

Homogeneous cooling state for a granular mixture

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The homogeneous cooling state for a binary mixture of inelastic hard spheres is studied using the Enskog kinetic theory. In the same way as for the one-component fluid, we propose a scaling solution in which the time dependence of the distribution functions occurs entirely through the temperature of the mixture. A surprising result is that the (partial) temperatures of each species are different, although their cooling rates are the same. Approximate forms for the distribution functions are constructed to leading order in a Sonine polynomial expansion showing a small deviation from Maxwellian, similar to that for the one-component case. The temperatures and overall cooling rate are calculated in terms of the restitution coefficients, the reduced density, and the ratios of mass, concentration, and sizes. [S1063-651X(99)00211-1]

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I. INTRODUCTION

Many features of granular media, particularly those associated with dissipation, can be represented by a fluid of hard spheres with inelastic collisions. The analysis of one-component systems via kinetic theory for such a fluid has been an active field of research in recent years. One important objective has been the derivation of fluid dynamic equations and corresponding expressions for the transport coefficients. For a given kinetic equation (e.g., the Boltzmann equation at low density or the revised Enskog equation at finite densities [1]) a “normal” solution is obtained approximately using the Chapman-Enskog method adapted to the case of inelastic collisions [2,3]. By “normal” solution is meant one whose space and time dependence occurs entirely through the hydrodynamic fields. For elastic collisions the solution is obtained as an expansion about the local Maxwellian, while for inelastic collisions the reference state is a local “cooling” solution with an inherent time dependence of the temperature due to the loss of energy in the collisions. For spatially homogeneous states the latter is referred to as the homogeneous cooling state (HCS). It qualifies as a normal solution since all time dependence appears only through the temperature. The distribution function is no longer Maxwellian but its form in the one-component case has been determined to a very good approximation [4]. The objective here is to extend this analysis of the HCS to the case of a binary mixture. Such an analysis is the essential first step needed for the derivation of hydrodynamic equations for mixtures, which will be discussed elsewhere.

It might appear that the extension from one to two components is simply a matter of increased complexity and additional parameters differentiating the two species. In fact, there are qualitative differences due to the cooling effects of the inelastic collisions. In the context of a normal solution, the distribution functions for the two species are defined such that all time dependence occurs through the overall temperature of the mixture. Nevertheless, due to the inelasticity,

these distributions yield two *different* partial temperatures with the same cooling rates. This surprising result is consistent with recent studies of tracer dynamics [5] and Brownian motion [6] with inelastic collisions, where the temperature of the tagged particle distribution was found to be different from the surrounding bath. This effect is generic for multi-component systems and is illustrated in detail here. Some important consequences for hydrodynamics are discussed in the final section. The analysis is based on the Enskog kinetic equation, which for homogeneous states is the same as the more accurate revised Enskog equation [1]. The coupled set of kinetic equations is recalled in Sec. II, the homogeneous cooling solutions are defined, and the condition of equal cooling rates is deduced. An approximate solution is obtained in Sec. III by expanding the distribution functions in a complete set of polynomials. Since quite accurate results are obtained in the one component case by truncation at first order, a similar approximation is considered here. The global temperature (which is the relevant one at a hydrodynamic level) and those for the two species are determined as a function of the restitution coefficients, mass ratio, composition, density, and ratio of hard-sphere diameters. The description applies for an arbitrary degree of inelasticity and it is not restricted to specific values of particle masses, molar fractions, and/or particle sizes. The results are discussed in Sec. IV and the relationship to previous work on mixtures [7–9] is also given there.

II. KINETIC THEORY AND HOMOGENEOUS COOLING STATE

Consider a binary mixture of smooth hard spheres of masses m_1 and m_2 and diameters σ_1 and σ_2 . In general, collisions among all pairs are inelastic and are characterized by three independent constant coefficients of normal restitution α_{11} , α_{22} , and $\alpha_{12} = \alpha_{21}$, where α_{ij} is the restitution coefficient for collisions between particles of species i and j . For spatially homogeneous isotropic states, the set of nonlinear Enskog kinetic equations determines the velocity distribution functions $f_i(\mathbf{v}_i; t)$ for $i = 1, 2$,

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$$\partial_t f_i(\mathbf{v}_1; t) = \sum_j J_{ij}[\mathbf{v}_1 | f_i(t), f_j(t)], \quad (1)$$

where $J_{ij}[\mathbf{v}_1 | f_i, f_j]$ is given by

$$J_{ij}[\mathbf{v}_1 | f_i, f_j] \equiv \chi_{ij} \sigma_{ij}^2 \int d\mathbf{v}_2 \int d\hat{\boldsymbol{\sigma}} \Theta(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) \\ \times [\alpha_{ij}^{-2} f_i(\mathbf{v}'_1) f_j(\mathbf{v}'_2) - f_i(\mathbf{v}_1) f_j(\mathbf{v}_2)]. \quad (2)$$

Here χ_{ij} is the pair distribution function for particles of types i and j when they are in contact, i.e., separated by $\sigma_{ij} = (\sigma_i + \sigma_j)/2$. Also, $\hat{\boldsymbol{\sigma}}$ is a unit vector directed along the line of centers from the sphere of species i to that of species j at contact, Θ is the Heaviside step function, $\mathbf{g}_{12} = \mathbf{v}_1 - \mathbf{v}_2$, and the precollisional velocities \mathbf{v}'_1 and \mathbf{v}'_2 are given by

$$\mathbf{v}'_1 = \mathbf{v}_1 - \mu_{ji}(1 + \alpha_{ij}^{-1})(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) \hat{\boldsymbol{\sigma}}, \\ \mathbf{v}'_2 = \mathbf{v}_2 + \mu_{ij}(1 + \alpha_{ij}^{-1})(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) \hat{\boldsymbol{\sigma}}, \quad (3)$$

where $\mu_{ij} = m_i / (m_i + m_j)$. The collision operators conserve particle number for each species and the total momentum

$$\int d\mathbf{v}_1 J_{ij}[\mathbf{v}_1 | f_i, f_j] = 0, \quad \sum_{i,j} \int d\mathbf{v}_1 m_i \mathbf{v}_1 J_{ij}[\mathbf{v}_1 | f_i, f_j] = 0. \quad (4)$$

