HOMOGENEOUS KÄHLER AND SASAKIAN STRUCTURES RELATED TO COMPLEX HYPERBOLIC SPACES

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Abstract We study homogeneous Kähler structures on a non-compact Hermitian symmetric space and their lifts to homogeneous Sasakian structures on the total space of a principal line bundle over it, and we analyze the case of the complex hyperbolic space.

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homogeneous almost contact Riemannian manifolds;
non-compact Hermitian symmetric spaces; Sasakian spaces

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1 Introduction

The general theory of homogeneous Kähler manifolds is well-known, as well as the relation between homogeneous symplectic and homogeneous contact manifolds (see e.g. Boothby and Wang [6], Díaz Miranda and Reventós [9], and Dorfmeister and Nakajima [10]).

As is also widely known, a connected, simply connected and complete Riemannian manifold is a symmetric space if and only if its curvature tensor field is parallel. Ambrose and Singer [2] extended this result to obtain a characterization of homogeneous Riemannian manifolds in terms of the existence of a tensor field S of type $(1, 2)$ on the manifold, called a homogeneous Riemannian structure (see Tricerri and Vanhecke [26], where a classification of such structures is also given), satisfying certain properties (see (2.1); if $S = 0$ one has the symmetric case). Moreover, Sekigawa [24] obtained the corresponding result for almost Hermitian manifolds, defining homogeneous almost Hermitian structures, which were classified by Abbena and Garbiero in [1] (among them the homogeneous Kähler structures). Its odd-dimensional version, the almost contact metric case, has been also studied (see, for instance, [8, 11, 14, 19]).

In Section 2, we give the basic results about homogeneous Riemannian and homogeneous Kähler structures. In particular we consider these structures on Hermitian symmetric spaces of non-compact type. Besides the trivial homogeneous structure $S = 0$ associated to the description of one such space as a symmetric space, other structures can be obtained associated to other descriptions as a homogeneous space and, in particular, to its description as a solvable Lie group given by an Iwasawa decomposition (§ 2.2).

We also give a construction of homogeneous Sasakian structures on the bundle space of a principal line bundle over a Hermitian symmetric space of non-compact type, endowed with a connection 1-form that is the contact form of a Sasakian structure on the total space (Proposition 2.5).

The complex hyperbolic space $\mathbb{C}\mathrm{H}(n) = SU(n,1)/S(U(n)\times U(1))$ with the Bergman metric is an irreducible Hermitian symmetric space of non-compact type, and, up to homotheties, is the simply-connected complete complex space form of negative curvature. It has been characterized in [12] in terms of the existence of certain type of homogeneous Kähler structure on it, and in [7] a Lie-theoretical description of its homogeneous structure of linear type is found. In Section 3 we study the homogeneous Kähler structures on $\mathbb{C}\mathrm{H}(n)$ from other point of view, which in particular provide an infinite number of descriptions of $\mathbb{C}\mathrm{H}(n)$ as non-isomorphic solvable Lie groups. Moreover, we consider the principal line bundle over $\mathbb{C}\mathrm{H}(n)$ with its Sasakian structure given in a natural way from a connection form on the bundle, and we obtain the families of homogeneous Sasakian structures on its bundle space following our previous general construction. Summarizing, we get:

(a) All the homogeneous Kähler structures on $\mathbb{C}\mathrm{H}(n) \equiv AN$. They are given in terms of some 1-forms related by a system of differential equations on the solvable Lie group AN (Theorem 3.1).

(b) The explicit description of a multi-parametric family of homogeneous Kähler structures on $\mathbb{C}\mathrm{H}(n)$, given by using the generators of $\mathfrak{a}+\mathfrak{n}$ (Proposition 3.6), and the corresponding subgroups of the full isometry group $SU(n,1)$ of AN (Theorem 3.7).

(c) The explicit description of a one-parametric family of homogeneous Sasakian structures on the bundle space of the line bundle $\bar{M} \to \mathbb{C}\mathrm{H}(n)$, given in terms of the horizontal lifts of the generators of $\mathfrak{a}+\mathfrak{n}$ and the fundamental vector field ξ on $\bar{M}$ (Proposition 3.9), and their associated reductive decompositions (Propositions 3.11 and 3.12). One of them describes $\bar{M}$ as the complete simply connected φ-symmetric Sasakian space $\widetilde{SU}(n,1)/SU(n)$, which is also a Sasakian space form.

