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Homogeneous Cosmological Models without Shear

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The paper presents some general differential relations on the dynamics of homogeneous rotating cosmological models with nonvanishing pressure. It appears from these that such models must in general be associated with shear if there be a linear relation between the pressure and the density except perhaps in the case $p=\rho/9$.

§ 1. Introduction

Schücking[1] proved that a spatially homogeneous rotating and expanding universe filled with incoherent dust must necessarily have shear. We propose now to investigate the case of nonvanishing pressure when the equation of state is

$$p=\alpha\rho\,, \tag{1}$$

α being a positive constant, which is 0 for dust, 1/3 for diffuse radiation and 1 for the extreme Zeldovich case.

Before taking up this problem in § 3, we shall first reduce some of the field equations for a rotating universe filled with a perfect fluid to a convenient form. We shall note that for a spatially homogeneous rotating universe the homogeneous varieties are not orthogonal to the world lines, so that although the pressure is uniform over the homogeneous varieties the pressure gradient has a nonvanishing component orthogonal to the world lines and this would lead to a deviation of the world lines from geodesic paths. This is essentially due to the noncoincidence of the two sets of 3-spaces: (1) the spaces defined by the Killing vectors and (2) the local spaces orthogonal to the world lines. The modification introduced by this in the acceleration equation of Raychaudhuri[2] will be studied in the next section.

§ 2. Field equations

The homogeneous varieties are defined as the t-constant spaces and the time-lines are chosen along the world lines of matter (i.e. we introduce a comoving coordinate system). In view of the above choice of the coordinate system and homogeneity, g_{44} is at most a function of t alone and can be reduced to unity by a suitable transformation of t. Therefore the line element is given by

$$ds^2=dt^2+2g_{4i}dtdx^i+g_{ik}dx^idx^k\,. \tag{2}$$

In this paper the latin indices run over the values 1 to 3 and the greek indices run from 1 to 4. Some at least of the g_{4i} are nonvanishing because when rotation is present the world lines do not form a normal congruence.

We now take the field equations in the form

$$\left.\begin{aligned} R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu}&=-8\pi T_{\mu\nu}\,,\\ \text{where}\qquad T_{\mu\nu}&=(\rho+p)v_\mu v_\nu-pg_{\mu\nu}\,,\end{aligned}\right\} \tag{3}$$

v^μ being the velocity vector of matter and is therefore in our case $=\delta_4{}^\mu$.

In this case the world lines of matter would not in general be geodesics. The relation $T^{\mu\nu};\nu=0$ gives

$$(\rho+p)v^\mu{}_{;\nu}v^\nu+p_{,\nu}(v^\mu v^\nu-g^{\mu\nu})=0 \tag{4}$$

and

$$\frac{\dot\rho}{p+\rho}=-3\frac{\dot G}{G}\,,\qquad \text{where } G^3=\sqrt{-g}\,. \tag{5}$$

The dot denotes the time derivative.

Let us split up the vector $p_{,\mu}/(p+\rho)$ into parts parallel and orthogonal to v_μ:

$$\frac{p_{,\mu}}{p+\rho}=\alpha v_\mu+\Pi_\mu\,, \tag{6}$$

where $\alpha=p_{,\mu}v^\mu/(p+\rho)$ and $\Pi_\mu v^\mu=0$.
From Eqs. (4) and (6) we obtain

$$\Pi_\mu=v_{\mu;\nu}v^\nu=(v_{\mu,\nu}-v_{\nu,\mu})v^\nu\,. \tag{7}$$

Equation (7) indicates that the deviation from the geodesic path is solely due to the component of the pressure gradient orthogonal to the velocity vector.