However, the total energy is not conserved. It is convenient to discuss energy transfer in terms of the partial temperatures T_i , defined by

$$\frac{3}{2} n_i k_B T_i = \int d\mathbf{v}_1 \frac{1}{2} m_i \mathbf{v}_1^2 f_i, \quad (5)$$

where k_B is the Boltzmann constant and

$$n_i = \int d\mathbf{v}_1 f_i \quad (6)$$

is the number density of species i . The temperature of the mixture is defined as

$$T = \frac{2}{3nk_B} \sum_i \int d\mathbf{v}_1 \frac{1}{2} m_i \mathbf{v}_1^2 f_i = \sum_i x_i T_i, \quad (7)$$

$x_i = n_i/n$ being the concentration of species i and $n = n_1 + n_2$ is the total number density. Also, the ‘‘cooling rates’’ (fractional energy changes per unit time) for these temperatures are defined by

$$\zeta_i \equiv -\partial_t \ln T_i, \quad \zeta \equiv -\partial_t \ln T, \quad \zeta = T^{-1} \sum_i x_i T_i \zeta_i, \quad (8)$$

where the last equality follows from the second equality of Eq. (7). The cooling rates are due to collisions, as follows from the Enskog equation:

$$\zeta_i = -\frac{2}{3n_i v_{0i}^2} \sum_j \int d\mathbf{v}_1 \mathbf{v}_1^2 J_{ij}[\mathbf{v}_1 | f_i, f_j], \quad (9)$$

where $v_{0i} = (2k_B T_i / m_i)^{1/2}$ is the thermal velocity for particles of species i . The first term on the right side with $i=j$ represents the rate of energy loss from collisions by particles of the same species. It vanishes for all f_i in the elastic limit but is nonzero for inelastic collisions. The second is nonzero in general, as it describes the transfer of energy between the different species which occurs for both elastic and inelastic collisions. However, in the special state of a Maxwellian distribution for each species *at the same temperature* this last term also vanishes in the elastic limit. This is due to detailed balance whereby the energy transfer between species is precisely balanced by energy conservation for this state.

The analog of the special detailed balance state for inelastic collisions is the homogeneous cooling state. Due to the energy loss on collisions, the ζ_i and ζ never vanish and the temperatures are always time dependent. In the same way as the single gas case, it is assumed that there is a special HCS normal solution for which all of the time dependence of $f_i(\mathbf{v}_1; t)$ is through the total temperature $T(t)$. It follows from dimensional analysis that $f_i(\mathbf{v}_1; t)$ has the form

$$f_i(\mathbf{v}_1; t) = n_i v_0^{-3}(t) \Phi_i(\mathbf{v}_1 / v_0(t)), \quad (10)$$

where $v_0^2(t) = 2k_B T(t)(m_1 + m_2)/(m_1 m_2)$ is a thermal velocity defined in terms of the temperature $T(t)$ of the mixture. It follows directly from this assumption and the definition of the partial temperature in Eq. (5) that all three temperatures are proportional to each other and their ratios are all constant. One possibility is that all three temperatures are equal, as in the case of elastic collisions. However, this cannot be assumed *a priori* and the proportionality constants must be determined from the solution to the kinetic equation. It is found below that the temperatures are in fact different. It is sufficient to determine the constant temperature ratio for the two different species $\gamma = T_1/T_2$ as all three temperatures are obtained from it via Eq. (7),

$$\gamma = \frac{T_1(t)}{T_2(t)} = \text{const}, \quad (11)$$

leading to

$$T_1(t) = \frac{\gamma}{1 + x_1(\gamma - 1)} T(t), \quad T_2(t) = \frac{1}{1 + x_1(\gamma - 1)} T(t). \quad (12)$$

An important further consequence of Eq. (11) is the equality of the cooling rates defined by Eq. (8),

$$\zeta_1(t) = \zeta_2(t) = \zeta(t). \quad (13)$$

The problem is therefore to solve the Enskog equation for a distribution function of the form (10) subject to the self-consistency constraint $\zeta_1(t) = \zeta_2(t)$.

In terms of the reduced velocity $\mathbf{v}_{1,2}^* = \mathbf{v}_{1,2} / v_0$, the dimensionless Enskog equations with Eq. (10) then become

$$\frac{1}{2} \zeta_i^* \nabla_{\mathbf{v}_1^*} \cdot (\mathbf{v}_1^* \Phi_i) = \sum_j J_{ij}^*[\mathbf{v}_1^* | \Phi_i, \Phi_j] \quad (14)$$

with $\nabla_{\mathbf{v}_1^*} = \partial / \partial \mathbf{v}_1^*$, $\zeta_i^* = \zeta_i / (n \sigma_{12}^2 v_0)$ and

$$\begin{aligned}
J_{ij}^*[v_1^*|\Phi_i, \Phi_j] &\equiv \frac{v_0^2}{n_i n \sigma_{12}^2} J_{ij}[v_1|f_i, f_j] \\
&= x_j \chi_{ij} \left(\frac{\sigma_{ij}}{\sigma_{12}} \right)^2 \int d\mathbf{v}_2^* \int d\hat{\boldsymbol{\sigma}} \Theta(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}^*) (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}^*) \\
&\quad \times [\alpha_{ij}^{-2} \Phi_i(v_1^*) \Phi_j(v_2^*) \\
&\quad - \Phi_i(v_1^*) \Phi_j(v_2^*)]. \tag{15}
\end{aligned}$$

Similarly, in dimensionless variables the cooling rates are given by

$$\zeta_i^*[\Phi_i, \Phi_j] = -\frac{2}{3} \lambda_i \sum_j \int d\mathbf{v}_1^* v_1^{*2} J_{ij}^*[v_1^*|\Phi_i, \Phi_j], \tag{16}$$

where $\lambda_i = (v_0/v_{0i})^2 = T/(T_i \mu_{ji})$. The use of ζ_i^* instead of ζ_i^* in Eq. (14) is permitted by Eq. (13). This choice is convenient since then the moments of (14) with respect to $1, \mathbf{v}$, and v^2 are automatically verified without further specification of Φ_i . Next, noting that $\gamma = (\mu_{12} \lambda_2)/(\mu_{21} \lambda_1)$, the constraint (13) determines the temperature ratio

$$\gamma = \frac{T_1(t)}{T_2(t)} = \frac{\sum_j \int d\mathbf{v}_1^* \frac{1}{2} \mu_{12} v_1^{*2} J_{1j}^*[v_1^*|\Phi_1, \Phi_j]}{\sum_j \int d\mathbf{v}_1^* \frac{1}{2} \mu_{21} v_1^{*2} J_{2j}^*[v_1^*|\Phi_2, \Phi_j]}. \tag{17}$$

Once Φ_1 and Φ_2 have been determined from Eqs. (14), the integrals on the right side of Eq. (17) can be performed to determine the temperatures.