On the other hand, complex hyperbolic space was the first target spacetime where Nishino's [21] alternative (i.e., neither necessarily hyper-Kähler nor quaternion-Kähler) $N=(4,0)$ superstring theory proved to work. This model has some interesting features, among them, not to have (which is a trait common to heterotic σ-models) the incompatibility between the torsion tensor and quaternion-Kähler manifolds found by de Wit and van Nieuwenhuizen [27]. Another peculiarity is that in this case, one of the two scalars of the relevant global multiplet is promoted to coordinates on $\mathbb{C}\mathrm{H}(n)$, while the other plays the role of a tangent vector under the holonomy group $S(U(n)\times U(1))$.

2 Homogeneous Riemannian Structures

Ambrose and Singer [2] proved that a connected, simply connected and complete Riemannian manifold is homogeneous if and only there exists a tensor field S of type $(1,2)$ on M such that the connection $\widetilde{\nabla} = \nabla - S$ satisfies the Eqs.

$$\widetilde{\nabla} g = 0, \quad \widetilde{\nabla} R = 0, \quad \widetilde{\nabla} S = 0\,, \tag{2.1}$$

where ∇ is the Levi-Civita connection of g and R its curvature tensor field, for which we adopt the conventions $R_{XY}Z = \nabla_{[X,Y]}Z - \nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z$, $R_{XYZW} = g(R_{XY}Z, W)$. Such a tensor field S is called a homogeneous Riemannian structure ([26]). We also denote by S the associated tensor field of type $(0,3)$ on M defined by $S_{XYZ} = g(S_X Y, Z)$.

2.1 Homogeneous Kähler structures

An almost Hermitian manifold (M, g, J) is said to be a homogeneous almost Hermitian manifold if there exists a Lie group of holomorphic isometries which acts transitively and effectively on M. Sekigawa proved the following

Theorem 2.1. ([24]) *A connected, simply connected and complete almost Hermitian manifold (M, g, J) is homogeneous if and only if there is a tensor field S of type $(1,2)$ on M which satisfies Eqs. (2.1) and $\widetilde{\nabla} J = 0$.*

A tensor S satisfying the Eqs. (2.1) and $\widetilde{\nabla} J = 0$ is called a homogeneous almost Hermitian structure. The almost Hermitian manifold (M, g, J) is Kähler if and only if J is integrable and the fundamental 2-form Ω on M, given by $\Omega(X,Y) = g(X, JY)$, is closed, or equivalently $\nabla J = 0$. In this case, a homogeneous almost Hermitian structure is also called a homogeneous Kähler structure, and we have

Proposition 2.2. *A homogeneous Riemannian structure S on a Kähler manifold (M,g,J) is a homogeneous Kähler structure if and only if $S \cdot J = 0$, or equivalently $S_{XYZ} = S_{X\,JY\,JZ}$ for all the vector fields X, Y, Z on M.*

Corollary 2.3. *A connected, simply connected and complete Kähler manifold (M, g, J) is a homogeneous Kähler manifold if and only if there exists a homogeneous Kähler structure on M.*

If $(M = G/H, g)$ is a homogeneous Riemannian manifold, where G is a connected Lie group acting transitively and effectively on M as a group of isometries and H is the isotropy group at a point $o \in M$, then the Lie algebra $\mathfrak{g}$ of G may be decomposed into a vector space direct sum $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m}$ is an $\mathrm{Ad}(H)$-invariant subspace of $\mathfrak{g}$. If G is connected and M is simply connected then H is connected, and the condition $\mathrm{Ad}(H)\,\mathfrak{m} \subset \mathfrak{m}$ is equivalent to $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. The vector space $\mathfrak{m}$ is identified with $T_o(M)$ by the isomorphism $X \in \mathfrak{m} \to X^*_o \in T_o(M)$, where X^* is the Killing vector field on M generated by the one-parameter subgroup $\{\exp tX\}$ of G acting on M. If $X \in \mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, we write $X = X_{\mathfrak{h}} + X_{\mathfrak{m}}$, $(X_{\mathfrak{h}} \in \mathfrak{h}, X_{\mathfrak{m}} \in \mathfrak{m})$. The canonical connection $\widetilde{\nabla}$ of $M = G/H$ (with regard to the reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$) is determined by

$$(\widetilde{\nabla}_{X^*} Y^*)_o = [X^*, Y^*]_o = -[X,Y]^*_o = -([X,Y]_{\mathfrak{m}})^*_o\,, \qquad X, Y \in \mathfrak{m}, \tag{2.2}$$

