Homoclinic orbits in reversible systems and their applications in mechanics, fluids and optics [1]

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Abstract This survey article reviews the theory and application of homoclinic orbits to equilibria in reversible continuous-time dynamical systems, where the homoclinic orbit and the equilibrium are both reversible. The focus is on even-order reversible systems in four or more dimensions. Local theory, generic argument, and global existence theories are examined for each qualitatively distinct linearisation. Several recent results, such as coalescence caused by non-transversality and the reversible orbit-flip bifurcation are covered. A number of open problems are highlighted. Applications are reviewed to systems arising from a variety of disciplines. With the aid of numerical methods, three examples are presented in detail, one of which is infinite dimensional.

1 Introduction

Classical Hamiltonian dynamical systems with quadratic kinetic energy are reversible in the sense that they are invariant under a reversal of time and all momentum variables. The concept of reversible systems in their own right, rather than an extra property of Hamiltonian systems, goes back to Devaney [56]. For even dimensional systems, the definition of a reversibility we shall adopt here is that there is a linear involution R that fixes half the phase variables and under which the system is invariant after time-reversal. That is

$$\dot{x} = f(x), \quad x \in \mathbb{R}^{2n}, \quad Rf(x) = -f(Rx), \quad R^2 = \mathrm{Id}, \quad \mathcal{S} = \mathrm{fix}(R) \cong \mathbb{R}^n. \quad (1)$$

The linear subspace $\mathcal{S}$ is sometimes referred to as the *symmetric section* of the reversibility. There are more general definitions of non-linearly reversible systems [56] and reversibility for odd-order systems [9,140], but these will not concern us here.

[1] submitted to Physica D special issue for Warwick Symposium on Reversible Symmetries in Dynamical Systems, Dec 1996. Revised Version July 1997.

Many theorems concerning Hamiltonian systems have counterparts for reversible systems. One example of such a result is that *homoclinic* trajectories which are themselves symmetric under the reversibility are of codimension-zero, that is, they persist under generic perturbation that preserves reversibility [56]. Such a homoclinic orbit $\Gamma = \{\gamma(t) | t \in I\!R\}$ is defined such that $\gamma(t)$ is a solution to (1) satisfying

$$\gamma(t) \to x^* \text{ as } t \to \pm\infty, \quad \gamma(0) \in \mathcal{S}, \quad \text{where} \quad f(x^*) = 0, \; x^* \in \mathcal{S}.$$

Homoclinic orbits of dynamical systems are important in applications for a number of reasons. First, they may be 'organising centres' for the dynamics in their neighbourhood. From their existence one may, under certain conditions, infer the existence of chaos nearby (e.g. shift dynamics associated with Smale horseshoes [144,141,142]) or the bifurcation behaviour of periodic orbits. Second, if the dynamical system arises as the travelling wave equation for a partial differential equation or system, then homoclinic solutions of it describe solitary waves, which are of importance in many fields. Examples of reversible systems of this type arise in water wave theories and optical pulse propagation; see Section 2.1 for examples and Sections 5.1 and 5.3 for two particular case studies. Finally, Hamiltonian and reversible dynamical systems may arise as (spatial) boundary value problems on the real line. Homoclinic solutions of such equations are often fundamental as they represent 'ground states' or 'localised modes'. Typical examples occur in structural mechanics; again see Section 2 for some references and see Section 5.2 for a detailed case study on an infinite-dimensional example.

The aim of this article is to review the current state of the theory of homoclinic orbits in reversible systems and also to list where this theory is of importance in applications. It will be a broad commentary on what is known via normal form theory, generic argument, existence theory and numerics. We aim to stress what generically occurs regarding multiplicity of secondary homoclinic solutions and the dynamics in a neighbourhood of a primary homoclinic orbit. We shall not state general theorems, but it will be noted where the rigourous theory requires important non-degeneracy assumptions. At all stages we shall keep applications in mind. If nothing else, it is hoped that this will be a first search point for both the pure and applied literature on homoclinic bifurcations in reversible systems. We also highlight open problems.

Since reversibility and Hamiltonian structure are closely related (systems may possess either, both or neither properties) we shall aim to highlight similarities and differences between the two cases. In practice many Hamiltonian systems are reversible and *vice versa*. Almost all the examples we touch upon will posses both properties. For such systems the reversibility can be more useful than the Hamiltonian structure, e.g. in defining approximating boundary-value problems for numerical investigation of homoclinic orbits (see Section 4.3). In fact, the only property of Hamiltonian systems we shall be interested in is that they conserve a first integral (the Hamiltonian function). Therefore, from now on, we shall consider the more general concept of 'conservative' rather than 'Hamiltonian' systems.

Principally, this article shall consider only symmetric homoclinic orbits to symmetric equilibria in even-order continuous-time reversible systems. Other phenomena — such as heteroclinic orbits, homoclinic orbits to periodic orbits (equivalently homoclinic orbits in non-autonomous or discrete-time dynamical systems), systems with additional symmetry and odd-order reversible systems — are not treated systematically, if at all. We do however remark on some intricate results on homoclinic, heteroclinic, and associated periodic orbits for a third-order reversible system arising as the travelling wave equation for the Kuramoto-Sivashinsky Equation [103,143,126,94,93,95,107]. There are also interesting results concerning homoclinic orbits to infinity in the Falkner-Skan equation [74,149].

We shall also restrict ourselves to the lowest possible dimension of system displaying the phenomena of interest. This may appear a gross simplification, but recently a so-called 'homoclinic center-manifold' theorem has been proved by Sandstede [135,136]. Here it is shown that given an homoclinic orbit in an arbitrary (even infinite) dimensional system, then there exits an invariant manifold along the homoclinic solution that is at least class C^1 and which contains all recurrent dynamics in a neighbourhood of the homoclinic orbit. The dimension of this manifold depends on the linearisation at the equilibrium and on the nature of the homoclinic trajectory itself. Roughly speaking, it is the dimension of the smallest possible phase space in which the particular homoclinic solution may generically arise.

The rest of the paper is outlined as follows. Section 2 introduces a fourth-order equation with general nonlinearity that serves as a paradigm for a wide class of reversible systems arising in applications. Its linearisation at the origin contains all possible qualitatively distinct combinations of eigenvalues. This includes four open parameter regions and four codimension-one curves, three of which correspond to local bifurcations. The normal form theory close to each of these local bifurcations is reviewed, with emphasis on consequent existence of homoclinic solutions. Section 3 then goes on to consider what is known about the nature of homoclinic orbits in each of the four open parameter regions. Section 4 concerns certain global issues, such as methods for proving existence (and stability of solitary waves) and the disappearance of homoclinic solutions as parameters vary. In applications, one usually has to turn to numerical methods for deciding what precisely (out of a generic set of alternatives) happens globally. Therefore, Section 4 concludes by considering numerical methods for homoclinic solutions in reversible systems. The theory and numerics from the preceding sections are applied to three examples in some detail in Section 5: a travelling-wave system derived as an approximation to the water wave problem with surface tension; the buckling of long cylindrical shells; and coupled nonlinear Schrödinger (NLS) equations arising in nonlinear optics with quadratic, or $\chi^{(2)}$, nonlinearity.

2 A canonical fourth-order equation; local bifurcations

In what follows we shall adopt the abbreviated notation that *reversible* applied to a system of ODEs means that the system is invariant under a given reversibility, and that an orbit is said to be *symmetric* if it is itself invariant under the reversibility. If there is any confusion as to which reversibility we are referring to then we shall say *R-reversible* etc.

2.1 The equation and its applications

For illustrative purposes, for much of this paper we shall tailor our discussion to a fourth-order reversible system that can be written as the single equation

$$u'''' - bu'' + au = f(u, u', u'', u'''), \tag{2}$$

where a and b are real parameters and f is a nonlinear function whose Taylor series vanishes at the origin together with its first derivatives, and the dependence of f on u' and u''' occurs as sums of even order products of these two variables. Viewed as a dynamical system in the phase-space variables (u, u', u'', u'''), (2) is then reversible under the standard reversibility of classical mechanical systems where the sign of time, velocity and all odd-order derivatives are reversed:

$$t \to -t \quad \text{and} \quad R : (u, u', u'', u''') \mapsto (u, -u', u'', -u'''). \tag{3}$$

If, in addition, f has only odd-order terms in its Taylor series then (2) has odd symmetry and is additionally reversible under the convolution of R with minus the identity $-R : (u, u', u'', u''') \mapsto (-u, u', -u'', u''')$.

Another possibility is that f is a pure function of u, independent of its derivatives. Then (2) can be rewritten as a conservative system with conserved first integral

$$H = u'u''' - \frac{b}{2}u'^2 - \frac{1}{2}u''^2 + \frac{a}{2}u^2 - \int_0^u f(v)\,\mathrm{d}v. \tag{4}$$

In fact, such a system can be written in classical Hamiltonian form with the first two terms of (4) representing kinetic energy and the last three terms potential energy. Note that there are other choices of the function $f(u, u', u'', u''')$ that lead to conservative systems (see, for example, [96, eq. (2)]).

We shall be interested in symmetric homoclinic orbits to the origin of (2). While for certain f there may be other equilibria, these will not be of concern and 'homoclinic' will be used synonymously with 'homoclinic to the origin' and $W^{s,u}$ will be used exclusively to denote the stable and unstable manifolds of the origin.

Equations of the form (2) arise in many contexts. The simplest case is when $f = u^2$ which comes from (at least) two distinct fields. One derivation is in describing travelling wave solutions of the Korteweg-de Vries (KdV) equation with an additional fifth-order dispersive term, which can be written in a form commensurate with (2) as

$$v_t = v_{xxxxx} - bv_{xxx} + 2vv_x. \tag{5}$$

Specifically, upon setting $v = u(x - at)$, taking a first integral and choosing the constant of integration to be zero (a necessary condition for solitary waves which asymptote to zero) one recovers (2). This 5th-order KdV equation ((5) or scalings thereof) has been used to describe chains of coupled nonlinear oscillators [130], hydro-magnetic wave propagation in plasmas [90,92] and most notably gravity-capillary shallow water waves [73,83,161]. In fact, Buffoni, Groves & Toland [25] have shown that (5) arises as the centre manifold reduction of the exact formulation, using the Euler equation and free surface condition, near certain critical parameter values of this water wave problem. Extended 5th-order KdV equations, (5) with extra nonlinear terms in u, u_x and u_{xx} on the right-hand side, have been considered by a number of authors [97,96,124,34,53,37]. Derivations arise in various more accurate approximations to waves on shallow water (see the references in [124] and [96]). Section 5.1 below reviews some recent results on an extended 5th-order KdV equation. A related derivation of (2) by Christov *et al* [50] is via a Bousinesq equation with second, fourth and sixth-order spatial derivatives, describing longitudinal vibrations in nonlinear chains.

The other derivation of (2) with $f = u^2$ is in describing the displacement $u(x)$ of a compressed strut with bending stiffness resting on a nonlinear elastic foundation with dimensionless restoring force proportional to $u - u^2$ [79,81]. Other related structural models which do not have precisely the form (2) are those modelling struts with geometric nonlinearity [82], torsionally strained rods [8,128,43,154] and thin cylindrical shells [80,122]. The latter of these is discussed in Section 5.2 below.