In summary, the HCS, solution is defined by the two equations (14) and the condition that the temperature ratio, Eq. (17), is time independent. These three equations must be solved self-consistently for the two distribution functions Φ_i and the temperature ratio γ . An approximate solution is described in the next section.

III. APPROXIMATE SOLUTION

To solve the equations for the HCS, the distribution functions first are expanded in a complete set of polynomials $\{P_q\}$ with a Gaussian measure. The coefficients c_q of such an expansion are polynomial moments of the distribution functions. In practice, Sonine polynomials are used. This representation is then substituted into Eq. (14) and the equation is multiplied by $P_q(v_1)$ and integrated over v_1 , giving an infinite hierarchy of equations for the coefficients c_q . Approximate solutions are obtained by selecting a finite subset of terms in the expansion. This approach is similar to the usual moment method for solving kinetic equations in the elastic case. It has been applied as well for inelastic collisions in the one-component case where an excellent approximation is obtained by retaining only the first two terms [4]. A similar approximation is assumed here. The parameters of the Gaussian prefactor are chosen such that it is normalized to unity and provides the exact second moment (5). In this case it becomes a Maxwellian distribution at the temperature T_i . This is consistent with the normal form (14) since

$T_i(t)/T(t)$ is a constant, and it has the advantage that the leading polynomial in the expansion is of degree 4. In the dimensionless variables of Sec. II the approximate form for the solution is

$$\begin{aligned}
\Phi_i(v_1^*) &= \left(\frac{\lambda_i}{\pi} \right)^{3/2} e^{-\lambda_i v_1^{*2}} \left[1 + \frac{c_i}{4} \left(\lambda_i^2 v_1^{*4} - 5\lambda_i v_1^{*2} + \frac{15}{4} \right) \right] \\
&\equiv \Phi_i^{(0)}(v_1^*) + c_i \Phi_i^{(1)}(v_1^*), \tag{18}
\end{aligned}$$

where the coefficients c_i must be determined from the Enskog equation. These coefficients measure the deviation of Φ_i from the chosen reference Maxwellian.

It is useful at this point to see the implications of the requirement of equal cooling rates, Eq. (13) or Eq. (17), by estimating Φ_i as $\Phi_i^{(0)}$. This estimate is suggested by the fact that it is correct in the elastic limit and is known to be a very good approximation for inelastic collisions in the one-component case. The integrals of Eq. (17) are readily evaluated with this approximation, and the details are described in Appendix A. The results provide an explicit expression for γ , which simplifies in the weak dissipation limit ($1 - \alpha_{ij} \ll 1$) to

$$\begin{aligned}
\frac{T_1(t)}{T_2(t)} &\rightarrow 1 + \frac{1}{2\mu_{12}\mu_{21}} \left\{ (\mu_{12}x_1 - \mu_{21}x_2)(1 - \alpha_{12}) \right. \\
&\quad + \frac{1}{\chi_{12}\sqrt{2}} \left[\left(\frac{\sigma_{22}}{\sigma_{12}} \right)^2 x_2 \chi_{22} \sqrt{\mu_{12}} (1 - \alpha_{22}) \right. \\
&\quad \left. \left. - \left(\frac{\sigma_{11}}{\sigma_{12}} \right)^2 x_1 \chi_{11} \sqrt{\mu_{21}} (1 - \alpha_{11}) \right] \right\}. \tag{19}
\end{aligned}$$

Here, μ_{ij} and x_i are the reduced mass and concentration, respectively, for species i . This shows that the assumption of different temperatures for each species is in fact necessary, except for mechanically equivalent particles or elastic collisions. The three terms on the right side proportional to $(1 - \alpha_{ij})$ represent three different types of inelastic collisions providing independent mechanisms to enforce this temperature difference. A particularly simple example is the tracer limit for a single particle in a surrounding bath (i.e., $x_1 \rightarrow 0, x_2 \rightarrow 1$), in which case Eq. (19) becomes

$$\frac{T_1(t)}{T_2(t)} \rightarrow 1 - \frac{1}{2\mu_{12}} (1 - \alpha_{12}) + \frac{\chi_{22}}{2\mu_{21}\sqrt{2\mu_{12}}} \left(\frac{\sigma_{22}}{\sigma_{12}} \right)^2 (1 - \alpha_{22}). \tag{20}$$

This agrees with the weak dissipation limit of the exact results derived in the tracer particle problem [5] and in the Fokker-Planck description [6].

Returning to the solution (18), the coefficients c_i are determined by substitution of Eq. (18) into the Enskog equation, multiplying that equation by v_1^{*4} , and integrating over the velocity. From experience with the one-component case, it is expected that the c_i are very small so that only linear terms in c_i are retained. The coupled set of equations is found to be

$$-\frac{15}{2\lambda_i^2}\zeta_i^*\left(1+\frac{1}{2}c_i\right)=\Lambda_i^{(0)}+\Lambda_{ii}^{(1)}c_i+\Lambda_{ij}^{(1)}c_j \quad (21)$$

with the definitions

$$\Lambda_i^{(0)}=\sum_j\int d\mathbf{v}_1^*v_1^{*4}J_{ij}^*[v_1^*|\Phi_i^{(0)},\Phi_j^{(0)}], \quad (22)$$

$$\Lambda_{ii}^{(1)}=\int d\mathbf{v}_1^*v_1^{*4}\{J_{ii}^*[v_1^*|\Phi_i^{(1)},\Phi_i^{(0)}]+J_{ii}^*[v_1^*|\Phi_i^{(0)},\Phi_i^{(1)}]+J_{ij}^*[v_1^*|\Phi_i^{(1)},\Phi_j^{(0)}]\}, \quad (23)$$

$$\Lambda_{ij}^{(1)}=\int d\mathbf{v}_1^*v_1^{*4}J_{ij}^*[v_1^*|\Phi_i^{(0)},\Phi_j^{(1)}]. \quad (24)$$

Similarly, the cooling rates can be evaluated from Eq. (16) as

$$\zeta_i^*=\zeta_i^{*(0)}+\zeta_{ii}^{*(1)}c_i+\zeta_{ij}^{*(1)}c_j, \quad (25)$$

with the definitions

$$\zeta_i^{*(0)}=\frac{2}{3}\lambda_i\sum_j\int d\mathbf{v}_1^*v_1^{*2}J_{ij}^*[v_1^*|\Phi_i^{(0)},\Phi_j^{(0)}], \quad (26)$$

$$\zeta_{ii}^{*(1)}=\frac{2}{3}\lambda_i\int d\mathbf{v}_1^*v_1^{*2}\{J_{ii}^*[v_1^*|\Phi_i^{(1)},\Phi_i^{(0)}]+J_{ii}^*[v_1^*|\Phi_i^{(0)},\Phi_i^{(1)}]+J_{ij}^*[v_1^*|\Phi_i^{(1)},\Phi_j^{(0)}]\}, \quad (27)$$

$$\zeta_{ij}^{*(1)}=\frac{2}{3}\lambda_i\int d\mathbf{v}_1^*v_1^{*2}J_{ij}^*[v_1^*|\Phi_i^{(0)},\Phi_j^{(1)}]. \quad (28)$$