and $S = \nabla - \widetilde{\nabla}$ satisfies the Ambrose-Singer Eqs. (2.1), and it is the homogeneous Riemannian structure associated to the reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. If (M, g) is endowed with a compatible almost complex structure J invariant by G (so that $(M = G/H, g, J)$ is a homogeneous almost Hermitian manifold), restricting J to $T_o(M) \equiv \mathfrak{m}$, we obtain a linear endomorphism J_o of $\mathfrak{m}$ such that $J_o^2 = -1$, and $J_o \operatorname{ad}_{\mathfrak{h}} = \operatorname{ad}_{\mathfrak{h}} J_o$. Moreover, J is integrable if and only if

$$[J_oX, J_oY]_{\mathfrak{m}} - [X, Y]_{\mathfrak{m}} - J_o[X, J_oY]_{\mathfrak{m}} - J_o[J_oX, Y]_{\mathfrak{m}} = 0$$

for all $X, Y \in \mathfrak{m}$ ([18], Ch. 10, Prop. 6.5).

Conversely, suppose that (M, g) is a connected, simply connected and complete Riemannian manifold, and let S be a homogeneous Riemannian structure on (M, g). We put $\mathfrak{m} = T_o(M)$, where $o \in M$. If $\widetilde{R}$ is the curvature tensor of the connection $\widetilde{\nabla} = \nabla - S$, the holonomy algebra $\tilde{\mathfrak{h}}$ of $\widetilde{\nabla}$ is the Lie subalgebra of the Lie algebra of antisymmetric endomorphisms $\mathfrak{so}(\mathfrak{m})$ of $(\mathfrak{m}, g_o)$ generated by the operators $\widetilde{R}_{XY}$, where $X, Y \in \mathfrak{m}$. A Lie bracket is defined (Nomizu [20]) in the vector space direct sum $\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} + \mathfrak{m}$ by

$$\begin{aligned} &[U, V] = UV - VU, && U, V \in \tilde{\mathfrak{h}}, \\ &[U, X] = U(X), && U \in \tilde{\mathfrak{h}},\ X \in \mathfrak{m}, \\ &[X, Y] = \widetilde{R}_{XY} + S_XY - S_YX, && X, Y \in \mathfrak{m}\,, \end{aligned} \tag{2.3}$$

and $\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} + \mathfrak{m}$ is the reductive decomposition corresponding to the homogeneous Riemannian structure S. Let $\widetilde{G}$ be the connected simply connected Lie group whose Lie algebra is $\tilde{\mathfrak{g}}$ and $\widetilde{H}$ the connected Lie subgroup of $\widetilde{G}$ whose Lie algebra is $\tilde{\mathfrak{h}}$. Then $\widetilde{G}$ acts transitively on M as a group of isometries and M is diffeomorphic to $\widetilde{G}/\widetilde{H}$. If Γ is the set of the elements of $\widetilde{G}$ which act trivially on M, then Γ is a discrete normal subgroup of $\widetilde{G}$, and the Lie group $G = \widetilde{G}/\Gamma$ acts transitively and effectively on M as a group of isometries, with isotropy group $H = \widetilde{H}/\Gamma$. Then M is diffeomorphic to G/H. Now, if J is a compatible almost complex structure on (M, g) and S is a homogeneous almost Hermitian structure, then the holonomy algebra $\tilde{\mathfrak{h}}$ is a subalgebra of the Lie algebra $\mathfrak{u}(\mathfrak{m}) = \{\, A \in \mathfrak{so}(\mathfrak{m}) : A \cdot J = 0 \,\}$ of the unitary group, and $M \approx \widetilde{G}/\widetilde{H} \approx G/H$ is a homogeneous almost Hermitian manifold.

2.2 Hermitian symmetric spaces of non-compact type

Suppose that $(M = G/K, g, J)$ is a connected Hermitian symmetric space of non-compact type, where $G = I_0(M)$ is the identity component of the group of (holomorphic) isometries and K is a maximal compact subgroup of G. Then M is simply connected and the Hermitian structure is Kähler. We consider a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of G, and the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$, where $\mathfrak{k}$ is the Lie algebra of K, $\mathfrak{a} \subset \mathfrak{p}$ is a maximal $\mathbb{R}$-diagonalizable subalgebra of $\mathfrak{g}$, and $\mathfrak{n}$ is a nilpotent subalgebra. Let A and N be the connected abelian and nilpotent Lie subgroups of G whose Lie algebras are $\mathfrak{a}$ and $\mathfrak{n}$, respectively. The solvable Lie group AN acts simply transitively on M, so M is isometric to AN equipped with the left-invariant Riemannian metric defined by the scalar product $\langle\ ,\ \rangle$, induced on $\mathfrak{a} + \mathfrak{n} \cong \mathfrak{g}\,/\,\mathfrak{k} \cong \mathfrak{p}$ by a positive multiple of $B_{|\,\mathfrak{p} \times \mathfrak{p}}$, where B is the Killing form of $\mathfrak{g}$.