Now in our coordinate system

$$\Pi_i=g_{i4,4}\,,\qquad \Pi_4=0\,. \tag{7a}$$

We define now the velocity vector, the shear and the scalar of expansion respectively by (Gödel,[3] Ehlers[4])

$$\omega^\mu=\tfrac{1}{2}\eta^{\mu\nu\rho\sigma}v_\nu v_{\rho;\sigma} \tag{8}$$

(where $\eta^{\mu\nu\rho\sigma}$ is the permutation tensor),

$$\varphi_{\mu\nu}=\tfrac{1}{2}(v_{\mu;\nu}+v_{\nu;\mu})-\tfrac{1}{3}(g_{\mu\nu}-v_\mu v_\nu)v^\gamma{}_{;\gamma}-\tfrac{1}{2}(\Pi_\mu v_\nu+\Pi_\nu v_\mu)\,, \tag{9}$$

$$\theta=v^\gamma{}_{;\gamma}\,. \tag{10}$$

From Eq. (3) we have

$$R_{\mu\sigma}v^\mu v^\sigma=-4\pi(\rho+3p)\,. \tag{11}$$

Further we can write using Eqs. (7) and (10)

$$R_{\mu\sigma}v^\mu v^\sigma = (v^\gamma{}_{;\gamma\sigma} - v^\gamma{}_{;\sigma\gamma})v^\sigma = \theta_{,\sigma}v^\sigma - \Pi^\gamma{}_{;\gamma} + v_{\gamma;\sigma}v^{\sigma;\gamma}\,. \tag{12}$$

Separating the symmetric and skewsymmetric parts of $v_{\gamma;\sigma}$ we obtain from Eq. (9)

$$v_{\gamma;\sigma} = \varphi_{\gamma\sigma} + \tfrac{1}{3}(g_{\gamma\sigma} - v_\gamma v_\sigma)\theta + \tfrac{1}{2}(\Pi_\gamma v_\sigma + \Pi_\sigma v_\gamma) + \tfrac{1}{2}(v_{\gamma;\sigma} - v_{\sigma;\gamma})\,. \tag{13}$$

Equation (8) gives

$$-\eta_{\mu\nu\gamma\sigma}\omega^\mu = \tfrac{1}{2}(v_{\gamma;\sigma} - v_{\sigma;\gamma})v_\nu + \tfrac{1}{2}(v_{\sigma;\nu} - v_{\nu;\sigma})v_\gamma + \tfrac{1}{2}(v_{\nu;\gamma} - v_{\gamma;\nu})v_\sigma\,. \tag{14}$$

Combining (13) and (14) we obtain

$$v_{\gamma;\sigma} = \varphi_{\gamma\sigma} + \tfrac{1}{3}(g_{\gamma\sigma} - v_\gamma v_\sigma)\theta + \Pi_\gamma v_\sigma - \eta_{\mu\nu\gamma\sigma}\omega^\mu v^\nu\,. \tag{15}$$

Substituting this value in Eq. (12) we obtain finally

$$\theta_{,\sigma}v^\sigma + \tfrac{1}{3}\theta^2 = \Pi^\mu{}_{;\mu} + 2\omega^2 - \varphi^2 - 4\pi(\rho + 3p)\,, \tag{16}$$

where $g_{\mu\nu}\omega^\mu\omega^\nu = -\omega^2$ and $\varphi_{\mu\nu}\varphi^{\mu\nu} = \varphi^2$.
In our coordinate system

$$\theta_{,\sigma}v^\sigma + \frac{1}{3}\theta^2 = 3\,\frac{\ddot{G}}{G}\,. \tag{17}$$

Using Eqs. (6), (16) and (17) we have

$$3\,\frac{\ddot{G}}{G} = -\frac{1}{G^3}\left[\frac{\dot{p}}{p+\rho}\,G^3(\delta_4{}^\mu - g^{\mu 4})\right]_{,\mu} + 2\omega^2 - \varphi^2 - 4\pi(\rho + 3p)\,. \tag{16a}$$

Equation (16a) is a generalization of the acceleration equation obtained by Raychaudhuri.[2)] Besides the usual occurrence of $(\rho+3p)$ instead of ρ in the source term, there is the new term involving the time derivative of p coupled with a characteristic factor, which vanishes when the homogeneous varieties are orthogonal to the world lines.

