HOMOGENEOUS EINSTEIN METRICS ON $SU(n)$ MANIFOLDS, HOOP CONJECTURE FOR BLACK RINGS, AND ERGOREGIONS IN MAGNETISED BLACK HOLE SPACETIMES

A Dissertation

by

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ABSTRACT

This Dissertation covers three aspects of General Relativity: inequivalent Einstein metrics on Lie Group Manifolds, proving the Hoop Conjecture for Black Rings, and investigating ergoregions in magnetised black hole spacetimes. A number of analytical and numerical techniques are employed to that end.

It is known that every compact simple Lie Group admits a bi-invariant homogeneous Einstein metric. We use two ansätze to probe the existence of additional inequivalent Einstein metrics on the Lie Group $SU(n)$. We provide an explicit construction of $2k+1$ and $2k$ inequivalent Einstein metrics on $SU(2k)$ and $SU(2k+1)$ respectively.

We prove the Hoop Conjecture for neutral and charged, singly and doubly rotating black rings. This allows one to determine whether a rotating mass distribution has an event horizon, that it is in fact a black ring.

We investigate ergoregions in magnetised black hole spacetimes. We show that, in general, rotating charged black holes (Kerr-Newman) immersed in an external magnetic field have ergoregions that extend to infinity near the central axis unless we restrict the charge to $q = amB$ and keep B below a maximal value. Additionally, we show that as B is increased from zero the ergoregion adjacent to the event horizon shrinks, vanishing altogether at a critical value, before reappearing and growing until it is no longer bounded as B becomes greater than the maximal value.

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Chapter 3 of this Dissertation details work carried out in collaboration with Dr. C. N. Pope and Chapter 4 work in collaboration with Dr. C. N. Pope and Dr. G. W. Gibbons.

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1. INTRODUCTION

The General Theory of Relativity has been a cornerstone of Physics since the theory was put forward in 1916. Over the years the theory has been extensively studied and tested. However, the mathematical formulation of the theory in terms of coupled non-linear differential equations makes the search for analytical solutions extremely difficult. It is remarkable, therefore, how many closed form solutions have been discovered. Most of these have been based on symmetry arguments, intuitive ansätze and analytical extensions of previous solutions.

Even with all this success analytically, the field has relied heavily on numerical analysis whenever analytical attempts have failed; which has been often. The sub-field of Numerical Relativity is entirely dedicated to the use of high-speed computers and optimized algorithms to study General Relativity.

The General Theory of Relativity has been further enriched by its extension to more exotic environments such as higher dimensions, additional scalar and vector fields and quantum mechanics. Consequently, a vast amount of seemingly disparate work has been conducted under the umbrella of General Relativity.

This Dissertation is one such work; an investigation of three particular aspects of the ever evolving, increasingly complex General Theory of Relativity.

1.1. Einstein Metrics on Lie Algebras

An Einstein metric is a metric which obeys the equation

$$R_{\mu\nu} = \lambda\, g_{\mu\nu}\,. \tag{1.1}$$

Einstein metrics are of particular interest to Physicists, in part because they are particular vacuum solutions to Einstein's Field Equations (with cosmological constant), that is they correspond to source-free spacetimes for particular cosmological constants.

Lie Algebras are the underlying structure of Lie Groups. They embody the essence of continuous symmetries. Each member of a Lie Group is a symmetry operation and can be classified as a particular point on the (smooth) Lie Algebra Manifold.

Since very smooth manifold is a candidate spacetime one can add additional structure to a Lie Algebra manifold by embodying it with a metric. The construction of a metric on a Lie Algebra manifold is a partially constrained process. The metric has to obey the underlying group structure of the manifold but is still left with enough freedom that a variety of metrics can be constructed.

The question then arises, is it possible to construct Einstein metrics on Lie Algebra manifolds? The question has been answered for a number of Lie Algebras [1][2][3][4][5][6][7][8]. This Dissertation answers it, in the affirmative, for $SU(n)$ manifolds.

1.2. Hoop Conjecture for Black Rings

It has been shown that the only allowed topology for event horizons in $3+1$-dimensional spacetimes is spherical. Additionally it has been shown that, again in $3+1$-dimensional spacetime, the parameters M, Q, and J (mass, electrical charge and angular momentum) uniquely specify black holes.

However in higher dimensions this is no longer the case. $(4+1)$-dimensional spacetime admits solutions to the Einstein's Field Equations which have event horizons with 2-torus $(S^1 \times S^2)$ topology. These solutions have consequently been dubbed "Black Rings." Not only is this the first example of a non-spherical event horizon topology but black rings also violate the uniqueness theorem by allowing more than one, unique, solution of the Einstein's Field Equations to have the same parameters M, Q, and J.

Since their discovery black rings have garnered a great deal of attention. Some of it has been focused on proving the well-known results for black holes for the case of black rings. One such result is the Hoop Conjecture.

The hoop conjecture was first presented by Kip Thorne for astrophysical black holes [9]. It states

> Horizons form when and only when a mass M gets compacted into a region whose circumference in every direction is $C \leq 4\pi M$.

The conjecture has since been modified allowing it to be applied to black holes in higher-dimensional spacetimes. This Dissertation formulates and proves the conjecture for neutral and charged, singly and doubly rotating Black Rings in $4+1$-dimensional spacetime.

1.3. Ergoregions in Magnetised Black Hole Spacetimes

Ergoregions have been a prominent feature of General Relativity ever since the discovery of the Kerr solution to Einstein's Field Equations, which corresponds to a neutral rotating black hole. The Kerr solution features a region adjacent and outside of the (spherical) event horizon where the g_{00} component of the metric becomes positive, known as the ergoregion. This results in non-stationary geodesics which means no object can possibly stay at rest in the ergoregion, but is forced to move along with the rotating black hole. This constitutes "frame-dragging".

The Kerr metric is an example of a compact ergoregion, one that is geographically bounded (does not extend to spatial infinity). This is not guaranteed when one studies more exotic spacetimes. The magnetised black hole spacetimes, in particular, suffer from ergoregions that extend to infinity; where the "magnetised black hole spacetime" consists of an electrically charged rotating black hole (Kerr-Newman) immersed in an external magnetic field of strength B.

2. EINSTEIN METRICS ON $SU(n)$

2.1. Introduction

An Einstein Metric, by definition, obeys the Einstein equation $R_{\mu\nu} = \lambda\, g_{\mu\nu}$ which constrains the Ricci and through it the Riemann Curvature Tensor. In general, in d dimensions the Ricci Tensor, like the (symmetric) Metric Tensor, has $\frac{1}{2}d(d+1)$ algebraically independent components while the Riemann Curvature Tensor has $\frac{1}{12}d^2(d^2-1)$ algebraically independent components [10]. It therefore follows that the Einstein equation places less constraints on the curvature of a metric as the dimension increases. In fact for $d \geq 4$ the number of independent components of the Ricci tensor is less than that of the Riemann tensor and the gap widens as d increases.

These considerations leads one to expect that as the number of dimensions d increases the number of (inequivalent) Einstein Metrics should also increase [3]. In this Dissertation we search for these increasing number of Einstein Metrics on $SU(n)$ group manifolds.

The number of independent components of an $n \times n$ unitary matrix with unit-determinant are $n^2 - 1$. This means the Lie Group $SU(n)$ and its associated manifold, which comprises of the set of all unit-determinant unitary $n \times n$ matrices, has dimension $d = n^2 - 1$ for a given value of n. Consequently, the number of possible Einstein metrics on the $SU(n)$ group manifold increases rapidly with increasing n.

2.1.1. Construction of Metrics

It is easier, when calculating the Riemann Curvature tensor, to work with the vielbiens σ_a (1-forms) coupled with a flat metric (a symmetric metric with constant components that do not vary with the parameters of the Lie-Algebra) as compared to a general metric [11]. Given a Lie Group G with generators T_a if $g \in G$ (g is a group element of G) then the left-invariant 1-forms σ^a are given by

$$g^{-1}dg = \sigma^a\, T_a\,. \tag{2.1}$$

The material presented in this section formed the basis of the paper *"Homogeneous Einstein metrics on $SU(n)$"* by A. H. Mujtaba in J. Geom. Phys., Volume 62, Issue 5, 2012 (arXiv:1110.1978). The material has been used with permission from Elsevier.

The motivation behind this construction is given in Appendix A.1.

With the 1-forms (vielbiens) in hand we can construct a general metric on the group manifold using a constant symmetric rank-2 tensor (the flat metric) [11], namely

$$ds^2 = g_{ab}\,\sigma^a\,\sigma^b. \tag{2.2}$$

Note that this construction leaves us free to choose any flat metric g_{ab} (as long as it is symmetric; and needs to have positive eigenvalues if the metric is required to be Riemannian) while the 1-forms (vielbiens) σ^a are fixed. In fact the 1-forms are derived from the group elements themselves and therefore inherit information pertaining to the structure of the group manifold and how it affects the curvature of the metric space we are constructing on top of it.

Therefore, our task is to find, given the 1-forms σ^a, flat metrics g_{ab} such that the metric ds^2 (as defined in (2.2)) is Einstein.

2.1.2. Additional Einstein Metrics

Every simply compact Lie group admits a bi-invariant metric of the form $\mathrm{tr}(g^{-1}dg)^2$ which in a suitable choice of basis for the generators T_a can be expressed as

$$ds^2 = c\,\sigma^a\sigma_a, \tag{2.3}$$

where c is a constant [3]. This corresponds to the flat metric $g_{ab} = c\,\delta_{ab}$.

D'Atri and Ziller [1] have shown that every simple compact Lie group, with the exception of $SU(2)$ and $SO(3)$, admits at least one additional homogeneous Einstein metric. These additional Einstein metrics, though not bi-invariant, are still invariant under the transitive G action (left or right as chosen by convention; we have chosen, by virtue of the construction in (2.1) and Appendix A.1, to preserve the full G_L) [3].

In particular cases homogeneous Einstein metrics have been known to exist in addition to the bi-invariant and D'Atri and Ziller cases. For example six inequivalent homogeneous Einstein metrics have been found, explicitly, on the exceptional group G_2 [2]; and $(3k-4)$ and $(3k-3)$ inequivalent Einstein metrics on $SO(2k)$ and $SO(2k+1)$ respectively [3]. These successes motivate the search for additional inequivalent Einstein metrics on $SU(n)$.

2.1.3. Inequivalence of Einstein Metrics

A recurring issue when searching for new Einstein metrics is determining whether a newly found Einstein metric is truly new or whether it is equivalent to an already known metric, possibly by a change of basis. A very useful tool for evaluating this possibility is to calculate some dimensionless invariant quantity (see Appendix A.2) which is constructed from the metric and its curvature. Two examples of these dimensionless invariants are [3][2]

$$I_1 = \lambda^{d/2}\, V\,, \tag{2.4}$$

and

$$I_2 = R_{abcd}\, R^{abcd}\, \lambda^{-2} = |\mathrm{Riem}|^2\, \lambda^{-2}\,, \tag{2.5}$$

where $R_{ab} = \lambda\, g_{ab}$, $V = \sqrt{(\det g)}\, \sigma^1 \wedge \sigma^2 \wedge ... \wedge \sigma^d$ is the volume of the space and d is the dimension of the Lie group.

For any two Einstein metrics one calculates the value of one of the invariants. If the calculated values are unequal the two Einstein metrics are clearly inequivalent. If the calculated values of the invariant are the same then it is not certain that they are equivalent but the likelihood is that they are. Thus it is easy to establish the inequivalence of Einstein metrics; while establishing equivalence is a more ambiguous proposition [2]. For practical reasons we choose to use the latter dimensionless invariant quantity (I_2) to establish the inequivalence of the Einstein metrics we discover on $SU(n)$.

2.1.4. Calculating the Ricci Tensor

The mechanism for calculating the Ricci Tensor follows [11]. If the metric is given in terms of the 1-forms and the flat metric by (2.2) then the following sequence of calculations will give the Ricci Tensor.

$$
\begin{aligned}
d\sigma^a &\equiv -\frac{1}{2} c_{bc}{}^a \, \sigma^b \, \sigma^c \\
c_{abc} &= g_{cd} \, c_{ab}{}^d \\
\omega_{ab} &\equiv \frac{1}{2}(c_{abc} + c_{acb} + c_{cba}) \, \sigma^c \\
\omega^a{}_b &= g^{ac} \, \omega_{cb} \\
\theta^a{}_b &\equiv d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \\
\theta^a{}_b &\equiv R^a{}_{bcd} \, \sigma^c \wedge \sigma^d \\
R_{ab} &\equiv R^c{}_{acb}
\end{aligned}
\tag{2.6}
$$

Having calculated the Ricci Tensor, R_{ab}, one can establish that a metric is Einstein by checking whether it is proportional to the Ricci Tensor (1.1).

2.2. Metrics on $SU(n)$

The construction and manipulation of metrics on $SU(n)$ is considerably simplified if one uses the left-invariant 1-forms $L_A{}^B$ where $1 \leq A \leq n$. These have the property $L_A{}^{B\dagger} = L_B{}^A$ and obey the algebra [2]

$$
dL_A{}^B = i \, L_A{}^C \wedge L_C{}^B. \tag{2.7}
$$

For details see Appendix A.3.

The 1-forms $L_A{}^B$ are n^2 in number while the Lie Algebra $\mathfrak{su}(n)$ of the Lie Group $SU(n)$ has $n^2 - 1$ generators which are by definition hermitian and traceless. Each “construction” of the hermitian traceless 1-forms from the $L_A{}^B$ constitutes a scheme and becomes one part of a particular construction of a metric as defined by (2.2).

The total number of possible metrics is large given the freedom to construct the 1-forms σ^a as well as the symmetric flat metric g_{ab}. In our search for Einstein Metrics on $SU(n)$ we choose to study certain ansätze in the hope of simplifying our task. These ansätze take the form of schemes for the construction of hermitian traceless 1-forms from the $L_A{}^B$ as well as particular choices of the flat metric.

2.2.1. Scheme 1

The Generators The scheme consists of the construction of n^2-1 traceless Hermitian 1-forms K_i from the ${L_A}^B$ (Appendix A.3). Let $m \equiv n(n-1)/2$. We begin by creating m "traceless Hermitian" 1-forms of the form

$$K_i = {L_A}^B + {L_B}^A \,, \tag{2.8}$$

where $A \neq B$, for example $K_1 = {L_1}^2 + {L_2}^1$. The next m 1-forms are of the form

$$K_{m+i} = i\,({L_A}^B - {L_B}^A)\,, \tag{2.9}$$

where $A \neq B$. This leaves us with the 1-forms created using the diagonal 1-forms ${L_A}^B$ where $A = B$. It is necessary that the 1-forms we create from the diagonal ${L_A}^B$ be "traceless".

Since the 1-forms K_i are created by taking linear combinations of the ${L_A}^B$ one can describe the construction using matrices. Let $\vec{l}$ be the vector with entries $l_i = {L_i}^i$ (for instance $l_2 = {L_2}^2$) and let $\vec{k}$ be the vector with entries $k_i = K_{2m+i}$ for $1 \leq i \leq n$, then the construction of the k_i from the l_i is given by

$$\vec{k} = \mathbf{P}\,\mathbf{Q}\,\vec{l}, \tag{2.10}$$

where

$$\mathbf{P} = \begin{pmatrix} (\frac{2}{n-1}-1) & \frac{2}{n-1} & \frac{2}{n-1} & \cdots & \frac{2}{n-1} & 0 \\ \frac{2}{n-1} & (\frac{2}{n-1}-1) & \frac{2}{n-1} & \cdots & \frac{2}{n-1} & 0 \\ \vdots & \vdots & \ddots & & \vdots & 0 \\ \frac{2}{n-1} & \frac{2}{n-1} & \cdots & \frac{2}{n-1} & (\frac{2}{n-1}-1) & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

and

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 & \cdots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \cdots & \vdots \\ \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \cdots & \cdots & \frac{1}{\sqrt{n(n-1)}} & \frac{-(n-1)}{\sqrt{n(n-1)}} \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \cdots & \cdots & \frac{1}{\sqrt{n}} \end{pmatrix}. \tag{2.11}$$

For details see Appendices A.3.4 and A.3.5.

For example in the case of $n = 4$ we have

$$\mathbf{P} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{-3}{\sqrt{12}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{2.12}$$

Note that the last K_i so defined is K_{n^2} which is **not** a 1-form of $SU(n)$ and has non-zero trace to boot (it corresponds to the unit matrix, that is the generator of $\mathfrak{u}(1)$). We include its construction for the sake of completeness so that the transformation is invertible, a property needed by the algorithm we will implement.

This construction, in particular the form of the matrix $\mathbf{P}$ was chosen to keep the K_{2m+i} on a symmetric footing with respect to the ${L_i}^i$. The need for symmetry arises from the consideration of the number of metric constants in our calculations. If we are able to place multiple 1-forms on a symmetric footing we can bundle them together and assign the same metric constant to them. This will reduce the number of metric constants in our calculations reducing the computational complexity of the problem.

The Metric Given the specific construction of the generators (previous section) with the particular emphasis on symmetry between the generators (with regards to the ${L_A}^B$) it is possible to choose as our ansatz an extremely simply flat metric, one which will greatly reduce the computational complexity of the calculations necessary for identifying an Einstein metric.

If we define $m = \frac{n(n-1)}{2}$ and use the definitions of K_i in (2.8) through (2.11) the metric that constitutes the ansatz for scheme 1 is given by

$$ds^2 = x_1 \sum_{i=1}^{m} K_i^2 + x_2 \sum_{i=m+1}^{2m} K_i^2 + x_3 \sum_{i=2m+1}^{n^2-1} K_i^2 \,. \tag{2.13}$$

With this construction the task of finding an Einstein metric is reduced to finding the values of the (just) three variables x_i that make the metric defined above Einstein. In addition note how the metric is diagonal, which adds to the computational simplicity of the algorithm.