Another nonlinearity of note is $f(u) = u^3$. Here a scaling of (2) arises as steady state solutions of

$$u_t = -\gamma u_{xxxx} + u_{xx} + u - u^3, \tag{6}$$

which has been given the name Extended Fischer Kolmogorov (EFK) owing to its limit as $\gamma \to 0$ [131,132]. When $\gamma > 0$, the solutions of interest are kinks, that is heteroclinic solutions connecting the equilibria at $u = \pm 1$. The ODE on the right-hand side of (6) (in fact a trivial scaling of it) also describes travelling wave solutions of the nonlinear Shrödinger equation with an additional fourth-order dispersion term

$$\mathrm{i}v_z + \alpha v_{tt} - \epsilon v_{tttt} + |v|^2 v = 0 \tag{7}$$

[28,91]. The ODE is obtained by setting $v(z,t) = u(t)\mathrm{e}^{\mathrm{i}cz+\phi_0}$, assuming u to be real.

Another important nonlinearity in (2) is the piecewise-linear function $f(u) = (u+1)H(-u-1)$, where H denotes the usual Heaviside function. This equation, describing travelling waves of the second-order-in-time PDE

$$u_{tt} + u_{xxxx} + u = f(u), \tag{8}$$

was proposed as a model of a non-linearly suspended beam such as a suspension bridge [125,47,41]. The associated ODE also arises in studies on pipeline buckling with the piecewise linearity modelling the effect of lift-off from an elastic bed [17], and in the compression of a railway line under the movement of a train [114]. Chen & McKenna [47] also describe a smooth approximation of this equation with $f = 1 + u - e^u$, solitary wave solutions of which appear to have soliton-like interaction behaviour.

2.2 Linearisation and normal forms

In order to examine the structure of homoclinic solutions to (2) the first step is to consider its linearisation. Eigenvalues λ of the linear problem satisfy the characteristic equation $\lambda^4 - b\lambda^2 + a = 0$, from which it can be inferred that the linearisation is as depicted in Figure 1. Note that the spectrum is symmetric under reflection about the imaginary axis. It is immediate from the definition of reversibility that this symmetry in the spectrum holds for all symmetric equilibria [56]. In the figure, four distinct regions of the parameter plane have been identified corresponding to qualitatively distinct linear dynamics. The four regions are bounded by the following codimension-one curves (see also

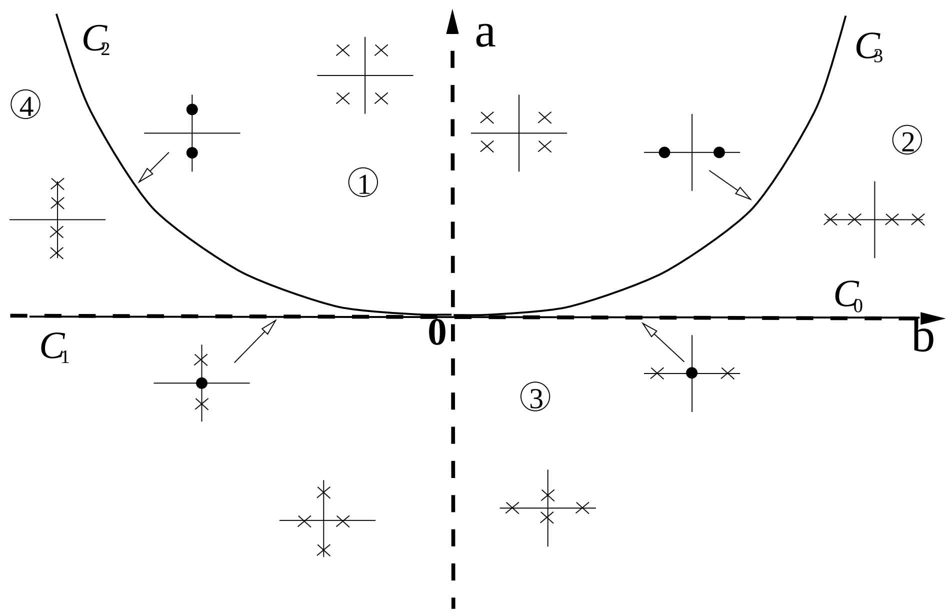

Fig. 1. Linearisation at the origin of (2)

Table 1): C_0 given by $a = 0$, $b > 0$ on which there are two zero eigenvalues and two real; C_1 given by $a = 0$, $b < 0$ on which there are two zero eigenvalues and two imaginary; C_2 given by $b = 2\sqrt{a}$, on which there is a double complex conjugate pair of imaginary eigenvalues $\pm \mathrm{i}\sqrt{b/2}$; and C_3 given by $b = -2\sqrt{a}$, on which there are two double real eigenvalues, symmetric with respect to the imaginary axis $\pm\sqrt{b/2}$.

The four curves meet at a codimension-two point $a = b = 0$ where the linearisation of the system corresponding to (2) has four zero eigenvalues with geometric multiplicity one. Iooss [85] has derived and analysed a normal form respective to such a codimension-two point arising in general fourth-order reversible vector fields for which the origin is a persistent symmetric fixed point. In particular, he computes the singular scaling that has to be applied to the (codimension-one) normal forms with respect to C_i, $i = 0, 1, 2$, originally analysed by Iooss and Kirchgassner [88] and Iooss and Peroueme [89] (see also [63] for the techniques involved in normal form reduction).

Let us briefly review these three one-parameter normal forms (more generally than in specific application to (2)) and what may be gleaned from them concerning the existence of homoclinic orbits. First, however, note that in regions 1 and 2 displayed in Fig. 1, the origin is hyperbolic. This then implies that homoclinic orbits to the origin are of co-dimension zero. More precisely, a homoclinic orbit is formed by the intersection of the unstable manifold of the origin W^u and the *symmetric section* $\mathcal{S} := \text{fix}(R)$. If such an intersection is *transverse*, then homoclinic orbits must persist under small reversible perturbation. Moreover, transversality of two 2-dimensional manifolds in $\mathbb{R}^4$ is a generic condition (but often hard to prove in examples, although see Section

4.1 below for a weaker requirement).

Near C_0; the eigendirections associated with the non-zero eigenvalues are unimportant in describing bifurcating solutions and hence center-manifold reduction can be used to reduce to a planar system. On the centre-manifold, reversible systems with this linearisation and no further degeneracy can be reduced by normal form analysis to the nonlinear oscillator system [88, Sect. 3.1]

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \sum_{j=1}^{\infty} c_j(\mu) x_1^j. \tag{9}$$

Here μ is a single parameter unfolding the degeneracy with $\mu > 0$ corresponding to the near-zero eigenvalues being real (region 2). Upon truncation to lowest order (assuming appropriate normal form coefficients $c_j(\mu)$ are non-zero) and application of a rescaling in which x_1 and x_2 become $\mathcal{O}(\mu)$, (9) becomes

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \mathrm{sign}(\mu)\, x_1 - \frac{3}{2} x_1^2. \tag{10}$$

For $\mu > 0$, (9) possesses a unique, symmetric homoclinic solution $x_1(t) = \mathrm{sech}^2(t/2)$. Note that the reversibility R in (9) and (10) has been reduced to $x_2 \to -x_2$.

A further reversibility of the form $-R$ would be reflected in the right-hand-side of the second equation of (9) being odd. Hence the truncated normal form would read

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \mathrm{sign}(\mu)\, x_1 - \beta x_1^3. \tag{11}$$

where $\beta = \pm 1$. If $\beta = -1$ then there is an R-symmetric pair of homoclinic solutions $x_1(t) = \pm\mathrm{sech}(t)$ existing for $\mu > 0$. Given the other sign of β there are no homoclinic solutions of the normal form.

In general, it is not true that solutions of a normal form truncated at any order persist under a perturbation that returns to the original system, unless it can be proved that the truncated normal form is a *topological normal form* (or versal unfolding); see, for example, [106, Sect. 2.5]. For the planar normal forms (10) or (11) it is not hard to show the persistence of the homoclinic solution for $\mu > 0$ since the equilibrium is hyperbolic and W^u and $\mathcal{S}$ intersect transversally (see [88, Sect. 4], [98, Prop. 5.1]).

Near C_1; Iooss and Kirchgassner [88, Sect. 3.2] derive the four-dimensional normal form in two real variables x_1 and x_2 and one complex variable z

$$\begin{aligned} \dot{x}_1 &= x_2, & \dot{x}_2 &= \Phi(\mu, x_1, |z|^2), \\ \dot{z} &= iz\Psi(\mu, x_1, |z|^2), & \dot{\bar{z}} &= -\mathrm{i}z\Psi(\mu, x_1, |z|^2), \end{aligned} \tag{12}$$

where Φ and Ψ are arbitrary order polynomials in their arguments. The system has two conserved integrals

$$K := |z|^2 \quad \text{and} \quad H := x_1^2 - \int_0^{x_1} \Phi(\mu, s, K)\, \mathrm{d}s.$$

A truncated normal form is obtained upon taking

$$\Phi(\mu, x_1, |z|^2) = c_1(\mu)x_1 + c_2(\mu)x_1^2 + d_2(\mu)|z|^2, \quad \Psi(\mu, \alpha_0, |z|^2) = \gamma_0(\mu) + \gamma_1(\mu)x_1.$$

If the choice $K = 0$ is taken for the first integral, then under a combination of the signs of the coefficients of c_1 and c_2 which corresponds to region 3 of Fig. 1, there is a sech^2 homoclinic solution (or a pair of sech solutions if the system additionally has $(-R)$-reversing symmetry and the correct sign of the cubic coefficient of Φ.). However, in this case the question of persistence is much harder because the equilibrium is not hyperbolic in region 3 and in general we do not expect these solutions to persist when we add small reversible perturbations to the normal form. See section 3.3 below for a further discussion of homoclinic orbits in this region. However, it is known that the normal form possesses a large class of solutions that are homoclinic to periodic orbits, which may be parametrised by the amplitude of the limiting periodic orbit. However, for a class of fourth-order equations including (2) with $f = u^2$ it is known that the limiting periodic orbit of amplitude zero (the equilibrium at the origin) does not have a homoclinic connection [4,120]. For (2) it has been shown both via asymptotics and rigourously that the amplitude of the periodic orbits to which homoclinic connections occur is an exponentially small function of μ [134,7,66,147] and hence cannot be captured by the normal form (12) truncated to any order. These results have been generalised by Lombardi [118,119] to cover generic reversible systems whose normal forms are given by (12), which includes the exact formulation of capillary-gravity water waves (see also [11,146,148]).