The condition for equal cooling rates is

$$\zeta_1^{*(0)}+\zeta_{11}^{*(1)}c_1+\zeta_{12}^{*(1)}c_2=\zeta_2^{*(0)}+\zeta_{22}^{*(1)}c_2+\zeta_{21}^{*(1)}c_1. \quad (29)$$

The problem now has been reduced to quadratures. All integrals of Eqs. (22)–(24) and (26)–(28) can be performed exactly as explicit functions of γ . The coupled set of linear equations (21) is first solved for c_1 and c_2 . Next, these are substituted into Eq. (29) to get a nonlinear function determining γ . This provides entirely all parameters of the distribution functions Φ_i and the temperatures T_i . The analysis is complex and is summarized in the Appendixes.

Before studying the general dependence of c_i and γ on the parameters of the problem, it is instructive to consider some special limit cases. In the elastic limit $\alpha_{11}=\alpha_{22}=\alpha_{12}=1$, the general results lead to $\gamma=1$ and $c_1=c_2=0$ corresponding to Maxwellians at the same temperature for the homogeneous equilibrium state [10]. In the case of mechanically equivalent particles ($m_1=m_2, \alpha_{11}=\alpha_{22}=\alpha_{12}\equiv\alpha, \sigma_{11}=\sigma_{22}$), the results of the single gas are recovered [4], namely, $\gamma=1$ and

$$c_1=c_2=\frac{32(1-\alpha)(1-2\alpha^2)}{81-17\alpha+30\alpha^2(1-\alpha)}. \quad (30)$$

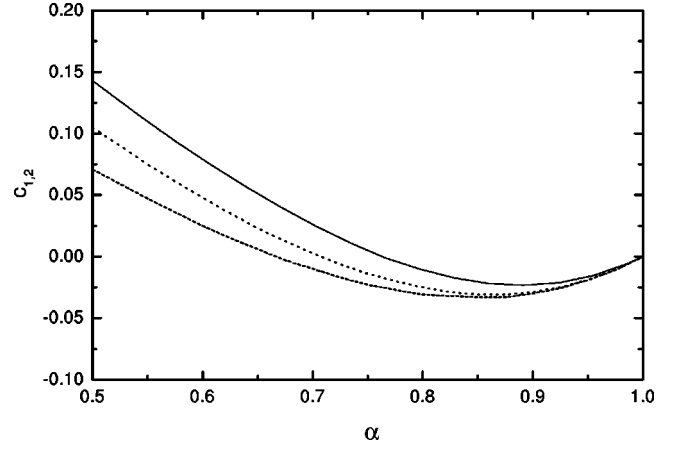


FIG. 1. Plot of the coefficients c_i versus the restitution coefficient $\alpha \equiv \alpha_{11}=\alpha_{22}=\alpha_{12}$ for $n^*=0$, $\sigma_{11}=\sigma_{22}=\sigma_{12}$, $x_1/x_2=1$, and $m_1/m_2=2$. The solid line refers to c_1 while the dashed line corresponds to c_2 . The dotted line is the common value in the single component case.

Also, in the tracer limit ($x_1 \ll x_2$) with $\alpha_{22}=1$, the solution is $c_1=0$ and

$$\gamma=\frac{1+\alpha_{12}}{2+(1-\alpha_{12})(\mu_{21}/\mu_{12})} \quad (31)$$

which agrees with the results derived in Ref. [5]. Finally, in the Fokker-Planck limit ($x_1 \ll x_2$ and $m_2 \ll m_1$), the results obtained in Ref. [6] follow from the present results. All this shows the self-consistency of the present description.

A full presentation of the results is difficult as there are many characteristic parameters: α_{ij} , m_1/m_2 , x_1/x_2 , σ_{11}/σ_{22} , and $n^*=n\sigma_{12}^3$. For the sake of concreteness, consider the case $\alpha_{11}=\alpha_{22}=\alpha_{12}\equiv\alpha$. The velocity dependence of the distribution functions is given explicitly by Eq. (18) but is parametrized by c_1 , c_2 , and γ . The primary feature of c_1 and c_2 is that they remain small for all relevant values of α , as illustrated in Fig. 1 for the typical case $m_1/m_2=2$, $x_1/x_2=1$, $\sigma_{11}=\sigma_{22}$, and $n^*=0$. Also shown is the corresponding result for a one-component system (mechanically equivalent particles). The small values of these coefficients support the assumption of a low-order truncation in the polynomial expansion of the distribution function. Further details of c_1 and c_2 will not be considered here. Instead, it is of interest to see the dependence of the temperature ratio γ on dissipation α . Figure 2 shows the dependence of γ on α for $x_1/x_2=2$, $\sigma_{11}=\sigma_{22}$, and $n^*=0$ for several values of the mass ratio. For large differences in the mass ratio the temperature differences are significant, even for moderate dissipation (say $\alpha \approx 0.9$). The temperature of the excess particles is larger (smaller) than that of the defect particles when the excess species is heavier (lighter) than the defect species. The influence of the concentration ratio on the temperature ratio is not as strong as that observed with the mass ratio, as is shown in Fig. 3, but is still quite important. Finally, in Fig. 4, $\gamma(\alpha, n^*)/\gamma(\alpha, 0)$ is plotted as a function of the reduced density n^* for $m_1/m_2=2$, $x_1/x_2=0.5$, $\sigma_{11}=2\sigma_{22}$, and for $\alpha=0.8$ and 0.6 . For a given value of the density, the relative temperature ratio decreases as the degree of inelasticity in-

