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HOMOCLINIC ORBITS FOR A CLASS OF HAMILTONIAN SYSTEMS

W. OMANA and M. WILLEM
Institut Mathematique, Universita de Louvain
Chemin du Cyclotron, 2, B-1348 Louvain-la-Neuve, Belgium

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Abstract. Using a new compact imbedding theorem, we prove the existence of infinitely many solutions in $H^1(\mathbb{R})$ of the system

$$\ddot{q} - L(t)q + W_q(t,q) = 0$$

when $W_q(t,\,\cdot\,)$ is even, superquadratic at infinity and subquadratic at the origin.

1. Introduction. This paper deals with the existence of nontrivial solutions $q \in H^1(\mathbb{R},\mathbb{R}^n)$ of the system

$$\ddot{q} + V_q(t,q) = 0. \tag{HS}$$

Those solutions are called homoclinic orbits emanating from 0.

P.H. Rabinowitz and K. Tanaka [4] have recently shown that the Hamiltonian system (HS) possesses a homoclinic orbit emanating from 0 by using a variant of the "Mountain Pass" theorem relying on Ekeland's Variational Principle. The use of this method is due to the difficulty of verifying the Palais-Smale condition. In this paper, we show that the Palais-Smale condition is satisfied and we use the usual Mountain Pass Theorem to prove the result of Rabinovitz and Tanaka. Moreover, if $W(t,\,\cdot\,)$ is an even function, we prove the existence of an unbounded sequence of homoclinic orbits of (HS) emanating from 0 by using the "symmetric" mountain pass theorem.

Throughout the paper, it will be assumed that V satisfies

(V1) $V(t,x) = -\frac{1}{2}L(t)x\cdot x + W(t,x)$,

(V2) $L(t) \in C(\mathbb{R},\mathbb{R}^{n^2})$ is a positive definite symmetric matrix for all $t \in \mathbb{R}$, and there is continuous function $\alpha\colon \mathbb{R} \to \mathbb{R}$ such that $\alpha(t) > 0$ for all $t \in \mathbb{R}$ and $L(t)x\cdot x \geq \alpha(t)|x|^2$,

(V3) $W \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ and there is a constant $\mu > 2$ such that

$$0 < \mu W(t,x) \leq x\cdot W_q(t,x)$$

for all $x \in \mathbb{R}^n \setminus \{0\}$ and $t \in \mathbb{R}$,

(V4) $W_q(t,x) = o(|x|)$ as $x \to 0$ uniformly in $t \in \mathbb{R}$.

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Remarks. Assumptions (V3)–(V4) imply

(1) $W(t,x) = o(|x|^2)$ as $x \to 0$ uniformly in $t \in \mathbb{R}$. Hence, by (V1)–(V4), $x = 0$ is a local maximum of $V(t,x)$ for all $t \in \mathbb{R}$; however, it is not a global maximum since, by (V3), for each $t \in \mathbb{R}$ there is $\alpha_1(t) > 0$ such that

(2) $W(t,x) \geq \alpha_1(t)|x|^\mu$ for each $|x| \geq 1$.

1. Compact imbedding and Palais-Smale condition. Let

$$X = \{q \in H^1(\mathbb{R},\mathbb{R}^n) \mid \int_{\mathbb{R}} [|\dot{q}|^2 + L(t)q \cdot q]\, dt < \infty\}.$$

The space X is a Hilbert space with the inner product

$$(x,y) = \int_{\mathbb{R}} (\dot{x} \cdot \dot{y} + L(t)x \cdot y)\, dt$$

and the corresponding norm

$$\|x\|^2 = (x,x).$$

Lemma 1. *Suppose V satisfies* (V2). *If*

(V5) $\quad \alpha(t) \to +\infty \quad$ *as* $\quad |t| \to \infty$

then the imbedding of X in $L^2(\mathbb{R})$ is compact.