Near C_2; Iooss and Perouème [89] analysed solutions of the following normal form originally derived in [63]. It is expressed in terms of two complex variables

z_1 and z_2

$$\begin{aligned} \dot{z}_1 &= \mathrm{i}\omega_0 z_1 + z_2 + \mathrm{i} z_1 \Phi_0(\mu, |z_1|^2, (i/2)(z_1\overline{z}_2 - \overline{z}_1 z_2)), \\ \dot{z}_2 &= \mathrm{i}\omega_0 z_2 + \mathrm{i} z_2 \Phi_0(\mu, |z_1|^2, (i/2)(z_1\overline{z}_2 - \overline{z}_1 z_2)) + z_1 \Phi_1(\mu, |z_1|^2, (i/2)(z_1\overline{z}_2 - \overline{z}_1 z_2)). \end{aligned} \tag{13}$$

Here Φ_0 and Φ_1 are arbitrary order real polynomials in their arguments, $\pm\omega_0$ are the values of the imaginary eigenvalues on C_2 and μ is an unfolding parameter, with $\mu < 0$ corresponding to region 2 (four complex eigenvalues) and $\mu > 0$ to region 1 (four imaginary eigenvalues). This normal form is rotationally symmetric, that is, invariant under $(z_1, z_2) \to (z_1 \mathrm{e}^{\mathrm{i}\phi}, z_2 \mathrm{e}^{i\phi})$ as well as the reversibility $z_2 \to -z_2$, $t \to -t$. Moreover, (13) has two conserved integrals

$$K := z_1\overline{z}_2 - \overline{z}_1 z_2, \quad \text{and} \quad H := |z_2|^2 - \int_0^{|z_1|^2} \Phi_1(\mu, s, K)\, \mathrm{d}s.$$

A truncated normal form is derived by replacing P and Q by the first-order terms in a Taylor expansion

$$\Phi_0(\mu, u, v) = p_1\mu + p_2 u + p_3 v, \quad \Phi_1(\mu, u, v) = q_1\mu + q_2 u + q_3 v. \tag{14}$$

By further reduction to polar co-ordinates and seeking periodic solutions, Iooss & Perouème argue that if q_2 is negative, the so-called *subcritical* case, then a 1-parameter family (orbit of the rotation group) of homoclinic solutions bifurcates for $\mu < 0$. These solutions are of the form of 'envelope' homoclinic solutions with oscillating tails $z_1(t) = A\,\mathrm{sech}(Bt)\,\exp(\mathrm{i}\omega\theta)$, for constants A, B and ω related to the coefficients of the normal form. In contrast, in the supercritical case there is the bifurcation of homoclinic orbits to periodic orbits for $\mu > 0$.

Now, in the subcritical case, the homoclinic solutions exist in the parameter region where the origin is hyperbolic. Hence, we should expect that transverse homoclinic solutions to the truncated version of (13) persist. It is not difficult to see that given a single reversibility, two solutions among the one-parameter family from transverse intersections between W^u and $\mathcal{S}$. However, to show that these transverse symmetric solutions persist is actually non-trivial since the size of the terms ignored in the truncated normal form are not small compared with the bifurcating solutions [89, Sect. IV.3]. If there is another reversibility $-R$ then there is another symmetric section and two more homoclinic solutions which are $(-R)$-symmetric will also persist.

Note that each of the normal forms (9), (12) and (13) is equivalent to a *completely integrable* conservative system no matter to what order they are

	codim	eigs.	name	theory	application
C_0	one	$0, 0, \pm\lambda$	Hamiltonian pitchfork; reversible Takens-Bogdanov	[88] [98]	water waves [15,3,37]; twisted rods [154]
C_1	one	$0, 0, \pm i\omega$	'Hamiltonian pitchfork-Hopf'; 'reversible Takens-Bogdanov-Hopf'	normal form [88]; homoclinics to periodics [118]	5th-order KdV [7,66,147]; water waves [11,146]
C_2	one	$\pm i\omega, \pm i\omega,$	Hamiltonian-Hopf; reversible 1:1 resonance Hopf	[87], [89], [24]	5th-order KdV [67,37]; water waves [24]; nonlinear struts [79]; twisted rods [154]; cylindrical shells [80]
C_3	one	$\pm\lambda, \pm\lambda$	'Belyakov-Devaney'	generic dynamical systems [14,13]; special Hamiltonians [44]	water waves [25]; 5th-order KdV [23]
0	two	$0, 0, 0, 0$	quadruple zero point (4×4 Jordan block)	[85]	water waves [98]; twisted rods [43,154]
1	zero	$\pm\lambda \pm i\omega$	saddle-focus	general reversible [71,72,36]; conservative [55,57,108]; variational [26]	5th-order KdV [161,23,37]; water waves [58,60]; struts [81,82]; beams [41]; EFK [131,132]; 4th-order NLS [28,91]
2	zero	$\pm\lambda_1, \pm\lambda_2$	saddle	uniqueness, e.g. [5]; orbit flip [139]; Hamiltonian multiplicity [77,152]	KdV and NLS e.g. [1]; 5th-order KdV [6,23,37]; optical fibres [139]; $\chi^{(2)}$ optics [30].
3	zero	$\pm\lambda, \pm i\omega$	saddle-center	Hamiltonian case [129,109,101,102]	5th-order KdV [70,4]; Hénon-Heiles [51,113]; double pendulum [129]; three-body problem [112,129]; string dynamics [129].
4	zero	$\pm i\omega_1, \pm i\omega_2$	focus	no homoclinics known; general dynamics [9,140]	

Table 1
A summary of the different parameter regimes represented in Figure 1. Names in quotations are proposed new names. In each case we give a (partial) list of references to both theory and application of homoclinic orbits in reversible systems with the given linearisation

truncated. This explains why, at least for the non-planar normal forms (12) and (13), they can never be topological normal forms for general reversible systems with the given degeneracy and why one has to look at the persistence of homoclinic orbits separately in each case. Note also that the normal form (9) always gives the small amplitude bifurcation of homoclinic solutions, whereas for (11) and (13) there is a condition to check. Which brings us to our first open problem

Problem 1 *Analyse an unfolding of the transition between super and sub-critical cases for the normal forms* (11) *and* (13)*.*

For the bifurcation near C_2, this question has been partially addressed by Dias and Iooss [59] and Iooss [86] who consider the degenerate case when q_2 vanishes in (14).

Near C_3; there is no small-amplitude bifurcation since the equilibrium remains hyperbolic. The nature of the equilibrium changes from being a saddle to being a saddle-focus (from four real to four complex eigenvalues) as one crosses C_3 from region 2 to 1. Moreover, owing to hyperbolicity there is no barrier to finite-amplitude homoclinic orbits making such a crossing. Nonetheless, as we shall see in the example in section 5.1 below, there is a dramatic non-local bifurcation which occurs causing the immediate birth of an infinite multiplicity of homoclinic orbits. This bifurcation has been partially analysed for a specific form of Hamiltonian system in [44]. For the non-Hamiltonian, non-reversible case, the equivalent (codimension-two) bifurcation was analysed by Belyakov [14]. In [13] this was given the name 'broom' bifurcation owing to the infinity of loci of homoclinic orbits becoming tangent to the primary branch (e.g. see Fig. 4 below, for P close to -2). In the conservative case, the dynamics close to a homoclinic orbit in region 1 was originally analysed by Devaney [55] (see Section 4.1 below), therefore a suitable name for the bifurcation upon crossing C_3 (at least for conservative systems) might be 'Belyakov-Devaney'.

In the following section, we review more generally what is known about homoclinic orbits to equilibria with the linearisation in each of the parameter regions of Figure 1. Table 1 summarises this information and the normal form results outlined above, together with a (partial) list of references to applications.

3 The four parameter regions

Recall that homoclinic orbits to hyperbolic equilibria are of codimension zero in reversible systems. Furthermore, it has been proved by Devaney [56,57] that each transverse homoclinic orbit must be accompanied by a one-parameter family of periodic orbits at fixed parameter values. The periodic orbits accumulate on the homoclinic orbit with period approaching infinity. That homoclinic orbits are generic and accompanied by families of symmetric periodic orbits is also true for conservative systems, reversible or otherwise [156]. However, the transversality required for persistence in the conservative case is different; that the stable and unstable manifolds intersect transversally within a level set of the conserved integral.

We shall now consider the existence of homoclinic orbits and the implications thereof in each of regions 1-4. Although we have systems of the form (2) in mind, our considerations will apply more generally to reversible systems in $\mathbb{R}^4$ having the given linearisation at the origin. At the same time, upon appealing to the Homoclinic Centre Manifold Theorem, much of what we say will apply to higher-dimensional systems also.

3.1 The saddle-focus case; multiplicity

In region 1, where the origin is a saddle focus, Härterich [71,72] shows that the existence of one transverse symmetric homoclinic orbit implies the existence of infinitely many others (see also [36]). The extra homoclinic solutions are like multiple copies of the primary orbit separated by finitely many oscillations close to the equilibrium. Specifically, there are infinitely many symmetric N-pulses for each $N > 1$. Here, an *N-pulse*, or N-modal, homoclinic orbit is defined to be contained in a tubular neighbourhood of the primary orbit Γ in phase space, crossing a transverse section to Γ N times.

The method of proof in [71] is to employ a Shil'nikov-type analysis (see, e.g. [142,158,106]). Specifically, one assumes the existence of a primary homoclinic orbit and then C^1 linearises the system in a neighbourhood of the equilibrium at the origin (which is possible in this case due to a theorem of Belitski [12]). The spiraling of the linear dynamics caused by the complex eigenvalues is then used to show that the a neighbourhood of the primary homoclinic orbit in the unstable manifold W^u cannot fail to intersect appropriate pieces of the symmetric section $\mathcal{S}$ infinitely many times.

It is possible to label each N-pulse orbit so constructed by a string of integers $N(i_1, i_2, \ldots, i_{N-1})$, $i_n \in \mathbb{N}$, where each i_n counts the number of oscillations near the equilibrium between passages close to Γ. For a class of conservative reversible systems including (2) with $f = u^2$, it is possible to make this labeling precise [44,23]. An interesting question is for which strings do there necessarily exist orbits. Due to reversibility, all N-pulses constructed in this way will have symmetric labels; $i_n = i_{N-n}$ for all $1 \leq n \leq N-1$. Moreover, there are further restrictions. The lowest N for which the the combined results of [71,36] do not give N-pulses corresponding to every possible symmetric string of N positive integers is $N = 6$. Specifically, it was argued in [36] that given i_1 and i_2 then it can only be guaranteed that there are 6-pulses of the form $6(i_1, i_2, i_3, i_2, i_1)$ for finitely many i_3, unless $i_1 = i_2$, $i_1 = i_3$ or $3(i_1, i_2)$ exists for some other reason. That this non-existence is not a property of the construction, was backed up by careful numerical experiments in [36] on the equation (2) with $f = \frac{1}{2}(u^2 + u'^2)$. Note the corresponding dynamical system is reversible but not conservative. Recent work by Sandstede [137], using using a Liapunov-Schmidt type method due to X.B. Lin [115], gives a more precise statement. For 6-pulses this gives that for each $k_1, k_2 \in \mathbb{N}$, there exists $M(k_1, k_2)$ such that 6-pulses with labels $6(n, n + k_1, n + k_2, n + k_1, n)$ exist for all $n > M(k_1, k_2)$.

If the system is conservative, then there is a different theory describing the dynamics in a neighbourhood of a saddle focus homoclinic orbit, due originally to Devaney [55], see also [158,109]. This theory shows that given transversality (in the conservative sense mentioned above) then within the level set of H containing the equilibrium there is a Smale horseshoe; specifically shift dynamics associated with an infinite number of symbols. Although not explicitly stated in Devaney's analysis, inherent in this construction is the existence of infinitely many N-pulses, one orbit corresponding to *each* possible string $N(i_1, i_2, \ldots, i_{N-1})$ [13,23]. Hence for systems which are conservative in addition to being reversible (like (2) with f a pure function of u) there are a greater multiplicity of multi-pulse orbits than could be inferred from the reversible structure alone, both asymmetric *and* symmetric. Upon adding a perturbation that breaks the conserved quantity but preserves reversibility, clearly the asymmetric orbits should instantaneously disappear because they are of codimension one for the reversible system. However, there is an interesting open question about what happens to the the extra symmetric solutions, because they should persist if they involve the transverse intersection of W^u and $\mathcal{S}$:

Problem 2 *Consider the reversible system* $u'''' - bu'' + au = (1-\alpha)u^2 + \alpha u'^2$. *When* $\alpha = 0$, *this is a conservative system (with conserved quantity* (4)*) and existence of a primary homoclinic solution is known for all* a, b *such that the equilibrium is a saddle-focus [26]. Moreover, near* C_3 *the primary solution is transverse. What is the unfolding of all symmetric* 6*-pulse homoclinic solutions, for small* $\alpha > 0$*?*

A related question is what happens to the existence of N-pulses upon perturbing away one reversibility in a system with two reversibilities such as that equivalent to (2) with $f = u^3$, while the system remains conservative.

