us conclude this section with some remarks about our motivations. Luk'yanov (1982) and Schecter (1987) first studied the homoclinic bifurcation with a saddle-node equilibrium point for planar systems. Chow and Lin (1988) then generalized their results to any finite-dimensional case, using a great variety of techniques, including exponential dichotomy, Melnikov function, smooth foliation, and Sil'nikov's central ideal, called parametrization in their terminology, which also gives rise to the emergence of our method presented here. They found the periodic orbit in a rather geometrical way, not by a Lyapunov–Schmidt reduction technique as we do here. Using their different method, they were the first to realize the necessity of the admissible normal form for the exponential expansion. However, Chow and Lin's technique for the expansion is applicable for only zero-center eigenvalues at the bifurcation point, ruling out the important class of Hopf bifurcation, where the center eigenvalues are nonzero in general. Also, as discussed in Chow, Deng, and Fiedler (1988), to use exponential dichotomy together with the Melnikov function is essentially to ignore the homoclinic doubling bifurcations that are very likely to occur when the center manifold is two-dimensional or the homoclinic orbit is degenerate. Homoclinic doubling bifurcations do occur in some hyperbolic cases (see, e.g., Yanagida (1987) and Chow, Deng, and Fiedler (1988)). Moreover, instead of separated apparatus to the homoclinic and periodic bifurcations, only one bifurcation equation derived from the Lyapunov–Schmidt reduction is needed in our strategy. More important, our main objectives in this paper are to unify the method for homoclinic bifurcation problems regardless of the nature of the equilibria, and to lay the foundation for our future investigations into other, more complicated problems, in particular, chaotical problems.

**2. Admissible variables.** From now on, we will let $u=(x, y, z)$ with $x\in\mathbb{R}^m$, $y\in\mathbb{R}^n$, $z\in\mathbb{R}^l$, and $m+n+l=d$ such that

$$DF(0)=\operatorname{diag}(A, B, C),$$

where $A$ and $B$ have the same meanings as in § 1 and all the eigenvalues of the $l\times l$ matrix $C$ lie on the imaginary axis of the complex plane. Let $W^{cs}$, $W^{cu}$, and $W^{c}$ denote an $(m+l)$-dimensional center-stable manifold, an $(n+l)$-dimensional center-unstable manifold, and an $l$-dimensional center manifold, respectively. Then, by the theory of invariant manifolds (see Hirsch, Pugh, and Shub (1977), Vanderbauwhede and van Gils (1987), Wells (1976), and Chow and Lu (1988)), these are $C^r$ manifolds with $r\neq\infty$. Moreover, up to a $C^r$ change of variables, we may assume

$$W^{cs}=\{y=0\},\quad W^{cu}=\{x=0\},\quad W^{c}=\{x=0, y=0\}$$

locally. Also, when (1.1) is written in terms of such a $C^r$ coordinate, it takes the form

$$\dot{x}=Ax+f(x, y, z),\quad \dot{y}=By+g(x, y, z),\quad \dot{z}=\theta(z)+h(x, y, z) \tag{2.1}$$

with $D\theta(0)=C$ and the nonlinear higher-order terms satisfying

$$\begin{aligned} &f(0, y, z)=0,\quad g(x, 0, z)=0,\quad h(0, 0, z)=0,\\ &Df(0, 0, 0)=0,\quad Dg(0, 0, 0)=0,\quad Dh(0, 0, 0)=0. \end{aligned} \tag{2.1a}$$

Moreover, the functions $\theta, f, g$, and $h$ are $C^{r-1}$.

DEFINITION 2.1. The coordinate $(x, y, z)$ is admissible if, in addition to condition (2.1a), we have $h(0, y, z)=h(x, 0, z)=0$. A change of variables is admissible if the new variables are admissible.

LEMMA 2.2. *There exists a $C^{r-2}$ admissible change of variables for* (2.1).

*Proof.* The proof is based on an idea by Ovsyannikov and Sil'nikov (1986) and Deng (1988c), using the Center Manifold Theorem. Let us rewrite (2.1) satisfying

(2.1a) as follows:

$$\begin{aligned} \dot{x} &= Ax + f_1(x, y, z)x, \\ \dot{y} &= By + g_1(x, y, z)y, \\ \dot{z} &= Cz + \bar{\theta}(z) + h_1(x, y, z)x + h_2(x, y, z)y, \end{aligned} \tag{2.2}$$

where $\bar{\theta}(z) = \theta(z) - Cz$ is $C^{r-1}$, but $f_1$, $g_1$, $h_1$, and $h_2$ are $C^{r-2}$. Consider a change of variables

$$x = x, \quad y = y, \quad \zeta = z - p(\zeta, x)x - q(\zeta, y)y$$

with some $C^{r-2}$ functions $p$ and $q$ to be determined satisfying $p(0,0) = 0$ and $q(0,0) = 0$. Note that such a change of variables necessarily preserves condition (2.1a). Substituting the new variables $\zeta$ into (2.2), we have

$$\begin{aligned} \dot{\zeta} &= \dot{z} - \dot{p}x - p\dot{x} - \dot{q}y - q\dot{y} \\ &= C(\zeta + px + qy) + \bar{\theta}(\zeta + px + qy) + h_1x + h_2y - \dot{p}x - p(Ax + f_1x) - \dot{q}y - q(By + g_1y), \end{aligned}$$

where $h_1$, $h_2$, $f_1$, and $g_1$ are understood in the new variables $x$, $y$, and $\zeta$. Also, $\dot{p}$ and $\dot{q}$ here are derivatives along the solutions of the new equations. For this reason $p$ and $q$ may also be regarded as variables from $\mathbb{R}^{l\times m}$ and $\mathbb{R}^{l\times n}$, respectively.

Let

$$\bar{\theta}(\zeta + px + qy) - \bar{\theta}(\zeta) = \theta_1(x, y, \zeta, p, q)px + \theta_2(x, y, \zeta, p, q)qy$$

for some $C^{r-2}$ functions $\theta_1$ and $\theta_2$. It is easy to see that

$$\theta_1(0,0,0,0,0) = \theta_2(0,0,0,0,0) = 0.$$

Moreover,

$$\theta_1(x, 0, \zeta, p, q) = \theta_1(x, 0, \zeta, p, 0), \qquad \theta_2(0, y, \zeta, p, q) = \theta_2(0, y, \zeta, 0, q). \tag{2.3}$$

Collecting like terms in the equation for $\dot{\zeta}$ above yields

$$\begin{aligned} \dot{\zeta} = C\zeta + \bar{\theta}(\zeta) &+ [Cp + \theta_1 p - \dot{p} - pA - pf_1 + h_1]x \\ &+ [Cq + \theta_2 q - \dot{q} - qB - qg_1 + h_2]y. \end{aligned}$$

Now it is easy to see that, for the new variable to be admissible, it suffices for the first bracket term above to be zero when $y = 0$ and for the second to be zero when $x = 0$. For the first case, this is equivalent to saying that on the center-stable manifold $y = 0$ the following coupled equations must be satisfied:

$$\begin{aligned} \dot{x} &= Ax + f(x, 0, \zeta + px), \\ \dot{\zeta} &= C\zeta + \bar{\theta}(\zeta), \\ \dot{p} &= Cp - pA + \theta_1(x, 0, \zeta, p, q)p - pf_1(x, 0, \zeta + px) + h_1(x, 0, \zeta + px). \end{aligned} \tag{2.4}$$

Note that these equations do not actually depend on the $q$ variable since $\theta_1(x, 0, \zeta, p, q) = \theta_1(x, 0, \zeta, p, 0)$ according to (2.3). The linearization of this vector field of $(m + l + ml)$ equations at the trivial equilibrium point of the origin has a lower triangular form whose diagonal blocks consist of the stable matrix $A$, the center matrix $C$, and the matrix for the linear operator $Lp = Cp - pA$ for all $l \times m$ matrices $p$. Thus the set of eigenvalues consists of $\Sigma(A)$, $\Sigma(C)$, and $\Sigma(L)$, where $\Sigma(A)$ is the set of eigenvalues of a given linear operator $A$. Let us determine $\Sigma(L)$. It is easy to check directly that if $\lambda$ is an eigenvalue for the transpose matrix $A^*$ and $v$ is a corresponding eigenvector, and likewise, if $\mu \in \Sigma(C)$ with a corresponding eigenvector $w$, then $wv^*$ is an eigenvector of $L$ for the eigenvalue $\mu - \lambda$, whose real parts are positive for all $\lambda \in \Sigma(A)$ and $\mu \in \Sigma(C)$. Moreover, $\mu - \lambda$ are the only eigenvalues, since the dimension of the generalized eigenvector space corresponding to $\mu - \lambda$ is the product of those of $\lambda$ and $\mu$ (see, e.g., Lancaster (1969)).

Because of such a separation of the eigenvalues, the theory of invariant manifolds (see the same references above) applies. Thus, there exists a $C^{r-2}$ function $p=p(\zeta, x)$ whose graph gives rise to the center-stable manifold of this $(x, \zeta, p)$ system of (2.4). The same argument yields the function $q$. $\square$

*Remark* 2.3. (a) If (1.1) is $C^r$ differentiably depending on a parameter, then the admissible change of variables also smoothly varies with the parameter. This can be directly achieved by the lemma, treating the parameter as an additional center flow.

(b) Under the admissible variables, the function $\theta$ in (2.1) remains unchanged and thus is $C^{r-1}$, but $f$ and $g$ are reduced to $C^{r-2}$ and $h$ is $C^{r-3}$.

(c) As mentioned earlier in the Introduction, we obtain the "straight" invariant foliations $W^{cs}=\bigcup_{|z_0|\ll 1}\{z=z_0, y=0\}$ and $W^{cu}=\bigcup_{|z_0|\ll 1}\{z=z_0, x=0\}$ as a corollary to the admissible change of variables. In particular, the local stable and local unstable manifolds are, respectively, the $x$-axis ($y=0$, $z=0$) and the $y$-axis ($x=0$, $z=0$) locally. For different approaches to achieve the same foliation there exists a geometric proof based on the graph transformation method by Hirsch, Pugh, and Shub (1977) extensively for diffeomorphisms and Fenichel (1979) for flows, and an analytic proof based on the variation of constants formula by Henry (1981) and later by Chow, Lin, and Lu (1988). In contrast to our approach, the admissible variables can also be obtained through their invariant foliations.

**3. Exponential expansion with center flows.** Let $z^c(t)$ with $z^c(0)=z_0$ (or $z^c(\tau)=z_1$) be any solution on the center manifold defined for $0\leqq t<\tau^c$ with respect to a certain neighborhood of the origin, where $\tau^c$ could be infinity. Given such a center solution and a triplet $(\tau, x_0, y_1)$ with $0\leqq\tau<\tau^c$, a solution $(x, y, z)(t)$ of (2.1) is called a Sil'nikov solution if the Sil'nikov conditions $x(0)=x_0$, $y(\tau)=y_1$, and $z(0)=z^c(0)$ are satisfied. This is sometimes referred to as the first type of Sil'nikov problem. The second type of Sil'nikov problem is, of course, the same as the first one except that the last condition $z(0)=z^c(0)$ is replaced by $z(\tau)=z^c(\tau)$. Indeed, they are identical up to the time reversal $(t\to -t)$. Suppose that the Sil'nikov solution exists and is unique with respect to the Sil'nikov conditions for all $0\leqq\tau<\tau^c$ and sufficiently small $|x_0|$, $|y_1|$, and $|z_0|$ (or $|z_1|$) and that the function $(x, y, z)(t)\overset{\text{def}}{=}(x, y, z)(t, \tau, x_0, y_1, z_0)$ (or $(x, y, z)(t, \tau, x_0, y_1, z_1)$) is $C^k$ for all the arguments. Then the solution is said to admit an exponential expansion of regularity $k$ if there exists a $C^k$ function $R$ of $(t, \tau, x_0, y_1, z_0)$ (or $(t, \tau, x_0, y_1, z_1)$) such that the following is satisfied:

$$z(t)=z^c(t)+R(t, \tau, x_0, y_1, z_0) \quad (\text{or } R(t, \tau, x_0, y_1, z_1)) \tag{3.1a}$$

with

$$R(0, \tau, x_0, y_1, z_0)=0 \quad (\text{or } R(\tau, \tau, x_0, y_1, z_1)=0), \tag{3.1b}$$

and there exist constants $\lambda_0<\lambda<0<\mu<\mu_0$ and $K$ independent of $t$, $\tau$, $x_0$, $y_1$ and $z_0$ (or $z_1$) such that

$$|D^i x(t)|\leqq K e^{\lambda t}, \quad |D^i y(t)|\leqq K e^{\mu(t-\tau)}, \quad |D^i R(t)|\leqq K e^{\lambda t+\mu(t-\tau)} \quad \text{for } 0\leqq t\leqq\tau, \tag{3.1c}$$

where $D^i$ denotes the $i$th derivative in all the arguments up to the order $0\leqq i\leqq k$.