As an example consider the case $n = 3$ where the metric for scheme 1 is given by

$$ds^2 = x_1\,(K_1^2 + K_2^2 + K_3^2) + x_2\,(K_4^2 + K_5^2 + K_6^2) + x_3\,(K_7^2 + K_8^2)\,. \tag{2.14}$$

Solutions Using the metric (2.13) we use the sequence of calculations described in (2.6) to calculate the corresponding Ricci Tensor in terms of the so-called "metric constants" x_1, x_2 and x_3. Applying the Einstein equation $R_{ab} = \lambda\, g_{ab}$ gives us a system of equations which can be solved for the unknowns x_i. These solutions correspond to Einstein metrics on $SU(n)$. The metrics can be reconstructed from the x_i using (2.13).

We implemented the calculation algorithm as a computer program and analyzed the results to motivate an analytical solution. The Einstein equation led to a system of three unique simultaneous equations in 4 variables, the 3 equations corresponding to the three classes of generators and metric constants. These equations are valid for $n \geq 2$.

$$\begin{gathered}\frac{n}{4} - \frac{n-2}{8}\frac{x_2}{x_1} + \frac{1}{4}\frac{x_1^2}{x_2 x_3} - \frac{1}{4}\frac{x_3}{x_2} - \frac{1}{4}\frac{x_2}{x_3} = \lambda\, x_1 \\ \frac{n+6}{16} + \frac{n-2}{16}\frac{x_2^2}{x_1^2} + \frac{1}{4}\frac{x_2^2}{x_1 x_3} - \frac{1}{4}\frac{x_3}{x_1} - \frac{1}{4}\frac{x_1}{x_3} = \lambda\, x_2 \\ \frac{n}{8}(2 - \frac{x_2}{x_1} - \frac{x_1}{x_2} + \frac{x_3^2}{x_1 x_2}) = \lambda\, x_3\end{gathered} \tag{2.15}$$

We choose to normalize the metric constants by setting $x_2 = 1$ (since if g_{ab} is an Einstein metric then so is any multiple of g_{ab}). With this choice of normalization we have the following solutions.

The homogeneous bi-invariant metric (for $n \geq 2$):

$$x_1 = x_2 = x_3 = 1 \qquad \lambda = \frac{n}{8} \qquad \frac{|\mathrm{Riem}^2|}{\lambda^2} = n^2 - 1 \tag{2.16}$$

and the left-invariant metric (for $n \geq 3$):

$$x_1 = x_3 = \frac{3n+2}{n-2} \qquad x_2 = 1 \qquad \lambda = \frac{n(n-2)(5n+6)}{8(3n+2)^2} \tag{2.17}$$

with

$$\frac{|\mathrm{Riem}^2|}{\lambda^2} = \frac{(2n^2+3n+2)(n-1)(3n+4)}{n(5n+6)}\,.$$

Thus we have discovered two inequivalent homogeneous Einstein metrics for each value of $n \geq 3$ using the ansatz outlined in Scheme 1.

2.2.2. Scheme 2

The ansatz in scheme 2 is based on the decomposition method from [2] and [3]. For $n \geq 2$ and any $0 \leq p \leq n$ we define $q \equiv n - p$. We study the decomposition

$$SU(p) \times SU(q) \subset SU(p+q)\,. \tag{2.18}$$

The Generators Once again our task is to construct 1-forms corresponding to traceless Hermitian Generators from the n^2 1-forms $L_A{}^B$. Using the decomposition method we split the Generators in to four classes.

Class 1:

From the p^2 $L_a{}^b$ where $a, b \in \{1, 2, \ldots, p\}$ we construct (p^2-1) 1-forms according to the construction given in Scheme 1, that is we have $\frac{p(p-1)}{2}$ 1-forms of the form $(L_a{}^b + L_b{}^a)$ for $a \neq b$, $\frac{p(p-1)}{2}$ 1-forms of the form $i(L_a{}^b - L_b{}^a)$ for $a \neq b$ and $(p-1)$ 1-forms constructed from a "symmetric" mixing of the diagonal $L_a{}^a$ as in (2.10) and (2.11).

Class 2:

From the q^2 $L_\alpha{}^\beta$ where $\alpha, \beta \in \{p+1, p+2, \cdots, n\}$ we construct (q^2-1) 1-forms according to Scheme 1 and analogous to Class 1.

Class 3:

We construct 1-forms from the off-diagonal $L_a{}^\beta$ ($a \in \{1, 2, \cdots, p\}$ and $\beta \in \{p+1, p+2, \cdots, n)\}$) as follows

$$(L_a{}^\beta + L_\beta{}^a) \qquad\qquad i(L_a{}^\beta - L_\beta{}^a) \tag{2.19}$$

This results in $2\,p\,q$ 1-forms in this class.

Class 4:

So far we have constructed $(p^2-1)+(q^2-1)+2pq=(p+q)^2-2$ 1-forms leaving one un-constructed. Since we have only constructed $(p-1)+(q-1)=(p+q)-2$ 1-forms from the diagonal ${L_A}^A$ we are still left with one missing. It must be traceless and it must mix diagonal 1-forms from both $SU(p)$ and $SU(q)$. The single 1-form in this class is

$$q\sum_{a=1}^{p} {L_a}^a - p\sum_{\beta=p+1}^{n} {L_\beta}^\beta\,. \tag{2.20}$$

The Metric The construction of the flat metric associated with the Scheme 2 ansatz treats each class of generators as a unit and associates a single metric constant to it. For classes 1 and 2 this corresponds to choosing the bi-invariant metric from Scheme 1.

The metric for scheme 2 is

$$ds^2 = x_1 \sum_{i_1\in \mathrm{C1}} K_{i_1}^2 + x_2 \sum_{i_2\in \mathrm{C2}} K_{i_2}^2 + x_3 \sum_{i_3\in \mathrm{C3}} K_{i_3}^2 + x_4\, K_{i_4}^2 \tag{2.21}$$

An example of this construction for $n=5$ and $p=2$ is given in Appendix A.4.

Solutions Using the metric (2.21) we use the sequence of calculations described in (2.6) to calculate the corresponding Ricci Tensor in terms of the metric constants x_1, x_2, x_3 and x_4. We implemented the calculation algorithm as a computer program and analyzed the results to motivate an analytical solution. The Einstein equation $R_{ab}=\lambda\, g_{ab}$ led to a system of 4 unique simultaneous equations in 5 variables, the 4 equations corresponding to the four classes of generators and metric constants. These equations are valid for $n\geq 2$ and $p\geq 0$ with $q\equiv n-p$.

$$\begin{aligned}
\frac{p}{8}+\frac{q}{8}\frac{x_1^2}{x_3^2} &= \lambda\, x_1 \\
\frac{q}{8}+\frac{p}{8}\frac{x_2^2}{x_3^2} &= \lambda\, x_2 \\
\frac{p+q}{4}-\frac{(p-1)(p+1)}{8p}\frac{x_1}{x_3}-\frac{(q-1)(q+1)}{8q}\frac{x_2}{x_3}-\frac{(p+q)^2}{16}\frac{x_4}{x_3} &= \lambda\, x_3 \\
\frac{pq(p+q)^2}{16}\frac{x_4^2}{x_3^2} &= \lambda\, x_4
\end{aligned} \tag{2.22}$$

We choose to normalize the variables by setting $x_3 = 1$ which results in the following set of solutions:

$$x_1 = 1 \qquad x_2 = 1 \qquad x_4 = \frac{2}{pq(p+q)} \qquad \lambda = \frac{p+q}{8} = \frac{n}{8} \tag{2.23}$$

This solution for every decomposition $SU(p) \times SU(q) \subset SU(n)$ corresponds to the bi-invariant homogeneous Einstein Metric which we have already discovered in Scheme 1. The other set of solutions is:

$$\begin{aligned} x_1 &= \frac{pq(p+q) \pm \sqrt{pq(p^2-1)(q^2-1)}}{q(p^2+pq+q^2-1)} \qquad & x_2 &= \frac{q}{p}\,x_1 \\ x_4 &= 2\,\frac{2p(p+q) + ((1-p^2)+(1-q^2))}{1+pq}\,x_1 \qquad & \lambda &= \frac{q}{16p(p+q)^2}\,x_4 \end{aligned} \tag{2.24}$$

The solutions can be classified by the decomposition being used.

Case 1 : $q = 0$**,** $p = n$

This corresponds to Scheme 1 and gives **two** inequivalent metrics one of which is the bi-invariant one.

Case 2 : $q = 1$**,** $p = n - 1$

When $q = 1$ the number of generators of $SU(q) = SU(1)$ is $q^2 - 1 = 0$ which means x_2 no longer shows up in the metric at all. The decomposition works out to be $p^2 - 1$ generators of $SU(p)$, $2p$ generators from the off-diagonal ${L_A}^B$ and 1 generator of class 4.

Substituting $q = 1$ in (2.24) gives us $x_1 = 1$, $x_4 = \frac{2}{p(p+1)}$ and $\lambda = \frac{p+1}{8}$ which corresponds exactly to the first solution set, equivalent to the bi-invariant metric. Thus the case $q = 1$ gives us **no** new inequivalent metrics on $SU(n)$.

Case 3 : $q = p$

Substituting $q = p$ in (2.24) makes the solution for x_1 degenerate and so we get textbfone inequivalent metric rather than the usual two. Note that this case is only possible if n is even.

Case 4 : *Everything Else*

If the values of p and q do not correspond to any of the earlier cases we have the default situation where (2.23) results in the homogeneous bi-invariant metric and (2.24) leads to two additional inequivalent metrics. Thus this case generates **two** additional inequivalent metrics.

2.3. Conclusion

When counting the number of inequivalent metrics generated by our ansätze we must take in to consideration the symmetry of the solutions under p - q exchange. Since the order in which the generators appear in the metric (2.21) is irrelevant exchanging p and q gives an equivalent metric. Thus when counting we only need to consider the case $q \leq p$.

As discussed earlier we need to distinguish between n being even and odd. In the case of $n = 2k + 1$ (odd) we have two inequivalent Einstein metrics for each value of $1 < q \leq k$ (for higher values of q the metrics repeat under p and q exchange). In addition we have the two inequivalent metrics from the case $q = 0$ which corresponds to Scheme 1. Thus the count is $2k - 2 + 2 = 2k$; with the -2 coming from the case $q = 1$ which gives no additional inequivalent Einstein metrics.

In the case of $n = 2k$ (even) we have two inequivalent Einstein metrics for each value of $1 < q < k$, one inequivalent Einstein metric for $q = k = p$ and two inequivalent Einstein metrics for $q = 0$ (corresponding to Scheme 1). Thus the count is $2k - 2 + 1 + 2 = 2k + 1$.

Thus using the ansätze described we have provided an explicit construction of $(2k + 1)$ inequivalent Einstein metrics on $SU(2k)$ and $2k$ inequivalent Einstein metrics on $SU(2k+1)$. It is important to note that we have only tested two ansätze; ones with limited complexity at that. It is unlikely that we have discovered all possible inequivalent Einstein metrics on $SU(n)$.

3. HOOP CONJECTURE FOR BLACK RINGS

3.1. Introduction

The Hoop Conjecture was first presented by Kip Thorne for astrophysical black holes [9]. It states

> *Horizons form when and only when a mass M gets compacted into a region whose circumference in every direction is $C \leq 4\pi M$.*

The conjecture hypothesises that a mass distribution is a black hole (has an event horizon) only when the mass has been squeezed in to a volume of space such that a hoop of circumference $C \leq 4\pi M$ can be rotated about said volume in all directions without touching it.

It is easily shown that the Schwarzschild metric black hole saturates the inequality. The conjecture then implies that all other black holes, those with charge and angular momentum, better obey the inequality, that is are more confined and so cannot be larger than the Schwarzschild black hole.

3.1.1. Reformulation

This formulation is imprecise in several ways. First off, it is unclear which mass M the conjecture is referring to since General Relativity allows for a number of formulations of mass. Secondly, the conjecture fails to specify which horizon one should be using. Thirdly, the concept of "circumference" is associated with the ability to define a circle on the horizon, a construction that becomes imprecise when the spacetime becomes complicated. And finally, the conjecture cannot be naturally extended to higher dimensions.

All of these concerns were addressed by Gibbons [12] as follows. The mass is taken to be the ADM Mass [13], useful since it is a global feature of the spacetime. The horizon is taken to be the apparent horizon and instead of the circumference the author introduces the Birkhoff's invariant.

3.1.2. Apparent Horizon

The apparent horizon is the outermost marginally trapped surface on the spacetime. It is a local feature of the spacetime, which means that unlike the event horizon, which requires knowledge of the entire history of spacetime to be accurately specified, the apparent horizon can be specified at any given instant of time [14].

We are able to use apparent horizons for the purpose of investigating the hoop conjecture because "if the cosmic censorship conjecture[15] holds and the null energy condition[14] is satisfied then the presence of an apparent horizon implies the existence of an event horizon that lies outside, or coincides with, the apparent horizon" [14]. So if we can show that the mass distribution is bounded by "hoops" on the apparent horizon it follows automatically that it is bounded by hoops on the event horizon (which coincides with or lies outside the apparent horizon). So one can prove the hoop conjecture by working with the apparent horizon.

3.1.3. Birkhoff's Invariant

Let S be the sub-manifold corresponding to the apparent horizon in the spacetime. Let $f : S \to \mathbb{R}$ be a real-valued function on S with just two critical points, a maximum and a minimum. For any $c \in \mathbb{R}$, $f^{-1}(c)$ is a level curve on S. Let $l(c)$ be the length of said level curve, calculated using the metric that S inherits from the spacetime. Note that S is foliated by the level curves $f^{-1}(c)$, that is S comprises of the union of the inherently non-intersecting level curves [12].

For any given function f we define

$$\beta(f) = \max_c l(c)\,. \tag{3.1}$$

We now define the Birkhoff invariant β by minimising $\beta(f)$ over all possible functions,

$$\beta = \inf_f \beta(f)\,. \tag{3.2}$$

This construction is motivated by [16]. To visualize the foliation and how it corresponds to the passage of a hoop over the surface of the apparent horizon read Appendix B.

3.1.4. Hoop Conjecture in 3 + 1 Dimensions

When considering 3+1 dimensional spacetime the apparent horizon is a 2-dimensional spatial manifold and so the level curves for a real-valued function are 1-dimensional curves. This makes the Birkhoff's invariant a length (rather than an area or a volume) in this case. The hoop conjecture is then formulated as

$$\beta \leq 4\pi \, M_{ADM} \, . \tag{3.3}$$

[12] proved the conjecture for the Schwarzschild black hole as well as the charged and rotating Kerr-Newman black hole. [17] extended the successful tests of the conjecture to more general four-dimensional black holes such as those with a cosmological constant, and a variety of rotating, possibly (multi)charged, black holes in gauged and ungauged supergravities.

3.1.5. Hoop Conjecture in N + 1 Dimensions

Using real-valued functions $f : S \to \mathbb{R}$ defined over the $N-1$ dimensional apparent horizon S the level-"curves" $f^{-1}(c)$ are $N-2$ dimensional sub-manifolds and so the Birkhoff's invariant will be defined in terms of the "volume" of these sub-manifolds. If we define $\mathcal{A}(c)$ to be the volume of the $N-2$ dimensional level-hypersurface $f^{-1}(c)$ then the Birkhoff's invariant is defined by

$$\beta(f) = \max_c \mathcal{A}(c) \qquad \text{and} \qquad \beta = \inf_f \beta(f) \, . \tag{3.4}$$

The constant of proportionality between β and M_{ADM} in the hoop inequality depends on the number of dimensions of the spacetime. It takes the form $\beta \leq \alpha_N \, M_{ADM}$, where $\alpha_N \in \mathbb{R}$. The value of α_N is calculated by testing the conjecture for the Schwarzschild metric in $N+1$ dimensional spacetime and assuming that the inequality is saturated in this case. In $4+1$ dimensions the 3-dimensional apparent horizon is foliated by $S^1 \times S^1$ sweepouts and the hoop conjecture takes the form [17]

$$\beta \leq \frac{16\pi}{3} \, M_{ADM} \, . \tag{3.5}$$

General tests of the conjecture for higher-dimensional black holes were carried out in [17]. The authors foliated the $(N-2)$-sphere horizon of the N-dimensional black hole spacetime using $(N-3)$-spheres. They defined the Birkhoff's invariant as the volume of the smallest equatorial $(N-3)$-sphere amongst all possible such foliations (the equatorial $(N-3)$-sphere is the largest sphere in any given foliation, analogous to the "Great Circles" on a 2-sphere).

Foliations other than $(N-3)$-spheres are possible. [17] also investigated the foliation of S^3 horizons in 5-dimensional black holes using $S^1 \times S^1$ Clifford tori, with the Birkhoff's invariant defined as the smallest "equatorial" torus amongst all possible such foliations. This procedure informs the one we use to foliate black ring horizons.

Our task is made much easier by realizing that since β is the smallest possible $\beta(f)$, $\beta \leq \beta(f)$ for any given function f. So if we can show for a judicious choice of function f that $\beta(f) \leq \alpha_N M_{ADM}$ we can verify the hoop conjecture without having to calculate β itself. We are saved the task of studying all possible foliations in our search for $\beta = \inf_f \beta(f)$, a considerable simplification of our task.

3.1.6. Black Rings

Black holes in four spacetime dimensions are constrained by a number of classical theorems which state that a stationary, asymptotically flat, vacuum black hole is completely characterized by its mass and spin, and event horizons in four dimensions are only allowed to have spherical topologies [18].