The entire *dynamics in a neighbourhood* of a transverse saddle-focus homoclinic orbit in reversible systems are not completely understood. To date, it is not known whether Smale horseshoes necessarily exist or not. However, we do know (see the comments at the start of this section) that there is a one-parameter family of symmetric periodic orbits on *each* homoclinic orbit (primary or N-pulsed). Moreover, Devaney [57] has shown that, due to the complex eigenvalues at the equilibrium, the manifold of periodic orbits spirals as it approaches the primary homoclinic Γ. Furthermore, for systems in $\mathbb{R}^4$ he shows that in this manifold the periodic orbits change from being *hyperbolic orientable* to *elliptic* to *hyperbolic non-orientable* and back again infinitely often. (With the Homoclinic Center Manifold Theorem on board, it is likely that similar results apply for reversible systems in $\mathbb{R}^{2n}$ for any $n \geq 2$.) The spectrum σ of a reversible periodic orbit necessarily has two Floquet multipliers at $+1$ and must be symmetric. In $\mathbb{R}^4$ this implies $\sigma = \{s, s^{-1}, 1, 1\}$ for some $s \in \mathbb{C}$. The elliptic case corresponds to non-real s, with $|s| = 1$ and $s^{-1} = \overline{s}$. The hyperbolic case corresponds to real $s \neq \pm 1$ and is *orientable* if $s < 0$ and *non-orientable* otherwise (the names being derived from the orientability of the strong stable manifold of the periodic orbit). For elliptic periodic orbits, each time s passes through a root of unity there is a sub-harmonic *period-adding bifurcation*, as described by Vanderbauwhede [155]. So in each of the (infinite number of) elliptic intervals on the manifold of periodic orbits accumulating on each of the (infinite number of) homoclinic orbits there will be an infinite number of sub-harmonic bifurcations of periodic orbits.

Problem 3 *Describe the complete dynamics in a neighbourhood of transverse saddle-focus homoclinic orbit in a reversible system. In particular determine whether or not shift dynamics necessarily occur.*

3.2 The saddle case; uniqueness or multiplicity

In region 2 of Fig. 1 the eigenvalues of the origin are real. Here, there is no *a priori* reason for multiplicity of homoclinic solutions. In certain cases, such as for (2) with $f = u^2$, one can prove global existence and uniqueness of a symmetric homoclinic orbit in this region [44,23]. The dynamics in a neighbourhood of the homoclinic orbit is, in contrast to the saddle-focus case, rather simple. In particular, the one-parameter family of periodic orbits accumulating on a non-degenerate symmetric homoclinic orbit to a saddle are all of hyperbolic type. However, see the work by Fiedler & Turaev [64] and Section 4.2 below for a codimension-one degeneracy caused by non-transversality that gives rise to elliptic periodic orbits. Note that homoclinic orbits to saddle equilibria can occur in planar reversible systems, where the interpretation of the one-parameter family of periodic orbits accumulating on a homoclinic orbit is intuitive. Consider for example the planar systems (10) and (11) which are equivalent to the undamped, unforced quadratic and cubic Duffing Oscillators; see, e.g., [68, Fig. 2.2.3] for the phase portrait of the cubic case.

In $\mathbb{R}^4$ or higher, however, there are other codimension-one mechanisms which

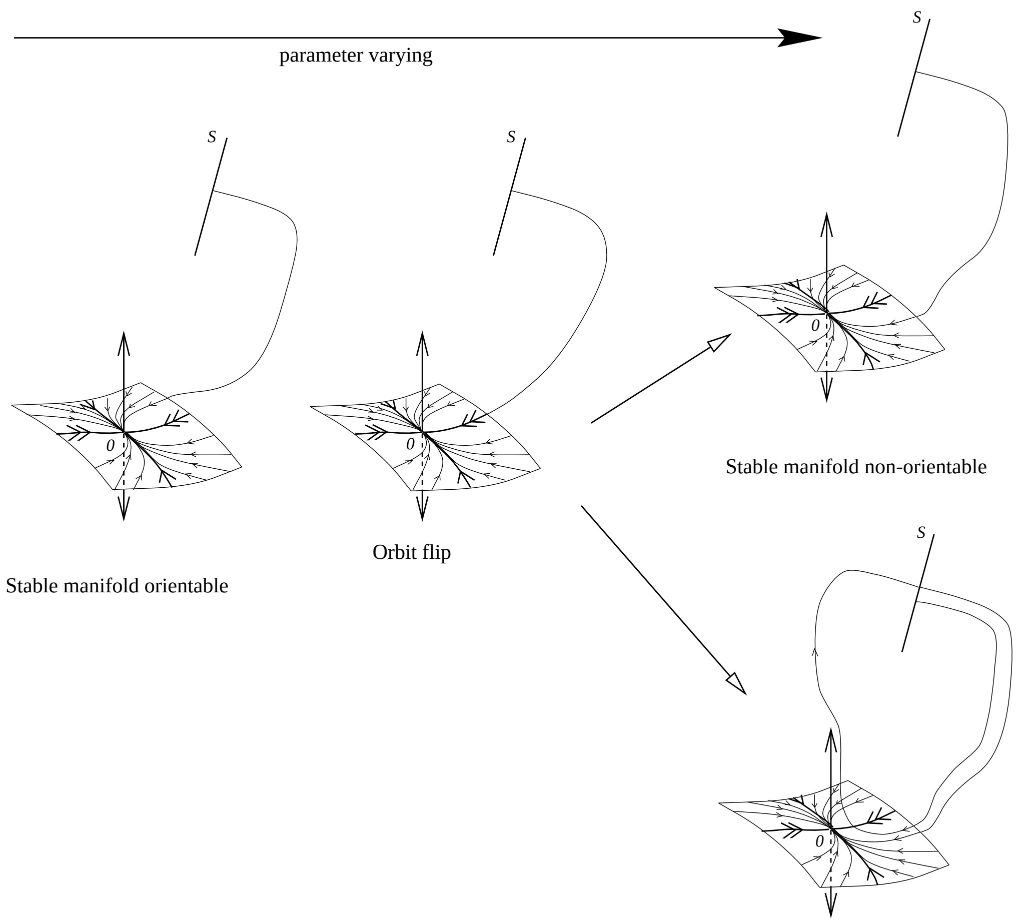

Fig. 2. The orbit flip bifurcation for reversible systems in four-dimensions. Half of each homoclinic orbit is depicted from the symmetric section $\mathcal{S}$ onwards, by projecting out the strong unstable direction.

can make the dynamics in the neighbourhood of a saddle homoclinic orbit non-trivial. These are associated with global changes to the homoclinic orbit and its stable and unstable manifolds occuring, without losing transversality or the linearisation at the origin changing.

The first of these is the so-called *orbit flip* bifurcation, analysed for reversible systems by Sandstede *et al.* [139], and applied to a model of pulse propagation in optical fibres with loss and periodic amplification [104,105,2]. The travelling-wave equation for this model is fourth-order, non-conservative, and is reversible under two reversibilities R and $-R$. An orbit flip occurs when the homoclinic orbit is tangent to the strong stable eigenvector of the equilibrium as $t \to \infty$ (and, due to reversibility, to the strong unstable eigenvector as $t \to -\infty$). An unfolding of this situation for reversible systems is depicted in Fig. 2, which depicts the generic situation that the orbit flip causes the tangent vector to the homoclinic orbit at $t = +\infty$ to flip between components of the weak stable eigenvector. Using Lin's method, Sandstede *et al* [139] show that for the sign of a perturbing parameter corresponding to the stable and unstable manifolds around the homoclinic orbit being non-orientable there is a bifurcation to infinitely many homoclinic orbits; a unique N-pulse for each N. If the system has an additional reversibility $-R$ then the N-pulses are not unique and there exist pulses which are symmetric under $-R$ as well as R, given a R-symmetric primary orbit.

One of the non-degeneracy conditions for the reversible orbit flip is that the unique bounded solution of the adjoint variational equations around the homoclinic orbit $\psi(t)$ behaves generically. That is, it converges to zero according to the weak stable and unstable eigenvalues as $t \to \pm\infty$. It was remarked in [139, Rem. 2.1] that this condition is automatically violated for conservative systems which can be written in Hamiltonian co-ordinates, because $\psi(t) = \nabla H(\gamma(t))$ and hence the adjoint and the homoclinic orbit must both undergo orbit-flips simultaneously. The analysis of orbit flips in the Hamiltonian reversible case is incomplete at present, but an example (see Section 5.1 below and [37]) suggests that the bifurcation of N-pulses may occur on *both* sides of the critical parameter value. Which brings us to another open question.

Problem 4 *Are there examples of orbit flips for non-reversible Hamiltonian systems or non-symmetric homoclinic orbits of Hamiltonian systems? If so, presumably the property that the orbit switches between components of both the weak stable and unstable manifolds simultaneously is broken. An analysis would be interesting.*

The switching of components of the weak stable eigenvectors of the adjoint linear problem by $\psi(t)$ in the limit $t \to \pm\infty$ is referred to as an *inclination-flip* bifurcation. Our statement about orbit flips in the Hamiltonian reversible case is that inclination and orbit flips must occur simultaneously. For general (non-reversible, non-conservative) systems, orbit-flip and inclination-flip homoclinic bifurcations are independent codimension-two events. However, their analyses [54,99,78,135] lead to almost identical conclusions about the number of separate cases, the existence of horseshoe dynamics and the bifurcation N-pulses.

Problem 5 *Can inclination flips occur for reversible systems? If so are there examples and will an analysis reveal a similar bifurcation result to the reversible orbit flip?*

Finally in this section, we note that the orbit flip is a codimension-one bifurcation that gives rise to an open set of parameter values for which there exist infinitely many homoclinic orbits. This suggests that there may be some codimension-zero property of a primary homoclinic orbit which is 'activated' by the orbit flip and that whenever this property is true then it can be inferred that there is infinite multiplicity. We have already mentioned that orbit flips cause the stable and unstable manifolds (specifically their tangent bundles around the homoclinic trajectory) to switch from being orientable to non-orientable. However, the planar center manifold around the homoclinic orbit remains orientable on both sides of the flip point [139].

For the conservative case, there is a codimension-zero theory that gives multiplicity and shift dynamics, namely that of Holmes ([77], see also Wiggins [158, Sect. 3.2e(ii)]). Here, the assumption is the existence of *at least two* homoclinic orbits satisfying certain conditions on the directions in which they asymptote to the saddle. Then, it can be shown that there is a Smale horseshoe in the dynamics in the level set containing the saddle. Although not explicitly stated in the theory, within this construction it is clear that there must be embedded N-pulse homoclinic orbits for each $N \in \mathbb{N}$. See also Turaev and Shil'nikov [152] for the dynamics in the neighbourhood of multiple homoclinic orbits to a saddle in conservative systems.