For definiteness, let us consider the first type of Sil'nikov problem in the following lemma. Necessary modifications for the second type are given in the remarks after the proof.

LEMMA 3.1. *Let the variable* $(x, y, z)$ *for* (2.1) *be* $C^{r-2}$ *admissible as in Lemma* 2.2. *Let* $\beta>0$, $\lambda<0<\mu$ *be arbitrary but fixed constants satisfying* $\lambda_0+\beta(r-2)<\lambda<0<\mu<\mu_0-\beta(r-2)$ *and* $\lambda+\mu-\beta(r-2)>0$. *Then there exist positive constants* $M$, $K$, *and small* $\delta_0$ *depending on the choices of* $\beta$, $\lambda$, *and* $\mu$ *only such that as long as* $|z^c(t)|\leqq\delta_0$

*for* $0 \leqq t < \tau^c$, *there exists a unique Sil'nikov solution for all* $0 \leqq \tau < \tau^c$, $|x_0|$, $|y_1|$, *and* $|z_0| \leqq \delta_0$, *which admits a* $C^{r-3}$ *exponential expansion. In particular, for the solution itself the constant* $K$ *in* (3.1c) *can be replaced by* $2M\delta_0$.

*Proof.* The proof is based on the uniform contraction mapping principle. Let

$$\sigma = \min\{\lambda - \lambda_0 - \beta(r-2), \mu_0 - \mu - \beta(r-2), \lambda + \mu - \beta(r-2)\}$$

and $M$ be large enough so that

$$|e^{At}| \leqq M\, e^{(\lambda_0+\beta)t} \quad \text{for } t \geqq 0, \qquad |e^{Bt}| \leqq M\, e^{(\mu_0-\beta)t} \quad \text{for } t \leqq 0,$$
$$|e^{Ct}| \leqq M\, e^{\beta|t|} \quad \text{for all } t.$$

Let $\|f\|$ be the $C^{r-2}$ norm of a given function $f$ in the neighborhood of the origin for which the admissible form (2.1) is valid. Let

$$\delta_0 = \left[\frac{4M^2}{\sigma}(\|f\| + \|g\| + \|\theta\| + \|h\|)\right]^{-1}. \tag{3.2}$$

Let $R(t) = z(t) - z^c(t)$. Let us consider the equations for $x$, $y$, and $R$. We have

$$\dot{R}(t) = \dot{z}(t) - \dot{z}^c(t) \overset{\text{def}}{=} CR(t) + L(z^c(t), R(t))R(t) + h(x(t), y(t), R(t) + z^c(t)),$$

where

$$L(z^c, R) = \int_0^1 \frac{d\bar{\theta}}{dz}(sR + z^c)\, ds$$

with $\bar{\theta}(z) = \theta(z) - Cz$ as in (2.1). Now, it is easy to check that the existence of the Sil'nikov solution is equivalent to the existence of the solutions to the following integral equations:

$$\begin{aligned} x(t) &= e^{At}x_0 + \int_0^t e^{A(t-s)} f(x, y, R + z^c)\, ds, \\ y(t) &= e^{B(t-\tau)}y_1 + \int_\tau^t e^{B(t-s)} g(x, y, R + z^c)\, ds, \\ R(t) &= \int_0^t e^{C(t-s)}[L(z^c, R)R + h(x, y, R + z^c)]\, ds, \end{aligned} \tag{3.3}$$

where $x = x(s)$, $y = y(s), \cdots$, etc., in the integrals. Let $\Sigma$ be the set of continuous functions $(x, y, R)(t)$ defined on $0 \leqq t \leqq \tau < \tau^c$ satisfying $|x(t)| \leqq 2\delta_0 M\, e^{\lambda t}$, $|y(t)| \leqq 2\delta_0 M\, e^{\mu(t-\tau)}$, and $|R(t)| \leqq 2\delta_0 M\, e^{\lambda t + \mu(t-\tau)}$. Equip $\Sigma$ with a weight norm

$$\|(x, y, R)\|_\Sigma = \sup_{0 \leqq t \leqq \tau} (|x(t)\, e^{-\lambda t}| + |y(t)\, e^{-\mu(t-\tau)}| + |R(t)\, e^{-\lambda t - \mu(t-\tau)}|).$$

Let $T$ denote the operator defined by the right-hand side of the integral equation (3.3). Then $T$ is a contraction mapping on $\Sigma$ with the contraction constant

$$\rho = \frac{2M^2}{\sigma}(\|f\| + \|g\| + \|\theta\| + \|h\|)\delta_0 = \frac{1}{2}.$$

Indeed, for $(x, y, R) \in \Sigma$ and $(\bar{x}, \bar{y}, \bar{R}) = T(x, y, R)$, we have

$$\begin{aligned} |\bar{x}(t)| &\leqq \delta_0 M\, e^{\lambda t} + \int_0^t M\, e^{(\lambda-\sigma)(t-s)} 4M^2\delta_0^2 \|f\|\, e^{\lambda s}\, ds \\ &\leqq \delta_0 M\, e^{\lambda t} + \frac{4M^3\|f\|}{\sigma}\delta_0^2\, e^{\lambda t} \leqq 2\delta_0 M\, e^{\lambda t}, \end{aligned}$$

because of (3.2) and $|f|=O(|x|+|y|+|z|)|x|$ and $\lambda_0+\beta\leqq\lambda-\sigma$. Similarly, we have $|\bar{y}|\leqq 2\delta_0 Me^{\mu(t-\tau)}$. Moreover,

$$|\bar{R}(t)|\leqq\int_0^t Me^{\beta(t-s)}[4M^2\|\theta\|\delta_0^2\, e^{\lambda s+\mu(s-\tau)}]\, ds,$$

since $|L|=O(|R|+|z^c|)$, $|h|=O(|x\|y|)$. Thus,

$$\begin{aligned}|\bar{R}(t)|&\leqq\frac{4M^3(\|\theta+\|h\|)\delta_0^2}{\lambda+\mu-\beta}\, e^{\lambda t+\mu(t-\tau)}\\&\leqq 2\delta_0 Me^{\lambda t+\mu(t-\tau)}\end{aligned}$$

because of (3.2) and $\lambda+\mu-\beta\geqq\sigma$. Hence, $T(\Sigma)\subset\Sigma$. To show $T$ is contractive, observe the following trivial estimates:

$$|f(x,y,R+z^c)-f(\tilde{x},\tilde{y},\tilde{R}+z^c)|\leqq 2M\delta_0\|f\|\, e^{\lambda t}\|(x,y,R)-(\tilde{x},\tilde{y},\tilde{R})\|_\Sigma,$$
$$|g(x,y,R+z^c)-g(\tilde{x},\tilde{y},\tilde{R}+z^c)|\leqq 2M\delta_0\|g\|\, e^{\mu(t-\tau)}\|(x,y,R)-(\tilde{x},\tilde{y},\tilde{R})\|_\Sigma$$

for $(x,y,R)$ and $(\tilde{x},\tilde{y},\tilde{R})\in\Sigma$, since $|f|=O(|x|+|y|+|z|)|x|$ and $|g|=O(|x|+|y|+|z|)|y|$. Also

$$\begin{aligned}&|L(z^c,R)R+h(x,y,R+z^c)-(L(z^c,\tilde{R})\tilde{R}+h(\tilde{x},\tilde{y},\tilde{R}+z^c))|\\&\quad\leqq 2M\delta_0(\|\theta\|+\|h\|)\, e^{\lambda t+\mu(t-\tau)}\|(x,y,k)-(\tilde{x},\tilde{y},\tilde{R})\|_\Sigma\end{aligned}$$

since $h(x,y,z)=O(|x||y|)$. Now, similarly to the estimates for $\bar{x}$, $\bar{y}$, and $\bar{R}$ above, it is easy to check that

$$\|(\bar{x},\bar{y},\bar{R})-(\bar{\tilde{x}},\bar{\tilde{y}},\bar{\tilde{R}})\|_\Sigma\leqq\rho\|(x,y,R)-(\tilde{x},\tilde{y},\tilde{R})\|_\Sigma$$

with $\rho=\frac{1}{2}$ as above. Thus, by the Uniform Contraction Mapping Theorem, there exists a unique function $\zeta^*$ in $\Sigma$ such that $\zeta^*=T(\zeta^*)$. Moreover, $\zeta^*(t)\overset{\text{def}}{=}\zeta^*(t,\tau,x_0,y_1,z_0)$ is $C^{r-3}$ in the parameters $x_0$, $y_1$, and $z_0$ since the admissible change of variables is $C^{r-2}$. To show it is also $C^{r-3}$ at $\tau=\tau_0<\tau^c$, we simply replace the interval $[0,\tau_0]$ on which all the functions of the space $\Sigma$ are defined by a larger one $[0,\tau_0+\varepsilon]$ and then show that the same operator $T$ has a unique fixed point $\zeta^*(t,\tau,x_0,y_1,z_0)$ in the new function space. Thus $\zeta^*(t,\tau,x_0,y_1,z_0)$ is $C^{r-3}$ in $\tau_0$ as well. The differentiability in time $t$ simply follows the standard argument for the smoothness of solutions to the initial value problem found in textbooks, for instance, Hale (1978) and Irwin (1980). Since all the partial derivatives are continuous, $\zeta^*$ is $C^{r-3}$ differentiable by a standard fact from calculus.

The estimates for the derivatives follow the same technique as for the exponential bounds above. To begin with, we observe first that the growth rate for all the variational flow $D^jz^c(t,z_0)$ on the center manifold cannot be greater than $O(e^{j\beta|t|})$ for all $0\leqq j\leqq r-3$ and all $t$ as defined. Indeed, this can be proved directly by using the same arguments as above, using an appropriate weight norm for all the functions $z(t)$, i.e., the maximum of the exponentially scaled function $|e^{-j\beta t}z(t)|$ over $0\leqq t<\tau^c$ (see also, e.g., Vanderbauwhede and van Gils (1987)). When these estimates are used for the variational integral equations for the mixed system obtained by differentiating (3.3), we clearly see that the corresponding derivatives on $x$, $y$, and $R$ will not exceed the orders $e^{(\lambda_0+(j+1)\beta)t}$, $e^{(\mu_0-(j+1)\beta)(t-\tau)}$, and $e^{(\lambda_0+(j+1)\beta)t+(\mu_0-(j+1)\beta)(t-\tau)}$, respectively. Again, the desired estimates are obtained by the uniform contraction mapping principle together with some appropriate weighted Banach spaces of functions. We note that all the contraction constants are the same number $\rho=\frac{1}{2}$ but the constant $2M\delta_0$ may vary for each $0\leqq j\leqq r-3$. By the choices of $\lambda$, $\mu$, $\beta$ and of a sufficiently large $K$( since $r$ is finite), the proof is completed. $\square$

*Remark* 3.2. (a) For the second type of Sil'kinov problem with $z(\tau)=z^c(\tau)=z_1$, Lemma 3.1 is still valid by changing the inequality $\lambda+\mu-\beta(r-2)>0$ into $\lambda+\mu+\beta(r-2)<0$. This is obtained by directly applying Lemma 3.1 to the time-reversed system. We will actually use this second type of lemma in § 5.