These restrictions do not apply in higher dimensions. In [18] the authors demonstrated the first $(4+1)$-dimensional spacetime metric whose event horizon has a non-spherical topology, in particular an $S^1 \times S^2$ topology. This has led to these spacetimes being named "black rings".

[18] describes the first black ring metric discovered which corresponds to a single rotating uncharged black ring. [19] and [20] describe a revised single rotating uncharged black ring metric while [21] describes a charged version. [22] describes a doubly rotating uncharged black ring while [23] and [24] describe a charged version.

3.1.7. Hoop Inequality with $S^1 \times S^1$ Sweepouts

[17] conjectured that the hoop inequality for a 5-dimensional black hole whose topologically S^3 horizon has been foliated with $S^1 \times S^1$ sweepouts is the same as that for a foliation using S^2 sweepouts, namely (3.5). The inequality is saturated by the Schwarzschild metric.

In contrast to [17], where the horizon was topologically S^3, the black ring horizon is topologically $S^1 \times S^2$ and it is this horizon that must be foliated using $S^1 \times S^1$ sweepouts. In keeping with [17], we conjecture that the hoop inequality satisfied by (5-dimensional) black rings (foliated by $S^1 \times S^1$ sweepouts) is still (3.5).

3.2. Neutral Single Rotating Black Ring

The metric for a single rotating (electrically) uncharged metric was first described in [20]. The original metric, although ground-breaking, was remarkably complicated. The metric was revised to a more tractable form in [19] and [20],

$$ds^2 = -\frac{F(y)}{F(x)}\left(dt - CR\,\frac{1+y}{F(y)}d\psi\right)^2 + \frac{R^2}{(x-y)^2}F(x)\left[-\frac{G(y)}{F(y)}d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi^2\right]\,, \quad (3.6)$$

where $F(\zeta) = 1 + \lambda\zeta, \quad G(\zeta) = (1-\zeta^2)(1+\nu\zeta) \quad \text{and} \quad C(\nu,\lambda) = \sqrt{\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}}.$

The apparent horizon is located at $y = -\frac{1}{\nu}$. The coordinates are restricted to the domains $-1 \leq x \leq 1$ and $y < -1$. The requirement that $G(\zeta)$ have real and distinct roots implies that the parameters obey the restriction $0 < \nu \leq \lambda < 1$. In addition, avoiding conical singularities at $x = \pm 1$ and $y = -1$ requires that the angular variables ϕ and ψ be periodic with period

$$\Delta\phi = \Delta\psi = 2\pi\frac{1+\lambda}{1+\nu}\,, \quad (3.7)$$

along with the restriction that λ and ν be related by

$$\lambda = \frac{2\nu}{1+\nu^2}\,. \quad (3.8)$$

For further details see Appendix C.4.

The apparent horizon, at $y = -\frac{1}{\nu}$, is spanned by the variables x, ϕ and ψ and has topology $S^1 \times S^2$. The S^1 corresponds to ψ which is the angular variable associated with the rotation of the black ring (since dt and $d\psi$ mix in the metric, that is $g_{t\psi} \neq 0$). The S^2 is spanned by x and ϕ, with x being the latitudinal coordinate (analogous to $\cos\theta$ on S^2) and ϕ being the azimuthal coordinate. Thus we can foliate the $S^1 \times S^2$ apparent horizon using surfaces of constant x ($S^1 \times S^1$ sweepouts spanned by ψ and ϕ). An example of such a sweepout is given in Figure 1.

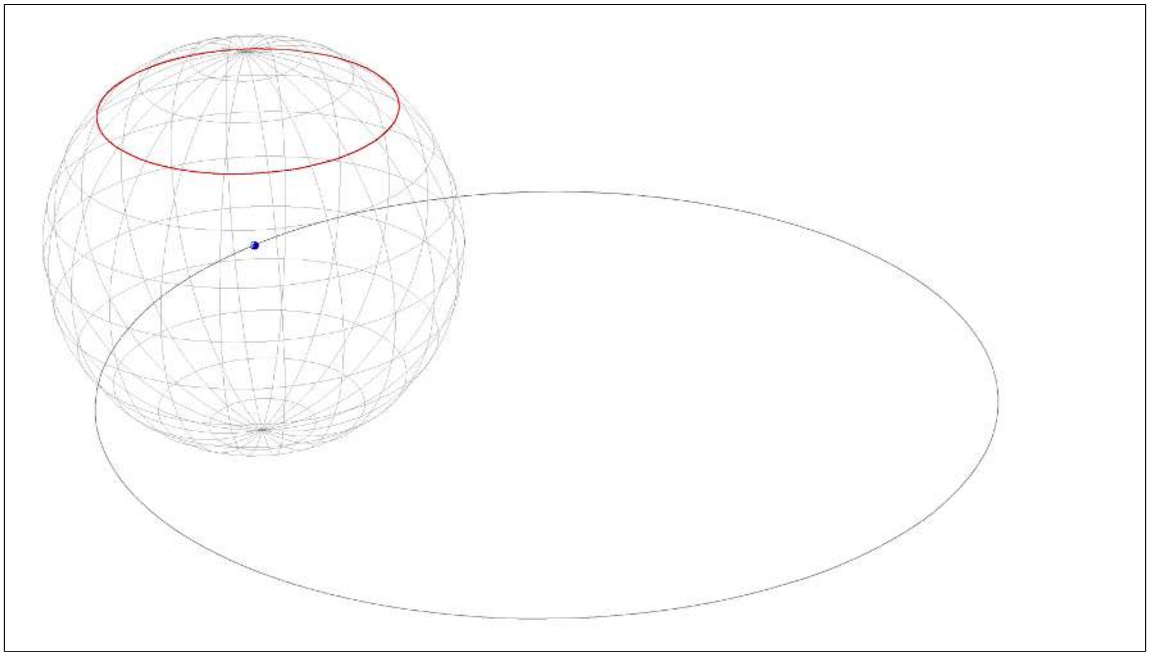

Figure 1: $S^1 \times S^1$ sweepout on the $S^1 \times S^2$ apparent horizon of a black ring.

The toroidal sweepout has area (volume)

$$\mathcal{A}(x) = \int d\phi \int d\psi \sqrt{g_{\phi\phi}\, g_{\psi\psi} - g_{\phi\psi}^2}\Big|_{y=-\frac{1}{\nu}} = \Delta\phi\, \Delta\psi \sqrt{g_{\phi\phi}\, g_{\psi\psi} - g_{\phi\psi}^2}\Big|_{y=-\frac{1}{\nu}} , \qquad (3.9)$$

since the metric is ϕ and ψ invariant. $\mathcal{A}(x)$ is a function of x, with $-1 \leq x \leq 1$ and ν as a parameter.

Recall from (3.4) that to prove the hoop conjecture it is sufficient to prove, for any given foliation (described by the level curves $f^{-1}(c)$ with volume $\mathcal{A}(c)$), that $\max_c\{\mathcal{A}(c)\} \leq \frac{16\pi}{3}M$ (in 5-dimensional spacetime). Since we are foliating the apparent horizon using $S^1 \times S^1$ sweepouts, corresponding to surfaces of constant x, all we have to show is that $\max_x\{\mathcal{A}(x)\} \leq \frac{16\pi}{3}M$ to verify the hoop conjecture. This amounts to

$$\frac{3\, max_x\{\mathcal{A}(x)\}}{16\pi M} \leq 1 \quad \Leftrightarrow \quad \left[\frac{3\, max_x\{\mathcal{A}(x)\}}{16\pi M}\right]^2 \leq 1. \qquad (3.10)$$

Substituting in the components of the metric and using the simplification $G(y = -\frac{1}{\nu}) = 0$, we have

$$\mathcal{A}(x) = 4\pi R^2 \frac{(1-\lambda)}{(1-\nu)} \left[\nu \frac{C(\nu,\lambda)^2}{(\lambda-\nu)} \frac{G(x)}{(1+\nu x)^2} \right]^{1/2} . \tag{3.11}$$

The ADM mass is given by [19, 20]

$$M = \frac{3\pi R^2}{4} \frac{\lambda}{(1-\nu)} , \tag{3.12}$$

and so verifying the hoop conjecture (3.5)(3.10) amounts to proving that

$$Z \equiv 1 - \left[\frac{3\mathcal{A}(x)}{16\pi M} \right]^2 = 1 - \nu \frac{(1-\lambda^2)}{\lambda} \frac{(1-x^2)}{(1+\nu x)} \geq 0 , \tag{3.13}$$

where λ is given by (3.8),

$$-1 \leq x \leq 1 \qquad \text{and} \qquad 0 < \nu \leq 1 . \tag{3.14}$$

Since $\nu \leq \lambda$, the inequality (3.13) will be established if we can show that

$$P(x) \equiv (1-\nu^2)(x^2-1) + \nu x + 1 + \nu x \geq 0 . \tag{3.15}$$

Writing this as

$$P(x) = \frac{1}{4}(3-\nu x)(x+\nu)^2 + \frac{1}{4}(1-\nu x)(x-\nu)^2, \tag{3.16}$$

and noting that $(3-\nu x) > 0$ and $(1-\nu x) \geq 0$ when the conditions in (3.14) are met, we see that $P(x) \geq 0$. Therefore the uncharged single rotating black ring obeys the hoop inequality (3.5).

We present an alternate proof here to illustrate an analogous technique we shall use when dealing with a more complicated black ring metric later on. We take the expression for Z in (3.13), with λ given by (3.8), and parameterise x and ν as

$$x = -1 + \frac{2}{1+u} \qquad \text{and} \qquad \nu = \frac{1}{1+v} , \tag{3.17}$$

where $u \geq 0$ and $v \geq 0$ (in order to cover the domains in (3.14)). Making the substitutions changes Z to

$$Z = \alpha(4 + 4u + 10v + 12uv + 2u^2v + 10v^2 + 6uv^2 + 4u^2v^2 + 5v^3 + 3u^2v^3 + v^4 + u^2v^4) , \tag{3.18}$$

where α is the non-negative expression

$$\alpha^{-1} = (1+v^2)(1+vx)(1+u)^2(1+v)^4. \tag{3.19}$$

Since the coefficient of every term in (3.18) is positive, and $u \geq 0$ and $v \geq 0$, it follows that $Z \geq 0$ and the hoop conjecture is verified (again).

3.3. Single Rotating Charged Black Ring

The solution for the single rotating charged black ring was first obtained in [21]. It is convenient to parameterise the solution in a form that reduces to (3.6) when the charge is set to zero. To that end we start with the solution (3.29) for the general doubly rotating charged black ring and turn off one of the rotations. We then make the appropriate parameter and coordinate transformations (with the exception of charge) that reduces (3.29) to (3.6) as outlined by [24]. This is achieved by performing the transformations

$$\nu = 0 \qquad \lambda \to \nu \qquad k \to \frac{1}{\sqrt{2}}\frac{R}{\sqrt{1+\nu^2}} \qquad \phi \to -\sqrt{1+\nu^2}\,\psi \qquad \psi \to \sqrt{1+\nu^2}\,\phi \tag{3.20}$$

on the metric in (3.29). This gives us the metric

$$\begin{aligned} ds^2 &= -D^{-2/3}\frac{F(y)}{F(x)}\left(dt + C\frac{(1+y)}{F(y)}R\,c\,d\psi\right)^2 \\ &\quad + D^{1/3}\frac{R^2}{(x-y)^2}F(x)\left[-\frac{G(y)}{F(y)}d\psi^2 + \frac{G(x)}{F(x)}d\phi^2 + \frac{dx^2}{G(x)} - \frac{dy^2}{G(y)}\right], \end{aligned} \tag{3.21}$$

$$D = 1 + s^2\frac{2\nu(x-y)}{(1+\nu^2)F(x)}, \tag{3.22}$$

where $c \equiv \cosh\delta$, $s \equiv \sinh\delta$, and δ parameterises the charge. The functions $F(\zeta)$ and $G(\zeta)$ are the same as those defined for the neutral single rotating black ring in (3.6), and the constant C is equal to $C(\nu,\lambda)$ (as defined in (3.6)) but with every instance of λ given by (3.8).

The mass and conserved electric charge for this solution are given by

$$M = (3+2s^2)\frac{\pi R^2\,\nu}{2(1-\nu)(1+\nu^2)} = (1+\frac{2}{3}s^2)M_0, \tag{3.23}$$

and

$$Q = -\frac{\pi R^2\, \nu}{(1-\nu)(1+\nu^2)} cs\,, \tag{3.24}$$

where M_0 is the mass for the neutral single rotating black ring (3.12).

The apparent horizon is (again) located at $y = -\frac{1}{\nu}$. The area of the $S^1 \times S^1$ sweepout of the horizon, corresponding to a surface of constant x, is given by

$$\mathcal{A}(x) = \frac{c}{D^{1/6}} \mathcal{A}_0(x)\,, \tag{3.25}$$

where $\mathcal{A}_0(x)$ is the area of the sweepout in the uncharged case ($\delta \to 0$) and is therefore given by (3.11). Thus the crucial ratio is

$$\frac{3\,\mathcal{A}(x)}{16\pi M} = \frac{c}{D^{1/6}(1+\frac{2}{3}s^2)} \left[\frac{3\,\mathcal{A}_0(x)}{16\pi M_0}\right]. \tag{3.26}$$

Since the hoop inequality (3.5) has already been verified for the uncharged case, we need only show that

$$\frac{c}{D^{1/6}\,(1+\frac{2}{3}s^2)} \leq 1\,, \tag{3.27}$$

for $-1 \leq x \leq 1$ and all δ, in order to verify the hoop conjecture for the charged case as well. We note that

$$x \geq -1 \quad \text{and} \quad y \leq -\nu^{-1} < -1 \qquad \Rightarrow \qquad (x-y) > 0\,,$$

and

$$x \geq -1 \quad \text{and} \quad 0 < \nu \leq 1 \qquad \Rightarrow \qquad F(x) = 1 + \frac{2\nu x}{(1+\nu^2)} \geq 0\,,$$

hence, from (3.22), we get $D > 1$. It only remains to show that $\frac{c}{(1+\frac{2}{3}s^2)} \leq 1$, which is evident if we re-express it as

$$\frac{c}{1+\frac{2}{3}s^2} = \frac{3c}{1+2c^2} = 1 - \frac{(c-1)(2c-1)}{1+2c^2} \leq 1\,, \tag{3.28}$$

and note that $c \equiv \cosh\delta \geq 1$. Thus the hoop inequality is satisfied by the single rotating charged black ring.

3.4. Doubly Rotating Charged Black Ring

The first doubly rotating black ring metric was described in [22]. This was generalised in [23] to a 2-charged doubly-rotating black ring, and in [24] to the more general 3-charge doubly rotating black ring solution in $\mathcal{N} = 2$ STU supergravity theory. It was shown in [24] that to avoid conical singularities at the poles of the x - ϕ S^2 sub-surfaces, two of the three charges must be set to zero. The metric is then given by

$$ds^2 = -D^{-2/3}\frac{H(y,x)}{H(x,y)}(dt + c\,\Omega)^2 + D^{1/3}\Bigg(- \frac{F(x,y)}{H(y,x)}d\phi^2 - 2\frac{J(x,y)}{H(y,x)}d\phi d\psi + \frac{F(y,x)}{H(y,x)}d\psi^2 + \frac{2k^2 H(x,y)}{(x-y)^2(1-\nu)^2}\left[\frac{dx^2}{G(x)} - \frac{dy^2}{G(y)}\right]\Bigg), \tag{3.29}$$

where

$$\begin{aligned}
G(x) &= (1-x^2)(1+\lambda x+\nu x^2)\,,\\
H(x,y) &= 1+\lambda^2-\nu^2+2\lambda\nu(1-x^2)y+2x\lambda(1-y^2\nu^2)+x^2y^2\nu(1-\lambda^2-\nu^2)\,,\\
J(x,y) &= \frac{2k^2(1-x^2)(1-y^2)\lambda\sqrt{\nu}}{(x-y)(1-\nu)^2}[1+\lambda^2-\nu^2+2(x+y)\lambda\nu-xy\nu(1-\lambda^2-\nu^2)]\,,\\
F(x,y) &= \frac{2k^2}{(x-y)^2(1-\nu)^2}\Bigg\{G(x)(1-y^2)\Big([(1-\nu)^2-\lambda^2](1+\nu)+y\lambda(1-\lambda^2+2\nu-3\nu^2)\Big)\\
&\quad +G(y)\Big[2\lambda^2+x\lambda[(1-\nu)^2+\lambda^2]+x^2[(1-\nu)^2-\lambda^2](1+\nu)\\
&\quad +x^3\lambda(1-\lambda^2-3\nu^2+2\nu^3)-x^4(1-\nu)\nu(-1+\lambda^2+\nu^2)\Big]\Bigg\}\,,
\end{aligned} \tag{3.30}$$

and

$$\Omega = -\frac{2k\lambda\sqrt{(1+\nu)^2-\lambda^2}}{H(y,x)}\left[(1-x^2)y\sqrt{\nu}d\psi+\frac{1+y}{1-\lambda+\nu}[1+\lambda-\nu+x^2y\nu(1-\lambda-\nu)+2\nu x(1-y)]d\phi\right],$$

$$D = 1+s^2\,\frac{2\lambda(1-\nu)(x-y)(1-\nu xy)}{H(x,y)}\,, \tag{3.31}$$

where $-1 \le x \le 1$, $y \le -1$ and ϕ and ψ each have period 2π. The parameters are restricted to $0 \le \nu < 1$ and $2\sqrt{\nu} \le \lambda < 1+\nu$. The apparent horizon is located at

$$y_h = \frac{-\lambda+\sqrt{\lambda^2-4\nu}}{2\nu}\,, \tag{3.32}$$

and the ADM mass is given by

$$M = k^2 \pi \frac{(3+2s^2)\,\lambda}{(1-\lambda+\nu)}\,. \tag{3.33}$$

Per the standard procedure, we consider the family of $S^1 \times S^1$ sweepouts of the apparent horizon at (3.32) corresponding to surfaces of constant x. The area of the sweepout is given by

$$\mathcal{A}(x) = (2\pi)^2 \, \sqrt{g_{\phi\phi}\, g_{\psi\psi} - g_{\phi\psi}^2}\Big|_{y=-\frac{1}{\nu}}\,. \tag{3.34}$$

Expressing

$$\left[\frac{3\,\mathcal{A}(x)}{16\pi\, M}\right]^2 \equiv D^{-1/3}\, Y\,, \tag{3.35}$$

the verification of the hoop conjecture (3.5) comes down to showing that

$$D^{-1/3}\, Y \le 1\,. \tag{3.36}$$

The expression for Y is too complicated to present. The procedure that proves the hoop inequality involves reparameterising the constants ν, λ and δ, and the latitude coordinate x on the 2-spheres, so that each parameter ranges over 0 to ∞ (with no restrictions). We do so by defining

$$\nu = \tanh^2\beta \qquad \text{and} \qquad \lambda = 2\tanh\beta\,\cosh\alpha\,. \tag{3.37}$$

The parameter β should therefore lie in the range $0 \le \beta \le \infty$, while α should range over $0 \le \alpha \le \alpha_{\rm max}$, where

$$e^{\alpha_{\rm max}} = \coth\beta\,. \tag{3.38}$$

The purpose of the ranges covered by β and α is to recreate the ranges that need to be covered by the original parameters, in this case $0 \le \nu < 1$ and $2\sqrt{\nu} \le \lambda < 1+\nu$.