Problem 6 *Is there a simple topological characterisation that can differentiate between homoclinic orbits to saddles in reversible systems that are locally unique and those that are accompanied by horseshoes and N-pulses?*

3.3 The saddle-centre case; isolation or cascades

In region 3 of Fig. 1 the origin is no longer hyperbolic because of the pair of imaginary eigenvalues. Hence questions of persistence of homoclinic orbits are more subtle. A complete analysis of homoclinic orbits with this linearisation for reversible systems is not known, but Mielke, Holmes & O'Reilly [129] and also Lerman and co-workers [109,101] have analysed the situation for Hamiltonian systems in $\mathbb{R}^4$. These results have been partially extended to arbitrary dimensions by Koltsova & Lerman [102]. Mielke *et al.* show that given a primary homoclinic orbit to a saddle-center that is symmetric under a reversibility then, under certain conditions, for each N there are infinitely many N-pulses occuring *at isolated parameter values.* The structure of homoclinic orbits is termed a *cascade*, as the parameter values of all MN-pulses, for $M > 0$ accumulate on those of each corresponding N-pulse for $N > 0$. Another consequence of the analysis is the existence of n-periodic orbits for all n and small horseshoes in nearby level sets of the Hamiltonian function. The results rely heavily on some non-degeneracy assumptions which in [129] are expressed in a specific form assumed of the Hamiltonian. The key assumption is a condition on coefficients of the quadratic part of the Hamiltonian, *viz* if

$$H = \frac{\omega}{2}(p_1^2 + q_1^2) + \frac{\lambda}{2}(p_2^2 - q_2^2) + \text{h.o.t},$$

then $\omega\lambda > 0$. Under the opposite sign of $\omega\lambda$, which is the case for Hamiltonian versions of (2) and for capillary-gravity water waves [3,127], then reversible homoclinic orbits are not accompanied by a cascade unless other symmetries force this. Furthermore, for (2) with $f = u^2$, Amick and McLeod ([4], see also [70]) have proved that there are no symmetric homoclinic orbits to the equilibrium anywhere in region 3.

One thing to note about the analysis in $\mathbb{R}^4$ is that homoclinic orbits to the origin must connect the one-dimensional (strong) stable and unstable manifolds W^s and W^u of the saddle-center. In general for conservative systems, such a connection (the identification of two lines in a level set $\cong \mathbb{R}^3$) would be of codimension two. However, reversible homoclinic orbits require an intersection between one of these one-dimensional manifolds and the two-dimensional $\mathcal{S}$ in $\mathbb{R}^4$. Hence, symmetric homoclinic orbits are of codimension-one in this parameter regime. Since the analysis of [129] describes homoclinic orbits occuring as one parameter varies, then reversibility is equally important in their construction as is the Hamiltonian structure. In fact, Koltsova & Lerman [101] describe an unfolding of a codimension-two non-symmetric homoclinic orbit to a saddle-center in four-dimensional Hamiltonian systems. We are left with the interesting question:

Problem 7 *Does the analysis of Mielke, Holmes & O'Reilly apply in the reversible case, independent of Hamiltonian structure?*

By analogy with the known results on the dynamics near a saddle-focus homoclinic orbit, a natural conjecture would be that the reversible case also leads to cascades, but the multiplicity (and existence of horseshoes) may be different from the conservative case.

It has already been remarked that the normal form (12) gives rise to homoclinic orbits to periodic orbits in a neighbourhood of the curve C_1 in region 3. Given that there are imaginary eigenvalues at the origin, the reversible Liapunov Centre Theorem [56] gives the existence of a one-dimensional family of periodic orbits surrounding the origin for each parameter value in region 3. Homoclinic orbits to periodic orbits connect the unstable manifold of one of these periodic orbits to themselves. (Note that symmetric *heteroclinic* orbits connecting different periodic orbits are not allowed by reversibility. Also, in conservative systems generically each periodic orbit will belong to a different level set, in which case asymmetric heteroclinic orbits are ruled out also.) Note the codimension of such homoclinic connections. In the symmetric case we require the two-dimensional (strong) unstable manifold of the periodic orbit to intersect the two-dimensional set $\mathcal{S}$. Hence symmetric homoclinic orbits to any given periodic orbit are persistent (codimension zero). It is not difficult to see that non-symmetric homoclinic orbits in the conservative case are also codim 0. Also, for the conservative case, a simple argument sketched in Champneys and Lord [40, Sect. 2.3.] shows that if such a homoclinic orbit to a periodic orbit is transverse (in the conservative sense) then the Smale-Birkoff Homoclinic Theorem applies, giving rise to a homoclinic tangle and consequently infinitely many N-pulse homoclinic orbits to the periodic orbit.

Problem 8 *Are symmetric homoclinic orbits to periodic orbits accompanied by N-pulses and horseshoes in the reversible non-conservative case also?*

3.4 The focus case; non-existence

In this case, at least while the linearisation at the origin is not strongly resonant and the system remains in normal form, then it is known that there cannot be any homoclinic orbits. Nonetheless, the dynamics in a neighbourhood of the origin may be quite complex, due to there being families of elliptic periodic orbits with consequent resonant sub-harmonic bifurcations, see [9,140] for the details. The situation when there exist strong resonances (eigenvalues $\pm\omega_1$, $\pm\omega_2$ where $n = 1, 2, 3$ or 4) is more subtle and remains open.

4 Global phenomena

4.1 Existence, transversality and stability of travelling waves

The work reviewed in the previous section generally concerns the dynamics and N-pulses given the *a priori* assumption of a primary symmetric homoclinic orbit. In order to apply such theory rigourously to an example, one needs first to prove existence of this primary orbit. Then it must be shown that this solution satisfies the required non-degeneracy assumptions, the key property being transversality.

The simplest form of existence proof is an explicit closed form solution. Such solutions were already obtained for the normal form equations (10), (11), (12) and (13). Also, in the work by Sandstede *et al* [139] and Mielke *et al* [129] cited above, the theory of orbit flips and homoclinic cascades were rigorously applied to example systems for which there is an explicit solution, calculations on which enabled the non-degeneracy conditions to be checked also.

Another method for showing existence of homoclinic solutions is analytic shooting. This has been used to great effect by Peletier & Troy [131,132] to show existence of infinitely many homoclinic and heteroclinic steady solutions of the EFK equation (6). Amick and McLeod [4] also used shooting arguments (in the complex plane) to show non-existence of symmetric homoclinic orbits of (2) with $f = u^2$ in region 3. A method due to Hofer & Toland [76] was used in [44,20] to show the bifurcation of infinitely many 2-pulse homoclinic orbits given a transverse homoclinic orbit crossing the curve C_3 for a class of Hamiltonian systems including (2) with $f = u^2$. In all of these examples, although the problem was Hamiltonian, explicit use was made of the symmetric section. For example, in [44] initial conditions were varied in the unstable manifold of the origin to hit the symmetric section after a finite time. But it was the particular Hamiltonian structure that was used to argue that such intersections must occur infinitely many times. It was also possible to show the required transversality of the primary orbit for that example [23] using a shooting method originally used to show uniqueness of symmetric homoclinic orbits in region 2 for a similar class of equations [6,5]. See also [27,21,74] for other multiplicity results obtainable via shooting.

Conservative systems usually arise via some kind of variational principle. The calculus of variations applied to such a weak formulation of the problem can often lead to the inference of at least one symmetric homoclinic orbit. One such approach is to find solutions as critical points of a functional using the Mountain Pass Lemma [19] and the idea of 'Concentration-Compactness' [116,117,52]. For (2) with $f = u^2$, for example, homoclinic solutions are given precisely by critical points in the Sobolev space $H^2(\mathbb{R})$ of the functional

$$J(u) = \tfrac{1}{2} \int_{\mathbb{R}} \{|u''|^2 + b|u'|^2 + au^2 + (2/3)\, u^3\} \mathrm{d}x. \tag{15}$$

Buffoni [22] has used this structure to show the existence of at least one homoclinic orbit for all a, b inside region 1. These results were substantially improved by Buffoni and Séré [26] who showed that there exist infinitely many homoclinic solutions for a class of functionals including (15) at parameter values for which the origin of the associated equation is a saddle-focus. The infinite multiplicity is in the sense of an alternative theorem. Either the stable and unstable manifolds around the primary homoclinic orbit are topologically transverse (a weaker non-degeneracy condition than the analytic transversality required of Devaney's theory [55]), then shift-dynamics occur with the inherent infinity N-pulse orbits described in Section 3.1. The other possibility is that the manifolds are identified, so that there is a continuum of primary orbits. Buffoni & Groves [24] have used these methods to show infinite multiplicity to the right of C_2 in Fig. 1 for any Hamiltonian perturbation of the normal form (13). Mountain Pass arguments have also been used to show existence for other fourth-order equations [47,37] and systems of equations [45].

Another variational technique is to pose a constrained optimisation problem. For example, homoclinic solutions of (2) with $f = u^2$ are described by minimizers of the functional

$$E(u) = \int_{\mathbb{R}} \{\||u''|^2 + b|u'|^2 + (2/3)\, u^3\}$$

subject to the Lagrange multiplier constraint $\int_{\mathbb{R}} u^2 \mathrm{d}x = \lambda$ [110].

Other formulations treat PDEs such as (5) directly [111,110,84,10]. Constrained minima of the appropriate functionals give solitary wave solutions which are homoclinic orbits of the appropriate ODEs. The real power of this constrained minimisation method is that it also allows one to prove stability of the travelling-wave solutions of the PDE. However, the method always describes 'ground states' of the ODE and therefore cannot find N-pulses.

Problem 9 *For reversible conservative systems for which existence can be proved by variational methods, is the primary homoclinic orbit or 'ground state' necessarily symmetric?*

Both the concept and calculation of stability of solitary waves depends crucially on the particular PDE setting in which the travelling wave ODE system arises. For example, stability of homoclinic solutions of (2) is different depending on whether they arise from (5), (6), (7), (8) or from the buckling of struts [137]. A general review of the stability of solitary waves is beyond the scope of this article. We do however mention recent work by Sandstede [138,137] that if a primary homoclinic orbit is stable, then in certain settings, a Belyakov-Devaney-type bifurcation can cause half of the bifurcating N-pulses to also be stable. A similar result was also proved by Sandstede *et al* [139] for the model optical system at the point at which it undergoes an orbit flip. Finally we remark on some recent numerical results by Chen & McKenna [47,46] suggesting that the primary and certain stable N-pulse solutions of (8) with the exponential nonlinearity collide inelastically like solitons of integral PDEs.

4.2 Coalescence and Bifurcation

We mentioned earlier that the transversality required for the persistence of homoclinic orbits is different for conservative and reversible systems. Given a symmetric homoclinic orbit Γ of a system that is *both conservative and reversible*, there are two codimension-one ways to lose transversality of W^s and W^u (within a level set), without a local bifurcation occuring. The best way to see this is to look at a Poincaré section containing the symmetric section $\mathcal{S}$, see Fig. 3. The consequences of these losses of transversality were analysed heuristicly in Buffoni *et al* [23], and subsequently put on a rigourous foundation by Knobloch [100].