Now, let $\hat{G}$ be a connected closed Lie subgroup of G which acts transitively on M. The isotropy group of this action at $o = K \in M$ is $H = \hat{G} \cap K$. Then $M = G/K$ has also the description $M \equiv \hat{G}/H$, and $o \equiv H \in \hat{G}/H$. Let $\hat{\mathfrak{g}} = \mathfrak{h} + \mathfrak{m}$ be a reductive decomposition of the Lie algebra $\hat{\mathfrak{g}}$ of $\hat{G}$ corresponding to $M \equiv \hat{G}/H$.

We have the isomorphisms of vector spaces

$$\mathfrak{p} \cong \mathfrak{g} \,/\, \mathfrak{k} \cong \hat{\mathfrak{g}}/\,\mathfrak{h} \cong \mathfrak{m} \cong T_o(M) \cong \mathfrak{a} + \mathfrak{n},$$

with

$$\xi\colon\ \mathfrak{p} \xrightarrow{\cong} \mathfrak{m}, \quad \mu\colon\ \mathfrak{m} \xrightarrow{\cong} T_o(M), \quad \zeta\colon T_o(M) \xrightarrow{\cong} \mathfrak{a} + \mathfrak{n},$$

given by

$$\xi^{-1}(Z) = Z_{\mathfrak{p}}, \quad \mu(Z) = Z^*_o, \quad \zeta^{-1}(X) = X^*_o, \qquad Z \in \mathfrak{m},\ X \in \mathfrak{a} + \mathfrak{n}\,.$$

For each $X \in \mathfrak{g}$, we have $(X_{\mathfrak{k}})^*_o = 0$ and $(\nabla(X_{\mathfrak{p}})^*)_o = 0$, and since the Levi-Civita connection ∇ has no torsion, for each $X, Y \in \mathfrak{g}$, we have

$$(\nabla_{X^*} Y^*)_o = (\nabla_{(X_{\mathfrak{p}})^*}(Y_{\mathfrak{k}})^*)_o = [(X_{\mathfrak{p}})^*, (Y_{\mathfrak{k}})^*]_o = -[X_{\mathfrak{p}}, Y_{\mathfrak{k}}]^*_o. \tag{2.4}$$

The reductive decomposition $\hat{\mathfrak{g}} = \mathfrak{h} + \mathfrak{m}$ defines the homogeneous Riemannian structure $S = \nabla - \widetilde{\nabla}$, where $\widetilde{\nabla}$ is the canonical connection of $M \equiv \hat{G}/H$ with respect to $\hat{\mathfrak{g}} = \mathfrak{h} + \mathfrak{m}$, which is $\hat{G}$-invariant and uniquely determined by $(\widetilde{\nabla}_{X^*} Y^*)_o = -[X, Y]^*_o$, for $X, Y \in \mathfrak{m}$ (2.2). The tensor field S is also uniquely determined by its value at o because $M \equiv \hat{G}/H$ and S is $\hat{G}$-invariant. Since J is $\hat{G}$-invariant, from [18], Ch. 10, Prop. 2.7, it follows that $\widetilde{\nabla} J = 0$, and by Theorem 2.1, S is a homogeneous Kähler structure.

We have

$$(S_{X^*} Y^*)_o = (\nabla_{X^*} Y^*)_o + [X, Y]^*_o = \nabla_{Y^*_o} X^*, \quad X, Y \in \mathfrak{m}\,. \tag{2.5}$$

By (2.4) and (2.5), S is given by

$$S_{X^*_o} Y^*_o = [X_{\mathfrak{k}}, Y_{\mathfrak{p}}]^*_o, \qquad X, Y \in \mathfrak{m}\,.$$

Then, for each $X, Y \in \mathfrak{a} + \mathfrak{n}$, we have

$$S_{X^*_o} Y^*_o = S_{\xi(X_{\mathfrak{p}})^*_o} \xi(Y_{\mathfrak{p}})^*_o = [(\xi(X_{\mathfrak{p}}))_{\mathfrak{k}}, Y_{\mathfrak{p}}]^*_o.$$