It is interesting to examine this term a little more closely. Using Eq. (5) one can reduce this term to

$$(1-g^{44})\left[3\left\{\frac{\ddot{G}}{G} + 2\left(\frac{\dot{G}}{G}\right)^2\right\}\frac{dp}{d\rho} - 9\,\frac{d^2p}{d\rho^2}\,(p+\rho)\left(\frac{\dot{G}}{G}\right)^2\right]$$

$$-3\,\frac{dp}{d\rho}\,\frac{\dot{G}}{G}(g^{44})_{,4} - \frac{3}{G^3}\,\frac{dp}{d\rho}\,\frac{\dot{G}}{G}(g^{i4}G^3)_{,i}\,,$$

so that one can write (16a) as

$$\frac{\ddot{G}}{G}\left[1 - \frac{dp}{d\rho}(1-g^{44})\right] = \frac{1}{3}(2\omega^2 - \varphi^2) - \frac{4\pi}{3}(\rho + 3p)\,,$$

+ terms involving $\dot{G}/G$, $dp/d\rho$ and $d^2p/d\rho^2$ but not involving $\ddot{G}/G$. In general owing to the negative definite character of the space metric $g_{ik} - g_{i4}g_{k4}$, $g^{44} \leqslant 1$ and vanishes if the homogeneous variety has null lines, and when the homogeneous varieties have time-like lines g^{44} becomes negative. Thus according as g^{44}

is positive, zero or negative, the pressure gradient which is orthogonal to the homogeneous varieties is a time-like, null or space-like vector respectively. When g^{44} becomes negative the possibility exists that with high enough values of pressure, the coefficient of $\ddot{G}/G$ may become negative. Under such circumstances the role of rotation, energy density and pressure may be quite the opposite of what one usually gets.

It is interesting to consider the Gödel solution. Here the coefficient of $\ddot{G}/G$ turns out to be -1 (as the Gödel universe may alternatively be supposed to have a pressure $p=\rho$ and cosmological constant $\lambda=0$), so that if one considers a universe starting out slowly from the Gödel state ($\dot{G}/G$ small) the tendency of the $(\rho+3p)$ term would be to accelerate an expansion while the rotation would try to arrest an expansion. This is exactly the opposite of what one would expect, say, in the case of perturbations of an Einstein universe.

Again we have from Eq. (3)

$$R_\mu{}^\gamma v^\mu = -4\pi v^\gamma(\rho+3p) \tag{18}$$

and obtain after a similar calculation as before

$$\begin{aligned} R_\mu{}^\gamma v^\mu = \theta_{,\sigma}\left(\tfrac{2}{3} g^{\gamma\sigma} + \tfrac{1}{3} v^\gamma v^\sigma\right) - \varphi^{\gamma\beta} + \tfrac{1}{3}\theta^2 v^\gamma - \Pi^\mu{}_{;\mu} v^\gamma \\ - \Pi_\beta \varphi^{\gamma\beta} - 2\omega^2 v^\gamma - \eta^{\mu\nu\beta\gamma}(\omega_{\mu,\beta} v_\nu - 2\omega_\mu v_\nu \Pi_\beta). \end{aligned} \tag{19}$$

We can further simplify this equation by using Eqs. (16), (18) and (6):

$$\frac{2}{3}\theta_{,\sigma}(g^{\sigma\gamma} - v^\sigma v^\gamma) = \varphi^{\gamma\beta}{}_{;\beta} + \varphi^{\gamma\beta}\Pi_\beta + \varphi^2 v^\gamma + \eta^{\mu\nu\beta\gamma}\left(\omega_{\mu,\beta} v_\nu - 2\omega_\mu v_\nu \frac{p_{,\beta}}{p+\rho}\right). \tag{20}$$

Equations (16) and (20) are not quite new as similar equations appear in Ehlers' paper.

§ 3. Special case: $p=\alpha\rho$, $\varphi^2=0$

Applying the condition of homogeneous density to Eq. (5) we have as in Schücking's paper

$$G = S(x^4)\, W(x^i). \tag{21}$$

The condition of shear-free expansion, i.e. $\varphi_{\mu\nu}=0$ gives from Eq. (9)

$$\bar{g}_{ik} \equiv g_{ik} - g_{i4} g_{k4} = G^2(x^\mu)\, \psi_{ik}(x^j), \tag{22}$$

if we remember Eq. (7a).