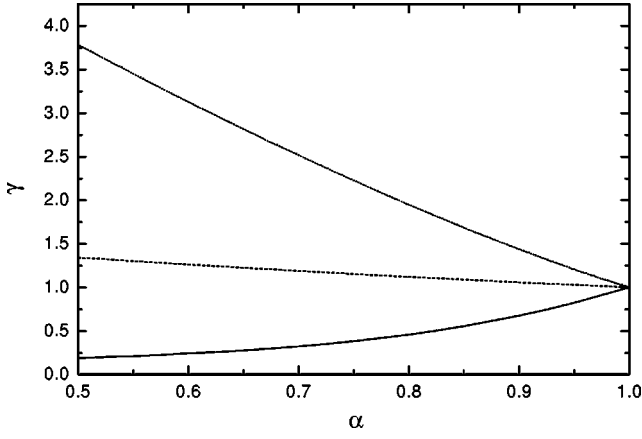


FIG. 2. Plot of the temperature ratio $\gamma = T_1/T_2$ versus the restitution coefficient $\alpha \equiv \alpha_{11} = \alpha_{22} = \alpha_{12}$ for $n^* = 0$, $\sigma_{11} = \sigma_{22} = \sigma_{12}$, $x_1/x_2 = 2$, and three different values of the mass ratio: $m_1/m_2 = 0.1$ (solid line), $m_1/m_2 = 2$ (dashed line), and $m_1/m_2 = 10$ (dotted line).

creases. In this last figure the extended Carnahan-Starling approximation for χ_{ij} as a function of n^* has been used [11],

$$\chi_{ij} = \frac{1}{1-\nu} + \frac{3}{2} \frac{\xi}{(1-\nu)^2} \frac{\sigma_{ii}\sigma_{jj}}{\sigma_{ij}} + \frac{1}{2} \frac{\xi^2}{(1-\nu)^3} \left(\frac{\sigma_{ii}\sigma_{jj}}{\sigma_{ij}} \right)^2, \quad (32)$$

where $\xi = (\pi n/6) \sum_i x_i \sigma_{ii}^2$, and $\nu = (\pi n/6) \sum_i x_i \sigma_{ii}^3$ is the volume packing fraction.

IV. DISCUSSION

The homogenous cooling state (HCS) for a binary mixture has been defined in Sec. II and evaluated to good approximation in Sec. III for a general degree of dissipation, composition, mass ratio, particle diameter, and a wide range of density (i.e., an expected accuracy comparable to that of the RET for elastic collisions). The distribution functions for each species have a scaling form with the time dependence

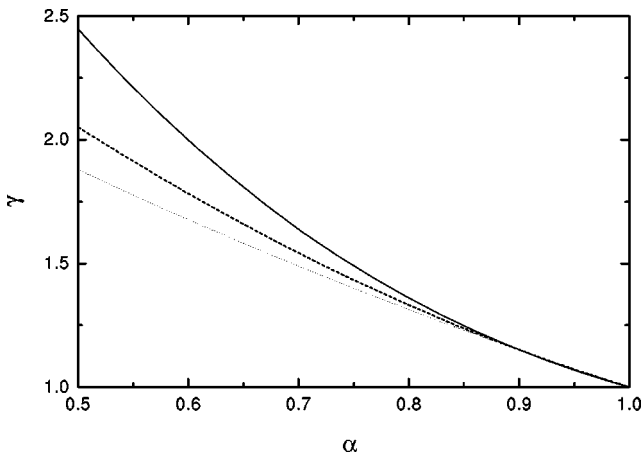


FIG. 3. Plot of the temperature ratio $\gamma = T_1/T_2$ versus the restitution coefficient $\alpha \equiv \alpha_{11} = \alpha_{22} = \alpha_{12}$ for $n^* = 0$, $\sigma_{11} = \sigma_{22} = \sigma_{12}$, $m_1/m_2 = 4$, and three different values of the concentration ratio: $x_1/x_2 = 0.25$ (solid line), $x_1/x_2 = 1$ (dashed line), and $x_1/x_2 = 4$ (dotted line).

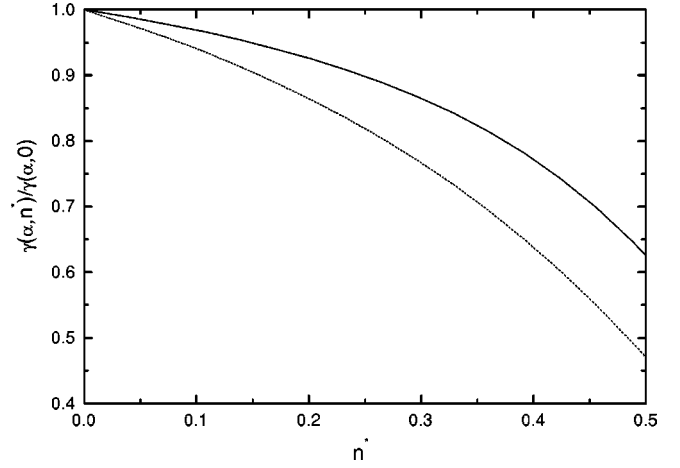


FIG. 4. Plot of the reduced temperature ratio $\gamma(\alpha, n^*)/\gamma(\alpha, 0)$ as a function of the reduced density n^* for $\sigma_{11} = 2\sigma_{22}$, $m_1/m_2 = 2$, $x_1/x_2 = 0.5$, and for two different values of the restitution coefficient: $\alpha = 0.8$ (solid line) and $\alpha = 0.6$ (dashed line).

determined by the temperature of the mixture $T(t)$, as required for a normal solution. A consequence of this scaling form is that all temperatures are proportional to each other with the same cooling rate. This does not imply that the temperatures themselves are the same, and indeed the analysis shows they are different. The present work extends previous analyses made in the limits of tracer dynamics [5] and Brownian motion [6].

The detailed mechanisms responsible for the resulting temperature differences are complex in general. However, some qualitative understanding can be obtained from the explicit weak dissipation expression (19). The cross-collisional contribution, proportional to $1 - \alpha_{12}$, tends to increase the temperature of the species with the greater mass density relative to that for the lower mass density. The self-collisional contributions tend to yield a higher temperature for the species with the weaker dissipation, larger particle size, larger concentration, and/or smaller mass. The number of variable parameters is large so that *a priori* prediction of the dominant mechanism controlling the temperature ratio is simple only in specific limiting cases.