The complex structure J on $M = G/K$ is defined by an element E_J in the center of $\mathfrak{k}$, and it defines the complex structure $J \in \mathrm{End}(\mathfrak{a} + \mathfrak{n})$ such that the following diagram is commutative, and $(\mathfrak{a} + \mathfrak{n}, \langle\, ,\, \rangle, J)$ becomes a Hermitian vector space isomorphic to $(T_o(M), g_o, J_o)$.

$$\begin{array}{ccccccc}
\mathfrak{p} & \xrightarrow{\ \xi\ } & \mathfrak{m} & \xrightarrow{\ \mu\ } & T_o(M) & \xrightarrow{\ \zeta\ } & \mathfrak{a} + \mathfrak{n} \\
{\scriptstyle \mathrm{ad}_{E_J}}\downarrow & & {\scriptstyle J_o}\downarrow & & {\scriptstyle J_o}\downarrow & & \downarrow{\scriptstyle J} \\
\mathfrak{p} & \xrightarrow{\ \xi\ } & \mathfrak{m} & \xrightarrow{\ \mu\ } & T_o(M) & \xrightarrow{\ \zeta\ } & \mathfrak{a} + \mathfrak{n}
\end{array}$$

Let A and N be the connected abelian and nilpotent Lie subgroups of G whose Lie algebras are $\mathfrak{a}$ and $\mathfrak{n}$, respectively. The solvable Lie group AN acts simply transitively on M. Then M is isometric to AN equipped with the left-invariant Riemannian metric defined by the scalar product induced on $\mathfrak{a} + \mathfrak{n} \cong \mathfrak{g} \,/\, \mathfrak{k} \cong \mathfrak{p}$ by a positive multiple of $B_{|\,\mathfrak{p} \times \mathfrak{p}}$, where B is the Killing form of $\mathfrak{g}$, so that AN equipped with the left-invariant almost complex structure defined by J is a Kähler manifold.

2.3 Homogeneous almost contact Riemannian manifolds

An almost contact structure on a $(2n+1)$-dimensional manifold $\bar{M}$ is a triple (φ, ξ, η), where φ is a tensor field of type $(1,1)$, ξ a vector field (called the characteristic vector field) and η a differential 1-form on $\bar{M}$ such that

$$\varphi^2 = -\mathrm{id} + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

Then $\varphi\xi = 0$, $\eta \circ \varphi = 0$, and φ has rank $2n$. If $\bar{g}$ is a Riemannian metric on $\bar{M}$ such that $\bar{g}(\varphi\tilde{X}, \varphi\tilde{Y}) = \bar{g}(\tilde{X}, \tilde{Y}) - \eta(\tilde{X})\eta(\tilde{Y})$ for all vector fields $\tilde{X}$ and $\tilde{Y}$ on $\bar{M}$ then $(\varphi, \xi, \eta, \bar{g})$ is said to be an almost contact metric structure on $\bar{M}$. In this case, $\bar{g}(\tilde{X}, \xi) = \eta(\tilde{X})$. The 2-form Φ on M defined by $\Phi(\tilde{X}, \tilde{Y}) = \bar{g}(\tilde{X}, \varphi\tilde{Y})$ is called the fundamental 2-form of the almost contact metric structure $(\varphi, \xi, \eta, \bar{g})$. If $d\eta(\tilde{X}, \tilde{Y}) = \tilde{X}\eta(\tilde{Y}) - \tilde{Y}\eta(\tilde{X}) - \eta([\tilde{X}, \tilde{Y}]) = 2\Phi(\tilde{X}\,\tilde{Y})$, then $(\phi, \xi, \eta, \bar{g})$ is called a contact metric (or contact Riemannian) structure; in particular, $\eta \wedge (d\eta)^n \neq 0$, that is, η is a contact form on $\bar{M}$. If

$$(D_{\tilde{X}}\varphi)\tilde{Y} = \bar{g}(\tilde{X}, \tilde{Y})\xi - \eta(\tilde{Y})\tilde{X}, \tag{2.6}$$

where D is the Levi-Civita connection of $\bar{g}$, then $(\varphi, \xi, \eta, \bar{g})$ is called a Sasakian structure, and the manifold $\bar{M}$ with such a structure is a Sasakian manifold. Sasakian manifolds can also be characterized as normal contact metric manifolds and they are in some sense odd-dimensional analogues of Kähler manifolds (see Blair [3, 4]).