In four dimensions, if W^u and W^s are non-transverse, then they must have a common tangent direction other than the tangent vector to the homoclinic orbit γ'. At the symmetric point $\gamma(0)$ this other tangent direction w must either be symmetric or anti-symmetric. That is either $Rw = w$ or $Rw = -w$. The former is termed a *coalescence* and leads to the disappearance of

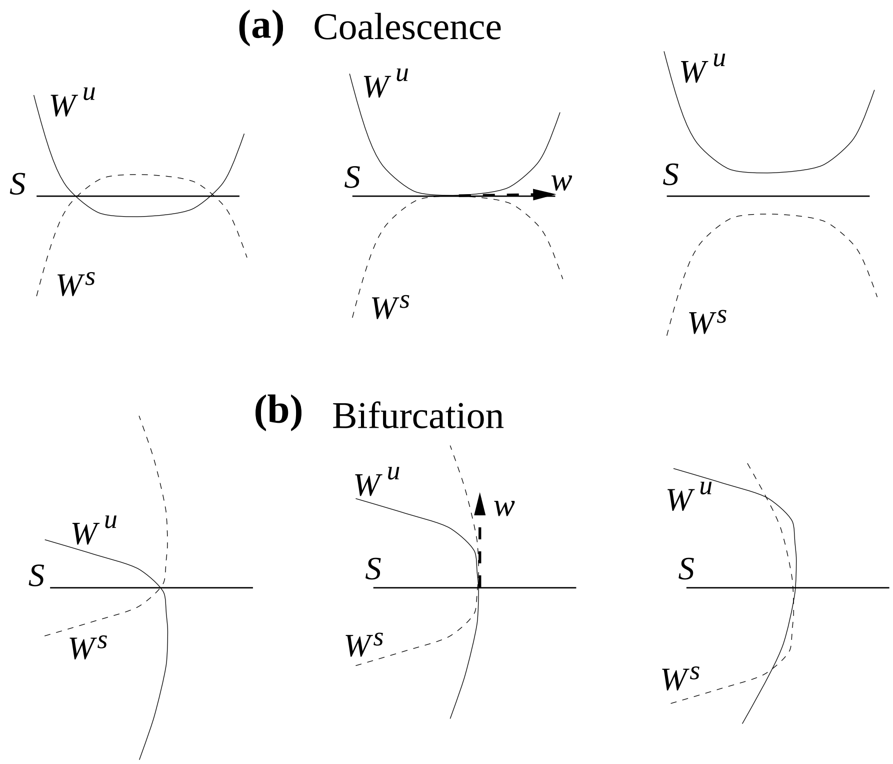

Fig. 3. Unfolding under parameter variation of coalescence and bifurcation for conservative reversible systems in $\mathbb{R}^4$. Depicted are the stable and unstable manifolds W^s and W^u and the symmetric section $\mathcal{S}$ in a two-dimensional Poincaré section contained in a level set

two symmetric homoclinic orbits under parameter variation (see Fig. 3). The latter case causes the *bifurcation* of two asymmetric homoclinic orbits from the primary symmetric one. That the primary orbit must persist through this bifurcation is apparent since W^u continues to intersect $\mathcal{S}$ transversally. For reversible systems that aren't conservative, then in general there is no bifurcation caused by the second form of non-transversality.

The coalescence and bifurcation just outlined assume that the equilibrium is hyperbolic. In the case where it is a saddle, then Fiedler & Turaev [64] have shown that the presence of a coalescence of homoclinic orbits in a reversible system causes the birth of periodic orbits of elliptic type. If the equilibrium is a saddle-focus, then the dynamics in the neighbourhood of a coalescence are more complex

Finally we remark that the two possibilities of the extra common tangent at $\mathcal{S}$ for non-transversality can be used as conditions to check when attempting to prove transversality. This approach was used by Champneys & McKenna [41] on a piecewise-linear equation for which calculations could be performed explicitly.

4.3 Numerics

The preceding theory is usually not enough on its own to give complete information on homoclinic solutions. In general one has to rely on numerical methods for finding solutions. These methods can be categorised into three approximate classes: locating homoclinic solutions at fixed parameter values, continuation of solutions as parameters vary, and numerical integration of the PDEs for which the homoclinic orbit is a travelling-wave solution.

There is now quite a large literature on methods for continuation of homoclinic (and heteroclinic) orbits in ODE systems (see [65,16,38,39] and references therein). The key idea is to truncate the infinite time interval and apply projection boundary conditions onto the stable and unstable eigenspaces of the equilibrium. Solutions of the resulting two-point boundary value problem can be shown to be exponentially close as the truncation interval increases to non-degenerate homoclinic solutions of the original problem. Champneys & Spence [42] proposed a simplification of this method for reversible systems. Here, the solution is sought over a truncation to of the half-interval with right-hand boundary conditions placing the solution in the symmetric section $\mathcal{S}$.

Also in [42] a numerical shooting method was proposed for locating solutions to this boundary-value problem at fixed parameter values. The shooting parameters are the truncation time and 'angle co-ordinates' which parametrise initial conditions at a fixed distance from the equilibrium in the unstable eigenspace, in order to satisfy the symmetric-section boundary conditions. This method has been successfully applied to (2) and related equations to systematically compute families of N-pulse homoclinic orbits [23,36,153]. For the solution of the initial value problems at each step of shooting, one could use a numerical integration method that preserves reversibility, e.g. [145], but in practice as solutions are sought over a finite time this is not necessary. Another possibility for finding locating solutions in conservative problems with variational structure, is to use numerical methods to compute extrema of the appropriate functionals, e.g. [47,48].

Direct time integration of PDEs is useful for providing numerical evidence on the stability and interaction properties of solitary waves. Finite difference methods are widely used, which should preferably respect the structure of conservative systems (e.g. [124,123,50]). Spectral methods have also been applied to good effect, e.g. [49,151].

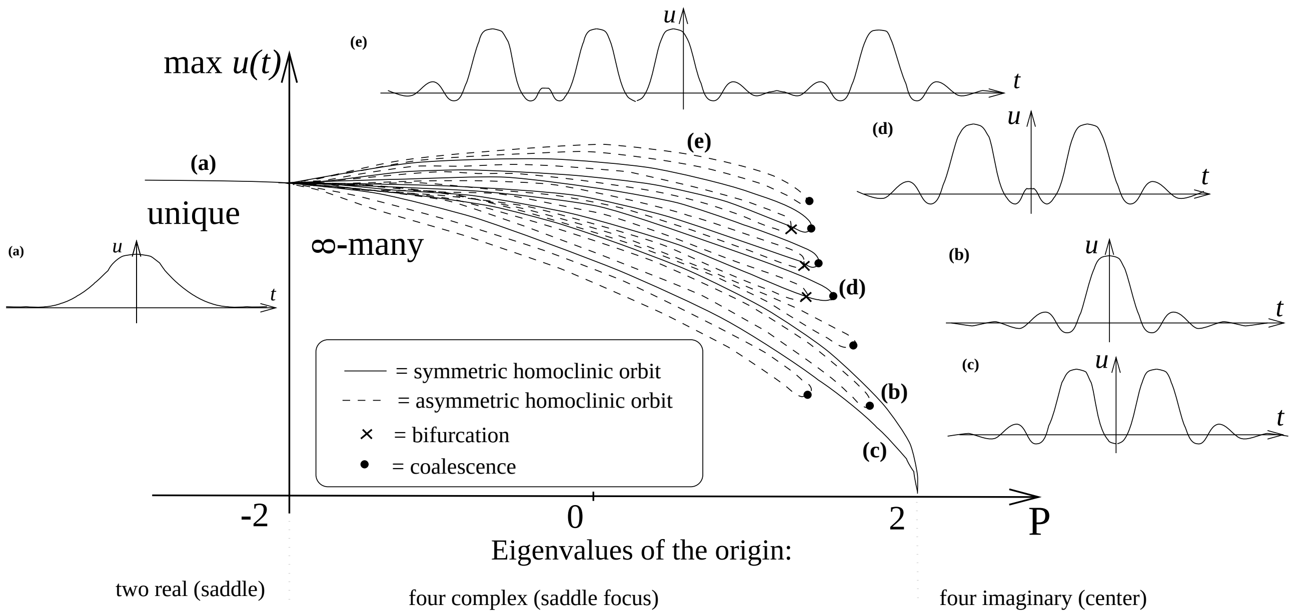

Fig. 4. Schematic summary of global information on homoclinic orbits to the origin of (17)

5 Three case studies

5.1 Example 1: A 5th-order KdV approximation for gravity-capillary water-waves

The ordinary differential equation

$$\frac{2}{15}u'''' - bu'' + au + \frac{3}{2}u^2 + \mu\left[\frac{1}{2}(u')^2 + (uu')'\right] = 0, \qquad (16)$$

was derived by Craig & Groves [53] as the travelling wave equation for an extended 5th-order KdV equation modelling weakly nonlinear long waves on the surface of a fluid with surface tension. It is analysed extensively in Champneys & Groves [37] using a mixture of the normal form analysis, generic argument, existence theory and numerical methods outlined above. Here we just summarise a few details as an illustration of the preceding discussions.

When $\mu = 0$, (16) reduces to a scaling of the canonical 5th-order KdV travelling-wave ODE, (2) with $f = u^2$. This equation can be further reduced by changes of variable to one of the following one-parameter equations

$$y'''' + Py'' + y - y^2 = 0 \quad \text{for} \quad a > 0 \qquad (17)$$
$$y'''' + Py'' - y - y^2 = 0 \quad \text{for} \quad a < 0, \qquad (18)$$

where $P = -b\sqrt{15/2|a|}$.

Equation (17) is written in the form of the dimensionless ODE modelling a strut on a nonlinear foundation [79,81], homoclinic solutions of which have been studied by a number of authors; the most complete account is in [23], see also Fig. 4. For $P < 2$ there is a unique, positive, even homoclinic solution. At $P = 2$ there is a Belyakov-Devaney bifurcation into an infinite multiplicity of N-pulses. Because the equation is conservative as well as reversible, these involve symmetric and asymmetric solutions. As $P \in (1.5, 2.0)$ is increased these N-pulses start disappearing, via a cascade of coalsences and bifurcations. It is a numerical observation that the coalescence and bifurcation points on symmetric branches occur at nearby parameter values. This is due to the spiked nature of the stable and unstable manifolds causing the two transitions sketched in Fig. 3 to occur one after the other. Similar pairs of bifurcations and coalescences in rapid succession were obtained asymptotically by Yang & Akylas [160] in the limit $P \to 2$. At $P = 2$ there is the small amplitude bifurcation of two homoclinic solutions as described by the normal form (13). No homoclinic solutions are known for $P > 2$.

In [29] numerical and asymptotic evidence is presented that half of the 2-pulse solutions of (17) for $P \in (-2, 2)$ are stable as solutions of the PDE. These calculations are in keeping with the more general theory of Sandstede concerning the stability of N-pulses [138,137]. Moreover, the interaction properties of some of these stable solitary waves was investigated in [123].

For equation (18) it has been proved that there are no symmetric homoclinic solutions to the origin for any $P < 0$ [4,70], and no homoclinic solutions at all for P sufficiently large and negative [4]. However, it is known that there are homoclinic solutions to periodic orbits, both single pulsed [134,7,18,66,147] and N-pulsed [40,35].