Note that the horizon is located at $y_h = -e^{-\alpha}\coth\beta$ in terms of the new parameters. Finally, we introduce new parameters u, v, w, and z, each lying in the unrestricted range 0 to ∞. The new parameters are defined as

$$e^\beta = 1+u\,, \qquad e^\alpha = 1 + \frac{2}{(1+v)[(1+u)^2-1]}\,, \qquad x = -1 + \frac{2}{1+w}\,,$$
$$e^\delta = 1+z \quad \text{or} \quad e^{-\delta} = 1+z\,. \tag{3.39}$$

This gives a complete covering of the parameter space and the x coordinate domain in terms of mutually independent and unrestricted non-negative variables. The two alternatives for the reparameterisation of δ correspond to positive and negative electric charge parameters on the black ring.

The transformations result in $(1-Y)$ being a rational function of u, v, w and z, in which every term in the numerator and denominator multinomials has a positive coefficient. (For example, the numerator is a multinomial with 3350 terms, all having positive coefficients.) This establishes that $(1-Y) > 0 \;\Rightarrow\; Y < 1$. Similarly the reparameterisation results in $(D-1)$ being a rational function with numerator and denominator multinomials such that all terms have a positive coefficient. This proves that $(D-1) > 0 \;\Rightarrow\; D > 1$.

Therefore, from (3.36), we see that the hoop conjecture is verified for the single-charged doubly rotating black ring.

4. ERGOREGIONS IN MAGNETISED BLACK HOLE SPACETIMES

4.1. Introduction

Studying the energetics of astrophysical black holes involves understanding the interaction of black holes with charged particles and **external** magnetic fields. There is a large body of work dedicated to this subject, both early work by Wald [25]; King, Kundt and Lasota [26]; and Blandford and Znajek [27]; and recent studies by Bičák [28]; Bičák, Karas and Ledvinka [29]; and Komissarov and McKinney [30].

In the case of rotating black holes the frame-dragging of the external magnetic field induces electric fields which in turn affect all charged particles in the vicinity of the black hole. It can be energetically favourable, in such cases, for an initially neutral rotating black hole to acquire electrical charge [25] and for currents to flow [27].

For all practical astrophysical purposes the gravitational back-reaction of the magnetic field may be neglected, and the electromagnetic field may be treated as a "test" field on the unperturbed, asymptotically flat, electrically neutral Kerr solution. The current flow that is responsible for lowering the energy of the black hole (by means of acquiring charge) may come about as conduction through the ambient plasma or as a breakdown of the vacuum through pair production. An analysis of the latter from the perspective of black hole thermodynamics and quantum field theory was given by [31], but gravitational back-reaction was **not** taken in to account.

Subsequent investigations have unveiled apparently appropriate exact solutions of the Einstein-Maxwell equations taking in to account the gravitational back-reaction [32][33], as well as an analysis of their properties [34][35][36][37][38]. However a full treatment of the thermodynamics of a rotating black hole immersed in an ambient magnetic field remains elusive.

It is anticipated [39] that a full treatment that takes in to account both the gravitational back-reaction and the torque exerted by the black hole on the source of the external magnetic field might be extremely illuminating. This view is further bolstered by the discovery that in the only case that has been studied so far, that of the Schwarzschild black hole immersed in a background Melvin solution [40], the black hole thermodynamics (the entropy and temperature of the black hole) are unaffected by the external magnetic field. The question then is, does the same occur in the case of a rotating black hole immersed in an external magnetic field.

The reason why we lack a full treatment is the complexity of the exact solutions which are apparently appropriate in this case (charged rotating black hole in a Melvin environment). The literature, to date, has assumed that such a metric will produce an "asymptotically Melvin" background at infinity. If this were true it would be straight-forward to calculate the total mass and angular momentum of the black hole, possibly using Komar Integrals.

In this Dissertation we will show that in addition to the inherent complexity of the exact solutions there is a more serious obstacle to calculating the conserved quantities; the relevant solutions are in general **not** asymptotic to the static Melvin solution. We will show that except when the charge parameter q of the Kerr-Newman metric is specifically given by $q = amB$ (where a is the rotation parameter, m is the mass parameter and B the external magnetic field strength) the spacetime contains an ergoregion that extends to infinity, with timelike boundary.

In other words, unless $q = amB$, "*the dragging of the inertial frames is so strong that even at infinity there is no Killing vector field which is everywhere timelike outside a compact set containing the black hole Killing horizon.*" – Pope[1]

4.2. Ergoregions

We use the metric described in (D.5) and set $p = 0$ (turn off the magnetic charge) to get the metric for an electrically charged rotating black hole immersed in an external magnetic field (charge parameter q, rotation parameter a, mass parameter m and external magnetic field strength B). Like the Kerr and Kerr-Newman metrics we expect there to be a compact ergoregion just outside the event horizon.

To investigate the ergoregion we study the g_{00} component of the metric. Points in the spacetime with $g_{00} > 0$ are part of the ergoregion. We note that g_{00} is an even function of z and is ϕ-invariant. Consequently we only need to look at the positive (ρ - z) quadrant cross-section of the spacetime to glean complete information about g_{00} across all of space.

We begin by looking at g_{00} at large r while keeping the polar angle θ fixed, in which regime

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it takes the form

$$g_{00} = -\frac{1}{16} B^4 r^4 sin^4\theta + \mathcal{O}(r^3)\,, \tag{4.1}$$

which is in asymptotic agreement with the Melvin universe.

This result is misleading because the neglected terms ($\mathcal{O}(r^3)$) become significant near the polar axis. To demonstrate this we use cylindrical coordinates

$$\rho = r\sin\theta \qquad \text{and} \qquad z = r\cos\theta\,, \tag{4.2}$$

and probe the region where ρ is small and z is large. The expansion of g_{00} in inverse powers of z is given by

$$g_{00} = \frac{16B^6(q-amB)^2\rho^2}{W} z^2 + \frac{4B^6(q-amB)[8qm + aB(q^2+4m^2)]\rho^2}{W} z + \mathcal{O}(z^0)\,, \tag{4.3}$$

where W is the clearly positive quantity

$$W = 16 + 8B^2\rho^2 + B^4(\rho^2+q^2)^2 + 24B^2(q - \tfrac{2}{3}amB)^2 + \frac{16}{3}a^2m^2B^2\,. \tag{4.4}$$

We observe that g_{00} becomes large and positive in this regime (small ρ large z) unless we choose $q = amB$. While it is easy to see that the ergoregion extends to infinity when $q \neq amB$, more analysis is required before we can assert that the ergoregion is compact (does not extend to infinity) when $q = amB$.

We begin by analyzing g_{00} in the small ρ large z regime after setting $q = amB$. The expansion of g_{00} in inverse powers of z is now give by

$$g_{00} = -\frac{F_+F_-}{16(4 + a^2m^2B^4 + B^2\rho^2)^2} + \mathcal{O}(z^{-1})\,, \tag{4.5}$$

where

$$\begin{aligned} F_\pm &= (4 + B^2\rho^2)^2 + 2a^2m^2B^4(4 + B^2\rho^2) + a^4m^4B^8 \pm 2am^2B^4(12 + a^2B^2)\rho \\ &= B^4\rho^4 + 2B^2(4 + a^2m^2B^4)\Big[\rho \pm \frac{aB^2m^2(12 + a^2B^2)}{2(4 + a^2m^2B^4)}\Big]^2 + \\ &\quad \frac{128 + 48a^2m^2B^4(2 - 3m^2B^2) - a^6m^4B^{10}(1 - 2m^2B^2)}{2(4 + a^2m^2B^4)}\,. \end{aligned} \tag{4.6}$$

If we can show that the last of the three terms in (4.6) is positive then by virtue of (4.5)

g_{00} will be negative and there will be no ergoregion in the small ρ large z region.

The event horizon for this spacetime is given by $\Delta(r) = 0$, the same condition as for the Kerr-Newman metric. Therefore, the parameters must satisfy the inequality $q^2 + a^2 \leq m^2$ to avoid a naked singularity. Substituting in $q = amB$ limits a to

$$a^2 \leq a_{max}^2 \equiv \frac{m^2}{1 + m^2 B^2} . \tag{4.7}$$

If we plug in the maximum allowed value of a, a_{max}, in to the final term of (4.6), we find (numerically) that it will be positive if

$$mB \leq 1.26015 \cdots . \tag{4.8}$$

Thus if the size/mass of the black hole is sufficiently small compared to the "Melvin scale" B^{-1} the ergoregions encountered at small ρ large z will be avoided as long as $q = amB$. In the next section we will provide an improved bound for the mass.

4.3. Detailed Analysis of Ergoregions

We now focus exclusively on the case $q = amB$. In Section 4.2 we showed that the axial ergoregions can be avoided if m is limited by (4.8). This limit can be improved because even when the third term in (4.6) is negative it can be countered by the first two positive terms.

We can express the rotation parameter a in terms of its maximal value, $a_{\max}$ and a newly defined parameter ε as

$$a = \varepsilon \, a_{max} = \frac{\varepsilon \, m}{\sqrt{1 + m^2 B^2}} , \tag{4.9}$$

where $0 \leq \varepsilon 1$ and we make use of the fact that we can choose a to be positive without loss of generality.

We know that at $B = 0$ the ergoregion is compact since the metric reduces to the Kerr-Newman metric. Numerical analysis of g_{00} suggests that if we hold m and ε fixed and increase B from 0 the, initially compact, ergoregion contracts until we reach a critical value (B_{crit}). At $B = B_{crit}$ the ergoregion vanishes completely. As B increases beyond B_{crit} the ergoregion reappears and expands but remains compact (geographically localised) until $B = B_{max}$. For $B > B_{max}$ the ergoregions extend upwards and downwards to infinity along the rotation axis (although g_{00} remains negative on the rotation axis itself).

Let us look at a concrete example. We set $q = amB$, $m = 1$ and $\varepsilon = 0.9$ and plot the ergoregion for various values of B (Figures 2, 3 and 4). $B_{crit} \approx 1.012131$ and $B_{max} \approx 1.37394$ for these values of m and ε. We go on to show how B_{crit} and B_{max} can be calculated.

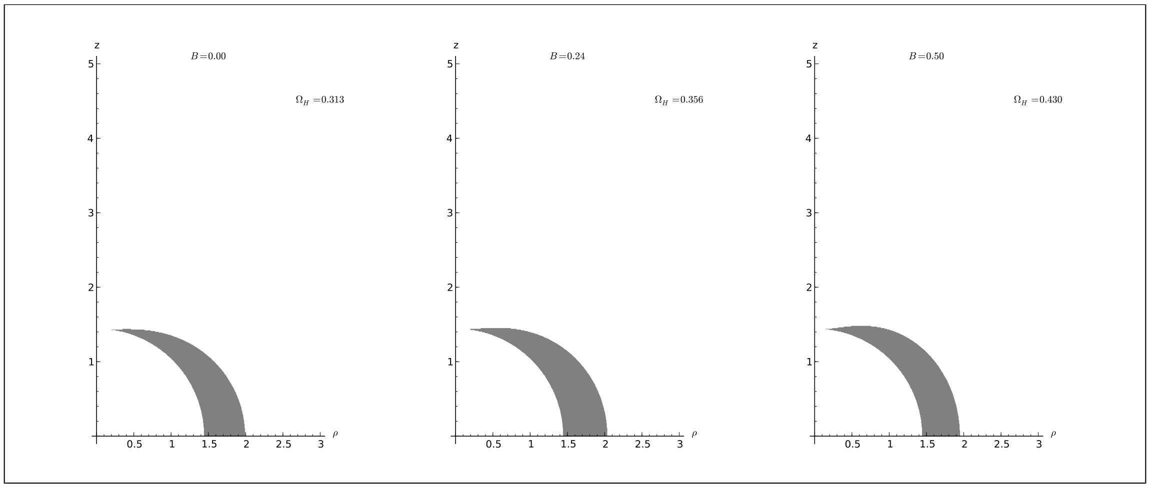

Figure 2: Positive quadrant of the ergoregion for a magnetised Kerr-Newman black hole with $q = amB$, $a = 0.9\, a_{max}$; for $B = 0$, 0.24 and 0.50 respectively.

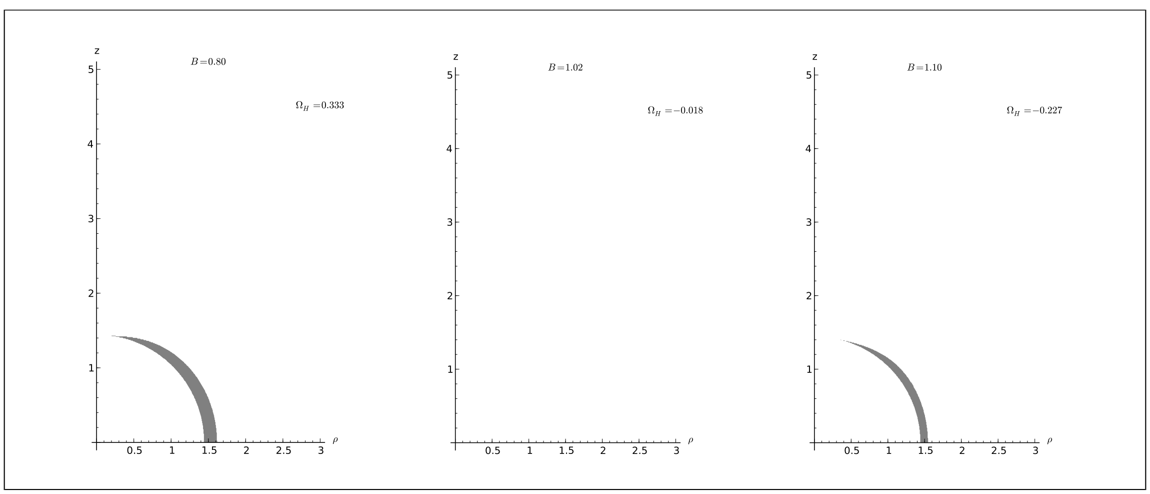

Figure 3: Positive quadrant of the ergoregion for a magnetised Kerr-Newman black hole with $q = amB$, $a = 0.9\, a_{max}$; for $B = 0.80$, 1.02 and 1.10 respectively.

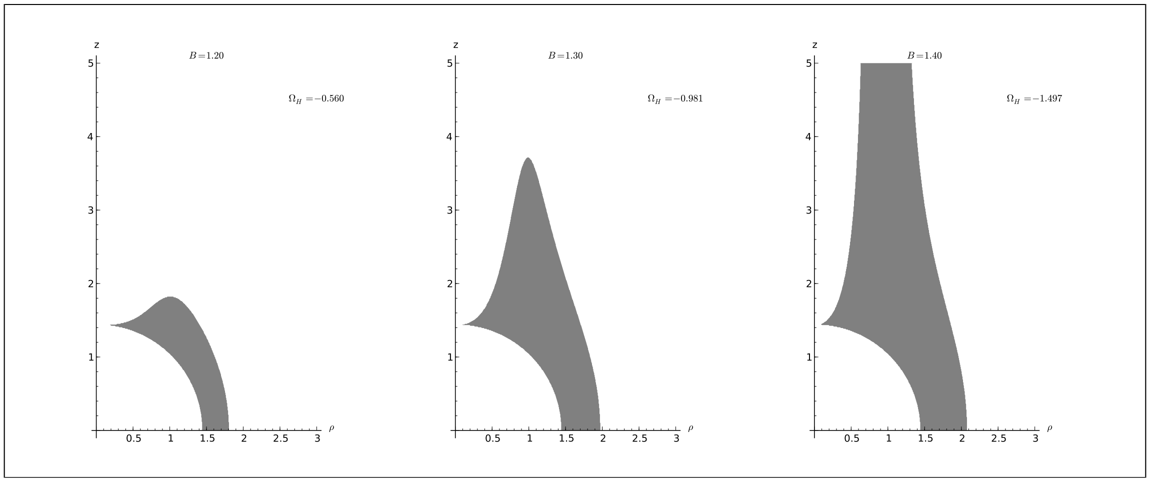

Figure 4: Positive quadrant of the ergoregion for a magnetised Kerr-Newman black hole with $q = amB$, $a = 0.9\, a_{max}$; for $B = 1.20$, 1.30 and 1.40 respectively.