The distribution function for each species is close to a Maxwellian at the temperature for that species. The corrections calculated within a first-order Sonine polynomial expansion are small and qualitatively similar to the small corrections found in the one-component case. However, the reference Maxwellians for the two species can be quite different due to the temperature differences. This leads to interesting new consequences for hydrodynamics. The Chapman-Enskog method for states with small spatial gradients is based on an expansion about the *local* HCS [2,3]. This is obtained from the HCS by replacing the temperature, densities, and flow velocity by their actual nonequilibrium values, e.g.,

$$f_{i,\neq}(\mathbf{r}, \mathbf{v}_i; t) = n_i(\mathbf{r}, t) v_0^{-3}(T(\mathbf{r}, t)) \Phi_i(V_i/v_0(T(\mathbf{r}, t))), \quad (33)$$

where $\mathbf{V}_i = \mathbf{v}_i - \mathbf{U}(\mathbf{r}, t)$, \mathbf{U} being the flow velocity of the mixture. The interesting new feature is an additional density

dependence due to the fact that the temperature ratio depends on the densities (or concentrations), i.e., for instance,

$$\lambda_1(\mathbf{r}, t) = \mu_{21}^{-1} \frac{T(\mathbf{r}, t)}{T_1(\mathbf{r}, t)} = \mu_{21}^{-1} [x_1(\mathbf{r}, t) + \gamma^{-1}(\mathbf{r}, t)x_2(\mathbf{r}, t)], \quad (34)$$

where the space and time dependence of $\gamma(\mathbf{r}, t)$ is through its dependence on the hydrodynamic variables [see Eq. (19) as a special example]. In the Chapman-Enskog expansion, spatial gradients of $f_{i,\ell}(\mathbf{r}, \mathbf{v}_i; t)$ occur so that completely new density gradient effects are generated by the dependence on $\lambda_i(\mathbf{r}, t)$. The details of this have not been investigated yet but it is expected to lead to interesting additional contributions to the transport coefficients. A study of these effects on mutual diffusion is in progress.

It is appropriate to comment at this point on the relationship of this work to previous studies of granular mixtures. There is no other study devoted to the homogeneous cooling state of which we are aware. There are related attempts at deriving hydrodynamic equations that implicitly assume a reference HCS [7,9]. In these cases the analysis was limited to asymptotically weak dissipation and the assumption of a local Maxwellian reference state. In addition, it is assumed that the reference states for both species are at the same global temperature. The analysis here applies for an arbitrary degree of dissipation and shows that the assumption of a common reference temperature is not justified even at weak dissipation. This suggests that the derivation of hydrodynamic equations and transport coefficients for granular mixtures should be revisited to account for these qualitatively new features.

The predicted temperature difference has a specific dependence on many parameters and is therefore susceptible to precise testing by Monte Carlo simulation of the RET and by molecular-dynamics simulations. We are not aware of any such studies of the HCS published to date (although simulations of driven mixtures appear to be underway by some groups) and hope that the present work will provide some motivation for both types of simulations. Such simulations would also provide important information about the stability of this state. This is important since it is known that the HCS for a one-component system is unstable to long-wavelength perturbations. Until the hydrodynamics for the binary mixture, described above, has been worked out, it will not be possible to say theoretically whether the mixture will be more or less stable and whether there are new mechanisms active (e.g., segregation). These interesting questions should be answered in the near future.

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APPENDIX A: COOLING RATES

In this appendix the cooling rates ζ_i^* given by Eq. (16) are evaluated by using the first Sonine polynomial approxima-

tion (18). In this appendix and the following one, it is understood that the dimensionless quantities of Sec. II are used, and the asterisk is deleted to simplify the notation. Moments of the collision integrals are evaluated using the identity

$$\begin{aligned} & \int d\mathbf{v}_1 v_1^\ell J_{ij}[\mathbf{v}_1 | \Phi_i, \Phi_j] \\ &= x_j \chi_{ij} \left(\frac{\sigma_{ij}}{\sigma_{12}} \right)^2 \int d\mathbf{v}_1 d\mathbf{v}_2 \int d\hat{\boldsymbol{\sigma}} \Theta(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) \\ & \quad \times (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) \Phi_i(\mathbf{v}_1) \Phi_j(\mathbf{v}_2) [v_1^{\ell'} - v_1^\ell], \end{aligned} \quad (A1)$$

with

$$\mathbf{v}_1^{\ell'} = \mathbf{v}_1 - \mu_{ji}(1 + \alpha_{ij})(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) \hat{\boldsymbol{\sigma}}. \quad (A2)$$

Use of Eqs. (A1) and (A2) in Eq. (16) allows the angular integrals to be performed. The calculation is straightforward but lengthy and only the result is given,

$$\begin{aligned} \zeta_i &= (1 - \alpha_{ii}^2) \frac{1}{12} \pi \lambda_i \left(\frac{\sigma_{ii}}{\sigma_{12}} \right)^2 x_i \chi_{ii} \int d\mathbf{v}_1 d\mathbf{v}_2 g_{12}^3 \Phi_i(\mathbf{v}_1) \Phi_i(\mathbf{v}_2) \\ & \quad + (1 - \alpha_{ij}^2) \frac{1}{3} \pi \lambda_i \mu_{ji}^2 x_j \chi_{ij} \int d\mathbf{v}_1 d\mathbf{v}_2 g_{12}^3 \Phi_i(\mathbf{v}_1) \Phi_j(\mathbf{v}_2) \\ & \quad + (1 + \alpha_{ij}) \frac{2}{3} \pi \lambda_i \mu_{ji} x_j \chi_{ij} \\ & \quad \times \int d\mathbf{v}_1 d\mathbf{v}_2 g_{12}(\mathbf{g}_{12} \cdot \mathbf{G}_{ij}) \Phi_i(\mathbf{v}_1) \Phi_j(\mathbf{v}_2), \end{aligned} \quad (A3)$$

where it is understood that $j \neq i$ and $\mathbf{G}_{ij} \equiv \mu_{ij} \mathbf{v}_1 + \mu_{ji} \mathbf{v}_2$. This expression for ζ_i is still exact.

The leading contribution $\zeta_i^{(0)}$ is obtained by the replacement $\Phi_i \rightarrow \Phi_i^{(0)}$,

$$\Phi_i^{(0)}(\mathbf{v}_1) = \left(\frac{\lambda_i}{\pi} \right)^{3/2} e^{-\lambda_i v_1^2}, \quad (A4)$$