(b) Later we will also use the proven fact that the variational flow $D^j z^c(t, z_0)$ cannot grow faster than $O(e^{j\beta t})$ for all $0\leqq t<\tau^c$ and $0\leqq j\leqq r-3$. More precisely, $|D^j z^c(t, z_0)|\leqq K e^{j\beta t}$ for $0\leqq t<\tau^c$, where the constant $K$ may be chosen as the same one in the lemma.

(c) All the results above can also be easily extended to systems depending on parameters by the same modification as in § 2 (see Remark 2.3(a))—that is, by considering the parameters as additional center directions.

**4. Strong $\lambda$-lemma.** We will continue to use the same notation and results from the two previous sections. In this section we consider the inclination behaviors of subsets when carried by the local flow forward in time. To be precise, let $D^n$ be an $n$-dimensional $C^{r-3}$ manifold intersecting the center-stable manifold $W^{cs}$ transversely at a point $p_0=(x_0, 0, z_0)$. Thus $D^n$ can be written as the graph of a $C^{r-3}$ function $(x, z)=(p, q)(y)$ over a small $\delta$-box $B^n(\delta)=\{y\,||y|\leqq\delta\}$ on the $y$-axis. Let $D^n_\tau$ denote the connected image of $D^n$ in the $\delta_0$-box $B^d(\delta_0)$ of the origin under the time $\tau$ mapping of the flow. We are interested in the asymptotic inclination behavior of $D^n_\tau$ as $\tau\to+\infty$ under the assumption that the center-stable trajectory through the point of intersection $p_0$ converges to the origin as $\tau\to+\infty$. As observed in the introduction, when $D^n$ happens to be one of the straight leaves $\{z=z_0, x=0\}$ on the center-unstable manifold, the asymptotic inclination behavior is self-evident: $D^n$ converges to the unstable manifold, with the tangent spaces identically equal to each other, so long as the center trajectory goes to the origin (cf. Fig. 1.4). But, in general, $D^n_\tau$ is said to be $C^{r-3}$ $\varepsilon$-close to the unstable manifold $W^u_{\text{loc}}$ in $B^d(\delta_0)$ by an arbitrarily small number $\varepsilon>0$ if there exists a time $\tau_*(\varepsilon)$ such that for every $\tau\geqq\tau_*(\varepsilon)$, $D^n_\tau$ is the graph of a $C^{r-3}$ function $(x, z)=(p_\tau, q_\tau)(y)$ over $|y|\leqq\delta_0$ satisfying $\|(p_\tau, q_\tau)\|<\varepsilon$, where $\|\cdot\|$ denotes the usual $C^{r-3}$ norm for all $C^{r-3}$ functions in $B^n(\delta_0)$. Note that $\delta_0$ is fixed but $0<\delta\leqq\delta_0$ is not, in general. Now we have Lemma 4.1.

LEMMA 4.1 (strong $\lambda$-lemma). *Given an $n$-dimensional $C^{r-3}$ disc $D^n$ transversely intersecting the center-stable manifold at a point $p_0$, if the solution through $p_0$ converges to the origin as $\tau\to+\infty$, then the image $D^n_\tau$ in $B^d(\delta_0)$ under the time $\tau$-mapping of the flow is $C^{r-3}$ $\varepsilon$-close to the unstable manifold. Moreover, the tangent space of $D^n_\tau$ at $p_\tau$ is exponentially close to the tangent unstable manifold at the origin.*

*Proof.* The idea of the proof is to use the Sil'nikov solution to express $D^n_\tau$, then the Implicit Function Theorem together with the exponential expansion to obtain the graph representation of $D^n_\tau$, and last, the expansion to estimate the rate of convergence.

Without loss of generality, we may assume that the center trajectory through the projection point $(0, 0, z_0)\in W^c_{\text{loc}}$ of the base point $p_0=(x_0, 0, z_0)$ converges to the origin forward in time. Indeed, this fact simply follows the straight invariant foliation on the center-stable manifold due to the admissible variable and the assumption. Thus, let us assume $|z^c(t, z_0)|\leqq\delta_0/4$ for all $t\geqq 0$.

By definition,

$$D^n_\tau=\{(x_1, y_1, z_1)\,|\,(x_1, y_1, z_1)=(x, y, z)(\tau, 0, p(y_0), y_0, q(y_0))$$
$$\text{for those } |y_0|\leqq\delta \text{ so that } |x_1|, |y_1| \text{ and } |z_1|\leqq\delta_0\}.$$

To use the Sil'nikov solution for the desired alternative representation of $D^n_\tau$, we need to estimate first the definition time for the center flow. Since $|z^c(t, q(0))|\leqq\delta_0/4$ with

$q(0)=z_0$ for all $t \geqq 0$, by the continuous dependence on the initial data we have that for every $\tau > 0$ there exists a small number $\gamma(\tau)$ such that $|z^c(t, q(y_0))| \leqq \delta_0/2$ for all $|y_0| \leqq \gamma(\tau)$ and $0 \leqq t \leqq \tau$. In fact, we can obtain a better approximation $\gamma(\tau) = (\delta_0/4K\|q\|)\, e^{-\beta\tau}$ by the following estimate:

$$|z^c(t, q(y_0))| \leqq |z^c(t, q(y_0)) - z^c(t, z_0)| + |z^c(t, z_0)|$$
$$\leqq K\, e^{\beta t} \|q\| \gamma(\tau) + \frac{\delta_0}{4} \leqq \frac{\delta_0}{2}$$

provided $t \leqq \tau$. Thus, the exponential expansion implies that for $|y_0| \leqq \gamma(\tau)$

$$y_0 = y(0, \tau, p(y_0), y_1, q(y_0))$$

holds true provided $|y_1| \leqq \delta_0$. By comparing the two sides of this formula we can easily show by the exponential bounds (3.1c) with $K = 2M\delta_0$ when $i = 0$ that for $\tau \geqq \tau_1 \stackrel{\text{def}}{=} 1/(\mu - \beta) \ln (8MK\|q\|)$, the relation $|y_1| \leqq \delta_0$ must imply $|y_0| < \gamma(\tau)$. Thus, in terms of the Sil'nikov solution, $D_\tau^n$ can be written as

$$D_\tau^n = \{(x_1, y_1, z_1) \,|\, x_1 = x(\tau, \tau, p(y_0), y_1, q(y_0)), y_0 = y(0, \tau, p(y_0), y_1, q(y_0)),$$
$$z_1 = z(\tau, 0, p(y_0), y_0, q(y_0)), \text{for those } |y_0| \leqq \gamma(\tau) \text{ such that } |x_1|, |y_1|, |z_1| \leqq \delta_0\}.$$

To express $D_\tau^n$ as the graph of a $C^{r-3}$ function over $|y_1| \leqq \delta_0$, we use the Implicit Function Theorem to solve the equation

$$\Psi_\tau(y_0, y_1) \stackrel{\text{def}}{=} y_0 - y(0, \tau, p(y_0), y_1, q(y_0)) = 0$$

for $y_0$ in terms of $y_1$. Since $\Psi_\tau(0, 0) = 0$ and the Jacobian

$$\left|\frac{\partial \Psi_\tau}{\partial y_0}\right| \geqq 1 - \left|\frac{\partial}{\partial y_0} y(0, \tau, p(y_0), y_1, q(y_0))\right| \geqq 1 - K(\|p\| + \|q\|)\, e^{-\mu\tau} > \frac{1}{2}$$

for $\tau \geqq \tau_2 \stackrel{\text{def}}{=} (1/\mu) \ln (2K(\|p\| + \|q\|))$ and all $|y_1| \leqq \delta_0$ and $|y_0| \leqq \delta$, we can solve $y_0 = \psi_\tau(y_1)$ from the equation for sufficiently small $|y_1|$. Moreover, $|\psi_\tau(y_1)| \leqq 2M\delta_0\, e^{-\mu\tau}$. Note that the last inequality actually implies that the domain of the solution $\psi_\tau$ can be extended to the entire $\delta_0$-box, while still maintaining the constraint $|y_0| \leqq \gamma(\tau)$ for all $\tau \geqq \tau_1 + \tau_2$. Furthermore, $\|\psi_\tau\| \leqq 2K\, e^{-\mu\tau}$. Let $p_\tau(y_1) = x(\tau, \tau, p(\psi_\tau(y_1)), y_1, q(\psi_\tau(y_1)))$ and $q_\tau(y_1) = z(\tau, 0, p(\psi_\tau(y_1)), \psi_\tau(y_1), q(\psi_\tau(y_1)))$ over $|y_1| \leqq \delta_0$. This completes the second part of the proof.

Last, let us estimate the rate of convergence. It is obvious that we have $\|p_\tau\| = O(e^{\lambda\tau}) < \varepsilon$ for large $\tau$. Moreover, by the expansion and Remark 3.2(b) on the growth rate of the center flows,

$$|q_\tau(y_1)| \leqq |z^c(\tau, q(\psi_\tau(y_1)))| + |R(\tau, \tau, p(\psi_\tau(y_1)), y_1, q(\psi_\tau(y_1)))|$$
$$\leqq |z^c(\tau, q(0))| + \left\|\frac{\partial z^c}{\partial z_0}(\tau, \cdot)\right\| \|q\| |\psi_\tau(y_1)| + 2M\delta_0\, e^{\lambda\tau}$$
$$\leqq \frac{\varepsilon}{2} + K\, e^{(r-3)\beta\tau} \|q\|\, e^{-\mu\tau} + 2M\delta_0\, e^{\lambda\tau}$$
$$\leqq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon$$

for sufficiently large $\tau$, provided $|z^c(\tau, q(0))| < \varepsilon/2$. The last inequality is true since $z^c(t, q(0)) \to 0$ as $t \to +\infty$. Finally, since $z^c(\tau, q(0))$ does not depend on $y_1$, all the derivatives for $p_\tau$ and $q_\tau$ in $y_1$ up to the order $r-3$ are exponentially small. $\square$

As we know, center-stable and center-unstable manifolds are not necessarily unique. However, we have the following corollary that will also be used in § 5.

COROLLARY 4.2. *Two given center-stable manifolds have the same tangent space at any common point whose trajectory converges to the equilibrium.*

*Proof.* Let $W_1^{cs}$ and $W_2^{cs}$ be two center-stable manifolds intersecting at a point $p$. If they do not have the same tangent space at $p$, then there must be at least one tangent direction, say $v$, of $W_1^{cs}$ normal to $W_2^{cs}$. Using $W_2^{cs}$ as the center-stable manifold in Lemma 4.1, the limiting direction of $v$ at the origin must be contained in the tangent unstable manifold at the origin that is normal to the center eigenvector space. This is a contradiction since both $W_1^{cs}$ and $W_2^{cs}$ have the same tangent space—the center eigenvector space—at the origin. □

**5. Homoclinic bifurcations with nonhyperbolic equilibria.** In this section we will classify homoclinic orbits with nonhyperbolic equilibria according to the strong inclination property from the previous section, and consider specifically three basic types of codimension-1 nonhyperbolic equilibria that undergo the saddle-node, transcritical, and pitchfork bifurcations, respectively. We will state and prove the corresponding theorems for the generic two-parameter unfoldings.

For definiteness, from now on we will explicitly assume that the vector field $F = F(u, \alpha)$ of (1.1) depends on two parameters $\alpha = (\alpha_1, \alpha_2)$ in the $C^r$ fashion. Also, for directness we will assume the parameter is generic in the sense that $\alpha_1$ governs the bifurcations of the equilibria while $\alpha_2$ governs the transverse crossing of the center-unstable manifold and the stable manifold. This will be made precise as we proceed.