Note that g_{00} will be positive at infinity if and only if F_- is negative (and F_+ is positive)[2]. Clearly F_- is positive at $\rho = 0$ and for (sufficiently) large ρ. $F_-(\rho)$ **just** manages to become negative when the curve $F_-(\rho)$ skims the ρ-axis, which happens when

$$F_-(\rho) = 0 \qquad \text{and} \qquad \frac{dF_-(\rho)}{d\rho} = 0\,. \tag{4.10}$$

We calculate the value of B for which this condition is met (for which $F_-(\rho)$ just manages to becomes negative) which gives us B_{max}. Solving the condition (4.10) for B requires us to solve

$$64\,\varepsilon^6 m^{12} B^{12} - 3\,\varepsilon^2(36 - 16\,\varepsilon + 3\,\varepsilon^2)(36 + 16\,\varepsilon + 3\,\varepsilon^2) m^{10} B^{10} - 24\,\varepsilon^2(196 - 5\,\varepsilon^2) m^8 B^8 + 16\,(256 + 141\,\varepsilon^2) m^6 B^6 + 3072\,(4 + \varepsilon^2) m^4 B^4 + 12288\, m^2 B^2 + 4096 = 0\,. \tag{4.11}$$

B_{max} is the smallest positive root of this equation, the lowest (positive) value of B for which $F_-(\rho)$ just manages to become negative. For an extremal black hole ($\varepsilon = 1$) this gives $mB_{max} \approx 1.33099$. For non-extremal black holes mB_{max} will become progressively larger as ε becomes smaller.

The ergoregion disappears at the critical value $B = B_{crit}$. This value can be calculated by looking at g_{00} in the equatorial plane ($z = 0$) at $r = r_+ = m(1 + \sqrt{1 - \varepsilon^2})$ at which

[2]It is clear from (4.6) that $F_+ > F_-$ for all values of B and that $F_-(B = 0) > 0$. We focus on increasing B from 0 and find the value of B where F_- first becomes negative.

location

$$g_{00} = \frac{(1-\tilde{\varepsilon})\,P^2}{16(1+\tilde{\varepsilon})(1+m^2B^2)^3(2+(1+\tilde{\varepsilon})m^2B^2)^2}\,, \tag{4.12}$$

where

$$P = (1+\tilde{\varepsilon})(21+2\tilde{\varepsilon}+\tilde{\varepsilon}^2)m^6B^6 + 2(1+6\tilde{\varepsilon}+\tilde{\varepsilon}^2)m^4B^4 - 4(7+3\tilde{\varepsilon})m^2B^2 - 8\,, \tag{4.13}$$

and $\tilde{\varepsilon}$ is the "co-extremality" parameter

$$\tilde{\varepsilon} = \sqrt{1-\varepsilon^2}\,. \tag{4.14}$$

Clearly the ergoregion vanishes on the equator ($z = 0$, $r = r_+$) when $P = 0 \;\Rightarrow\; g_{00}(z = 0, r = r_+) = 0$ (4.12). Thus B_{crit} is the smallest positive root of $P(B)$. For an extremal black hole ($\varepsilon = 1 \;\Rightarrow\; \tilde{\varepsilon} = 0$) we have $mB_{crit} \approx 1.11114$. For non-extremal black holes mB_{crit} becomes progressively smaller as ε becomes smaller.

If we calculate g_{00} on the horizon in general ($r = r_+$) we have

$$g_{00} = \frac{(1-\tilde{\varepsilon})P^2\sin^2\theta}{16\,(1+m^2B^2)^2\,W}\,, \tag{4.15}$$

where W is a certain (complicated) function of θ, mB and $\tilde{\varepsilon}$ that is never singular. Thus we observe that when $B = B_{crit}$, g_{00} vanishes everywhere on the horizon. This corresponds to the ergoregion vanishing completely.

It is important to note that in the case of the magnetised black hole spacetime there exists no **unique** choice of asymptotically timelike Killing vector. Therefore, it may be possible, for other choices of asymptotically timelike Killing vector (which take the form $\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial \phi}$) that there be no ergoregion at spatial infinity even when B is arbitrarily large.

4.4. Angular Velocity of the Horizon

Let us calculate the angular velocity of the horizon (Ω_H) of the black hole. If we consider the Killing Vector

$$\ell = \frac{\partial}{\partial t} - \Omega_H\frac{\partial}{\partial \phi} \tag{4.16}$$

then one solves for Ω_H by requiring that ℓ be null on the horizon. This gives us

$$\Omega_H = \frac{\sqrt{1-\tilde{\varepsilon}^2}\,P}{16\,(1+m^2B^2)^{3/2}(1+\tilde{\varepsilon})}\,, \tag{4.17}$$

where P is given by (4.13).

It immediately follows that when $B = B_{crit}$, P and consequently Ω_H vanish. Since Ω_H is the angular velocity with which the event horizon of the black hole is rotating (and frame-dragging) it means that at $B = B_{crit}$, when the ergoregion vanishes, the frame-dragging angular speed on the event horizon equator goes to zero as well. If B is increased beyond B_{crit} the sign of Ω_H is reversed, as evidenced by Figure 5.

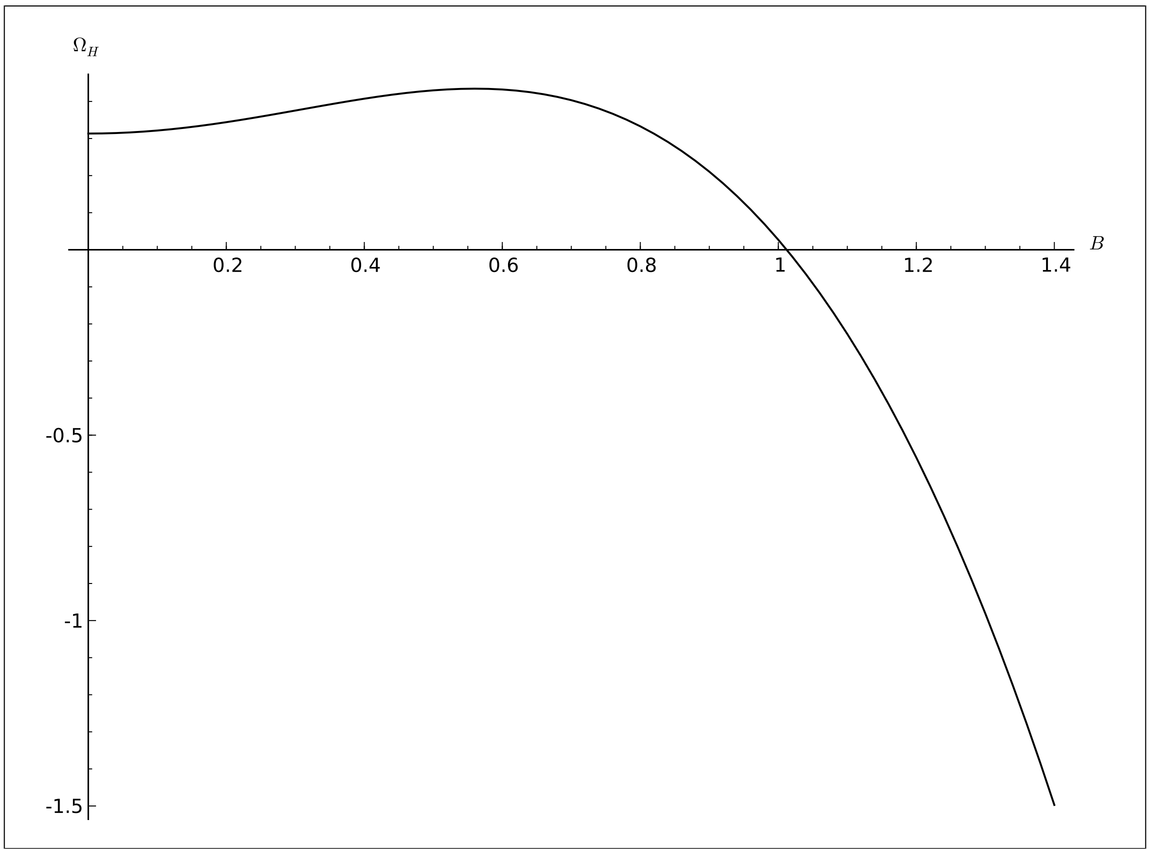

Figure 5: Ω_H vs B for a magnetised Kerr-Newman black hole with $q = amB$, $a = 0.9\,a_{max}$ and $m = 1$ for $0 \leq B \leq 1.4$ sampled at $\Delta B = 0.02$.

5. CONCLUSIONS

5.1. Einstein Metrics on $SU(n)$

We constructed two ansätze as candidate Einstein metrics on the $SU(n)$ manifold. We tested these ansätze and came up with a number of inequivalent Einstein metrics. Specifically, we provided an explicit construction for $(2k+1)$ inequivalent Einstein metrics on $SU(2k)$ and $2k$ inequivalent Einstein metrics on $SU(2k+1)$. These are the constructions from just the two ansätze. It is unlikely that we have discovered all possible inequivalent Einstein metrics on an $SU(n)$ manifold.

5.2. Hoop Conjecture for Black Rings

We analyzed the Black Ring metric for a number of cases: neutral and charged, singly and doubly rotating. We foliated the $S^1 \times S^2$ apparent horizons of these black rings using $S^1 \times S^1$ sweepouts. Using this foliation we were able to verify the hoop conjecture in all the stated cases.

The proof for the single rotating black ring (neutral and charged) was analytical and precise. The proof for the doubly rotating charged black ring was analytical but rather involved. Nonetheless, we have demonstrated that black ring event horizons obey the hoop conjecture.

5.3. Ergoregions in Magnetised Black Hole Spacetimes

We investigated the ergoregions in spacetimes corresponding to Kerr-Newman black holes immersed in Melvin backgrounds. We showed that in the case $q \neq amB$ the ergoregion extends to infinity near the axis of rotation. The special case $q = amB$ was investigated and we discovered that if we increase the magnetic field strength B from zero the initially compact ergoregion expands slightly before contracting until it vanishes at B_{crit}. Past B_{crit} the ergoregion expands (remaining compact) until $B = B_{max}$, after which the ergoregion extends to infinity.

We also studied the response of Ω_H, the angular speed of the event horizon at the equator, to changing the magnetic field strength, B. As B increases from zero Ω_H increases slightly before decreasing to zero at B_{crit}. It then becomes negative and keeps decreasing even as the value of B surpasses B_{max}. Thus, increasing B causes Ω_H to change direction. A very interesting result.

Therefore, we have shown that magnetised black hole spacetimes have compact ergoregions if and only if $q = amB$ and $B < B_{max}$.

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APPENDIX A

CONSTRUCTING METRICS ON $SU(n)$

A.1. Construction of 1-forms

Given a Lie Group G with generators of the Lie-Algebra T_a any group element $g \in G$ can be expressed in terms of the generators and the parameters x^a. That is there exist real numbers x^a such that

$$g = e^{ix^a T_a} \tag{A.1}$$

We then construct the object $g^{-1}dg$ which is left-invariant, that is for any constant element $a \in G$ the transformation $g \to a \circ g$ (that is multiplication of all group elements $g \in G$ by a from the left) clearly leaves the object $g^{-1}dg$ unchanged since

$$g^{-1}dg \to (a \circ g)^{-1} d(a \circ g) = g^{-1} \circ a^{-1} \circ a\, dg = g^{-1}dg \tag{A.2}$$

since a is a constant element and so its exterior derivative $da = 0$.

Analyzing the object $g^{-1}dg$ we observe

$$\begin{aligned} g^{-1}dg &= (e^{ix^a T_a})^{-1} d(e^{ix^b T_b}) \\ &= (e^{ix^a T_a})^{-1} e^{ix^b T_b} d(ix^c T_c) \\ &= iT_c dx^c \end{aligned} \tag{A.3}$$

where the third line comes from recognizing that the generators T_c are fixed for a given representation of the Lie-Algebra and that the exterior derivative acts only on the parameters x^c.

It is now clear that the object $g^{-1}dg$ consists of a sum over 1-forms multiplied by generators so that in general for any given representation of the Lie-Algebra one can construct 1-forms σ^a by declaring

$$g^{-1}dg = \sigma^a\, T_a\,. \tag{A.4}$$

A.2. Dimensionless Invariant Quantities

By dimensionless invariant quantities we mean mathematical objects constructed from the metric (and constructs derived from it) which have no units (dimensionless) and which are invariant under a change of basis. The idea being that if two metrics differ only by a change of basis and/or by certain multiplicative scalar factors in their definitions then both will give the same results when one calculates a dimensionless invariant quantity using them.

Two common examples are

$$I_1 = \lambda^{d/2}\, V \qquad \text{and} \qquad I_2 = R_{abcd}\, R^{abcd}\, \lambda^{-2}\,, \tag{A.5}$$

where $R_{ab} = \lambda\, g_{ab}$, $V = \sqrt{(\det g)}\, \sigma^1 \wedge \sigma^2 \wedge ... \wedge \sigma^d$ is the volume of the space and d is the dimension of the Lie group.

We will begin by proving that both these objects are dimensionless. Since we are working with metrics let us define the units of length to be L so that $[dx] = L$, where [...] denotes the units of an object. Since σ^a are 1-forms the must have units $[\sigma^a] = L$. Then from $ds^2 = g_{ab}\, \sigma^a \sigma^b$ and the fact that clearly $[ds] = L$ it follows that $[g_{ab}] = 1$ and since g^{ab} is the inverse of the metric, it has inverse the units, and so $[g^{ab}] = 1$ as well. This means that the act of raising or lowering an index does not change the units of an object and so we need not worry about doing so when calculating units.

Now we simply work our way through the deriviation of the various objects of interest, calculating units along the way. It is crucial to note that taking the exterior derivative of an object doesn't change its units that is $[dA] = [A]$ in general.

$$\begin{aligned}
d\sigma^c = \frac{1}{2}\, c^c{}_{ab}\, \sigma^a \sigma^b \;&\Rightarrow\; [c^c{}_{ab}] = [\sigma]^{-1} = L^{-1} \\
\omega_{ab} = \frac{1}{2}(c_{abc} + c_{acb} + c_{cba})\, \sigma^c \;&\Rightarrow\; [\omega_{ab}] = [c_{abc}]\, [\sigma^c] = [c^a{}_{bc}] \times L = 1 \\
\Theta^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \;&\Rightarrow\; [\Theta^a{}_b] = [\omega^a{}_b]^2 = 1 \\
\Theta^a{}_b = R^a{}_{bcd} \sigma^c \sigma^d \;&\Rightarrow\; [R^a{}_{bcd}] = [\Theta^a{}_b]\, [\sigma]^{-2} = L^{-2} \\
R_{ab} = R^c{}_{acb} \;&\Rightarrow\; [R_{ab}] = [R^a{}_{bcd}] = L^{-2} \\
R_{ab} = \lambda\, g_{ab} \;&\Rightarrow\; [\lambda] = [R_{ab}]\, [g_{ab}]^{-1} = L^{-2}
\end{aligned} \tag{A.6}$$

Therefore

$$
\begin{aligned}
[I_1] &= [\lambda]^{d/2}\,[V] \\
&= (L^{-}2)^{d/2}\ [\sqrt{(\det g)}\,\sigma^1 \wedge \sigma^2 \wedge ... \wedge \sigma^d] \\
&= L^{-d}\sqrt{[\det g]}\,[\sigma]^d \\
&= L^{-d} \times \sqrt{1} \times L^d \\
&= 1
\end{aligned}
\tag{A.7}
$$

and

$$
\begin{aligned}
[I_2] &= [R_{abcd}]\,[R^{abcd}]\,[\lambda]^{-2} \\
&= L^{-2} \times L^{-2} \times (L^{-2})^{-2} \\
&= 1\,,
\end{aligned}
\tag{A.8}
$$

proving that both quantities are dimensionless.

We next prove that the quantities are invariant under a change of basis. Since the metrics are based upon 1-forms (2.2), by a change of basis we mean a linear transformation of the 1-forms which we characterize by

$$
\sigma'^a = \Lambda^a{}_b\,\sigma^b\,. \tag{A.9}
$$

The inverse transformation is given by $\sigma^a = \Lambda_b{}^a\,\sigma'^b$ where $\Lambda^a{}_c\Lambda_b{}^c = \delta^a_b$. When transforming from the un-primed coordinates to primed coordinates (that is performing a change of basis) the $\Lambda^a{}_b$ transform raises indices and the $\Lambda_a{}^b$ transform lowers them. We are now set to study how I_1 and I_2 transform under a change of basis.

$$
\begin{aligned}
R'_{ab} &= \lambda'\,g'_{ab} \\
\Rightarrow\ \ \Lambda_a{}^c\Lambda_b{}^d R_{cd} &= \lambda'\,\Lambda_a{}^c\Lambda_b{}^d\,g_{cd} \\
\Rightarrow\ \ R_{cd} &= \lambda'\,g_{cd} \\
\Rightarrow\ \ \lambda' &= \lambda\,,
\end{aligned}
\tag{A.10}
$$

meaning λ is invariant under a change of basis.