to get

$$\begin{aligned} \zeta_i^{(0)} &= (1 - \alpha_{ii}^2) \frac{1}{12} \pi^{-2} \left(\frac{\sigma_{ii}}{\sigma_{12}} \right)^2 x_i \chi_{ii} \lambda_i^{-1/2} \\ & \quad \times \int d\mathbf{v}_1 d\mathbf{v}_2 g_{12}^3 e^{-(v_1^2 + v_2^2)} \\ & \quad + (1 - \alpha_{ij}^2) \frac{1}{3} \pi^{-2} \mu_{ji}^2 x_j \chi_{ij} \lambda_j^{-1/2} \frac{\lambda_i}{\lambda_j} \\ & \quad \times \int d\mathbf{v}_1 d\mathbf{v}_2 |(\lambda_j / \lambda_i)^{1/2} \mathbf{v}_1 - \mathbf{v}_2|^3 e^{-(v_1^2 + v_2^2)} \\ & \quad + (1 + \alpha_{ij}) \frac{2}{3} \pi^{-2} \mu_{ji} x_j \chi_{ij} \lambda_j^{-1/2} \frac{\lambda_i}{\lambda_j} \\ & \quad \times \int d\mathbf{v}_1 d\mathbf{v}_2 |(\lambda_j / \lambda_i)^{1/2} \mathbf{v}_1 - \mathbf{v}_2| \end{aligned} \quad (A5)$$

$$\begin{aligned} & \times [(\lambda_j / \lambda_i)^{1/2} \mathbf{v}_1 - \mathbf{v}_2] \cdot [(\lambda_j / \lambda_i)^{1/2} \mu_{ij} \mathbf{v}_1 + \mu_{ji} \mathbf{v}_2] \\ & \times e^{-(v_1^2 + v_2^2)}. \end{aligned} \quad (A6)$$

The first integral is performed by transforming to relative and center-of-mass variables. The second two integrals can be performed by the change of variables

$$\mathbf{x} = \left(\frac{\lambda_j}{\lambda_i} \right)^{1/2} \mathbf{v}_1 - \mathbf{v}_2, \quad (\text{A7})$$

$$\mathbf{y} = \left(\frac{\lambda_j}{\lambda_i} \right)^{-1/2} \mathbf{v}_1 - \mathbf{v}_2, \quad (\text{A8})$$

with the Jacobian $[(\lambda_j/\lambda_i)^{1/2} + (\lambda_j/\lambda_i)^{-1/2}]^3$. The final result for $\zeta^{(0)}$ is

$$\begin{aligned} \zeta_1^{(0)} \rightarrow & \frac{2}{3} \sqrt{2} \pi \left(\frac{\sigma_{11}}{\sigma_{12}} \right)^2 x_1 \chi_{11} \lambda_1^{-1/2} (1 - \alpha_{11}^2) \\ & + \frac{4}{3} \sqrt{\pi} x_2 \chi_{12} \mu_{21} \left(\frac{1 + \eta}{\eta} \right)^{1/2} (1 + \alpha_{12}) \lambda_2^{-1/2} \\ & \times [2 - \mu_{21}(1 + \alpha_{12})(1 + \eta)], \end{aligned} \quad (\text{A9})$$

where $\eta = \lambda_1/\lambda_2 = \mu_{12}/(\mu_{21}\gamma)$. The result for $\zeta_2^{(0)}$ is obtained from Eq. (A9) by interchanging 1 and 2, and $\eta \rightarrow \eta^{-1}$. The estimate of γ given by Eq. (19) is obtained by equating $\zeta_1^{(0)} = \zeta_2^{(0)}$ and evaluating γ to first order in $(1 - \alpha_{ij})$.

The expression for the coefficient of c_i , $\zeta_{ii}^{(1)}$, is given by Eq. (27). This is identified from Eq. (A3) as

$$\begin{aligned} \zeta_{ii}^{(1)} = & (1 - \alpha_{ii}^2) \frac{1}{12} \pi \lambda_i \left(\frac{\sigma_{ii}}{\sigma_{12}} \right)^2 x_i \chi_{ii} \\ & \times \int d\mathbf{v}_1 d\mathbf{v}_2 g_{12}^3 [\Phi_i^{(0)}(\mathbf{v}_1) \Phi_i^{(1)}(\mathbf{v}_2) \\ & + \Phi_i^{(1)}(\mathbf{v}_1) \Phi_i^{(0)}(\mathbf{v}_2)] + (1 - \alpha_{ij}^2) \\ & \times \frac{1}{3} \pi \lambda_i \mu_{ji}^2 x_j \chi_{ij} \int d\mathbf{v}_1 d\mathbf{v}_2 g_{12}^3 \Phi_i^{(1)}(\mathbf{v}_1) \Phi_j^{(0)}(\mathbf{v}_2) \\ & + (1 + \alpha_{ij}) \frac{2}{3} \pi \lambda_i \mu_{ji} x_j \chi_{ij} \\ & \times \int d\mathbf{v}_1 d\mathbf{v}_2 g_{12}(\mathbf{g}_{12} \cdot \mathbf{G}_{ij}) \Phi_i^{(1)}(\mathbf{v}_1) \Phi_j^{(0)}(\mathbf{v}_2). \end{aligned} \quad (\text{A10})$$

These are Gaussian integrals of polynomials similar to those of Eq. (A5) and can be performed in the same way. The result for $\zeta_{11}^{(1)}$ is

$$\begin{aligned} \zeta_{11}^{(1)} = & \frac{1}{8} \sqrt{\frac{\pi}{2}} \left(\frac{\sigma_{11}}{\sigma_{12}} \right)^2 x_1 \chi_{11} \lambda_1^{-1/2} (1 - \alpha_{11}^2) \\ & + \frac{1}{12} \sqrt{\pi} x_2 \chi_{12} \mu_{21} \frac{(1 + \eta)^{-3/2}}{\eta^{1/2}} (1 + \alpha_{12}) \lambda_2^{-1/2} \\ & \times [2(3 + 4\eta) - 3\mu_{21}(1 + \alpha_{12})(1 + \eta)]. \end{aligned} \quad (\text{A11})$$

In the same way $\zeta_{12}^{(1)}$ is given by

$$\begin{aligned} \zeta_{12}^{(1)} = & -\frac{1}{12} \sqrt{\pi} x_2 \chi_{12} \mu_{21} \left(\frac{1 + \eta}{\eta} \right)^{-3/2} (1 + \alpha_{12}) \lambda_2^{-1/2} \\ & \times [2 + 3\mu_{21}(1 + \alpha_{12})(1 + \eta)]. \end{aligned} \quad (\text{A12})$$

The expressions for $\zeta_{22}^{(1)}$ and $\zeta_{21}^{(1)}$ can be easily obtained from Eqs. (A11) and (A12).