If $(\varphi, \xi, \eta, \bar{g})$ is an almost contact metric structure on $\bar{M}$ and $(\bar{M} = \bar{G}/H, \bar{g})$ is a homogeneous Riemannian manifold such that φ is invariant under the action of the connected Lie group $\bar{G}$ (and hence so are ξ and η) then $(\bar{M}, \varphi, \xi, \eta, \bar{g})$ is called a homogeneous almost contact Riemannian manifold ([8, 14, 19]). Let $\bar{R}$ be the curvature tensor field of the Levi-Civita connection D of $\bar{g}$. Let S be a homogeneous Riemannian structure on $\bar{M}$, that is $\widetilde{D}\bar{g} = 0$, $\widetilde{D}\bar{R} = 0$ and $\widetilde{D}S = 0$, where $\widetilde{D} = D - S$. If S satisfies the additional condition $\widetilde{D}\varphi = 0$ (and hence $\widetilde{D}\xi = 0$ and $\widetilde{D}\eta = 0$), then S is called a homogeneous almost contact metric structure on $(\bar{M}, \varphi, \xi, \eta, \bar{g})$. From the results of Kiričenko in [17] on homogeneous Riemannian spaces with invariant tensor structure, it follows

Theorem 2.4. *A connected, simply connected and complete almost contact metric manifold $(\bar{M}, \varphi, \xi, \eta, \bar{g})$ is a homogeneous almost contact Riemannian manifold if and only if there exists a homogeneous almost contact metric structure on $\bar{M}$.*

A homogeneous almost contact metric structure on a Sasakian manifold will be also called a homogeneous Sasakian structure.

2.4 Principal 1-bundles over almost Hermitian manifolds

Let (M, g, J) be an almost Hermitian manifold and let $\bar{M}$ be the bundle space of a principal 1-bundle over M. Let η be a connection (form) on the principal bundle $\pi \colon \bar{M} \to M$, and let ξ be the fundamental vector field on $\bar{M}$ defined by the element 1 of the Lie algebra $\mathbb{R}$ of the structure group of the bundle. Then $\eta(\xi) = 1$. For each vector field X on M, we denote by X^H the horizontal lift of X with respect to η. If $\bar{X}$ is a vector field on $\bar{M}$, its vertical part is $\eta(\bar{X})\xi$. Then, for any vector fields X and Y on M, we have

$$[X^H, Y^H] = [X, Y]^H + \eta([X^H, Y^H])\xi.$$

Moreover, $[X^H, \xi] = 0$, because X^H is invariant under the action of the structural group. We define a tensor field φ of type $(1,1)$ and a Riemannian metric $\bar{g}$ on $\bar{M}$ by

$$\varphi X^H = (JX)^H, \ \ \varphi\xi = 0, \qquad \bar{g} = \pi^* g + \eta \otimes \eta, \tag{2.7}$$

where X and Y are vector fields on M. Clearly, $(\varphi, \xi, \eta, \bar{g})$ is an almost contact metric structure on $\bar{M}$, and we have $\bar{g}(X^H, Y^H) = g(X, Y) \circ \pi$, and $\bar{g}(X^H, \xi) = 0$. Let Φ be its 2-fundamental form. If Ω is the fundamental 2-form of the almost Hermitian manifold (M, g, J), then $\pi^*\Omega = \Phi$.

If ∇ and D are the the Levi-Civita connections of g and $\bar{g}$, respectively, then (Ogiue [22])

$$D_{X^H} Y^H = (\nabla_X Y)^H + \frac{1}{2}\eta([X^H, Y^H])\xi = (\nabla_X Y)^H - \frac{1}{2}\, d\eta(X^H, Y^H)\xi,$$

and $D_{X^H}\xi = D_\xi X^H = -\varphi X^H$. Now, if $2\Phi = d\eta$, Eq. (2.6) is satisfied as one can easily see by replacing $(\tilde{X}, \tilde{Y})$ by (X^H, Y^H), (X^H, ξ), and (ξ, Y^H), respectively. Then, if the almost contact metric structure $(\varphi, \xi, \eta, \bar{g})$ is a contact structure, it is also Sasakian.