In the transformed basis (primed) we have

$$
\begin{aligned}
I'_1 &= \lambda'^{d/2}\,V' \\
&= \lambda^{d/2}\,\sqrt{(\det g')}\,\sigma'^1 \wedge \sigma'^2 \wedge ... \wedge \sigma'^d\,.
\end{aligned}
\tag{A.11}
$$

Since g is the matrix corresponding to the metric g_{ab} it follows from $g'_{ab} = \Lambda_a{}^c\Lambda_b{}^d\,g_{cd}$ that in matrix form we have $g' = \Lambda^{-1}\, g\, (\Lambda^{-1})^T$ so that from (A.11)

$$\begin{aligned} I_1' &= \lambda^{d/2} \sqrt{(\det \Lambda^{-1}\, g\, (\Lambda^{-1})^T)}\, \sigma'^1 \wedge \sigma'^2 \wedge \ldots \wedge \sigma'^d \\ &= \lambda^{d/2} \sqrt{(\det \Lambda^{-1})\,(\det g)\,(\det(\Lambda^{-1})^T))}\, \sigma'^1 \wedge \sigma'^2 \wedge \ldots \wedge \sigma'^d \,. \end{aligned} \tag{A.12}$$

But in general for a matrix A, $\det A^T = \det A$ so

$$\begin{aligned} I_1' &= \lambda^{d/2} \sqrt{(\det g)} (\det \Lambda^{-1})\, \sigma'^1 \wedge \sigma'^2 \wedge \ldots \wedge \sigma'^d \\ &= \lambda^{d/2} \sqrt{(\det g)} (\det \Lambda^{-1})\, \Lambda^1{}_{i_1} \sigma^{i_1} \wedge \Lambda^2{}_{i_2} \sigma^{i_2} \wedge \ldots \wedge \Lambda^d{}_{i_d} \sigma^d \,. \end{aligned} \tag{A.13}$$

We note that the wedge-product is an inherently anti-symmetric construct and so no σ^a can be repeated in the wedge-product in the last line of (A.13). $\Lambda^1{}_{i_1}\sigma^{i_1} = \Lambda^1{}_1\sigma^1 + \Lambda^1{}_2\sigma^2 + \ldots + \Lambda^1{}_d\sigma^d$ and so when we take the wedge-product the only non-vanishing terms will be permutations of $\sigma^1 \wedge \sigma^2 \wedge \ldots \wedge \sigma^d$. When we collect all of these terms and transform all of the permutations to $\sigma^1 \wedge \sigma^2 \wedge \ldots \wedge \sigma^d$ (by multiplying by appropriate factors of -1) we will end up with a coefficient that will consist of a sum over products of $\Lambda^a{}_b$ with the particular feature that every term in the sum will have each index appearing only once in the upper indices and only once in the lower indices. In fact a little thought shows that the coefficient will be exactly equal to

$$\det \Lambda = \begin{vmatrix} \Lambda^1{}_1 & \Lambda^1{}_2 & \cdots & \Lambda^1{}_d \\ \Lambda^2{}_1 & \Lambda^2{}_2 & \cdots & \Lambda^2{}_d \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda^d{}_1 & \Lambda^d{}_2 & \cdots & \Lambda^d{}_d \end{vmatrix} \,. \tag{A.14}$$

The rules governing the calculation of the determinant exactly mirror the constraints imposed by the wedge-product, including the factors of -1 that come from permutations of the 1-forms in the wedge-product. This then results in

$$\begin{aligned} \Rightarrow\ I_1' &= \lambda^{d/2} \sqrt{(\det g)}\, (\det \Lambda^{-1}) (\det \Lambda)\, \sigma^1 \wedge \sigma^2 \wedge \ldots \wedge \sigma^d \\ &= \lambda^{d/2} \sqrt{(\det g)}\, \sigma^1 \wedge \sigma^2 \wedge \ldots \wedge \sigma^d \\ &= I_1 \,, \end{aligned}$$

thus proving that I_1 is invariant under a change of basis.

The proof for I_2 is much simpler since $R_{abcd}\, R^{abcd}$ is clearly invariant, being a scalar, that is a tensor of rank 0. Thus

$$\begin{aligned} I'_2 &= \lambda'^{-2} R'_{abcd} R'^{abcd} \\ &= \lambda^{-2} R_{abcd} R^{abcd} \\ &= I_2 \end{aligned} \tag{A.15}$$

Therefore both I_1 and I_2 are dimensionless invariant quantities by virtue of their construction.

A.3. Genesis of the Left-invariant 1-forms $L_A{}^B$

In the defining representation the elements of $SU(n)$ are $n \times n$ complex-valued matrices which are unitary and have unit determinant. This in turn requires that the generators of $SU(n)$, that is the elements of the corresponding Lie Algebra $\mathfrak{su}(n)$, be complex-valued, Hermitian and traceless.

A.3.1. Construction and Properties of the $T^A{}_B$

While one can work out the curvature of a metric on $SU(n)$ using a basis of Hermitian traceless generators our calculations are considerably simplified if we instead start off using a different basis. In the defining representation of $\mathfrak{su}(n)$ the generators manifest as $n \times n$ complex-valued Hermitian traceless matrices which act upon $\mathbb{C}^n$ (the n-dimensional vector space over the field of complex numbers). The generators can then be viewed as operators (matrices) which act upon the vector space. Given this view one can define the "generators/operators"

$$T^A{}_B \equiv \delta^A_B \tag{A.16}$$

according to their action upon vectors in $\mathbb{C}^n$ [41].

This means, for example, that $T^1{}_2$ manifests in the 3-dimensional vector space for $\mathfrak{su}(3)$ as

$$T^1{}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.17}$$

and maps the standard orthonormal unit vector $\vec{e}_2 \in \mathbb{C}^n$ to $\vec{e}_1$. It immediately follows from the definition (A.16) that

$${T^A{}_B}^\dagger = T^B{}_A \tag{A.18}$$

since the $T^A{}_B$ are real-valued and so their Hermitian conjugate becomes a straight-forward transpose.

Let v^C be the (abstract) state corresponding to the unit-vector $\vec{e}_C \in \mathbb{C}^n$ then the action of the generator/operator $T^A{}_B$ upon the state v^C is given by

$$T^A{}_B\, v^C = \delta^C_B\, v^A \tag{A.19}$$

since $T^A{}_B$ only maps v^B to v^A.

Since the states v^A span the space upon which the generators $T^A{}_B$ act we can calculate the commutator of the generators by observing how it acts on an arbitrary state v^E.

$$\begin{aligned}\left[T^A{}_B, T^C{}_D\right]\, v^E &= (T^A{}_B\, T^C{}_D - T^C{}_D\, T^A{}_B) v^E \\ &= T^A{}_B\, \delta^E_D\, v^C - T^C{}_D\, \delta^E_B\, v^A \\ &= \delta^E_D\, T^A{}_B\, v^C - \delta^E_B\, T^C{}_D\, v^A \\ &= \delta^E_D\, \delta^C_B\, v^A - \delta^E_B\, \delta^A_D\, v^C \\ &= \delta^C_B\, \delta^E_D\, v^A - \delta^A_D\, \delta^E_B\, v^C \\ &,\end{aligned}$$

at which point we use the converse of (A.16), so

$$\begin{aligned}\left[T^A{}_B, T^C{}_D\right]\, v^E &= \delta^C_B\, T^A{}_D\, v^E - \delta^A_D\, T^C{}_B\, v^E \\ &= (\delta^C_B\, T^A{}_D - \delta^A_D\, T^C{}_B)\, v^E \\ \Rightarrow\ \left[T^A{}_B, T^C{}_D\right] &= (\delta^C_B\, T^A{}_D - \delta^A_D\, T^C{}_B)\,.\end{aligned} \tag{A.20}$$

The space of the generators T_a is spanned by the basis $T^A{}_B$ with the caveat that the linear combination $T^A{}_A$ does not belong to this space since the generators of the Lie Algebra $\mathfrak{su}(n)$ are traceless matrices. In this fashion the n^2 "generators" $T^A{}_B$ can create the n^2-1 traceless Hermitian generators T_a.

A.3.2. Construction and Properties of the $L_A{}^B$

We can now construct the 1-forms $L_A{}^B$ associated with the "generators" $T^A{}_B$ in the standard way (A.4)

$$g^{-1}dg = \alpha\, T^A{}_B\, L_A{}^B\,, \tag{A.21}$$

where $\alpha \in \mathbb{C}$ is a normalization constant for the ${L_A}^B$ which we are free to choose since the ${L_A}^B$, so defined, span the space of 1-forms on the $SU(n)$ manifold for any choice of $\alpha \neq 0$.

This construction results in the exterior derivative of the 1-forms ${L_A}^B$ taking an extremely simple and elegant form. We begin with (A.21)

$$\begin{aligned} g^{-1}dg &= \alpha\, {T^A}_B\, {L_A}^B \\ \Rightarrow\ d(g^{-1}dg) &= d(\alpha\, {T^A}_B\, {L_A}^B) \\ \Rightarrow\ -g^{-2}dg \wedge dg + g^{-1}d^2g &= \alpha\, {T^A}_B\, d{L_A}^B \\ \Rightarrow\ -g^{-1}dg \wedge g^{-1}dg &= \alpha\, {T^A}_B\, d{L_A}^B \end{aligned} \tag{A.22}$$

where to get from the 2[nd] to the 3[rd] line we used the fact that the ${T^A}_B$ are constants and so $d{T^A}_B = 0$ while the ${L_A}^B$ being 1-forms do have exterior derivatives. To get from the 3[rd] to the 4[th] line we used the fact that in general taking the exterior derivative twice makes a term vanish so $d^2g = 0$. We continue by substituting in (A.21) so that

$$\begin{aligned} -(\alpha\, {T^C}_D {L_C}^D) \wedge (\alpha\, {T^E}_F {L_E}^F) &= \alpha\, {T^A}_B\, d{L_A}^B \\ \Rightarrow\ -\alpha\, {T^C}_D\, {T^E}_F\, {L_C}^D \wedge {L_E}^F &= {T^A}_B\, d{L_A}^B\,. \end{aligned}$$

Since the wedge-product ${L_C}^D \wedge {L_E}^F$ is inherently anti-symmetric it samples out just the anti-symmetric part of the term ${T^C}_D\, {T^E}_F$ which corresponds to the anti-commutator, so

$${T^A}_B\, d{L_A}^B = -\frac{1}{2}\alpha\, \left[{T^C}_D, {T^E}_F\right]\, {L_C}^D \wedge {L_E}^F\,.$$

We now make use of the expression for the commutator given in (A.20) to get

$${T^A}_B\, d{L_A}^B = -\frac{1}{2}\alpha\, (\delta^E_D\, {T^C}_F - \delta^C_F\, {T^E}_D){L_C}^D \wedge {L_E}^F\,.$$

We use delta functions to artificially introduce ${T^A}_B$ on the RHS, giving us

$$
\begin{aligned}
{T^A}_B \, d{L_A}^B &= -\frac{1}{2}\alpha \, (\delta_D^E \, \delta_A^C \, \delta_F^B \, {T^A}_B - \delta_F^C \, \delta_A^E \, \delta_D^B \, {T^A}_B) {L_C}^D \wedge {L_E}^F \\
&= -\frac{1}{2}\alpha \, {T^A}_B (\delta_D^E \, \delta_A^C \, \delta_F^B - \delta_F^C \, \delta_A^E \, \delta_D^B) {L_C}^D \wedge {L_E}^F \\
\Rightarrow \; d{L_A}^B &= -\frac{1}{2}\alpha (\delta_D^E \, \delta_A^C \, \delta_F^B - \delta_F^C \, \delta_A^E \, \delta_D^B) {L_C}^D \wedge {L_E}^F \\
&= -\frac{1}{2}\alpha \, ({L_A}^D \wedge {L_D}^B - {L_C}^B \wedge {L_A}^C) \\
&= -\frac{1}{2}\alpha \, ({L_A}^C \wedge {L_C}^B - {L_C}^B \wedge {L_A}^C) \\
&= -\frac{1}{2}\alpha \, ({L_A}^C \wedge {L_C}^B + {L_A}^C \wedge {L_C}^B) \\
\Rightarrow \; d{L_A}^B &= -\alpha \, {L_A}^C \wedge {L_C}^B \, .
\end{aligned}
\tag{A.23}
$$

Since we are free to choose, we set $\alpha = e^{-i\frac{\pi}{2}}$, which gives us the conventional form

$$
d{L_A}^B = i{L_A}^C \wedge {L_C}^B \, . \tag{A.24}
$$

A.3.3. Construction of Traceless Hermitian Generators

The ${T^A}_B$ we have been working with so far are not Hermitian (A.18) but we can construct traceless Hermitian generators from them via linear sums. This construction automatically generates the associated 1-forms. For example let us create the Hermitian (and obviously traceless) generators

$$
T_1 = {T^1}_2 + {T^2}_1 \qquad \text{and} \qquad T_2 = i \, ({T^1}_2 - {T^2}_1) \, , \tag{A.25}
$$

with the obvious inverse transforms

$$
{T^1}_2 = \frac{1}{2}(T_1 - iT_2) \qquad \text{and} \qquad {T^2}_1 = \frac{1}{2}(T_1 + iT_2) \, . \tag{A.26}
$$

The generators ${T^1}_2$ and ${T^2}_1$, and the corresponding generators T_1 and T_2, span the same subspace of the generators of $\mathfrak{su}(n)$. Let us call the corresponding subspace of $g^{-1}dg$, V', then $V' = \alpha \, ({T^1}_2 \, {L_1}^2 + {T^2}_1 \, {L_2}^1)$. If we define L_1 and L_2 as the 1-forms corresponding to T_1 and T_2 in the subspace V', we have $V' = T_1 L_1 + T_2 L_2$. We can easily calculate the relationship (transformation) between the 1-forms:

$$
\begin{aligned}
V' &= \alpha(T^1{}_2\, L_1{}^2 + T^2{}_1\, L_2{}^1) \\
&= \frac{1}{2}\alpha((T_1 - iT_2)L_1{}^2 + (T_1 + iT_2)L_2{}^1) \\
\Rightarrow\; T_1\, L_1 + T_2\, L_2 &= \frac{1}{2}\alpha\,(T_1\,(L_1{}^2 + L_2{}^1) + T_2\,(-i\,L_1{}^2 + i\,L_2{}^1))\,.
\end{aligned}
$$

It is clear by comparing the two sides and by realizing that the normalization of the 1-forms is irrelevant that the relationship (transformation) between the 1-forms, dual to (A.25), is

$$
L_1 = L_1{}^2 + L_2{}^1 \qquad \text{and} \qquad L_2 = -i\,(L_1{}^2 - L_2{}^1)\,. \tag{A.27}
$$

Note how the transformations for the 1-forms L_i are the complex conjugate of the transformations for the T_i. Thus, rather than creating linear sums of the $T^A{}_B$, which are Hermitian, and then calculating the corresponding linear sums of the $L_A{}^B$ one can directly create "Hermitian" sums of the $L_A{}^B$ which are guaranteed to correspond to Hermitian generators.

The exterior derivative of these constructed "Hermitian" 1-forms can be calculated by applying (A.24) to the constituent $L_A{}^B$.

A.3.4. Construction of Diagonal Traceless Hermitian Generators

(A.25) gives the construction of Hermitian Generators from the off-diagonal $T^A{}_B$. It remains to construct the traceless diagonal generators K'_i ($n-1$ in number) from the n diagonal $T^A{}_A$. The standard construction involves, for any $1 \le p < n$, a diagonal matrix with the first p diagonal entries being 1, the next entry being $-p$ and the remaining entries being 0. These matrices are traceless by construction. In addition the matrices are orthogonal under taking the trace after multiplying the matrices together. The next step is to (ortho-)normalize them such that

$$
tr(K'_i\, K'_j) = \delta_{ij}\,, \tag{A.28}
$$

which simply involves dividing each matrix by the square-root of the trace of the square of the matrix.

For instance for $n = 4$ we have, by this construction, the following traceless Hermitian matrices:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \qquad \frac{1}{\sqrt{6}}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \qquad \frac{1}{\sqrt{12}}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -3\end{pmatrix} \tag{A.29}$$

If we write down these newly constructed matrices in terms of the diagonal ${L_A}^B$ we get the matrix $\mathbf{Q}$ and its action on $\vec{l}$ as laid down in (2.10).

A.3.5. Symmetrization of the Diagonal Traceless Hermitian Generators

The construction we have come up with so far in Appendix A.3.4, although traceless and hermitian as well as orthonormal, is not symmetric with respect to the diagonal ${L_A}^B$; that is the matrices (generators) we have constructed do not have symmetric contribution from the diagonal ${L_A}^B$. In particular some of the matrices have no contribution from some of the diagonal ${L_A}^B$ at all.

For the sake of the algorithm we will be employing we wish to have all the diagonal generators on a roughly symmetric footing with respect to the diagonal ${L_A}^B$. To achieve this we implement a mixing scheme for the generators constructed in A.3.4. Since these generators form an orthonormal set with respect to taking the trace over multiplication (A.28) we come up with the mixing scheme by studying the transformation of one orthonormal basis over $\mathbb{R}^{n-1}$ to another.

It can be demonstrated that perfectly symmetric mixing is only possible when the number of objects/dimensions are a power of 2. In view of this we use a different scheme, one which mixes the generators, not perfectly, but enough to suffice. The scheme is based on the following manipulation of orthonormal basis vectors to construct a new basis.