APPENDIX B: COLLISION INTEGRALS

The collision integrals (22)–(24) are evaluated in this appendix. Consider first the general expression

$$\begin{aligned} \Lambda_i = & \sum_j \int d\mathbf{v}_1 v_1^4 J_{ij} [v_1 | \Phi_i, \Phi_j] \\ = & \sum_j x_j \chi_{ij} \left(\frac{\sigma_{ij}}{\sigma_{12}} \right)^2 \int d\mathbf{v}_1 d\mathbf{v}_2 \int d\hat{\boldsymbol{\sigma}} \Theta(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) \\ & \times (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) \Phi_i(v_1) \Phi_j(v_2) [v_1''^4 - v_1^4], \end{aligned} \quad (\text{B1})$$

where the identity (A1) has been used. Substitution of Eq. (A2) into Eq. (B1) allows the angular integral to be performed with the result

$$\Lambda_i = \sum_j x_j \chi_{ij} \left(\frac{\sigma_{ij}}{\sigma_{12}} \right)^2 \int d\mathbf{v}_1 d\mathbf{v}_2 \Phi_i(v_1) \Phi_j(v_2) F_{ij}(\mathbf{g}_{12}, \mathbf{G}_{ij}), \quad (\text{B2})$$

$$\begin{aligned} F_{ij}(\mathbf{g}_{12}, \mathbf{G}_{ij}) = & -\frac{\pi}{3} (1 - \alpha_{ij}^2) \mu_{ji}^4 (2 + \alpha_{ij}^2) g_{12}^5 + \pi (1 + \alpha_{ij}) \\ & \times \left[\frac{2}{3} \mu_{ji}^2 (2\alpha_{ij} - 1) g_{12}^3 G_{ij}^2 \right. \\ & - 2\mu_{ji} g_{12} G_{ij}^2 (\mathbf{g}_{12} \cdot \mathbf{G}_{ij}) + \mu_{ji}^2 (\alpha_{ij} - 3) \\ & \times g_{12} (\mathbf{g}_{12} \cdot \mathbf{G}_{ij})^2 - \frac{2}{3} \mu_{ji}^3 (4 - 3\alpha_{ij} + 2\alpha_{ij}^2) \\ & \left. \times g_{12} (\mathbf{g}_{12} \cdot \mathbf{G}_{ij})^2 \right]. \end{aligned} \quad (\text{B3})$$

The collision integrals (22)–(24) can now be identified as

$$\begin{aligned} \Lambda_i^{(0)} = & \sum_j x_j \chi_{ij} \left(\frac{\sigma_{ij}}{\sigma_{12}} \right)^2 \int d\mathbf{v}_1 d\mathbf{v}_2 \\ & \times \Phi_i^{(0)}(v_1) \Phi_j^{(0)}(v_2) F_{ij}(\mathbf{g}_{12}, \mathbf{G}_{ij}), \end{aligned} \quad (\text{B4})$$

$$\begin{aligned} \Lambda_{ii}^{(1)} = & x_i \chi_{ii} \left(\frac{\sigma_{ii}}{\sigma_{12}} \right)^2 \int d\mathbf{v}_1 d\mathbf{v}_2 [\Phi_i^{(0)}(v_1) \Phi_i^{(1)}(v_2) \\ & + \Phi_i^{(1)}(v_1) \Phi_i^{(0)}(v_2)] F_{ii}(\mathbf{g}_{12}, \mathbf{G}_{ii}), \end{aligned} \quad (\text{B5})$$

$$\Lambda_{ij}^{(1)} = x_j \chi_{ij} \int d\mathbf{v}_1 d\mathbf{v}_2 \Phi_i^{(0)}(v_1) \Phi_j^{(1)}(v_2) F_{ij}(\mathbf{g}_{12}, \mathbf{G}_{ij}). \quad (\text{B6})$$

These are Gaussian integrals that can be calculated by the same method as described in Appendix A. The integrations

can be done quite efficiently by using a computer package of symbolic calculation. Here, we have used MATHEMATICA [12]. The results are

$$\Lambda_1^{(0)} = -\sqrt{2\pi}\lambda_1^{-5/2} \left\{ x_1 \chi_{11} \left(\frac{\sigma_{11}}{\sigma_{12}} \right)^2 \frac{9 + 2\alpha_{11}^2}{2} (1 - \alpha_{11}^2) \right. \\ \left. - \sqrt{2} x_2 \chi_{12} (1 + \eta)^{-1/2} \mu_{21} (1 + \alpha_{12}) [-2(6 + 5\eta) \right. \\ \left. + \mu_{21} (1 + \alpha_{12}) (1 + \eta) (14 + 5\eta) - 8\mu_{21}^2 (1 + \alpha_{12})^2 \right. \\ \left. \times (1 + \eta)^2 + 2\mu_{21}^3 (1 + \alpha_{12})^3 (1 + \eta)^3 \right\}, \quad (\text{B7})$$

$$\Lambda_{11}^{(1)} = -\sqrt{2\pi}\lambda_1^{-5/2} \left\{ x_1 \chi_{11} \left(\frac{\sigma_{11}}{\sigma_{12}} \right)^2 \left[1 + \alpha_{11} + \frac{3}{64} (69 + 10\alpha_{11}^2) \right. \right. \\ \left. \left. \times (1 - \alpha_{11}^2) \right] - \frac{\sqrt{2}}{16} x_2 \chi_{12} (1 + \eta)^{-5/2} \mu_{21} (1 + \alpha_{12}) \right. \\ \left. \times [-2(90 + 231\eta + 184\eta^2 + 40\eta^3) + 3\mu_{21} (1 + \alpha_{12}) \right. \\ \left. \times (1 + \eta)(70 + 117\eta + 44\eta^2) - 24\mu_{21}^2 (1 + \alpha_{12})^2 \right. \\ \left. \times (1 + \eta)^2 (5 + 4\eta) + 30\mu_{21}^3 (1 + \alpha_{12})^3 (1 + \eta)^3 \right\}, \quad (\text{B8})$$

$$\Lambda_{12}^{(1)} = \frac{\sqrt{\pi}}{8} x_2 \chi_{12} \lambda_1^{-5/2} \eta^2 (1 + \eta)^{-5/2} \mu_{21} (1 + \alpha_{12}) [2(2 + 5\eta) \\ + 3\mu_{21} (1 + \alpha_{12}) (1 + \eta) (2 + 5\eta) - 24\mu_{21}^2 (1 + \alpha_{12})^2 \\ \times (1 + \eta)^2 + 30\mu_{21}^3 (1 + \alpha_{12})^3 (1 + \eta)^3]. \quad (\text{B9})$$

The corresponding expressions for $\Lambda_2^{(0)}$, $\Lambda_{22}^{(1)}$, and $\Lambda_{21}^{(1)}$ can be easily inferred from Eqs. (B7)–(B9) by interchanging 1 and 2 and setting $\eta \rightarrow \eta^{-1}$.

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