Proof. Let $(x_k) \subset X$ be a sequence such that $x_k \rightharpoonup x$ in X. We will show that $x_k \to x$ in L^2. Suppose, without loss of generality, that $x_k \rightharpoonup 0$ in X. The Banach-Steinhaus theorem implies that

$$A = \sup_k \|x_k\| < +\infty.$$

Let $\varepsilon > 0$; there is $T_0 < 0$ such that $\frac{1}{\alpha(t)} \leq \varepsilon$ for all t such that $t \leq T_0$. Similarly, there is $T_1 > 0$ such that $\frac{1}{\alpha(t)} \leq \varepsilon$ for all $t \geq T_1$. Since $\alpha(t) > c > 0$ on $]T_0, T_1[= I$, the operator defined by $S\colon X \to H^1(I)\colon u \to u|_I$ is a linear continuous map. So $x_k \rightharpoonup 0$ in $H^1(I)$. Sobolev's theorem (see e.g. [1]) implies that $x_k \to 0$ uniformly on $\overline{I}$, so there is a k_0 such that

$$\int_I |x_k(t)|^2\, dt \leq \varepsilon \quad \text{for all } k \geq k_0. \tag{i}$$

Since $\frac{1}{\alpha(t)} \leq \varepsilon$ on $]-\infty, T_0]$, we have

$$\int_{-\infty}^{T_0} |x_k(t)|^2\, dt \leq \varepsilon \int_{-\infty}^{T_0} \alpha(t)|x_k(t)|^2\, dt \leq \varepsilon A^2. \tag{ii}$$

Simiarly, since $\frac{1}{\alpha(t)} \leq \varepsilon$ on $[T_1, +\infty[$, we have

$$\int_{T_1}^{+\infty} |x_k(t)|^2\, dt \leq \varepsilon A^2. \tag{iii}$$

Combining (i), (ii) and (iii) we get $x_k \to 0$ in L^2.

Remark. The above result is due to C. De Coster and M. Willem see [5].

We now state the theorem of Rabinowitz and Tanaka. We shall use the following assumption:

(V6) There exists $\overline{W} \in C(\mathbb{R}^n, \mathbb{R})$ such that

$$|W(t,x)| + |W_q(t,x)| \leq |\overline{W}(x)|$$

for every $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$.

Theorem 1. *Suppose V satisfies* (V1)–(V6). *Then there exists a homoclinic orbit q of* (HS) *emanating from* 0 *such that*

$$0 < \int_{\mathbb{R}} \Big(\frac{1}{2}|\dot{q}|^2 - V(t,q)\Big)\,dt < +\infty. \tag{3}$$

To prove this theorem, we shall need some technical lemma's.

Lemma 2. *Suppose V satisfies* (V1)–(V6). *If $q_k \to q_0$ in X, then $W_q(t,q_k) \to W_q(t,q_0)$ in L^2.*

Proof. There is a $c_1 \geq 0$ such that

$$\sup_{k\in\mathbb{N}} \|q_k\|_{L^\infty} \leq c_1.$$

Assumptions (V4) and (V6) imply the existence of a $c_2 \geq 0$ such that

$$|W_q(t,q_k(t))| \leq c_2|q_k(t)|$$

for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$. Hence,

$$|W_q(t,q_k(t)) - W_q(t,q_0(t))| \leq c_2(|q_k(t)| + |q_0(t)|) \leq c_2(|q_k(t) - q_0(t)| + 2|q_0(t)|).$$

Since, by Lemma 1, $q_k \to q_0$ in L^2, passing to a subsequence if necessary, it can be assumed that

$$\sum_{k=1}^{\infty} \|q_k - q_0\|_{L^2} < +\infty.$$

But this implies $q_k(t) \to q_0(t)$ almost everywhere $t \in \mathbb{R}$ and

$$\sum_{k=1}^{\infty} |q_k(t) - q_0(t)| = v(t) \in L^2(\mathbb{R}).$$

Therefore,

$$|W_q(t,q_k(t)) - W(t,q_0(t))| \leq c_2(v(t) + 2|q_0(t)|).$$

Thus, using Lebesgue's convergence theorem, the lemma is proved.

Remark. The above argument is due to M. Ramos [2]. For $q \in X$, let

$$\varphi(q) = \int_{\mathbb{R}} \Big(\frac{1}{2}|\dot{q}|^2 - V(t,q)\Big)\,dt = \frac{1}{2}\|q\|^2 - \int_{\mathbb{R}} W(t,q)\,dt. \tag{4}$$

Then $\varphi \in C^1(X,\mathbb{R})$ and any critical point of φ is a classical solution of (HS) with $q(\pm\infty) = 0 = \dot{q}(\pm\infty)$; see e.g. [4].