Suppose now that the structural group of the principal 1-bundle $\pi\colon \bar{M} \to M$ is $\mathbb{R}$ and that the base manifold is a $2n$-dimensional connected Hermitian symmetric space of non-compact type $(M = G/K, g, J)$, so that M is isometric to the solvable Lie group AN as in § 2.2. Then M is holomorphically diffeomorphic to a bounded symmetric domain, i.e., to a simply connected open subset of $\mathbb{C}^n$ such that each point is an isolated fixed point of an involutive holomorphic diffeomorphism of itself ([15], Ch. VIII, Th. 7.1). Since $\pi\colon \bar{M} \to M$ is a principal line bundle over the paracompact manifold M, then it admits a global section ([18], Ch. I, Th. 5.7), so there exists a diffeomorphism $\bar{M} \to M \times \mathbb{R}$, and the bundle space $\bar{M}$ may be identified with $AN \times \mathbb{R}$, with π being the projection on AN. On the other hand, since the fundamental 2-form Ω associated to the Kähler structure (g, J) is closed, $\Omega = d\zeta$ for some real analytic 1-form ζ on AN. We consider the connection form $\eta = 2\pi^*\zeta + dt$ on $\bar{M}$, where t is the coordinate of $\mathbb{R}$. The vertical vector field ξ with $\eta(\xi) = 1$ can be identified with $\frac{d}{dt}$, and we consider φ and $\bar{g}$ given by (2.7). Then $2\Phi = 2\pi^*\Omega = 2\pi^* d\zeta = d\eta$, and hence $(\varphi, \xi, \eta, \bar{g})$ is a Sasakian structure on $\bar{M}$.

If $\bar{S}$ is a homogeneous almost contact metric structure on $\bar{M}$, and $\widetilde{D} = D - \bar{S}$, then $\widetilde{D}\xi = 0$, and hence $\bar{S}_{X^H}\xi = D_{X^H}\xi = -\varphi X^H$. We have

Proposition 2.5. *Let $(M = G/K, g, J)$ be a connected Hermitian symmetric space of non-compact type. Let $\pi\colon \bar{M} \to M$ be a principal line bundle with connection form η such that the almost contact metric structure $(\varphi, \xi, \eta, \bar{g})$ on $\bar{M}$ defined by (2.7) is Sasakian.*

(a) *If S is a homogeneous Kähler structure on M then the tensor field $\bar{S}$ on $\bar{M}$ defined by*

$$\bar{S}_{X^H} Y^H = (S_X Y)^H - \bar{g}(X^H, \varphi Y^H)\xi, \quad \bar{S}_{X^H}\xi = -\varphi X^H = \bar{S}_\xi X^H, \quad \bar{S}_\xi\xi = 0,$$

for all vector fields X and Y on M, is a homogeneous Sasakian structure on $\bar{M}$.

(b) *$\{S^t : t \in \mathbb{R}\}$ defined by*

$$S^t_{X^H} Y^H = -\bar{g}(X^H, \varphi Y^H)\xi, \quad S^t_{X^H}\xi = -\varphi X^H, \quad S^t_\xi X^H = -t\varphi X^H, \quad S^t_\xi \xi = 0,$$

is a family of homogeneous Sasakian structures on $\bar{M}$.

Proof. (a) If $\widetilde{D} = D - \bar{S}$, then since $\bar{S}_{X^H Y^H Z^H} = \bar{g}((S_X Y)^H, Z^H) = g(S_X Y, Z) \circ \pi = -g(Y, S_X Z) \circ \pi = -\bar{g}(Y^H, (S_X Z)^H) = -\bar{S}_{X^H Z^H Y^H}$, and $\bar{S}_{X^H Y^H \xi} = -\bar{S}_{X^H \xi Y^H}$, the condition $\widetilde{D}\bar{g} = 0$ is satisfied. On the other hand, if $\widetilde{\nabla} = \nabla - S$ we have

$$\widetilde{D}_{X^H} Y^H = (\widetilde{\nabla}_X Y)^H, \quad \widetilde{D}_{X^H} \xi = \widetilde{D}_\xi X^H = 0. \tag{2.8}$$