In this paper, we consider only the bifurcation of those homoclinic orbits at the bifurcation point $\alpha = 0$, $\Gamma = \Gamma(t)$ that are contained in either $W^{cu} \cap W^s$ or $W^{cs} \cap W^u$. Up to the time reversal $(t \to -t)$ we will always assume the first case. We will also assume that the homoclinic orbit is asymptotically tangent to the center eigenvector space of the linearization $DF(0, 0)$ as $t \to -\infty$. A homoclinic orbit $\Gamma$ satisfying these two conditions is called nondegenerate if, in addition, there exists an $n$-dimensional $C^1$ disc $D^n$ on the center-unstable manifold such that as a submanifold of $W^{cu}$ it transversely intersects the center manifold, while as a submanifold of $\mathbb{R}^d$ it transversely intersects the center-stable manifold $W^{cs}$ at a point from $\Gamma$. Observe that when the center dimension $l$ is 1, the nondegeneracy of a $\Gamma$ is equivalent to the transverse intersecting of the center-unstable and center-stable manifolds along $\Gamma$, i.e.,

$$\text{(5.1a)} \qquad \operatorname{span}\{T_pW^{cs}, T_pW^{cu}\} = \mathbb{R}^d \quad \text{for all } p \in \Gamma,$$

where $T_pW$ means the tangent space of a smooth manifold $W$ at a base point $p$ (see Fig. 5.1). Also note that, since the tangent spaces of center-stable and center-unstable manifolds are uniquely defined along a homoclinic orbit $\Gamma$ by Corollary 4.2, the definition of nondegeneracy is independent of the choices of these manifolds (for a different justification, see Chow and Lin (1988)).

Let $\Sigma$ be any $(d-1)$-dimensional small and closed Poincaré cross section transverse to the homoclinic orbit $\Gamma$. Let $W^{cu}(\alpha)$ and $W^s(\alpha)$ denote the parametrically dependent center-unstable and stable manifolds that also vary with the parameter $\alpha$ in the $C^r$ fashion. Let $d(\alpha_1, \alpha_2)$ be the distance between $\Sigma \cap W^{cu}(\alpha)$ and $\Sigma \cap W^s(\alpha)$, where $d$ is continuous and $d(0, 0) = 0$, which represents the existence of the original homoclinic orbit $\Gamma$. The crossing of the center-unstable and stable manifolds is said to be transverse along the $\alpha_2$-direction if the following condition is satisfied:

$$\text{(5.1b)} \qquad \lim_{\alpha_2 \to 0} \frac{d(0, \alpha_2)}{|\alpha_2|} \neq 0.$$

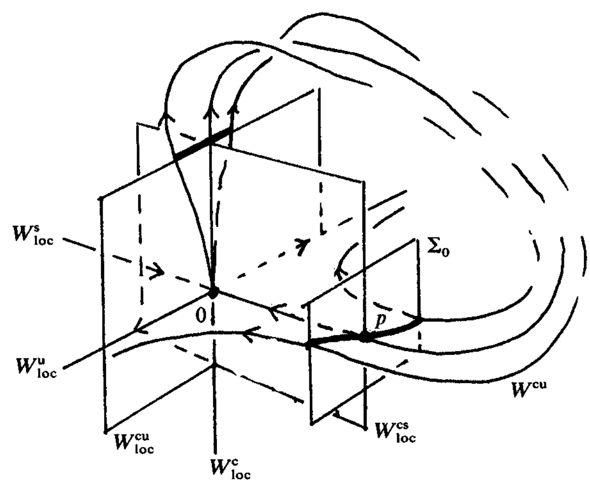


FIG. 5.1. *The phase portrait of a nondegenerate homoclinic orbit.*

Since the flow from one Poincaré cross section to another gives rise to a diffeomorphism, this property of nonzero limit, and consequently the definition of transverse crossing above, are independent of the choices of cross sections.

Next, let us introduce the types of codimension-1 bifurcations of the equilibria to be considered. Let us assume that the linearization $D_uF(0,0)$ has only one eigenvalue with zero real part, and that the equilibrium point $u=0$ at $\alpha=0$ is any of the following:

(5.2a) Saddle-node: $e_lD_u^2F(0,0)(e_r,e_r)>0, \quad e_lD_{\alpha_1}F(0,0)>0,$
$e_lD_{\alpha_2}F(0,0)=0,$

(5.2b) Transcritical: $F(0,\alpha)=0$ for all $\alpha$, $\quad e_lD_u^2F(0,0)(e_r,e_r)>0,$
$e_lD^2_{u\alpha_1}F(0,0)e_r>0, \quad e_lD^2_{u\alpha_2}F(0,0)e_r=0,$

(5.2c) Pitchfork: $F(0,\alpha)=0, \quad e_lD_u^2F(0,\alpha)(e_r,e_r)=0$ for all $\alpha$,
$e_lD_u^3F(0,0)(e_r,e_r,e_r)>0, \quad e_lD^2_{u\alpha_1}F(0,0)e_r>0, \quad e_lD^2_{u\alpha_2}F(0,0)e_r=0,$

where $e_r$ and $e_l$ are a right and a left eigenvector of the zero eigenvalue, respectively, with $e_l$ chosen so that $e_le_r>0$ (see Sotomayor (1974), Guckenheimer and Holmes (1983)). In terms of the center manifold, these intimidating and technical conditions can always be reinterpreted, respectively, in the following relatively explicit ways. Indeed, if we let $\theta=\theta(z,\alpha)$ be the vector field on the center manifold as in (2.1) of § 2, then we have, equivalent to (5.2a–c), respectively,

$$\theta(0,0)=\frac{\partial\theta}{\partial z}(0,0)=0, \quad \frac{\partial^2\theta}{\partial z^2}(0,0)>0, \quad \frac{\partial\theta}{\partial\alpha_1}(0,0)>0, \quad \frac{\partial\theta}{\partial\alpha_2}(0,0)=0, \tag{5.3a}$$

$$\theta(0,\alpha)=0 \text{ for all } \alpha, \quad \frac{\partial\theta}{\partial z}(0,0)=0, \quad \frac{\partial^2\theta}{\partial z^2}(0,0)>0, \tag{5.3b}$$
$$\frac{\partial^2\theta}{\partial z\,\partial\alpha_1}(0,0)>0, \quad \frac{\partial^2\theta}{\partial z\,\partial\alpha_2}(0,0)=0,$$

$$\theta(0,\alpha)=\frac{\partial^2\theta}{\partial z^2}(0,\alpha)=0 \text{ for all } \alpha, \quad \frac{\partial\theta}{\partial z}(0,0)=0, \tag{5.3c}$$
$$\frac{\partial^3\theta}{\partial z^3}(0,0)>0, \quad \frac{\partial^2\theta}{\partial z\,\partial\alpha_1}(0,0)>0, \quad \frac{\partial^2\theta}{\partial z\,\partial\alpha_2}(0,0)=0.$$

We emphasize once again the explicit roles forced on the parameters as in (5.1b) and (5.2a–c) are simply for definiteness and they can be achieved by following the procedure below. Take the first case (5.2a) as an example. $e_1 D_{\alpha_1} F(0,0) > 0$ and $e_1 D_{\alpha_2} F(0,0) = 0$ in (5.2a) can always be obtained by choosing $\alpha_1$ as the gradient direction of the scalar function $e_1 F(0, \cdot)$ and $\alpha_2$ the normal direction to the gradient vector at $\alpha = 0$. Once this is done, it only remains to check condition (5.1b) if an application problem ever arises. As another remark, let us point out that the last two bifurcations of steady states are not generically of codimension 1. They can always be perturbed into a saddle-node equilibrium point by making $\partial\theta/\partial\alpha_1(0,0) \neq 0$. But they do appear in many applications due to other mechanisms, e.g., certain types of symmetries adhering to the physical models considered will force the persistence of the transcritical or pitchfork steady states. See Guckenheimer and Holmes (1983), and in particular, Dangelmayr, Armbruster, and Neveling (1985) and Ju (1988) for specific examples. Nevertheless a two-parameter family of vector fields having a nondegenerate homoclinic orbit to a nonhyperbolic equilibrium point (of the preceding three types) is said to be generic (in our restrictive sense above and in this paper only) if up to a $C^r$ change of parameters the transverse crossing condition (5.1b) and one of the three nonhyperbolic conditions (5.2a–c) are satisfied.

We are now in a position to state our main theorems. Before doing so, we discuss preliminary results on the local bifurcations of steady states as preparation. For the original account of those results, see Sotomayor (1974). We should be aware that all the discussions are valid only in an implicitly small but fixed neighborhood of the origin $(u, \alpha) = (0, 0)$ in $\mathbb{R}^{d+2}$. Let us begin with the saddle-node case.

Solving the equation $\theta(z, \alpha) = 0$ and $\partial\theta/\partial z(z, \alpha) = 0$ simultaneously for $z$ and $\alpha_1$ by the Implicit Function Theorem (IFT), we obtain the continuation of the saddle-node equilibrium points $z = E_0(\alpha_2)$ along a curve $\alpha_1 = c_0(\alpha_2)$. Both functions are $C^{r-1}$ and satisfy

$$E_0(0) = c_0(0) = E_0'(0) = c_0'(0) = 0, \tag{5.4a}$$

because of $\partial\theta/\partial\alpha_2(0,0) = 0$. To find hyperbolic equilibria near $u = 0$ we solve the equation $\theta(z, \alpha) = 0$ alone this time for $\alpha_1$ by the IFT and obtain a $C^{r-1}$ function $\alpha_1 = \gamma(z, \alpha_2)$ satisfying

$$\begin{aligned} &\gamma(E_0(\alpha_2), \alpha_2) = c_0(\alpha_2), \quad \frac{\partial\gamma}{\partial z}(E_0(\alpha_2), \alpha_2) \equiv 0, \\ &\frac{\partial^2\gamma}{\partial z^2}(E_0(\alpha_2), \alpha_2) < 0, \quad \frac{\partial\gamma}{\partial\alpha_2}(0,0) = 0. \end{aligned} \tag{5.4b}$$

Thus, by the Taylor expansion at $z = E_0(\alpha_2)$ we can easily see that $\alpha_1 = \gamma(z, \alpha_2) < c_0(\alpha_2)$ for $z \neq E_0(\alpha_2)$. Indeed, expanding $\gamma$ at $z = E_0(\alpha_2)$ and taking the square root, we have

$$\pm\sqrt{c_0(\alpha_2) - \alpha_1} = \sqrt{-\tfrac{1}{2}\partial^2\gamma/\partial z^2(E_0(\alpha_2), \alpha_2) + O(|z - E_0(\alpha_2)|)}\,(z - E_0(\alpha_2)). \tag{5.4c}$$

Therefore, for every $\alpha_1 < c_0(\alpha_2)$ there are exactly two equilibria lying on both sides of $E_0(\alpha_2)$. Denote the one above $E_0$ by $E_+$ and the other by $E_-$. Note that $E_+$ and $E_-$ collide at $E_0$ when $\alpha_1 = c_0(\alpha_2)$. As we have mentioned earlier, in the Introduction, a number of people have also contributed to the following theorem.