Lemma 3. *Suppose V satisfies* (V1)–(V6). *Then φ satisfies the Palais-Smale condition.*

Proof. Let (q_k) be a sequence in X such that

$$\varphi(q_k) \to c, \quad \varphi'(q_k) \to 0 \quad \text{as } k \to \infty. \tag{5}$$

We show that (q_k) possesses a convergent subsequence. By (5), there is a constant $d \geq 0$ such that

$$\begin{aligned} d + \|q_k\| &\geq \varphi(q_k) - \frac{1}{\mu}\varphi'(q_k)q_k \\ &= \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|q_k\|^2 - \int_{\mathbb{R}} [W(t, q_k) - \frac{1}{\mu} W_q(t, q_k)q_k]\,dt \\ &\geq \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|q_k\|^2 \qquad \text{(by (V3))}. \end{aligned}$$

Hence, (q_k) is bounded. So passing to a subsequence if necessary, it can be assumed that $q_k \rightharpoonup q_0$ in X and hence, by Lemma 1, $q_k \to q_0$ in L^2. It follows from the definition of φ that

$$\begin{aligned} &(\varphi'(q_k) - \varphi'(q_0))(q_k - q_0) \\ &= \|q_k - q_0\|^2 - \int_{\mathbb{R}} [W_q(t, q_k) - W_q(t, q_0)](q_k - q_0)\,dt. \end{aligned} \tag{6}$$

Since $q_k \to q_0$ in $L^2(\mathbb{R})$, we have (see Lemma 2) $W_q(t, q_k) \to W_q(t, q_0)$ in $L^2(\mathbb{R})$. Hence, $\int_{\mathbb{R}} [W_q(t, q_k) - W_q(t, q_0)](q_k - q_0)\,dt \to 0$ in $\mathbb{R}$ and (6) implies that $q_k \to q_0$ in X.

Proof of Theorem 1. We use the usual Mountain Pass Theorem (see e.g. [1]) to prove the existence of a nontrivial critical point of φ. We already know that $\varphi \in C^1(X, \mathbb{R})$, $\varphi(0) = 0$ and φ satisfies the Palais-Smale condition. Hence, it suffices to prove that φ satisfies the following conditions:

1°) there are constants α and $\rho > 0$ such that $\varphi|_{\partial B_\rho} \geq \alpha$;
2°) there is a $q_0 \in X \setminus B_p$ such that $\varphi(q_0) \leq 0$.

By Lemma 1, there is a $c_0 > 0$ such that $\|q\|_{L^2} \leq c_0\|q\|$. On the other hand, there is $c_1 > 0$ such that $\|q\|_\infty \leq c_1\|q\|$. And by (V4), for all $\varepsilon > 0$, there is $\delta > 0$ such that $|W(t, x)| \leq \varepsilon|x|^2$ whenever $|x| \leq \delta$. Let $\rho = \frac{\delta}{c_1}$ and $\|q\| \leq \rho$; we have $\|q\|_\infty \leq \frac{\delta}{c_1} \cdot c_1 = \delta$. Hence, $|W(t, q(t))| \leq \varepsilon|q|(t)^2$ for all $t \in \mathbb{R}$. Integrating on $\mathbb{R}$, we get

$$\int_{\mathbb{R}} W(t, q)\,dt \leq \varepsilon\|q\|^2_{L^2} \leq \varepsilon C_0^2\|q\|^2.$$

So, if $\|q\| = \rho$, then

$$\varphi(q) = \frac{1}{2}\|q\|^2 - \int_{\mathbb{R}} W(t, q)\,dt \geq \Big(\frac{1}{2} - \varepsilon c_0^2\Big)\|q\|^2 = \Big(\frac{1}{2} - \varepsilon c_0^2\Big)\rho^2.$$

And it suffices to choose $\varepsilon = \frac{1}{4c_0^2}$ to get

$$\varphi \geq \frac{\rho^2}{4} = \alpha > 0. \tag{7}$$

Consider

$$\varphi(\sigma q) = \frac{\sigma^2}{2}\|q\|^2 - \int_{\mathbb{R}} W(t, \sigma q)\, dt$$

for all $\sigma \in \mathbb{R}$. By (V3), there is a continuous function $\alpha_1(t) > 0$ such that

$$W(t,x) \geq \alpha_1(t)|x|^\mu \text{ for all } |x| \geq 1.$$

Let $\overline{q} \in X$ be such that $|\overline{q}(t)| \geq 1$ on an open and non-empty interval $I \subset \mathbb{R}$. For any $\sigma \geq 1$, we have

$$\varphi(\sigma\overline{q}) \leq \frac{\sigma^2}{2}\|\overline{q}\|^2 - \int_I W(t, \sigma\overline{q})\, dt \leq \frac{\sigma^2}{2}\|\overline{q}\|^2 - \sigma^\mu \int_I \alpha_1(t)|\overline{q}(t)|^\mu\, dt.$$

Since $\mu > 2$, we can find a $\sigma \geq 1$ such that $\|\sigma\overline{q}\| \geq R > \rho$ and

$$\varphi(\sigma\overline{q}) \leq 0 = \varphi(0). \tag{8}$$

Theorem 2. *Suppose V satisfies* (V1)–(V6). *If*

(V7) $W(t,-x) = W(t,x)$ *for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$,*

then there exists an unbounded sequence in X of homoclinic ordits of (HS) *emanating from* 0.