Combining Eqs. (21) and (22) we have

$$\bar{g}_{ik} = S^2(x^4)\, \bar{\psi}_{ik}(x^j), \qquad \bar{\psi}_{ik} = W^2 \psi_{ik}. \tag{23}$$

Defining $\bar{g}^{kl}\bar{g}_{ik} = \delta_i{}^l$, we have

$$\bar{g}^{kl} = S^{-2}(x^4)\, \bar{\psi}^{kl}, \qquad \bar{\psi}_{ik}\bar{\psi}^{kl} = \delta_i{}^l. \tag{24}$$

Now we have the relations

$$g^{i4}g_{4k}+g^{il}g_{lk}=\delta_k{}^i \tag{25}$$

and

$$g^{i4}=-g^{ik}g_{k4}\,. \tag{26}$$

From Eqs. (25) and (26) we obtain

$$g^{il}(g_{lk}-g_{l4}g_{k4})=g^{il}\bar{g}_{lk}=\delta_k{}^i\,. \tag{27}$$

Therefore from the definition of $\bar{g}^{il}$ we have

$$\bar{g}^{il}=g^{il}\,. \tag{28}$$

Using Eqs. (1), (6) and (7a) and remembering that $p_{,i}=0$ we obtain in view of homogeneity

$$g_{i4}=G^{3\alpha}(x^\mu)\psi_{i4}(x^j)=S^{3\alpha}\bar{\psi}_{i4}(x^j)\,,\qquad \bar{\psi}_{i4}=W^{3\alpha}\psi_{i4}\,. \tag{29}$$

From Eqs. (22) and (29) we have

$$g_{ik}=S^2\bar{\psi}_{ik}+S^{6\alpha}\bar{\psi}_{i4}\bar{\psi}_{k4}\,. \tag{30}$$

From Eqs. (24), (26), (28) and (29) we have

$$g^{i4}=-S^{3\alpha-2}\bar{\psi}^{ik}\bar{\psi}_{k4}\,. \tag{31}$$

Again

$$g^{44}=1-g^{4i}g_{4i}=1+S^{6\alpha-2}\bar{\psi}^{ik}\bar{\psi}_{i4}\bar{\psi}_{k4}\,. \tag{32}$$

Now for shear-free expansion ($\varphi^2=0$), Eqs. (16a) and (20) reduce to

$$3\frac{\ddot{G}}{G}=(1-g^{44})\left[3\left\{\frac{\ddot{G}}{G}+2\left(\frac{\dot{G}}{G}\right)^2\right\}\frac{dp}{d\rho}-9\frac{d^2p}{d\rho^2}(p+\rho)\left(\frac{\dot{G}}{G}\right)^2\right]$$
$$-3\left(\frac{dp}{d\rho}\right)\frac{\dot{G}}{G}(g^{44})_{,4}-\frac{3}{G^3}\frac{dp}{d\rho}\frac{\dot{G}}{G}(g^{i4}G^3)_{,i}+2\omega^2-4\pi(\rho+3p) \tag{16b}$$

and

$$\frac{2}{3}\theta_{,\sigma}(g^{\sigma\gamma}-v^\sigma v^\gamma)=\eta^{\mu\nu\beta\gamma}\left(\omega_{\mu,\beta}v_\nu-2\omega_\mu v_\nu\frac{p_{,\beta}}{p+\rho}\right). \tag{20a}$$

We can now use Eqs. (21), (24), (28)-(32) to express all the terms in (16b) and (20a) as products of a time dependent and a space dependent part. The results are given below:

$$\omega^i=\frac{1}{2}\frac{S^{3\alpha-3}}{W^3}\epsilon^{i4kl}\bar{\psi}_{4k,l}\,,\qquad [\epsilon^{\mu\nu\rho\sigma}\equiv\text{Levi-Civita tensor density}] \tag{33a}$$

$$\omega^4=\frac{1}{2}\frac{S^{6\alpha-3}}{W^3}\epsilon^{4jkl}\bar{\psi}_{4j}\bar{\psi}_{4k,l}\,, \tag{33b}$$

$$-\omega^2=\frac{1}{4}\frac{S^{6\alpha-4}}{W^6}\epsilon^{i4kl}\epsilon^{m4np}\bar{\psi}_{im}\bar{\psi}_{4k,l}\bar{\psi}_{4n,p}\,. \tag{33c}$$

Since ω^2 is a scalar quantity, it must be a function of t alone in a spatially homogeneous universe. Hence

$$\omega^2 = AS^{6\alpha-4}, \qquad A \text{ being a positive constant.} \tag{34}$$