THEOREM 5.1 (Chow and Lin (1988)). *For a generic two-parameter family of vector fields satisfying conditions* (5.1a, b) *and* (5.2a) *for a nondegenerate homoclinic orbit to a saddle-node equilibrium there exists in a neighborhood* $\Lambda$ *of* $\alpha = 0$ *a* $C^{r-3}$ *curve*

*$\alpha_1 = c_1(\alpha_2)$ with a quadratic tangency to the $c_0$ curve at $\alpha = 0$ (i.e., $c_0(0) = c_1(0)$, $c_0'(0) = c_1'(0)$ but $c_0''(0) \neq c_1''(0)$) such that, up to possibly renaming the direction of $\alpha_2$, the following are satisfied in a small tubular neighborhood of the homoclinic orbit*:

(i) *For $\alpha \in \mathrm{I} = \{\alpha \in \Lambda \mid$ either $\alpha_1 > c_0(\alpha_2)$, $\alpha_2 \leqq 0$, or $\alpha_2 > 0$, $\alpha_1 > c_1(\alpha_2)\}$ there exists a unique hyperbolic periodic orbit having m Floquet multipliers inside the unit circle in the plane.*

(ii) *For $\alpha \in \mathrm{II} = \{\alpha \in \Lambda \mid \alpha_1 = c_1(\alpha_2), \alpha_2 > 0\}$ there exists a unique homoclinic orbit to $E_+$.*

(iii) *For $\alpha \in \mathrm{III} = \{\alpha \in \Lambda \mid \alpha_1 < c_1(\alpha_2), \alpha_2 \geqq 0$ or $\alpha_1 < c_0(\alpha_2), \alpha_2 \leqq 0\}$ there exists a unique global heteroclinic orbit from $E_+$ to $E_-$ in addition to the one connecting $E_+$ to $E_-$ from the local bifurcation of the saddle-node equilibrium. In particular, when $\alpha_1 < c_1(\alpha_2)$, $\alpha_2 > 0$, respectively, $\alpha_1 = c_1(\alpha_2)$, $\alpha_2 < 0$, respectively, $c_1(\alpha_2) < \alpha_1 \leqq c_0(\alpha_2)$, $\alpha_2 \leqq 0$, this orbit approaches $E_-$ forward in time asymptotically along the center manifold from above $E_-$, respectively, the strong stable manifold of $E_-$, respectively, the center manifold from below $E_-$* (see Fig. 5.2).

Next, we consider the transcritical case (5.3b). To find nonzero equilibrium points we solve equation $\tilde{\theta}(z, \alpha) = 0$ for $z$, where $\tilde{\theta} = \theta(z, \alpha)/z$. Again by the IFT we obtain a $C^{r-2}$ function $z = E_1(\alpha)$ satisfying

$$E_1(0) = 0, \quad \frac{\partial E_1}{\partial \alpha_1}(0) < 0, \quad \frac{\partial E_1}{\partial \alpha_2}(0) = 0. \tag{5.5}$$

Then we have Theorem 5.2 below.

THEOREM 5.2. *For a generic two-parameter family of vector fields satisfying* (5.1a, b) *and* (5.2b) *for a nondegenerate homoclinic orbit to a transcritical equilibrium there exists in a neighborhood $\Lambda$ of $\alpha = 0$ a $C^{r-2}$ curve $\alpha_1 = c_1(\alpha_2)$ satisfying $c_1'(0) < 0$ such that, up*

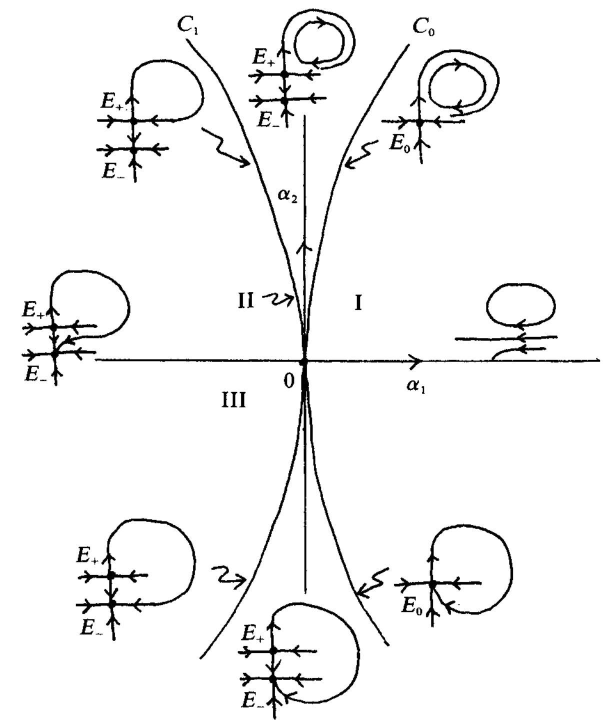


FIG. 5.2. *The bifurcation diagram for the saddle-node homoclinic bifurcation.*

*to possibly renaming the direction of* $\alpha_2$, *the following are satisfied in a small tubular neighborhood of the homoclinic orbit*:

(i) *For* $\alpha \in \mathrm{I} = \{\alpha \in \Lambda \mid \alpha_1 > c_1(\alpha_2),\ \alpha_2 > 0\}$ *there exists a unique periodic orbit having m Floquet multipliers inside the unit circle.*

(ii) *For* $\alpha \in \mathrm{II} = \{\alpha \in \Lambda \mid \alpha_1 = c_1(\alpha_2),\ \alpha_2 > 0\}$ *there exists a unique homoclinic orbit to* $E_1$.

(iii) *For* $\alpha \in \mathrm{III} = \{\alpha \in \Lambda \mid \alpha_1 < c_1(\alpha_2)\}$ *there exists a unique global heteroclinic orbit from* $E_1$ *to the origin in addition to the local connection due to the local transcritical bifurcation. In particular, it approaches the origin from different sides of the origin as the sign of* $\alpha_2$ *changes, and from the strong stable manifold of the origin when* $\alpha_2 = 0$.

(iv) *For* $\alpha \in \mathrm{IV} = \{\alpha \in \Lambda \mid \alpha_1 = 0,\ \alpha_2 \leqq 0\}$ *there exists a unique homoclinic orbit to the saddle-node origin.*

(v) *For* $\alpha \in \mathrm{V} = \{\alpha \in \Lambda \mid \alpha_1 > 0,\ \alpha_2 < 0\}$ *there exists a unique global heteroclinic orbit from the origin to* $E_1$ *in addition to the local connection. In particular, it approaches* $E_1$ *from its different sides on the center manifold as* $\alpha$ *crosses the curve* $\alpha_1 = c_1(\alpha_2)$, *and from the strong unstable manifold of* $E_1$ *on the curve.*

(vi) *For* $\alpha \in \mathrm{VI} = \{\alpha \in \Lambda \mid \alpha_2 = 0,\ \alpha_1 > 0\}$ *there exists a unique homoclinic orbit to the origin* (see Fig. 5.3).

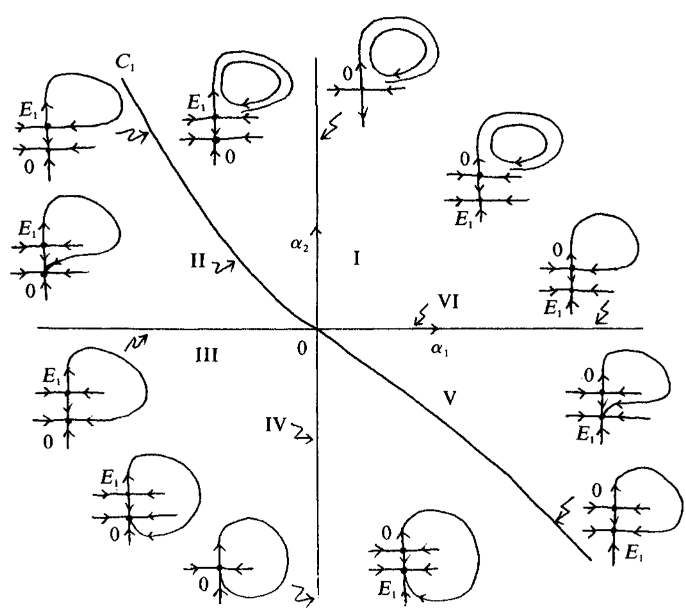


FIG. 5.3. *The bifurcation diagram for the transcritical homoclinic bifurcation.*

Finally, we consider the pitchfork case (5.3c). To find nonzero equilibrium points we solve the equation $\tilde{\theta}(z, \alpha) = 0$ for $\alpha_1$ by the IFT, where $\tilde{\theta} = \theta(z, \alpha)/z$ and obtain a $C^{r-2}$ curve $\alpha_1 = \gamma(z, \alpha_2)$ satisfying

$$\gamma(0, \alpha_2) = \frac{\partial \gamma}{\partial z}(0, \alpha_2) = 0, \qquad \frac{\partial^2 \gamma}{\partial z^2}(0, 0) < 0. \tag{5.6a}$$

Thus, by the Taylor expansion at $z = 0$, we can easily see that $\alpha_1 = \gamma(z, \alpha_2) < 0$ for $z \neq 0$. In fact, we have

$$\pm\sqrt{-\alpha_1} = \sqrt{-\tfrac{1}{2}(\partial^2 \gamma/\partial z^2)(0, \alpha_2) + O(|z|)}\, z. \tag{5.6b}$$

Therefore, for every $\alpha_1 < 0$ there exist exactly two nonzero equilibria lying on both sides of the zero on the $z$-axis. Denote the one above the origin by $E_+$ and the other by $E_-$. Note that $E_+$ and $E_-$ collide at the origin when $\alpha_1 = 0$. We have Theorem 5.3.

THEOREM 5.3. *For a generic two-parameter family of vector fields satisfying conditions* (5.1a, b) *and* (5.2c) *for a nondegenerate homoclinic orbit to a pitchfork equilibrium, there exists in a neighborhood* $\Lambda$ *of* $\alpha = 0$ *a* $C^{r-4}$ *curve* $\alpha_1 = c_1(\alpha_2)$ *with a quadratic tangency to the* $\alpha_2$*-axis* (*i.e.*, $c_1(0) = c_1'(0) = 0$ *but* $c_1''(0) \neq 0$) *such that, up to possibly renaming the direction of* $\alpha_2$, *the following are satisfied in a small tubular neighborhood of the homoclinic orbit*:

(i) *For* $\alpha \in \mathrm{I} = \{\alpha \in \Lambda \mid \alpha_1 > c_1(\alpha_2),\ \alpha_2 > 0\}$ *there exists a unique periodic orbit having* $m$ *Floquet multipliers inside the unit circle.*

(ii) *For* $\alpha \in \mathrm{II} = \{\alpha \in \Lambda \mid \alpha_1 = c_1(\alpha_2),\ \alpha_2 > 0\}$ *there exists a unique homoclinic orbit to* $E_+$.

(iii) *For* $\alpha \in \mathrm{III} = \{\alpha \in \Lambda \mid \alpha_1 < c_1(\alpha_2)\}$ *there exists a unique global heteroclinic orbit from* $E_+$ *to the origin, approaching the origin asymptotically along the center direction but from its different sides as the sign of* $\alpha_2$ *changes.*

(iv) *For* $\alpha \in \mathrm{IV} = \{\alpha \in \Lambda \mid \alpha_1 = c_1(\alpha_2),\ \alpha_2 < 0\}$ *there exists a unique global heteroclinic orbit from* $E_+$ *to* $E_-$.

(v) *For* $\alpha \in \mathrm{V} = \{\alpha \in \Lambda \mid \alpha_1 > c_1(\alpha_2),\ \alpha_2 < 0\}$ *there does not exist any global homoclinic, heteroclinic, or periodic orbit.*

(vi) *For* $\alpha \in \mathrm{VI} = \{\alpha \in \Lambda \mid \alpha_2 = 0,\ \alpha_1 > 0\}$ *there exists a unique homoclinic orbit to the origin which is the continuation of the original homoclinic orbit* (see Fig. 5.4).