The analytical and numerical evidence presented in [37] suggests that, broadly speaking, the structure of homoclinic solutions to (17) and (18) survives on adding the extra nonlinear terms to (16) obtained by setting $\mu = 1$. However, there is additional complexity. First notice that in region 3, where for $\mu = 0$ the dynamics are governed by (18) and there are no symmetric homoclinic solutions, there is now an explicit solution [97]

$$u(t) = 3\left(b + \frac{1}{2}\right)\operatorname{sech}^2\left(\sqrt{\frac{3(2b+1)}{4}}\,t\right) \tag{19}$$

defined along the curve

$$a = \frac{3}{5}(2b+1)(b-2), \qquad b \geq -1/2. \tag{20}$$

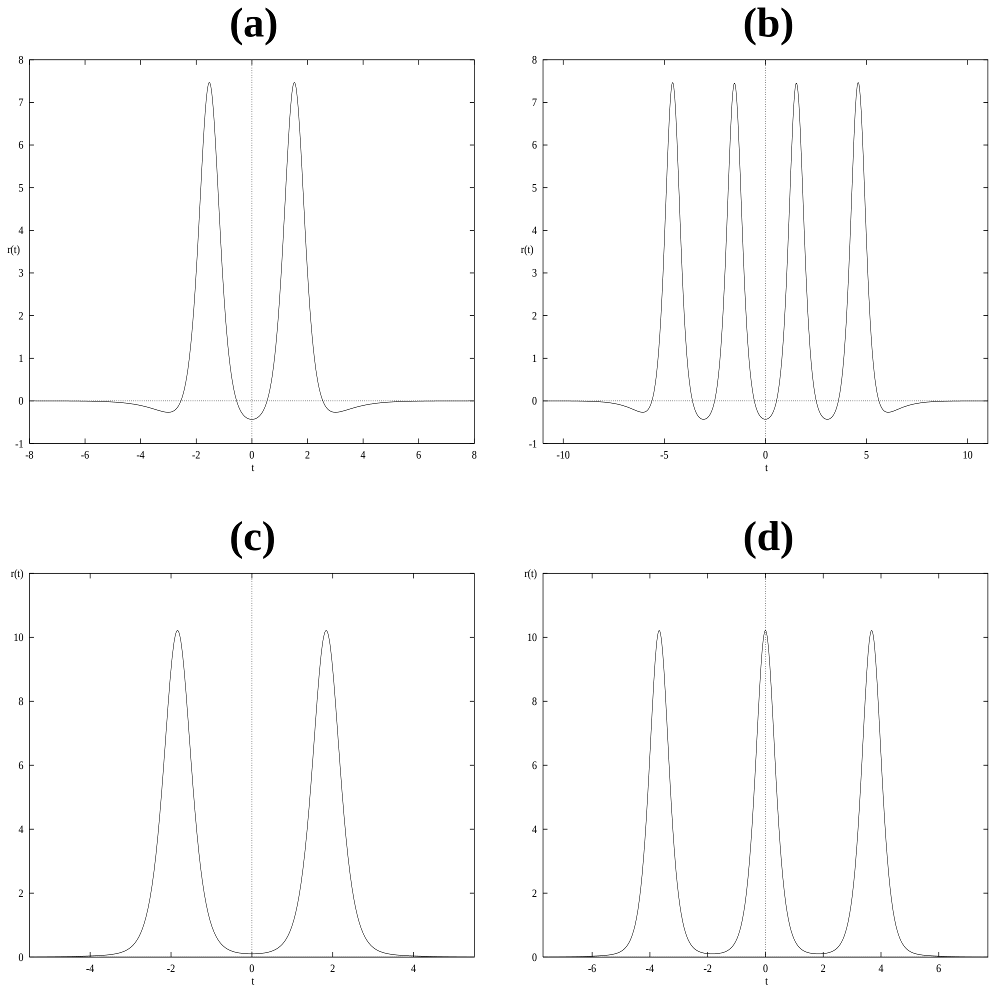

Fig. 5. Numerically computed N-pulses of (16) close to the orbit flip: for $a = 3.0$ and (a),(b) $b = 1.5$; (c),(d) $b = 3.0$.

Numerical evidence suggests that this is the first of a countable family of curves on which there are solitary wave solutions in region 3, only the first one of which is available in closed form [37, Sect. 4]. Now note that (20) extends into region 2, given by $a > 0, b > \sqrt{(8a/15)}$ for (16), where the origin is a saddle. Here (19) defines a homoclinic solution decaying to zero corresponding to the strong stable and unstable eigenvalues. Hence, generically, upon transversally crossing (20) (and each of the countably many similar curves) in the parameter plane, we should expect an orbit-flip bifurcation to occur. Fig. 5 shows some numerically computed primary and N-pulses in this parameter regime which suggests that the bifurcation to N-pulses occurs on both sides of the orbit flip for this system (see [37, Sect. 5.] for more details). A conjectured complete unfolding of homoclinic solutions to (16) is given in Fig. 6.

Problem 10 *How much of the structure of homoclinic solutions of* (16) *survives for the exact Euler equation formulation of capillary-gravity water waves?*

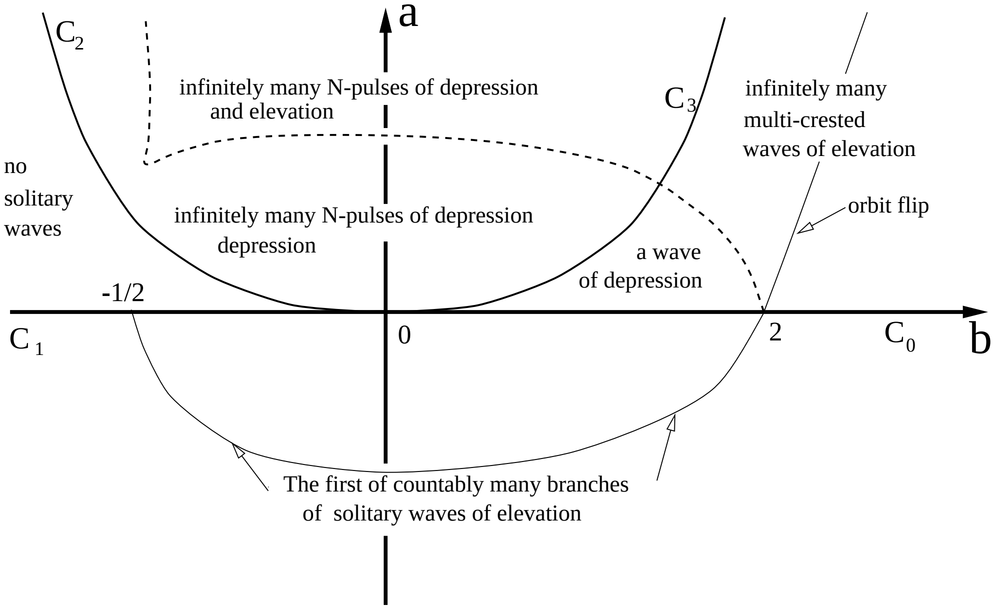

Fig. 6. Schematic summary of the global bifurcation diagram of homoclinic solutions to the origin of (16)

In partial answer to this question, we remark that Dias *et al* [60] have found N-pulses in a numerical investigation of the full problem.

5.2 Example 2: Spatially localised buckling of cylindrical shells

There is evidence to suggest that long structures that buckle elastically tend to do so in a manner that is localised to some portion of their length (see, for example, [79]). Such phenomena lend themselves naturally to a description in terms of homoclinic solutions of continuum equations posed over an infinite length. Owing to physical principles, these equations are typically both conservative and reversible. See the paper by van der Heijden & Thompson [154] in these Proceedings for an example describing the buckling of rods subject to end moment and tension (see also [128,8,150,43]). Here we summarise results on modelling localised buckling in thin cylindrical shells [80,122,121].

The classical equilibrium equations for the in–plane stress ϕ and displacement w in the post–buckling regime of an infinitely long cylinder with radius R and shell thickness t are given by the von Kármán–Donnell equations:

$$\kappa^2\nabla^4 w + \lambda w_{xx} - \rho\phi_{xx} = w_{xx}\phi_{yy} + w_{yy}\phi_{xx} - 2w_{xy}\phi_{xy}, \tag{21}$$

$$\nabla^4\phi + \rho w_{xx} = (w_{xy})^2 - w_{xx}w_{yy}. \tag{22}$$

Here ∇^4 denotes the two dimensional bi–harmonic operator, $x \in I\!R$ is the axial and $y \in [0, 2\pi R)$ is the circumferential co-ordinate. The parameters appearing in (21) and (22) are the curvature $\rho := 1/R$,

$$\kappa^2 := t^2/12(1-\nu^2),$$

where ν is Poisson's ratio and the bifurcation parameter

$$\lambda := P/Et,$$

where P is the compressive axial load applied per unit length and E is Young's modulus. The form of solutions we are concerned with impose periodic boundary conditions in y and homoclinic boundary conditions in the axial direction x:

$$(w,\phi)(x,0) = w(x,2\pi R), \qquad (w,\phi)(y,x) \to 0 \quad \text{as} \quad x \to \pm\infty.$$

In [122,121] these equations were discretised circumferentially using a Galerkin spectral method, assuming that the solutions are even periodic in y and remain within the subspace corresponding to invariance under rotation through $2\pi/s$. Hence the following cosine functions are used

$$w(y) = \sum_{k=0}^{M} a_k(x)\psi_k^s; \qquad \phi(y) = \sum_{k=0}^{M} b_k(x)\psi_k^s,$$

where

$$\psi_k^s = \cos(ks\rho y), \qquad k \in \mathbb{N} \cup \{0\}, s \in \mathbb{N}.$$

This leads to a large system of $8(M+1)$ equations

In the x-direction, it has been observed experimentally [159] that buckling modes have one of two symmetry properties about the central horizontal cross-section given by $x = T$, either *symmetric*

$$w(x,y) = w(2T-x,y) \qquad \& \qquad \phi(x,y) = \phi(2T-x,y).$$

or *cross-symmetric*

$$w(x,y) = w(2T-x, y+\pi R/s) \qquad \& \qquad \phi(x,y) = \phi(2T-x, y+\pi R/s).$$

When stated in terms of the ODEs obtained on Galerkin truncation, these symmetry conditions correspond to two different types of reversibility. Hence, the methods reviewed in Section 4.3 for computing homoclinic orbits in reversible systems can be used.