Proof. The condition (V7) implies that φ is even and by (V1)–(V6) and Lemma 3, we already know that $\varphi \in C^1(X,\mathbb{R})$, $\varphi(0) = 0$ and φ satisfies the Palais-Smale condition. To apply the symmetric Mountain Pass Theorem (see e.g. Theorem 9.12 in [3]), it suffices to prove that φ satisfies the following conditions.

1°) There is constants ρ and $\alpha > 0$ such that

$$\varphi|_{\partial B_\rho} \geq \alpha. \tag{9}$$

2°) For each finite dimensional subspace $\widetilde{X} \subset X$, there is $R = R(\widetilde{X})$ such that $\varphi \leq 0$ on $\widetilde{X} \setminus B_R$.

1°) is identically the same as in Theorem 1, so it is already proved.

We prove 2°). Let $\widetilde{X} \subset X$ be a finite dimension subspace. Consider $q \in \widetilde{X}$ with $q \neq 0$. By (2), there is a continuous function $\alpha_1(t) > 0$ such that

$$W(t,q(t)) \geq \alpha_1(t)|q(t)|^\mu \quad \text{for all } |q(t)| \geq 1. \tag{10}$$

Hence,

$$\int_{|q(t)|>1} W(t,q(t))\, dt \geq \int_{|q(t)|>1} \alpha_1(t)|q(t)|^\mu\, dt \tag{11}$$

and

$$\varphi(q) = \frac{1}{2}\|q\|^2 - \int_{|q(t)|>1} W(t,q(t))\,dt - \int_{|q(t)|\leq 1} W(t,q(t))\,dt.$$

Now for all $q \in \widetilde{X}$, we have $\|q\|^2 \leq c\|q\|_\infty^2$, where $c = c(\widetilde{X})$.

Let us define $m = \inf_{\substack{\|q\|_\infty = 2 \\ q \in X}} \int_{|q(t)|>1} \alpha_1(t)|q(t)|^2\,dt$. If $m = 0$, we will have $|q(t)| \equiv 0$ for all $t \in \{t\colon |q(t)| > 1\}$ which contradicts $\|q\|_\infty = 2$. Thus, $m > 0$ and we obtain

$$\begin{aligned}
\varphi(q) &\leq C\|q\|_\infty^2 - \int_{|q(t)|>1} W(t,q(t))\,dt - \int_{|q(t)|\leq 1} W(t,q(t))\,dt \\
&\leq C\|q\|_\infty^2 - \int_{|q(t)|>1} \alpha_1(t)|q(t)|^\mu\,dt \\
&\leq C\|q\|_\infty^2 - \frac{\|q\|_\infty^\mu}{2\mu} \int_{|q(t)|>1} \alpha_1(t) 2^\mu \frac{|q(t)|^\mu}{\|q\|_\infty^\mu}\,dt \\
&\leq C\|q\|_\infty^2 - \frac{m}{2\mu}\|q\|_\infty^\mu.
\end{aligned}$$

Since $\mu > 2$, we deduce that there is an $R = R(\widetilde{X})$ such that $\varphi(q) \leq 0$ whenever $\|q\|_\infty \geq \rho$. Hence, by the Theorem 9.12 in [3], φ possesses an unbounded sequence of critical values (c_j) with $c_j = \varphi(q_j)$, where q_j is such that

$$0 = \varphi'(q_j)q_j = \|q_j\|^2 - \int_{\mathbb{R}} W_q(t,q_j)q_j\,dt$$

so that

$$\int_{\mathbb{R}} W_q(t,q_j)q_j\,dt = \|q_j\|^2. \tag{12}$$

Thus, we have

$$c_j = \int_{\mathbb{R}} \left[\frac{1}{2}W_q(t,q_j)q_j - W(t,q_j)\right] dt. \tag{13}$$

Since $c_j \to \infty$ as $j \to \infty$, (V3), (12) and (13) imply that (q_j) is unbounded in X.

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