Before proving the theorems, let us draw heuristically the phase portraits in Fig. 5.5 for some oversimplified situations where $d = 2$, $\theta = \alpha_1 + z^2$ for the saddle-node case, $\theta = z(\alpha_1 + z)$ for the transcritical case, and $\theta = z(\alpha_1 + z^2)$ for the pitchfork case, respectively. In terms of the straight invariant foliation, the $c_1$ curves, for example, are given

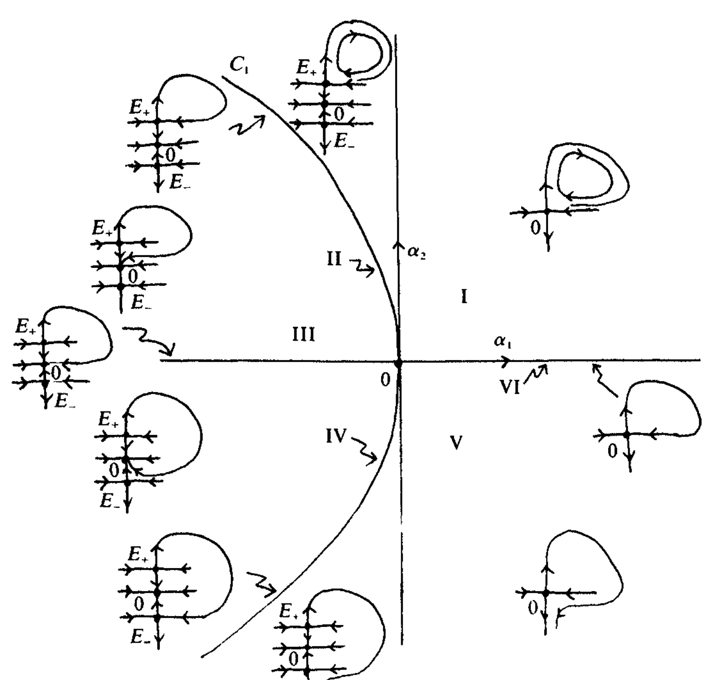


FIG. 5.4. *The bifurcation diagram for the pitchfork homoclinic bifurcation.*

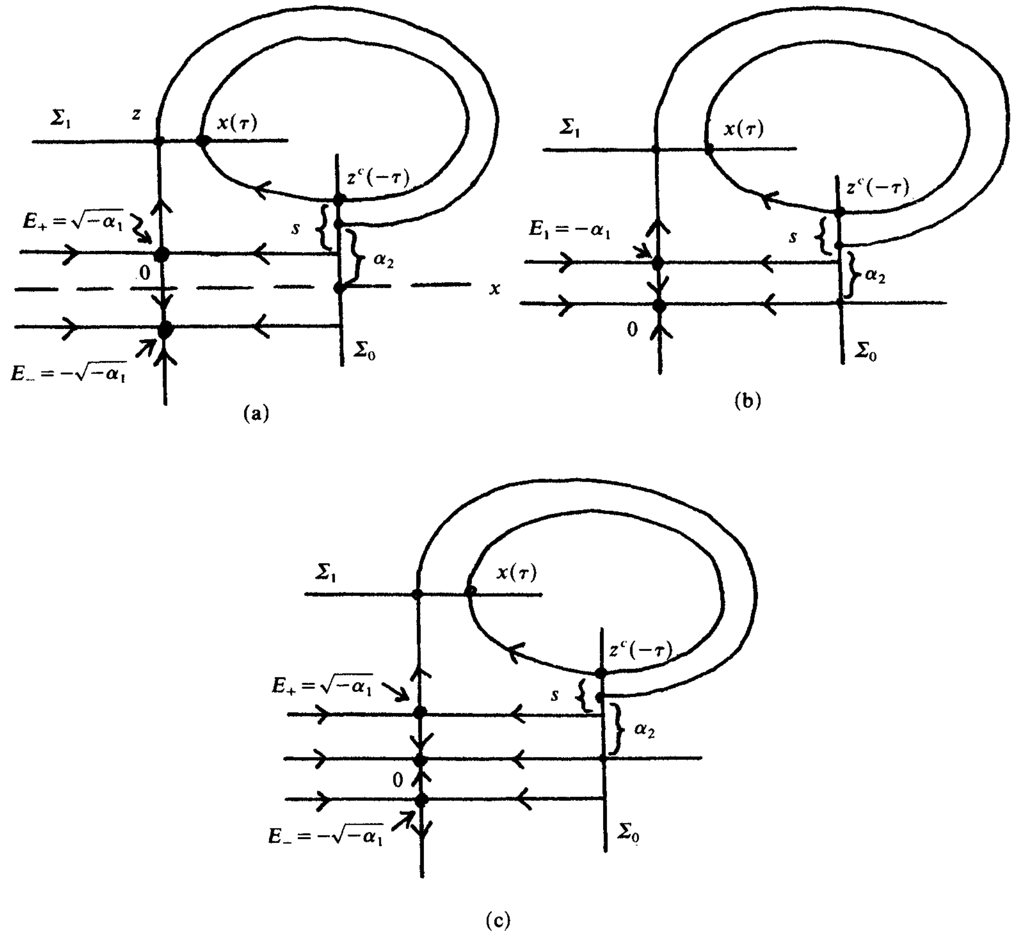


FIG. 5.5. *Some phase portraits for when periodic orbits take place.* (a) *Saddle-node.* (b) *Transcritical.* (c) *Pitchfork.*

as $\alpha_1 = -\alpha_2^2$ in the first and third cases and $\alpha_1 = -\alpha_2$ in the second case. Also, the existence of the periodic orbits is equivalent to solving the scalar equation $z^c(0) = \Pi_1(x(\tau), \delta_0)$ for the time $\tau$, where $(x(t), z^c(t))$ solves the second type of Sil'nikov problem $x(0) = x_0 = \delta_0$ and $z^c(\tau) = z_1 = \delta_0$. Let $E_+(\alpha_1)$ denote the bifurcated equilibrium point above the origin, if any, or zero otherwise, and let $s = z^c(0) - E_+$ be the distance of the "initial" point $z^c(0)$ on $\Sigma_0$ to $E_+$. Then $x(\tau)$ must be $O(s^2)$ by the exponential expansion (for more details concerning this estimate, see the proof below). Thus, we obtain the bifurcation equation

$$s = -E_+(\alpha_1) + \alpha_2 + O(s^2) \quad \text{for } s > 0 \tag{5.7}$$

by the Taylor expansion $\Pi_1(x, \delta_0) = \alpha_2 + O(x)$. We do not even have the constraint $s > 0$ in the saddle-node bifurcation case, where the equilibrium disappears completely for $\alpha_1 > 0$. Using this equation together with the straight foliation, it is not difficult to derive all the conclusions in the theorems. Not surprisingly, we will derive the same form of the bifurcation equation (5.7) for the general cases through a modified Lyapunov-Schmidt reduction. Motivated by these model examples, we can now prove Theorem 5.1.

*Proof of Theorem* 5.1. We will assume that our readers are familiar with the construction of the Poincaré cross sections $\Sigma_0$ and $\Sigma_1$ and the Poincaré return maps $\Pi_0$ and $\Pi_1$ from § 1. The necessary modifications are made as follows: the $y$-component there is now augmented into the $(y, z)$-component and, specifically, $\Sigma_1$ is given as $\{z=\delta_0\}$ in the $\delta_0$-box of the origin. To use the idea of Lyapunov-Schmidt reduction to the return map $\Pi_1 \circ \rho_1(\zeta)=\rho_0(\zeta)$ under the Sil'nikov variables, we need to normalize our variables in the following as preparations.

First, normalize the local coordinates on $\Sigma_0$ as $(\xi, y, z)$ such that $(\xi, y, z)=(0,0,0)$ represents the intersection point $\Gamma \cap \Sigma_0$. Similarly, use $(x, \eta)$ for $\Sigma_1$ so that $(x, \eta)=(0,0)$ corresponds $\Gamma \cap \Sigma_1$ because we have assumed the homoclinic orbit is asymptotically tangent to the center eigenvector space as $t \to -\infty$. See Fig. 5.6.

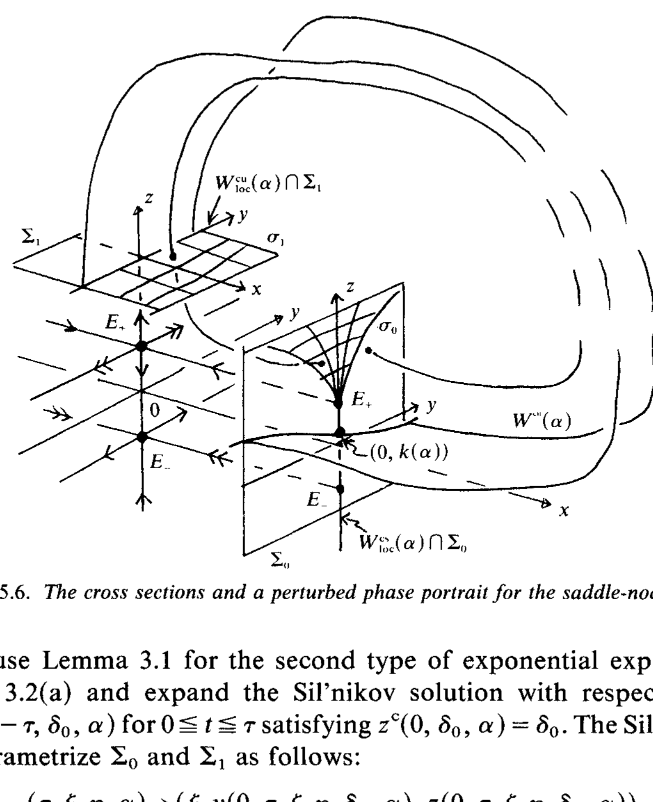


FIG. 5.6. *The cross sections and a perturbed phase portrait for the saddle-node case.*

Second, use Lemma 3.1 for the second type of exponential expansion together with Remark 3.2(a) and expand the Sil'nikov solution with respect to the center trajectory $z^c(t-\tau, \delta_0, \alpha)$ for $0 \leqq t \leqq \tau$ satisfying $z^c(0, \delta_0, \alpha)=\delta_0$. The Sil'nikov variables $(\tau, \xi, \eta, \alpha)$ parametrize $\Sigma_0$ and $\Sigma_1$ as follows:

$$(\tau, \xi, \eta, \alpha) \to (\xi, y(0, \tau, \xi, \eta, \delta_0, \alpha), z(0, \tau, \xi, \eta, \delta_0, \alpha)),$$

$$(\tau, \xi, \eta, \alpha) \to (x(\tau, \tau, \xi, \eta, \delta_0, \alpha), \eta, \delta_0).$$

Replace $\tau$ by a variable $s$, where $\tau$ and $s$ are related by

$$s=z^c(-\tau, \delta_0, \alpha)-s^*(\alpha),$$

where

$$s^*(\alpha)=\begin{cases} E_+(\alpha) & \text{if } \alpha_1 \leqq c_0(\alpha_2), \\ E_0(\alpha_2) & \text{otherwise.} \end{cases}$$

Since $\partial z^c/\partial \tau \neq 0$, we may solve for $\tau$ as a function of $(s, \alpha)$:

$$\tau=\tau(s, \alpha). \tag{5.8}$$

Recall from (3.1a) that

$$z(0, \tau, \xi, \eta, \delta_0, \alpha) = z^c(-\tau, \delta_0, \alpha) + R(0, \tau, \xi, \eta, \delta_0, \alpha).$$

Define

$$X(s, \xi, \eta, \alpha) = x(\tau, \tau, \xi, \eta, \delta_0, \alpha),$$

$$Y(s, \xi, \eta, \alpha) = y(0, \tau, \xi, \eta, \delta_0, \alpha),$$

$$R(s, \xi, \eta, \alpha) = R(0, \tau, \xi, \eta, \delta_0, \alpha),$$

where $\tau$ is defined by (5.8). Then the normalized Sil'nikov variables $(s, \xi, \eta, \alpha)$ parametrize $\Sigma_0$ and $\Sigma_1$ as follows:

$$\rho_0(s, \xi, \eta, \alpha) = (\xi, Y(s, \xi, \eta, \alpha), s + s^*(\alpha) + R(s, \xi, \eta, \alpha)),$$

$$\rho_1(s, \xi, \eta, \alpha) = (X(s, \xi, \eta, \alpha), \eta).$$

Clearly, the local map under these new variables is

$$\Pi_0 \circ \rho_0(s, \xi, \eta, \alpha) = \rho_1(s, \xi, \eta, \alpha).$$

Note that the change of variables $\tau \to s$ is $C^{r-1}$ in $\tau$ and at least continuous in $\alpha$. Actually we will see in a moment that it is $C^{r-3}$ in $\varepsilon$ and $\alpha_2$ if $\alpha_1 \leqq c_0(\alpha_2)$, where $\varepsilon = \sqrt{c_0(\alpha_2) - \alpha_1}$, and $C^{r-1}$ in $\alpha_1 > c_0(\alpha_2)$. Note also that when $\alpha_1 > c_0(\alpha_2)$ there is not any equilibrium point; therefore, the existence time of definition for the local center trajectory $z^c(-\tau, \delta_0, \alpha)$ is finite and $s$ can be both positive and negative depending on whether $\tau$ sufficiently large. On the contrary, we require $s > 0$ when $\alpha_1 \leqq c_0(\alpha_2)$.