Let the original orthonormal basis over $\mathbb{R}^N$ be $\hat{x_i}$ (obeying $\hat{x_i} \,.\, \hat{x_j} = \delta_{ij}$). We define $\hat{a} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \hat{x_i}$. In the $\hat{x_i}$ basis the unit vector $\hat{a}$ takes the form $\hat{a} = \frac{1}{\sqrt{N}}(1, 1, ..., 1)$. By definition $\hat{a}$ has equal contributions from each of the $\hat{x_i}$ as demonstrated by

$$\hat{x_i} \,.\, \hat{a} = \hat{x_i} \,.\, \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \hat{x_j} = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \hat{x_i} \,.\, \hat{x_j} = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \delta_{ij} = \frac{1}{\sqrt{N}} \equiv A \,. \tag{A.30}$$

We calculate the part of $\hat{x_i}$ perpendicular to $\hat{a}$,

$$\hat{b_i} \equiv \hat{x_i} - (\hat{x_i} \,.\, \hat{a})\hat{a} = \hat{x_i} - A\hat{a}\,. \tag{A.31}$$

We define the new basis vectors as

$$\hat{y_i} \equiv A\hat{a} - \hat{b_i} = A\hat{a} - (\hat{x_i} - A\hat{a}) = 2A\hat{a} - \hat{x_i}\,, \tag{A.32}$$

which simply corresponds to taking the part of x_i perpendicular to $\hat{a}$ and flipping it around. The $\hat{y_i}$ have the following properties

$$\begin{aligned} \hat{y_i} \,.\, \hat{y_j} &= (2A\hat{a} - \hat{x_i}) \,.\, (2A\hat{a} - \hat{x_j}) = 4A^2 - 2A^2 - 2A^2 + \hat{x_i}\hat{x_j} = \delta_{ij}\,, \\ \hat{y_i} \,.\, \hat{x_j} &= (2A\hat{a} - \hat{x_i}) \,.\, \hat{x_j} = 2A^2 - \delta_{ij} = \frac{2}{N} - \delta_{ij}\,. \end{aligned} \tag{A.33}$$

By studying how the y_i relate to the x_j we can reconstruct the transformation matrix that maps the x_j basis vectors to the y_i,

$$\begin{pmatrix} \frac{2}{N} - 1 & \frac{2}{N} & \cdots & \frac{2}{N} \\ \frac{2}{N} & \frac{2}{N} - 1 & \cdots & \frac{2}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{N} & \cdots & \frac{2}{N} & \frac{2}{N} - 1 \end{pmatrix}. \tag{A.34}$$

Note how each $\hat{y_i}$ has symmetric contributions from every x_j where $j \neq i$ and an asymmetric contribution from x_i itself. In addition the mixing becomes weak for large N. However this implementation suffices for the algorithm we are choosing to implement. This transformation matrix motivates the top-left $(n-1)\times(n-1)$ sub-matrix of the matrix $\mathbf{P}$ defined in (2.11).

The transformations described in this section and the previous one are both invertible, an invaluable property when it comes to the implementation of the algorithm.

A.4. Example of Decomposition Scheme (Scheme 2)

Let $n = 5$ and $p = 2$ ($q = 3$). We construct the $SU(2) \times SU(3) \subset SU(5)$ decomposition scheme. The $5^2 - 1 = 24$ 1-forms of $SU(5)$ divided in to the 4 classes of Scheme 2 are given as follows.

The Class 1 generators (analogous to the generators for $SU(2)$ and $2^2 - 1 = 3$ in number) constructed from ${L_a}^b$ where $a \neq b \in \{1,2\}$ are

$$\begin{aligned} K_1 &= {L_1}^2 + {L_2}^1 \\ K_2 &= i({L_1}^2 - {L_2}^1) \\ K_3 &= {L_1}^1 - {L_2}^2 \,. \end{aligned} \tag{A.35}$$

The Class 2 generators (analogous to the generators for $SU(3)$ and $3^2 - 1 = 8$ in number) constructed from ${L_\alpha}^\beta$ where $\alpha \neq \beta \in \{3,4,5\}$ are

$$\begin{aligned} K_4 &= {L_3}^4 + {L_4}^3 \\ K_5 &= {L_3}^5 + {L_5}^3 \\ K_6 &= {L_4}^5 + {L_5}^4 \end{aligned}$$

$$\begin{aligned} K_7 &= i({L_3}^4 - {L_4}^3) \\ K_8 &= i({L_3}^5 - {L_5}^3) \\ K_9 &= i({L_4}^5 - {L_5}^4) \end{aligned} \tag{A.36}$$

$$\begin{aligned} K_{10} &= (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}})\, {L_3}^3 + (-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}})\, {L_4}^4 - \frac{2}{\sqrt{6}}\, {L_5}^5 \\ K_{11} &= (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}})\, {L_3}^3 - (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}})\, {L_4}^4 + \frac{2}{\sqrt{6}}\, {L_5}^5 \,. \end{aligned}$$

The Class 3 generators ($2 \times 2 \times 3 = 12$ in number) correspond to the off-diagonal ${L_A}^B$ and are

$$\begin{aligned} K_{12} &= {L_1}^3 + {L_3}^1 \\ K_{13} &= {L_1}^4 + {L_4}^1 \\ K_{14} &= {L_1}^5 + {L_5}^1 \\ K_{15} &= {L_2}^3 + {L_3}^2 \\ K_{16} &= {L_2}^4 + {L_4}^2 \\ K_{17} &= {L_2}^5 + {L_5}^2 \end{aligned} \tag{A.37}$$

$$
\begin{aligned}
K_{18} &= i(L_1{}^3 + L_3{}^1) \\
K_{19} &= i(L_1{}^4 + L_4{}^1) \\
K_{20} &= i(L_1{}^5 + L_5{}^1) \\
K_{21} &= i(L_2{}^3 + L_3{}^2) \\
K_{22} &= i(L_2{}^4 + L_4{}^2) \\
K_{23} &= i(L_2{}^5 + L_5{}^2)\,.
\end{aligned}
\tag{A.38}
$$

The Class 4 generator (one in number) which institutes the mixing of diagonal elements of both $SU(2)$ and $SU(3)$ is

$$
K_{24} = 3\,(L_1{}^1 + L_2{}^2) - 2\,(L_3{}^3 + L_4{}^4 + L_5{}^5)\,. \tag{A.39}
$$

APPENDIX B

VISUALISING THE BIRKHOFF'S INVARIANT

To better understand the meaning of β note that $\beta(f)$ is the longest length curve in the foliation of S using level curves of f. If we consider the level curves $f^{-1}(c)$ as the spatial configurations of an elastic loop of maximal length $\beta(f)$, then the function f through its level curves describes a specific way of passing said loop across the surface S.

It is clear that one is able to cleanly pass a flexible loop of length $\beta(f)$ across S without intersecting with the enclosed region if one follows the configurations described by f. By varying the function f we sample the various ways of passing an elastic loop across the surface S.

So β being the smallest of the $\beta(f)$, which in turn are the largest length the hoop has during each passage defined by f, gives the smallest hoop length which can still pass over S without intersecting the interior. Thus β quantifies how compact the region enclosed by the apparent horizon S is.

An example should clarify. Given a 2-sphere S with radius R and using the standard spherical polar coordinates a particular foliation of S can be given by $f(\theta, \phi) = cos(\theta)$ in which case $f^{-1}(c)$ corresponds to circles of latitude (constant θ) on S. For this particular foliation the maximal length level curve occurs at $\theta = \pi/2$ and corresponds to the equator. And so $\beta(f) = 2\pi R$, the circumference of the equator. It turns out that this is the smallest that $\beta(f)$ can be for any f and so $\beta = 2\pi R$; as expected the Birkhoff's invariant for the 2-sphere is the length of the great circles on it.

Included below are a few example foliations of a 2-sphere illustrated by Figures 6, 7 and 8.

B.1. Example Foliations of the 2-sphere

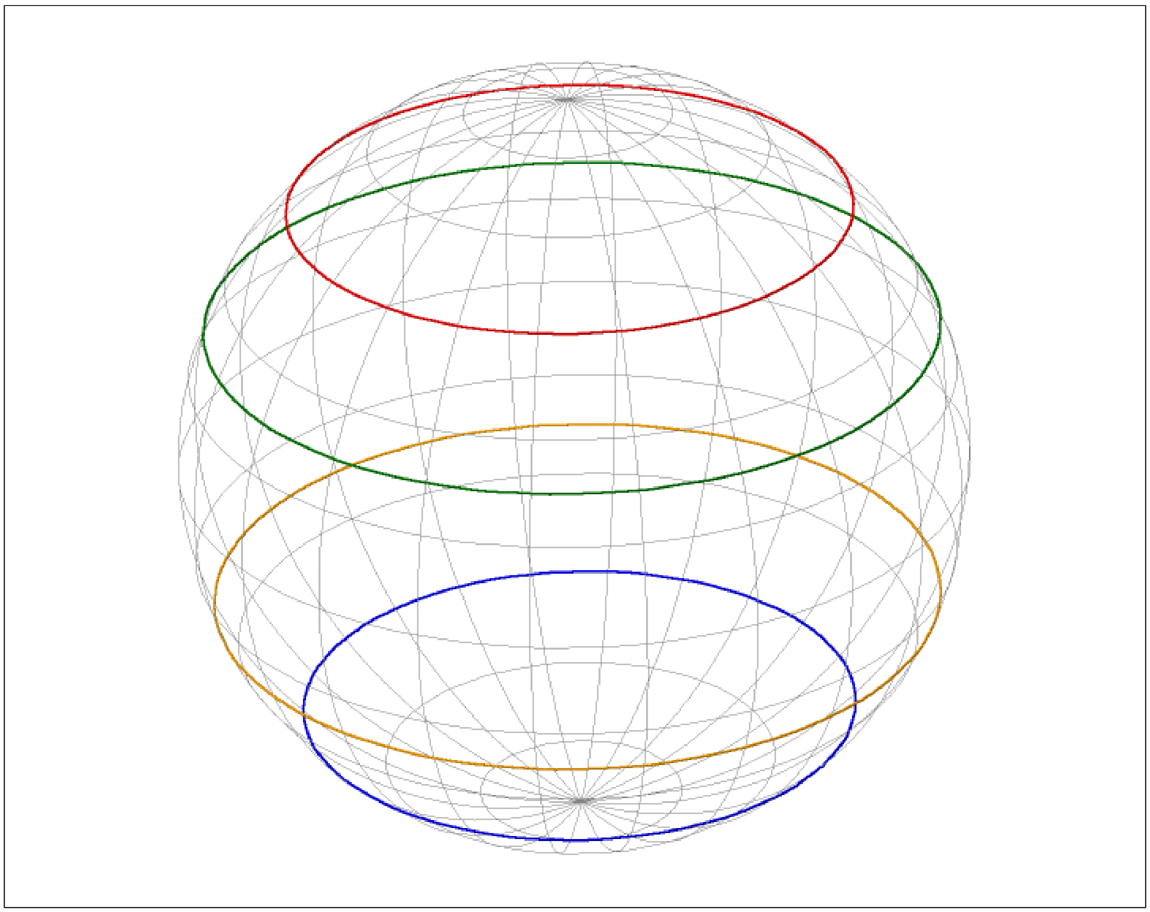

Figure 6: Level curves of $f(\theta, \phi) = \cos(\theta)$ *for a 2-sphere.*

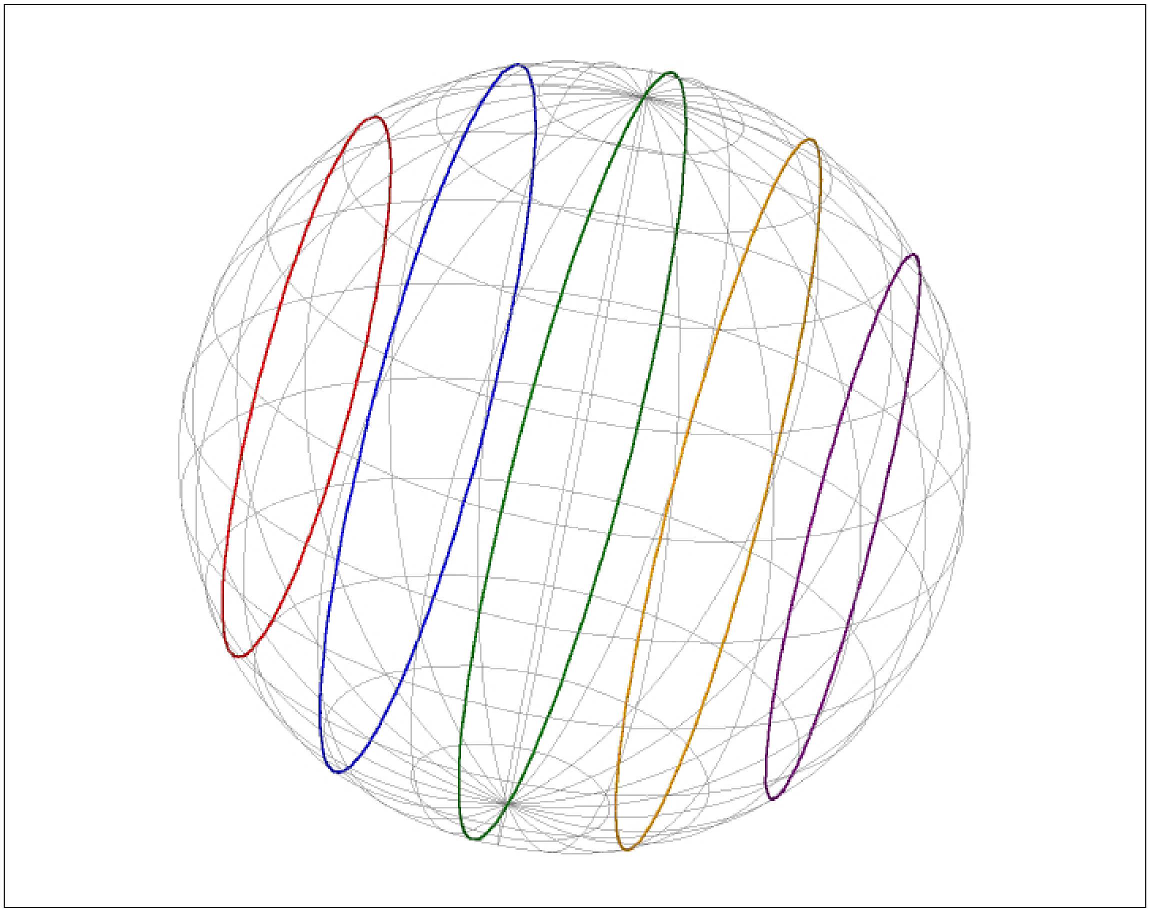

Figure 7: Level curves of $x = c$, $y = \sqrt{1-c^2}\cos(\psi)$, $z = \sqrt{1-c^2}sin(\psi)$ *for a 2-sphere.*

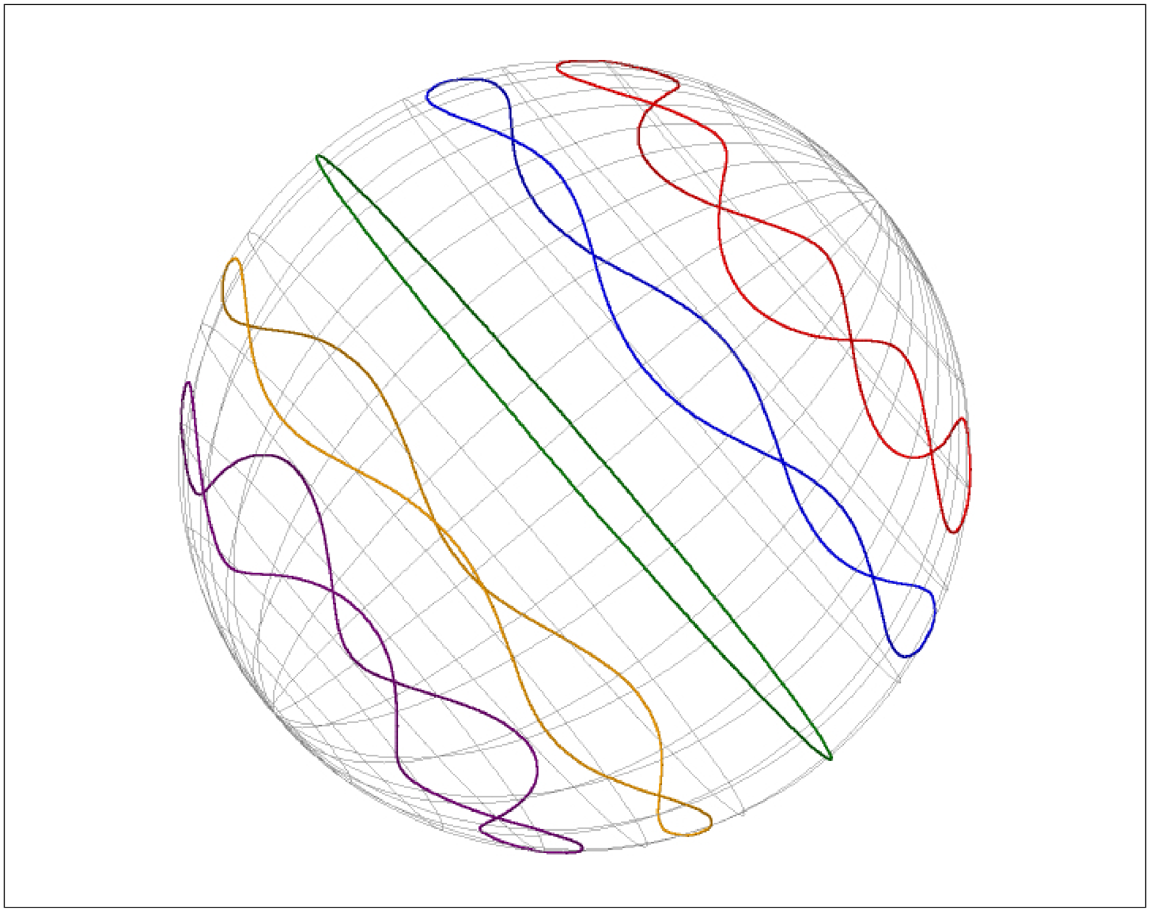

Figure 8: Level curves of $z = \cos(\theta_0)(1 + 0.2\,\sin(\theta_0)\,\sin(5\psi))$, $x = \sqrt{1 - z^2}\cos(\psi)$, $y = \sqrt{1 - z^2}\sin(\psi)$ *for a 2-sphere.*

APPENDIX C

TOPOLOGY OF BLACK RINGS

C.1. Limitations on the Black Ring Parameters and Coordinates

It can be easily shown that the metric corresponding to the neutral singly rotating black ring (3.6)[20] is Ricci-free, that it is a valid vacuum solution of the Einstein Field Equations. This is possible without unduly restricting the various parameters and coordinates that form part of the metric. However, there exist mathematical and physical requirements which significantly limit the range of these parameters and coordinates.