Using (34), Eq. (16a) becomes

$$3\frac{\ddot{S}}{S} - 2AS^{6\alpha-4} + 4\pi BS^{-3(1+\alpha)} = -3\alpha\overline{\phi}^{ik}\overline{\phi}_{i4}\overline{\phi}_{k4}[S^{6\alpha-3}\ddot{S} + 6\alpha S^{6\alpha-4}\dot{S}^2]$$
$$+3\alpha S^{3\alpha-3}\dot{S}W^{-3}(W^3\overline{\phi}^{ik}\overline{\phi}_{k4})_{,i},$$

where $\rho+3p=BS^{-3(1+\alpha)}$, B being a positive constant. Since the left-hand side is a function of t alone, $\overline{\phi}^{ik}\overline{\phi}_{i4}\overline{\phi}_{k4}$ and $W^{-3}(W^3\overline{\phi}^{ik}\overline{\phi}_{k4})_{,i}$ must both be constants. Therefore

$$3\frac{\ddot{S}}{S} - 2AS^{6\alpha-4} + 4\pi BS^{-3(1+\alpha)} = CS^{6\alpha-3}\ddot{S} + 6\alpha CS^{6\alpha-4}\dot{S}^2 + DS^{3\alpha-3}\dot{S}, \tag{35}$$

where $C=-3\alpha\overline{\phi}^{ik}\overline{\phi}_{i4}\overline{\phi}_{k4}$ and $D=3\alpha W^{-3}(W^3\overline{\phi}^{ik}\overline{\phi}_{k4})_{,i}$. The fourth equation of (20a) $(\gamma=4)$ becomes similarly

$$\frac{\ddot{S}}{S} - \left(\frac{\dot{S}}{S}\right)^2 = \frac{E}{S^2}, \qquad E \text{ being a constant,} \tag{36}$$

where $E=\epsilon^{ijk4}\epsilon^{m4np}\overline{\phi}_{4j}\{W^{-3}\overline{\phi}_{im}\overline{\phi}_{4n,p}\}_{,k}/4W^3(\overline{\phi}^{ik}\overline{\phi}_{i4}\overline{\phi}_{k4})$. The above equation gives on a single integration

$$\dot{S}^2 = FS^2 - E, \tag{37}$$

where F is a constant of integration.

Substituting this value in (35) we have

$$3F - S^{6\alpha-4}(2A-6\alpha CE) + 4\pi BS^{-3(1+\alpha)}$$
$$= (CF+6\alpha CF)S^{6\alpha-2} + D\sqrt{F}\,S^{3\alpha-2}\left(1-\frac{E}{FS^2}\right)^{1/2}. \tag{38}$$

If $F\neq 0$ we should have another term involving S^0 to balance it. Considering the different possibilities we can show that this is impossible for nonvanishing density and spin. Using $F=0$ we obtain the solution

$$S=\sqrt{-Et}\,. \tag{39}$$

Substituting this value in (35) we find that if $B\neq 0$ (i.e. $\rho\neq 0$) then the only two possibilities are $\alpha=0, \frac{1}{9}$. The first case has already been considered by Schücking. For $\alpha=\frac{1}{9}$ we have the conditions

$$D=0, \text{ i.e. } (W^3\overline{\phi}^{ik}\overline{\phi}_{k4})_{,i}=0, \text{ i.e. } (g^{i4}G^3)_{,i}=0 \tag{40}$$

and $3A-CE=6\pi B$

$$\text{i.e. } \epsilon^{ijk4}\epsilon^{m4np}W^{-3}[\overline{\phi}_{4j}(W^{-3}\overline{\phi}_{im}\overline{\phi}_{4n,p})_{,k} - 9W^{-3}\overline{\phi}_{im}\phi_{4j,k}\overline{\phi}_{4n,p}] = 72\pi B\,. \tag{41}$$

Further from Eq. (39) it follows that

$$E<0\,. \tag{42}$$

These three conditions must be satisfied if a spatially homogeneous rotating and expanding universe without shear exists for $p=\frac{1}{9}\rho$. We find no inconsistency in this case. Thus apparently a spatially homogeneous rotating universe must necessarily have shear except when $p=\frac{1}{9}\rho$. However, the complete set of field equations has not been considered because the other equations are too complicated. Hence the possibility remains that the case $p=\frac{1}{9}\rho$ may also be ruled out, which seems not unlikely.

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