Third, we also need to extend the functions $X$, $Y$, and $R$ differentiably to $s \leqq 0$ whenever $\alpha_1 \geqq c_0(\alpha_2)$ occurs. To do this, it suffices to show that these functions are of order $O(s^2)$, at least at this parameter range. Because of the exponential bounds $e^{\lambda\tau}$ and $e^{-u\tau}$ for the functions $x(\tau, \tau, x_0, y_1, z_1, \alpha)$ and $(y, z)(0, \tau, x_0, y_1, z_1, \alpha)$, respectively, it suffices to show $s = z^c(-\tau, \delta_0, \alpha) - E_+(\alpha) \geqq a\, e^{-b\varepsilon\tau}$ for some positive constants $a$ and $b$. To show the lower exponentially small bound on $\alpha_1 = c_0(\alpha_2)$, we take the Taylor expansion of $\theta$ at $z = E_0(\alpha_2) = E_+(\alpha)$ and use (5.3a) to obtain

$$\begin{aligned}\theta(z, \alpha) &= \frac{1}{2}\frac{\partial^2\theta}{\partial z^2}(E_0, \alpha)(z - E_0)^2 + O(|z - E_0|^3)\\ &\leqq a_1(z - E_0)^2\end{aligned}$$

for some positive constant $a_1$. Thus, by the comparison principle, the integral curve through the same value $\delta_0$ at $t = 0$ for the vector field $\theta(z, \alpha)$ lies above that of the vector field $a_1(z - E_0)^2$ for the negative time, that is,

$$z^c(-\tau, \delta_0, \alpha) - E_0(\alpha_2) \geqq \frac{k_1}{\tau + k_2} > 0$$

for some constants $k_i$. This certainly implies the desired lower bound. For the other case where $\alpha_1 < c_0(\alpha_2)$, we take the Taylor expansion of $\theta$ at $z = E_+(\alpha)$: $\theta(z, \alpha) = \partial\theta/\partial z(E_+, \alpha)(z - E_+) + O(|z - E_+|^2)$. Expanding $\partial\theta/\partial z$ further, we have

$$\begin{aligned}\theta(z, \alpha) &= \left[\frac{\partial^2\theta}{\partial z^2}(E_0, \alpha)(E_+ - E_0) + O(|E_+ - E_0|^2)\right](z - E_+) + O(|z - E_+|^2)\\ &\leqq b\varepsilon(z - E_+)\end{aligned}$$

for some positive constant $b$ since, by (5.4c), $E_+ - E_0 = O(\sqrt{c_0(\alpha_2) - \alpha_1}) = O(\varepsilon)$. (In fact, by the IFT we can solve $E_+ - E_0$ as a $C^{r-3}$ function of $\varepsilon$ and $\alpha_2$.) Therefore, by the comparison principle we again derive the desired lower bound for $s$. We also use $X$, $Y$, and $R$ to extend the functions.

As the last preparation, we write the global map $\Pi_1$ in the normalized coordinates for $\Sigma_0$ and $\Sigma_1$ as

$$\xi = P(x, \eta, \alpha), \quad y = Q(x, \eta, \alpha), \quad z = T(x, \eta, \alpha).$$

We are now ready to consider the equation $\rho_0(s, \xi, \eta, \alpha) = \Pi_1 \circ \rho_1(s, \xi, \eta, a)$ for periodic orbits running around the homoclinic loop once. This is equivalent to solving the equation $\Phi(s, \xi, \eta, \alpha) = 0$ for the normalized Sil'nikov variable $(s, \xi, \eta)$ with the constraint $s > 0$ only if $\alpha_1 \leqq c_0(\alpha_2)$, where

$$\Phi(s, \xi, \eta, \alpha) = \begin{pmatrix} \xi \\ Y(s, \xi, \eta, \alpha) \\ s + s^*(\alpha) + R(s, \xi, \eta, \alpha) \end{pmatrix} - \begin{pmatrix} P \\ Q \\ T \end{pmatrix} (X(s, \xi, \eta, \alpha), \eta, \alpha).$$

Certainly $\Phi(0, 0, 0, 0) = 0$ due to the existence of the homoclinic orbit. Compute the Jacobian with respect to $(s, \xi, \eta)$ at the origin in order to use the IFT; then we have

$$D\Phi(0,0,0,0) = \begin{bmatrix} 0 & I & -P_2'(0,0,0) \\ 0 & 0 & -Q_2'(0,0,0) \\ 1 & 0 & -T_2'(0,0,0) \end{bmatrix} \tag{5.9}$$

by (3.1c), Remark 3.2(b), and the definition of $s$. Since the first $m$-columns span the local center-stable tangent space in $\Sigma_0$, $T_p(W^{cs}_{loc} \cap \Sigma_0)$, at $p = \Gamma \cap \Sigma_0$ while the last $n$-columns span the global center-unstable tangent space in $\Sigma_0$, $T_p(W^{cu} \cap \Sigma_0)$, then by the nondegeneracy condition (5.1a) the Jacobian is nonsingular. Thus, by the IFT, a unique solution $(s, \xi, \eta) = (\bar{s}, \bar{\xi}, \bar{\eta})(\alpha)$ exists for $\alpha$ from a small neighborhood $\Lambda$ of the origin. To ensure that this solution indeed gives rise to a periodic orbit, we need to find only those $\alpha$ satisfying $\alpha_1 \leqq c_0(\alpha_2)$ such that the constraint $s > 0$ is satisfied because there is no restriction on $s$ when $\alpha_1 > c_0(\alpha_2)$. To do this, we need the following Lyapunov–Schmidt reduction procedure to obtain the bifurcation equation.

Because of the special structure of the Jacobian (5.9) of $\Phi$, we can first solve $\xi$ and $\eta$ in terms of $s$ and $\alpha$ from the first $m + n - 1$ equations of $\Phi = 0$ by the IFT. Thus, as far as only these equations are concerned, $\bar{\xi}(\alpha)$ and $\bar{\eta}(\alpha)$ are the solutions and $\bar{s}(\alpha)$ can be treated as an independent variable. Hence, by formally setting $\bar{s} = 0$ and plugging $\xi = \bar{\xi}(\alpha)$ and $\eta = \bar{\eta}(\alpha)$ in the function $\Phi$, the equation $\Phi = 0$ will be respected except for the last equation, that is,

$$\bar{\xi} = P(0, \bar{\eta}, \alpha), \quad 0 = Q(0, \bar{\eta}, \alpha), \quad k(\alpha) = T(0, \bar{\eta}, \alpha),$$

where the function $k(\alpha) \stackrel{\text{def}}{=} T(0, \bar{\eta}, \alpha)$. Note that the geometrical interpretation of this relation is that $(\bar{\xi}, 0, k(\alpha))$ is the unique intersection point of the global center-unstable manifold $\{(P, Q, T)(0, \eta, \alpha)\}$ with the local center-stable manifold $W^{cs}_{loc} = \{y = 0\}$ (cf. Fig. 5.6) in $\Sigma_0$. By the transversality condition (5.1a) the function $k(\alpha)$ must be at least $C^{r-2}$ (the same as the admissible variable; the same conclusion also holds for $\bar{\xi}(\alpha)$ and $\bar{\eta}(\alpha)$ as well). By means of the distance $d(\alpha)$ between $W^{cu}$ and $W^s_{loc}$ in $\Sigma_0$, we actually know more about the function $k(\alpha)$. Indeed, by definition, it must satisfy

$$0 \leqq d(\alpha) = \min_{|\xi|, |\eta| \leqq \delta_0} |(P, Q, T)(0, \eta, \alpha) - (\xi, 0, 0)| \leqq |k(\alpha)|,$$

$$\underline{\lim}_{\alpha_2 \to 0} \frac{|k((0, \alpha_2))|}{|\alpha_2|} > 0$$

by the transverse crossing condition (5.1b). Thus, for $\alpha_2>0$, $k((0,\alpha_2))$ must have a constant sign. Since $k(0)=0$, the inequalities above imply

$$\frac{\partial k(0)}{\partial \alpha_2}\neq 0.$$

For definiteness, we assume $\partial k(0)/\partial\alpha_2>0$, which corresponds to preserving the direction of $\alpha_2$ in the statement of the theorem.

Now, the desired bifurcation equation is simply the last equation of $\Phi=0$ at $(s,\xi,\eta)=(\bar{s},\bar{\xi},\bar{\eta})(\alpha)$ and $\alpha_1\leqq c_0(\alpha_2)$. Using Taylor expansion at $\bar{s}=0$, and the order estimates for the functions $X$, $Y$, and $R$ above, we have

$$\bar{s}=-E_+(\alpha)+k(\alpha)+O(\bar{s}^2),$$

which has the same form as (5.7). Thus $\bar{s}>0$ for $\alpha_1\leqq c_0(\alpha_2)$ if and only if

$$k(\alpha)>E_+(\alpha).$$

To describe this region $\{k(\alpha)>E_+(\alpha)\}\cap\{\alpha_1\leqq c_0(\alpha_2)\}$, let us begin with its boundary $k(\alpha)=E_+(\alpha)$. This is precisely when the homoclinic orbit to $E_+(\alpha)$ takes place, since the stable manifold of $E_+$ is $\{y=0, z=E_+\}$ by the "straight" foliation of the admissible variable mentioned in §§ 1 and 2 and the intersection point of the unstable manifold of $E_+$, where the center-stable manifold of the origin is $(\bar{\xi},0,k(\alpha))$. By substituting $z=E_+(\alpha)=k(\alpha)$ into the function $\alpha_1=\gamma(z,\alpha_2)$ in (5.4b) we can solve for a $C^{r-2}$ curve $\alpha_1=c_1(\alpha_2)$ satisfying $c_1'(0)=0$ from the equation $\alpha_1=\gamma(k(\alpha),\alpha_2)$, and $c_1(\alpha_2)<c_0(\alpha_2)$ is always true by the definition of $\gamma$. Thus $k(\alpha)=E_+(\alpha)$, or $E_-(\alpha)$ on $c_1$. Indeed we claim that $k(\alpha)=E_+(\alpha)$ is satisfied if and only if $\alpha_1=c_1(\alpha_2)$ and $\alpha_2>0$. To show this we need only to rule out $k(\alpha)=E_+(\alpha)$ on the lower half $c_1$ curve. Since $E_0'(0)=0$, $c_1'(0)=0$ and $d/d\alpha_2 k((c_1,\alpha_2))|_{\alpha_2=0}=\partial k/\partial\alpha_2(0)>0$, $k((c_1(\alpha_2),\alpha_2))=E_+((c_1(\alpha_2),\alpha_2))\geqq E_0(\alpha_2)$ if and only if $\alpha_2>0$. For exactly the same reason we see that $k((c_0(\alpha_2),\alpha_2))>E_+((c_0(\alpha_2),\alpha_2))=E_0(\alpha_2)$ for $\alpha_2>0$ and $k((c_0(\alpha_2),\alpha_2))<E_-((c_0(\alpha_2),\alpha_2))=E_0(\alpha_2)$ for $\alpha_2<0$. Moreover, since $E_+(\alpha)-E_0(\alpha_2)=O(\sqrt{c_0(\alpha_2)-\alpha_1})$ while $k(\alpha)-E_0(\alpha_2)$ is differentiable, it must be that $k(\alpha)<E_+(\alpha)$ for $\alpha_1<c_1(\alpha_2)$. Therefore, we can conclude that $\{k(\alpha)>E_+(\alpha)\}\cap\{\alpha_1\leqq c_0(\alpha_2)\}$ is the wedge-shaped region between the curves $c_1$ and $c_0$:

$$\{\alpha\,|\,c_1(\alpha_2)<\alpha_1\leqq c_0(\alpha_2),\ \alpha_2>0\}$$

(see Fig. 5.7).