Figure (7) shows examples of symmetric and cross-symmetric solutions posed using asymptotic boundary conditions at the top of the cylinder and symmetric-section conditions at the mid point. These solutions were obtained with $M = 5$,

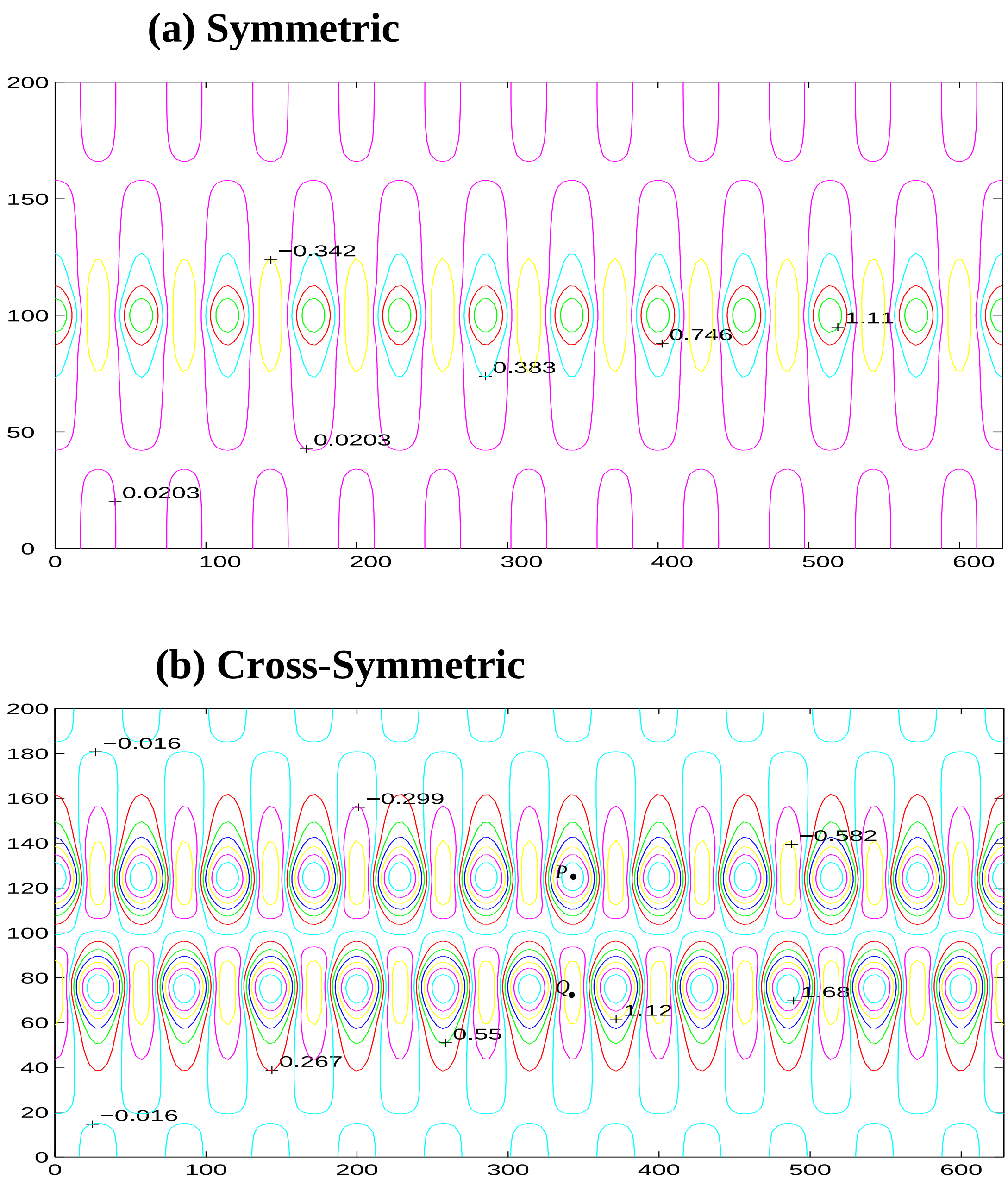

Fig. 7. Symmetric and cross-symmetric homoclinic solutions of the von Kármán–Donnell equations.

implying 48 ODEs. Solutions were continued as the load parameter P was varied using the methods outlined in Section 4.3, implemented in the code AUTO [61,62], and a good agreement with experiments was found. Actually the hard problem for (22) is to locate homoclinic solutions at fixed parameter values; see [121] for more details, including the computation of N-pulse solutions.

The theory of N-pulse homoclinic orbits of elliptic equations on infinite domains regarded as dynamical systems is still in its infancy, but see [133] for some results in this direction. Also see [127,157] and references therein for some general theory on homoclinic orbits in such systems.

5.3 *Example 3: coupled NLS equations*

Buryak & Kivshar [30] (see also [31,69,75]) consider the following system of two coupled NLS equations modelling spatial solitons in crystals with so-called $\chi^{(2)}$ nonlinearity

$$i\frac{\partial w}{\partial \zeta} + r\frac{\partial^2 w}{\partial t^2} - w + w^* v = 0, \quad i\frac{\partial v}{\partial \zeta} + s\frac{\partial^2 v}{\partial t^2} - \alpha v + \frac{1}{2}w^2 = 0. \tag{23}$$

The complex variables $w(\zeta, t)$ and $v(\zeta, t)$ represent respectively the first and second harmonics of the amplitude envelope of an optical pulse. Stationary solutions of (23) satisfy

$$rw'' - w + wv = 0, \quad sv'' - \alpha v + \frac{1}{2}w^2 = 0, \tag{24}$$

where $r, s = \pm 1$ and $w(t)$ and $v(t)$ may be taken to be real [30]. Note that the system (24) is invariant under two reversibilities

$$R : (w, w', v, v') \mapsto (w, -w', v, -v') \quad \text{and} \quad S : (w, w', v, v') \mapsto (-w, w', v, -v').$$

It is also Hamiltonian with total energy

$$H = r\frac{w'^2}{2} + s\frac{v'^2}{2} + \frac{1}{2}(w^2 v - \alpha v^2 - w^2).$$

Depending on the signs of r and s a variety of homoclinic and heteroclinic phenomena can occur. When $r = -1$ the phenomena of interest are principally 'dark solitons', heteroclinic orbits connecting non-trivial equilibria. Note, though, that in the case $s = +1$ these non-trivial equilibria have complex eigenvalues and there are infinite families of N-pulse orbits, by a slight adaptation of the theory reviewed in Section 3.1 (upon identifying the two equilibria which are images of each other under $\mathbb{Z}_2$ symmetry). These N-pulses include both heteroclinic and homoclinic orbits, referred to as *bound-states of dark solitons*, and evidence is presented in [30,31] that these may represent stable solutions of the travelling wave equations.

For the case $r = +1$, $s = -1$ the origin is a saddle-center. Numerical results in [30] indicate a cascade of 2-pulses occur at isolated parameter values, much as predicted by the theory of Mielke, Holmes & O'Reilly [129] reviewed in Section 3.3. It appears these solutions 'converge' on a primary homoclinic solution that exists only in the limit $\alpha \to \infty$. Presumably N-pulses may be found for higher N also.

The case $r = s = +1$ is perhaps the most interesting physically because the primary homoclinic solution most resembles a 'bright' soliton. Specifically at $\alpha = 1$ there is the exact solution

$$w = (3/\sqrt{2})\text{sech}^2(t/2), \qquad v = (3/2)\text{sech}^2(t/2). \tag{25}$$

It has been confirmed numerically in [30] that a branch of homoclinic solutions containing this solution exists for all $\alpha > 0$. Furthermore it was proved by variational methods in [45] that there exists at least one homoclinic solution to the origin for all $\alpha > 0$. Note that at $\alpha = 1$, when (25) exists, the eigenvalues of the origin form a double real pair, as on the curve C_3 in Fig. (1). However, the linearisation is semi-simple and hence the eigenvalues pass through each other on the real axis rather than becoming complex. For $\alpha < 1$ it has been shown numerically by a number of authors [75,69,45] that there are N-pulse orbits, one for each N, see Figure 8. The numerical results in [45] pose interesting questions as to the bifurcation of these N-pulses at $\alpha = 0$ and $\alpha = 1$. At $\alpha = 1$, it appears that a non-standard bifurcation occurs owing to the semi-simple double eigenvalues. An analysis of this via Lin's method is currently under investigation by A.C. Yew, but preliminary results suggest the bifurcation of N-pulses can be proved to occur for a class of systems containing this degeneracy. At $\alpha = 0$, the N-pulses all appear to occur as singular perturbations of the state where only v is non-zero.

Problem 11 *Describe theoretically the bifurcation of N-pulses for* (24) *with* $r = s = +1$ *at* $\alpha = 0$.

Unfortunately, these N-pulses are not likely to be physically important, because numerical calculations by Haelterman *et al* [69] reveal them to be unstable as solutions of the PDE system (23)

Finally we remark that there are other forms of reversible non-integrable travelling wave systems arising from couplings between NLS-type equations for which there are multiplicities of homoclinic solutions e.g. [32,33].

Acknowledgements

I am grateful to Bjön Sandstede (WIAS, Berlin) and Mark Groves (University of Loughborough) for helpful comments on an earlier draft.

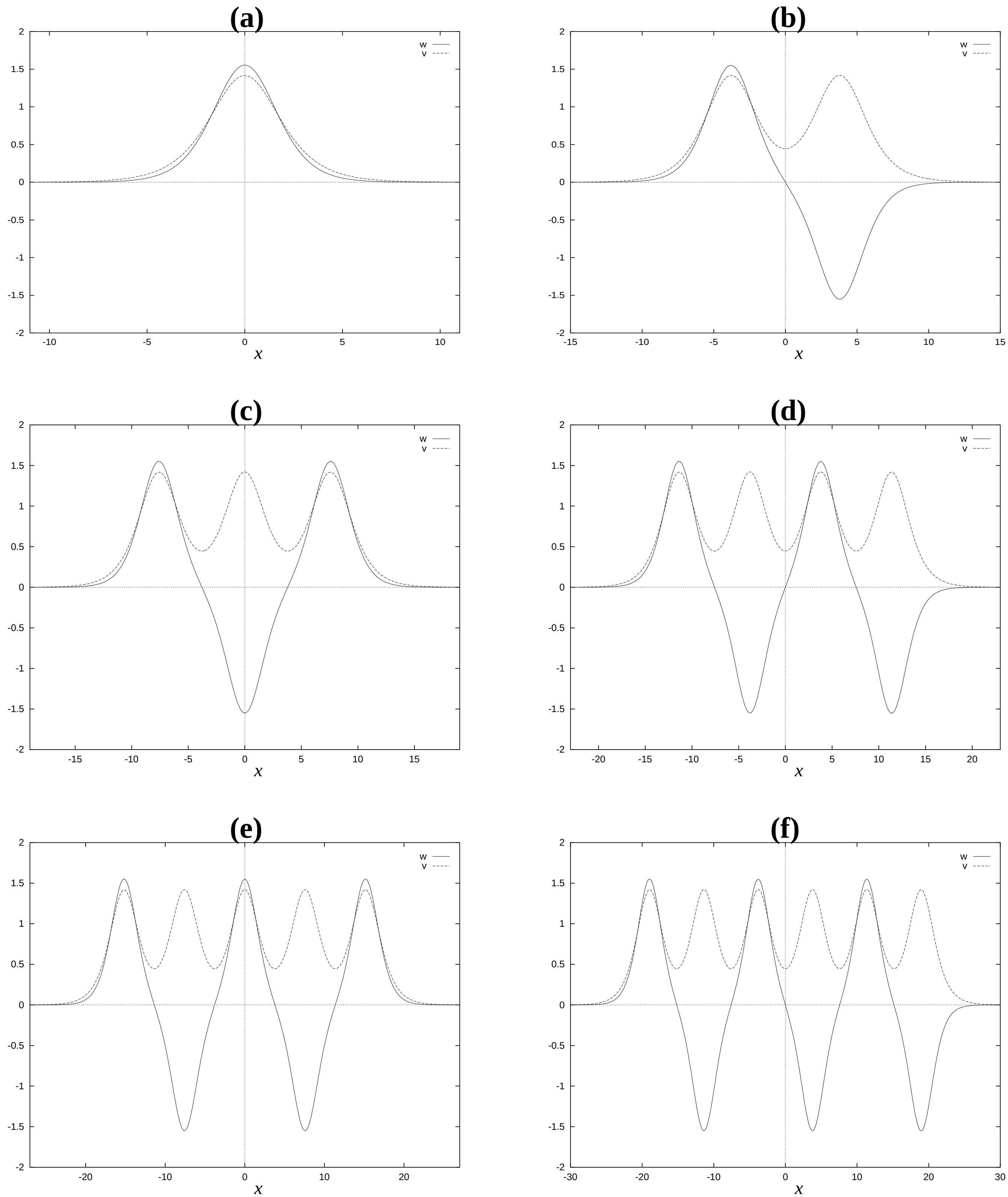

Fig. 8. Six homoclinic solutions of (24) for for $r = s = +1$ and $\alpha = 0.5$.

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