To begin we look at $G(x) = (1-x^2)(1+\nu x)$. Since $g_{xx} \sim \frac{1}{G(X)}$ the physical requirement that $g_{xx} > 0$, so that x be a spatial coordinate (we want t to be the only time-like coordinate), demands that $G(x) > 0$ over the range of x. This limits x to $-1 \leq x \leq 1$. Additionally, this requires that $0 < \nu < 1$ so that $G(x)$ has distinct roots and remains positive for $-1 \leq x \leq 1$. (This requires that the "third" root of $G(x)$ be less than -1).

To ensure that $C(\nu, \lambda) = \sqrt{\lambda(\lambda - \nu)\dfrac{1+\lambda}{1-\lambda}}$ is real we require $\lambda > 0$, $\lambda \geq \nu$ and $\lambda < 1$. Therefore, the dimensionless parameters λ and ν must lie in the range [20]

$$0 < \nu \leq \lambda < 1\,. \tag{C.1}$$

Since $-1 \leq x \leq 1$ and $0 < \lambda < 1$ implies that $F(x) = 1 + \lambda x > 0$; the physical requirement that $g_{tt} < 0$ (so that t be a time-like coordinate) requires $F(y) < 1$. This restricts y to $-\infty < y < -1$.

Finally, avoiding conical singularities requires that ϕ and ψ be periodic, with the periods calculated in Appendix C.4.

C.2. Topology of Black Rings

We begin by noting that t is the only time-like coordinate in the metric, the rest being space-like. The parameter R appears only in the spatial components of the metric and so clearly scales the spatial part of the metric. λ specifies the shape and ν the rotation velocity of the ring [20].

ϕ and ψ are angular variables with periodicity given by Appendix C.4. x is a latitudinal coordinate, analogous to $\cos(\theta)$ on S^2. The hypersurface Σ (the spatial portion of the entire metric, spanned by y, x, ϕ and ψ) can be split in to surfaces of constant y. Each constant-y surface is topologically $S^1 \times S^2$ spanned by ψ (S^1) and, x and ϕ (S^2).

The constant-y surfaces are nested like a Matryoshka doll, with the outermost surface corresponding to $y = -1$. At $x = -1$ we lie on the outward portion of the $S^1 \times S^2$ surface while at $x = 1$ we lie on the inward portion closer to the axis of rotation. Spatial infinity lies at $y = -1$, $x = -1$ while the axis of rotation corresponds to $y = -1$, $x = 1$.

As y increases from $-\infty$ the surfaces shrink (in a nested fashion). At $y = -1/\nu$, $G(y = -1/\nu) = 0$ and so $1/g_{yy} = 0$. This corresponds to the apparent horizon. Continuing to decrease y gives surfaces that lie inside the horizon with a coordinate singularity at $y = -\infty$.

C.3. Conical Singularities

To understand conical singularities we start by studying the flat (Euclidean) metric on a $2D$ space in polar coordinates

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2 \qquad (\theta \in [0, 2\pi] \text{ i.e. periodic with period } 2\pi) . \tag{C.2}$$

Surfaces of constant r are topologically S^1. As $r \to 0$ the circles limit to a single point but the space remains (locally) flat, so the origin is **not** singular. This is related to the fact that θ has period 2π. To understand why the period is crucial consider a related metric

$$ds^2 = dr^2 + \beta^2 r^2 d\theta^2 \qquad (\theta \in [0, 2\pi]) . \tag{C.3}$$

When $\beta = 1$ we get the original metric and the space is flat (non-singular) at $r = 0$ (the origin). Consider the coordinate transformation $\phi = \beta\,\theta$, which gives us

$$\theta \in [0, 2\pi] \qquad \Rightarrow \qquad \phi \in [0, 2\pi\,\beta] , \tag{C.4}$$

and

$$ds^2 = dr^2 + r^2 d\phi^2 , \tag{C.5}$$

so ϕ has period $2\pi\,\beta$.

If $\beta < 1$ then the period of $\phi = \beta\theta$ is $2\pi\beta < 2\pi$. Per usual, we identify the end-points of the angular domain, $\phi = 0$ and $\phi = 2\pi\beta$. Since $\beta < 1$ this is equivalent to cutting a slice out of a disc and pasting the cut edges together. The resulting space is topologically a cone. Consequently, this creates a "conical singularity" at $r = 0$, the tip of the cone. The singularity (locally non-flat region) at the tip of any conical surface is a general feature of the cone.

Therefore, for an arbitrary metric of the form $ds^2 = dr^2 + r^2 d\theta^2$, if θ does **not** have period 2π the space will have a conical singularity at $r = 0$. Imposing the correct periodicity on the relevant (angular) variable is a means of avoiding conical singularities at the point where the angular component of the metric goes to zero ($g_{\theta\theta} \to 0$).

C.4. Conical Singularities in Black Rings

We consider the revised metric (3.6). Let us begin by studying the ϕ variable. $g_{\phi\phi}$ is a function of x and y alone. Since the metric is independent of ϕ and ϕ happens to be a spatial coordinate, intuition suggests that ϕ is an angular coordinate and x is a latitudinal coordinate (analogous to $\cos\theta$ on S^2); the two spanning a topologically S^2 subspace. This immediately suggests the possibility of a conical singularity. We focus on the x - ϕ sub-metric

$$\frac{R^2}{(x-y)^2} F(x) \left[\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\phi^2 \right] . \tag{C.6}$$

The common factor of $\frac{R^2}{(x-y)^2} F(x)$ can be safely ignored as a scaling factor (since it is finite for the range of x and y). $G(x = -1) = 0$ so we investigate the possible conical singularity at $x = -1$. We ignore the scaling factor and focus on

$$d\hat{s}^2 = \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\phi^2 . \tag{C.7}$$

We define $x \equiv -1 + \rho$ and study the sub-metric (C.7) in the limit $\rho \to 0^+$. In this limit

$$G(x) = G(-1+\rho) \approx G(-1) + \rho\, G'(-1) + \mathcal{O}(\rho^2) = \rho\, G'(-1) \, , \tag{C.8}$$

where $G'(x) = \frac{d}{dx} G(x)$ and we made use of $G(-1) = 0$.

Similarly

$$F(x) = F(-1+\rho) \approx F(-1)\,, \tag{C.9}$$

since $F(-1) >> \rho\, F'(-1)$ in the limit $\rho \to 0^+$. Finally we note that $x = -1+\rho \implies dx = d\rho$. We can now express (C.7) in terms of the coordinate ρ as

$$d\hat{s}^2 = \frac{1}{\rho\, G'(-1)} d\rho^2 + \frac{\rho\, G'(-1)}{F(-1)} d\phi^2 = \frac{1}{G'(-1)} \left[\frac{1}{\rho} d\rho^2 + \frac{\rho\, G'(-1)^2}{F(-1)} d\phi^2 \right]. \tag{C.10}$$

We make the transformation $\rho = r^\alpha$

$$\frac{1}{\rho} d\rho^2 = \frac{1}{r^\alpha} (\alpha\, r^{\alpha-1} dr)^2 = \alpha^2 r^{\alpha-2}\, dr^2\,, \tag{C.11}$$

and choose $\alpha = 2$ so that $\frac{1}{\rho} d\rho^2 \sim dr^2$. This results in

$$d\hat{s}^2 = \frac{4}{G'(-1)} \left[dr^2 + \frac{G'(-1)^2}{4\, F(-1)}\, r^2\, d\phi^2 \right]. \tag{C.12}$$

Note that requiring $\frac{1}{\rho} d\rho^2 \sim dr^2$ automatically gives us $g_{\phi\phi} \sim r^2$. This suggests that the x - ϕ subspace is topologically like a pole of S^2 at $x = -1$. We once again ignore the scaling factor $\frac{4}{G'(-1)}$ and focus on

$$d\hat{s}^2 = dr^2 + \frac{G'(-1)^2}{4\, F(-1)} r^2 d\phi^2\,. \tag{C.13}$$

This takes the form $ds^2 = dr^2 + r^2 d\theta^2$ with

$$\theta \equiv \frac{G'(-1)}{2\,\sqrt{F(-1)}} \phi \qquad \Rightarrow \qquad \Delta\theta = \frac{G'(-1)}{2\,\sqrt{F(-1)}} \Delta\phi\,. \tag{C.14}$$

To avoid a conical singularity at $r = 0$ we must explicitly impose the periodicity

$$\frac{G'(-1)}{2\,\sqrt{F(-1)}}\, \Delta\phi = 2\pi \qquad \Rightarrow \qquad \Delta\phi = \frac{4\pi\sqrt{F(-1)}}{G'(-1)} = 2\pi \frac{\sqrt{1-\lambda}}{1-\nu}\,. \tag{C.15}$$

A similar calculation for the y - ψ sub-metric at $y = -1$ requires

$$\Delta\psi = \Delta\phi = 2\pi \frac{\sqrt{1-\lambda}}{1-\nu}\,. \tag{C.16}$$

And again, the x - ϕ sub-metric at $x = 1$ gives us

$$\Delta\phi = 2\pi \frac{\sqrt{1+\lambda}}{1+\nu} \,. \tag{C.17}$$

Reconciling (C.16) and (C.17) requires

$$\lambda = \frac{2\nu}{1+\nu^2} \,. \tag{C.18}$$

So the requirement that the black ring (neutral single rotating) metric be free of conical singularities shrinks the parameter space by requiring that λ be a function of ν.

APPENDIX D

MAGNETISED KERR-NEWMAN METRIC

The construction of the magnetised Kerr-Newman metric, as pertains to this Dissertation, was performed primarily by Gibbons[3] and Pope[4].

The magnetised Kerr-Newman metric describes the spacetime corresponding to a charged rotating black hole suspended in an ambient magnetic field. Its construction follows.

We start with the Kerr-Newman metric describing a rotating black hole carrying an electric charge q and magnetic charge p :

$$\begin{aligned} d\hat{s}_4^2 &= -f dt^2 + R^2\Big(\frac{dr^2}{\Delta} + d\theta^2\Big) + \frac{\Sigma\,\sin^2\theta}{R^2}\,(d\phi - \bar{\omega} dt)^2\,, \\ A &= \bar{\Phi}_0\,dt + \bar{\Phi}_3\,(d\phi - \bar{\omega} dt)\,, \end{aligned} \tag{D.1}$$

where

$$\begin{aligned} R^2 &= r^2 + a^2\cos^2\theta\,, \qquad \Delta = (r^2+a^2) - 2mr + q^2 + p^2\,, \\ \bar{\omega} &= \frac{a(2mr - q^2 - p^2)}{\Sigma}\,, \qquad f = \frac{R^2\Delta}{\Sigma}\,, \qquad \Sigma = (r^2+a^2)^2 - a^2\Delta\sin^2\theta\,, \end{aligned} \tag{D.2}$$

and

$$\begin{aligned} \bar{\Phi}_0 &= \frac{2qr(r^2+a^2)}{\Sigma} + \frac{2ap\Delta\cos\theta}{\Sigma}\,, \\ \bar{\Phi}_3 &= -\frac{2aqr\sin^2\theta}{R^2} - \frac{2p(r^2+a^2)\cos\theta}{R^2}\,. \end{aligned} \tag{D.3}$$

To magnetise this solution we apply an $SU(2,1)$ transformation that generates magnetised

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[4] C. N. Pope, *George P. & Cynthia Woods Mitchell Institute for Fundamental Physics and Astronomy, Texas A&M University, College Station, TX 77843, USA*

solutions from non-magnetised ones

$$U = \begin{pmatrix} 1 & 0 & 0 \\ \frac{B}{\sqrt{2}} & 1 & 0 \\ \frac{B^2}{4} & \frac{B}{\sqrt{2}} & 1 \end{pmatrix} . \tag{D.4}$$

Applying the transformation gives us the magnetised Kerr-Newman solution

$$\begin{aligned} d\hat{s}^2 &= H \left[-f dt^2 + R^2 \left(\frac{dr^2}{\Delta} + d\theta^2 \right) \right] + \frac{\Sigma \sin^2\theta}{H R^2} (d\phi - \omega dt)^2 \,, \\ A &= \Phi_0 dt + \Phi_3 (d\phi - \omega dt) \,, \end{aligned} \tag{D.5}$$

where

$$H = 1 + \frac{H_{(1)}\, B + H_{(2)}\, B^2 + H_{(3)}\, B^3 + H_{(4)}\, B^4}{R^2} \,, \tag{D.6}$$

with

$$\begin{aligned} H_{(1)} &= -\, 2aqr \sin^2\theta - 2p(r^2 + a^2)\cos\theta \,, \\ H_{(2)} &= \frac{1}{2} \left[(r^2 + a^2)^2 - a^2 \Delta \sin^2\theta \right] + \frac{3}{2} \tilde{q}^2 (a^2 + r^2 \cos^2\theta) \,, \\ H_{(3)} &= -\, pa^2 \Delta \sin^2\theta \cos\theta + \frac{qa\Delta}{r} [r^2(3 - \cos^2\theta)\cos^2\theta + a^2(1 + \cos^2\theta)] - \frac{aq(r^2 + a^2)^2 (1 + \cos^2\theta)}{2r} \\ &\quad - \tfrac{1}{2} p(r^4 - a^4)\sin^2\theta\cos\theta - \frac{q\tilde{q}^2 a[(2r^2 + a^2)\cos^2\theta + a^2]}{2r} - \tfrac{1}{2} p\tilde{q}^2 (r^2 + a^2)\cos^3\theta \,, \\ H_{(4)} &= \tfrac{1}{16} (r^2 + a^2)^2 R^2 \sin^4\theta + \tfrac{1}{4} m a^2 r (r^2 + a^2) \sin^6\theta + \tfrac{1}{4} m a^2 \tilde{q}^2 r (\cos^2\theta - 5)\sin^2\theta\cos^2\theta \\ &\quad + \tfrac{1}{4} m^2 a^2 [r^2 (\cos^2\theta - 3)^2 \cos^2\theta + a^2 (1 + \cos^2\theta)^2] \\ &\quad + \tfrac{1}{8} \tilde{q}^2 (r^2 + a^2)(r^2 + a^2 + a^2\cos^2\theta)\sin^2\theta\cos^2\theta + \tfrac{1}{16} \tilde{q}^4 [r^2 \cos^2\theta + a^2(1 + \sin^2\theta)^2]\cos^2\theta \,, \end{aligned} \tag{D.7}$$

and we have defined

$$\tilde{q}^2 = q^2 + p^2 \,. \tag{D.8}$$

Finally ω is given by

$$\omega = \frac{(2mr - \tilde{q}^2)a + \omega_{(1)}\, B + \omega_{(2)}\, B^2 + \omega_{(3)}\, B^3 + \omega_{(4)}\, B^4}{\Sigma} \,, \tag{D.9}$$

where

$$
\begin{aligned}
\omega_{(1)} &= 2qr(r^2+a^2) + 2ap\Delta\cos\theta\,,\\
\omega_{(2)} &= -\tfrac{3}{2}a\tilde{q}^2(r^2+a^2+\Delta\cos^2\theta)\,,\\
\omega_{(3)} &= -4qm^2a^2r + \tfrac{1}{2}ap\tilde{q}^4\cos^3\theta + \tfrac{1}{2}qr(r^2+a^2)[r^2-a^2+(r^2+3a^2)\cos^2\theta]\\
&\quad + \tfrac{1}{2}ap(r^2+a^2)[3r^2+a^2-(r^2-a^2)\cos^2\theta]\cos\theta - \tfrac{1}{2}q\tilde{q}^2r[(r^2+3a^2)\cos^2\theta-2a^2]\\
&\quad + \tfrac{1}{2}ap\tilde{q}^2[3r^2+a^2+2a^2\cos^2\theta]\cos\theta + am\tilde{q}^2(2aq-pr\cos^3\theta)\\
&\quad - qm[r^4-a^4+r^2(r^2+3a^2)\sin^2\theta] - apmr[2R^2+(r^2+a^2)\sin^2\theta]\,, \qquad \text{(D.10)}\\
\omega_{(4)} &= \tfrac{1}{2}a^3m^3r(3+\cos^4\theta) - \tfrac{1}{16}a\tilde{q}^6\cos^4\theta - \tfrac{1}{8}a\tilde{q}^4[r^2(2+\sin^2\theta)\cos^2\theta + a^2(1+\cos^2\theta)]\\
&\quad + \tfrac{1}{16}a\tilde{q}^2(r^2+a^2)[r^2(1-6\cos^2\theta+3\cos^4\theta) - a^2(1+\cos^4\theta)] - \tfrac{1}{4}a^3m^2\tilde{q}^2(3+\cos^4\theta)\\
&\quad + \tfrac{1}{4}am^2[r^4(3-6\cos^2\theta+\cos^4\theta) + 2a^2r^2(3\sin^2\theta-2\cos^4\theta) - a^4(1+\cos^4\theta)]\\
&\quad + \tfrac{1}{8}am\tilde{q}^4r\cos^4\theta + \tfrac{1}{4}am\tilde{q}^2r[2r^2(3-\cos^2\theta)\cos^2\theta - a^2(1-3\cos^2\theta-2\cos^4\theta)]\\
&\quad + \tfrac{1}{8}amr(r^2+a^2)[r^2(3+6\cos^2\theta-\cos^4\theta) - a^2(1-6\cos^2\theta-3\cos^4\theta)]\,.
\end{aligned}
$$