To show that $c_1$ has quadratic tangency to the $c_0$ curve, simply observe from (5.4c), $c_1'(0)=c_0'(0)=E_0'(0)=0$, and $\partial k/\partial\alpha_2(0)\neq 0$ that

$$(c_1-c_0)''(0)=\frac{1}{2}\frac{\partial^2\gamma}{\partial z^2}(0,0)\left[\frac{\partial k}{\partial\alpha_2}(0)\right]^2<0.$$

Next, to show that the periodic orbit is unique, we need to rule out the possibility that there might be solutions to the following cyclic equations other than the trivial one $(\bar{s},\bar{\xi},\bar{\eta})$ found above:

$$\begin{pmatrix}\xi_{i+1}\\ Y(s_{i+1},\xi_{i+1},\eta_{i+1},\alpha)\\ s_{i+1}+s^*(\alpha)+R(s_{i+1},\xi_{i+1},\eta_{i+1},\alpha)\end{pmatrix}-\begin{pmatrix}P\\ Q\\ T\end{pmatrix}(X(s_i,\xi_i,\eta_i,\alpha),\eta_i,\alpha)=0$$

for $i=0,\cdots,(\mathrm{mod}\ k)$. Note that their solutions with $s_i>0$ imply the existence of periodic orbits running around the loop $k$ times. By the IFT again, we can show that

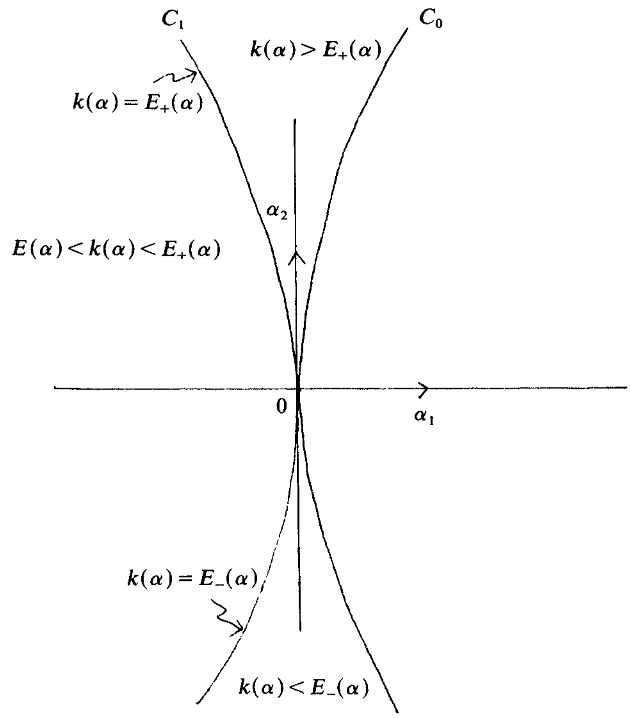


FIG. 5.7

the solutions are unique, which must be the repetition of $k$ copies of the $(\bar{s}, \bar{\xi}, \bar{\eta})$. Thus the uniqueness is established. Indeed, because the associated Jacobian consists of nonzero blocks only as does $D\Phi(0,0,0,0)$ above, the parameter range on which the existence and uniqueness conclusion holds for such periodic orbits can be chosen the same as $\Lambda$.

Thus to complete (i) it only remains to show that the periodic orbit has $m$ Floquet multipliers inside and $n$ outside the unit circle. To see this, consider the characteristic polynomial $\det |D(\rho_0^{-1} \circ \Pi \circ \rho_0(\bar{s}, \bar{\xi}, \bar{\eta}, \alpha)) - \lambda I| = 0$, where $\Pi = \Pi_1 \circ \rho_1 \circ \rho_0^{-1}$ is the Poincaré map in the old variables. It is equivalent to considering $\det |D\Pi_1 \cdot D\rho_1(\bar{s}, \bar{\xi}, \bar{\eta}, \alpha) - \lambda D\rho_0(\bar{s}, \bar{\xi}, \bar{\eta}, \alpha)| = 0$, which has the form $\det Q_2'(0,0,\alpha)\lambda^m + p(\lambda, \bar{s}, \bar{\xi}, \bar{\eta}, \alpha) = 0$, where $p$ is an $(m+n)$-degree polynomial with all the coefficients having the order at least $O(s)$. Thus as $s \to 0$ it has precisely $m$ roots inside and $n$ outside the unit circle.

Part (ii) has been proved above, since $k(\alpha) = E_+(\alpha)$ takes place exactly on the $c_1$ curve if and only if $\alpha_2 > 0$. To show (iii), note that when $k(\alpha) < E_+(\alpha)$, the point $(\bar{\xi}, 0, k)$ from the global unstable manifold of $E_+$ lies in the local stable manifold of $E_-$, which is $\{y = 0\}$. Note also that when $k(\alpha) = E_-(\alpha)$ the heteroclinic connection comes in along the strong stable manifold of $E_-$, which is $\{y = 0, z = E_-\}$ by the straight foliation of the admissible variable. This happens precisely on the curve $\alpha_1 = c_1(\alpha_2)$ for the same reasons as for $k(\alpha) = E_+(\alpha)$ above. But this time $\alpha_2 < 0$ since $k(\alpha) = E_-(\alpha) < E_+(\alpha)$. □

The proofs for Theorems 5.2 and 5.3 follow the same strategy as above. That is, use $s$, the distance of the center trajectory $z^c(-\tau, \delta_0, \alpha)$ to the bifurcated equilibrium above the origin; use the comparison principle to estimate the lower bounds of $s$ in terms of an exponentially small number $e^{-\varepsilon\tau}$; extend the functions $X$, $Y$, and $Z$ to $s \leqq 0$ differentiably; and use the IFT to obtain the bifurcation equation and the straight foliation of the admissible variable to establish the connections. In these two cases, $s$ is always positive because of the persistence of equilibria. We omit the details here because the proofs are not only similar but also much easier.

**Acknowledgment.** I acknowledge my postdoctoral fellowship at the Lefschetz Center for Dynamical Systems and the Program for Turbulence at Brown University from 1987 to 1988. Special thanks go to Professors J. K. Hale, J. Mallet-Paret, and L. Sirovich for providing me with a stimulating and carefree research environment.

## REFERENCES

S. N. CHOW, B. DENG, AND B. FIEDLER (1988), *Homoclinic bifurcations at resonant eigenvalues*, J. DDE, April 1990, to appear.

S. N. CHOW AND X.-B. LIN (1988), *Bifurcation of a homoclinic orbit with a saddle-node equilibrium*, preprint.

S. N. CHOW AND K. LU (1988), *$C^k$ center-unstable manifolds*, Proc. Roy. Soc. Edinburgh, 108A, 303-320.

S. N. CHOW, X.-B. LIN, AND K. LU (1988), *Smoothness of invariant foliation in infinite dimensional spaces*, preprint.

S. N. CHOW, B. DENG, AND D. TERMAN (1987), *The bifurcation of a homoclinic and periodic orbit from two heteroclinic orbits*, SIAM J. Appl. Math., 21 (1990), to appear.

P. COLLET AND J.-P. ECKMANN (1980), *Iterated Maps of the Interval as Dynamical Systems*, Birkhäuser, Boston.

B. DENG (1988a), *The Sil'nikov problem, exponential expansion, strong $\lambda$-lemma, $C^1$-linearization and homoclinic bifurcation*, J. Differential Equations, 79 (1989), pp. 189-231.

——— (1988b), *Exponential expansion with Sil'nikov's saddle-focus*, J. Differential Equations, 81 (1989).

——— (1988c), *Sil'nikov problem, invariant manifolds and $\lambda$-lemma*, preprint.

——— (1988d), *Exponential expansion with principal eigenvalues*, Proc. Roy. Soc. Edinburgh Sect. A, to appear.

——— (1988e), *The bifurcation of countable connections from a twisted heteroclinic loop*, SIAM J. Math. Anal., submitted.

G. DANGELMAYR, D. ARMBRUSTER, AND N. NEVELING (1985), *A codimension three bifurcation for the laser with saturable absorber*, Z. Phys. B, 59, pp. 365-370.

N. FENICHEL (1979), *Geometric singular perturbation theory for ordinary differential equations*, J. Differential Equations, 31, pp. 53-98.

J. GUCKENHEIMER AND P. HOLMES (1983), *Nonlinear Oscillations, Dyamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, Berlin, New York.

J. K. HALE (1978), *Ordinary Differential Equations*, Second edition, Krieger, Melbourne, FL.

D. HENRY (1981), *Geometric Theory of Parabolic Equations*, Lecture Notes in Math 840, Springer-Verlag, Berlin, New York.

M. HIRSCH, C. PUGH, AND M. SHUB (1977), *Invariant Manifolds*, Lecture Notes in Math. 583, Springer-Verlag, Berlin, New York.

M. C. IRWIN (1980), *Smooth Dynamical Systems*, Academic Press, New York.

H. K. JU (1988), *Bifurcation of symmetric planar vector fields*, Ph.D. thesis, Michigan State University, East Lansing, MI.

P. LANCASTER (1969), *Theory of Matrices*, Academic Press, New York.

V. I. LUK'YANOV (1982), *Bifurcations of dynamical systems with a saddle-point-separatrix loop*, Differentsial'nye Uravneniya, 18, pp. 1493-1506. (In Russian.) J. Differential Equations, 18, pp. 1049-1059. (In English.)

J. MOSER (1973), *Stable and random motions in dynamical systems*, Ann. of Math. Stud., 77, Princeton University Press, Princeton, NJ.

I. M. OVSYANNIKOV AND L. P. SIL'NIKOV (1986), *On systems with saddle-focus homoclinic curve*, Mat. Sb., 172 (1986). (In Russian.) Math. USSR-Sb., 58 (1987), pp. 557-574. (In English.)

S. SCHECTER (1987), *The saddle-node separatrix-loop bifurcation*, SIAM J. Math. Anal., 18, pp. 1142-1157.

L. P. SIL'NIKOV (1967), *On a Poincaré-Birkhoff problem*, Math. USSR-Sb., 3, pp. 353-371.

——— (1970), *A contribution to the problem of the structure of an extended neighborhood of a rough state of saddle-focus type*, Math. USSR-Sb., 10, pp. 91-102.

J. SOTOMAYOR (1974), *Generic one-parameter families of vector fields,* Inst. Hautes Etudes Sci. Publ. Math., 43, pp. 5–46.

A. VANDERBAUWHEDE AND S. VAN GILS (1987), *Center manifold and contractions on a scale of Banach spaces,* J. Funct. Anal., 72, pp. 209–224.

J. C. WELLS (1976), *Invariant manifolds of nonlinear operators,* Pacific J. Math., 625, pp. 285–293.

E. YANAGIDA (1987), *Branching of double pulse solutions from single pulse solutions in nerve axon equation,* J. Differential Equations, 66, pp. 243–262.