We can identify $M = G/K$ with the solvable Lie group AN in an Iwasawa decomposition $G = KAN$ and consider the Lie algebra $\mathfrak{a} + \mathfrak{n}$ of AN. If $\tilde{U}, \tilde{V}, \tilde{X}, \tilde{Y}, \tilde{Z}$ are horizontal lifts of elements of $\mathfrak{a} + \mathfrak{n}$ or some of them are the vertical vector field ξ, then

$$(\widetilde{D}_{\tilde{U}} \bar{R})_{\tilde{X}\tilde{Y}\tilde{Z}\tilde{V}} = -\bar{R}_{\tilde{X}\tilde{Y}\tilde{Z}\widetilde{D}_{\tilde{U}}\tilde{V}} + \bar{R}_{\tilde{X}\tilde{Y}\tilde{V}\widetilde{D}_{\tilde{U}}\tilde{Z}} - \bar{R}_{\tilde{Z}\tilde{V}\tilde{X}\widetilde{D}_{\tilde{U}}\tilde{Y}} + \bar{R}_{\tilde{Z}\tilde{V}\tilde{Y}\widetilde{D}_{\tilde{U}}\tilde{X}}, \tag{2.9}$$

since $\tilde{U}(\bar{R}_{\tilde{X}\tilde{Y}\tilde{Z}\tilde{V}}) = 0$. Now, if $X, Y, Z, V \in \mathfrak{a} + \mathfrak{n}$, then

$$\begin{aligned}
\bar{R}_{X^H Y^H Z^H V^H} = {} & \big(R_{XYZV} - 2g(X, JY)g(Z, JV) \\
& + g(X, JV)g(Y, JZ) - g(X, JZ)g(Y, JV)\big) \circ \pi, \\
\bar{R}_{X^H Y^H Z^H \xi} = {} & - \bar{g}([X, Y]^H, \varphi Z^H) \\
& + \bar{g}((\nabla_X Z)^H, \varphi Y^H) - \bar{g}((\nabla_Y Z)^H, \varphi X^H), \\
\bar{R}_{X^H \xi Z^H \xi} = {} & \bar{g}(D_{X^H}\xi, D_{Z^H}\xi).
\end{aligned} \tag{2.10}$$

By using (2.8) and (2.10), the conditions $\widetilde{\nabla} R = 0$ and $\widetilde{\nabla} J = 0$ for the homogeneous Kähler structure S on M, and the formula $\bar{R}_{\tilde{X}\tilde{Y}}\xi = \eta(\tilde{X})\tilde{Y} - \eta(\tilde{Y})\tilde{X}$ for the Sasakian manifold $(\bar{M}, \varphi, \xi, \eta, \bar{g})$ ([4], Prop. 7.3), one obtains from (2.9) that $\widetilde{D}\bar{R} = 0$. Now, $(\widetilde{D}_{U^H}\bar{S})_{X^H} Y^H = ((\widetilde{\nabla}_U S)_X Y)^H$, $(\widetilde{D}_{U^H}\bar{S})_{X^H}\xi = -((\widetilde{\nabla}_U J)X)^H$, and $\widetilde{D}_\xi S = 0$, then $\widetilde{D}S = 0$. Moreover, $(\widetilde{D}_{X^H}\varphi)Y^H = ((\widetilde{\nabla}_X J)Y)^H$ and $(\widetilde{D}_{X^H}\varphi)\xi = 0$, then $\widetilde{D}\varphi = 0$, and $\bar{S}$ is a homogeneous Sasakian structure on $\bar{M}$.

(b) If $t = 1$ the corresponding tensor S^1 coincides with $\bar{S}$ in (a) for $S = 0$. For arbitrary t, if $\widetilde{D}^t = D - S^t$ we have $\widetilde{D}^t_\xi X^H = (t-1)(JX)^H$, and we get $\widetilde{D}^t\bar{g} = 0$, $\widetilde{D}^t\bar{R} = 0$, $\widetilde{D}^t\bar{S}^t = 0$, $\widetilde{D}^t\varphi = 0$. □

3 The Complex Hyperbolic Space $\mathbb{C}\mathrm{H}(n)$

3.1 $\mathbb{C}\mathrm{H}(n)$ as a solvable Lie group

The complex hyperbolic space $\mathbb{C}\mathrm{H}(n)$, which may be identified with the unit ball in $\mathbb{C}^n$ endowed with the hyperbolic metric of constant holomorphic sectional curvature -4, may also be viewed as the irreducible Hermitian symmetric space of non-compact type $SU(n,1)/S(U(n) \times U(1))$.

The Lie algebra $\mathfrak{su}(n,1)$ of $SU(n,1)$ can be described as the subalgebra of $\mathfrak{sl}(n+1,\mathbb{C})$ of all matrices of the form

$$X = \begin{pmatrix} Z & P^T \\ \bar{P} & ic \end{pmatrix}, \tag{3.1}$$

where $Z \in \mathfrak{u}(n)$, $c \in \mathbb{R}$, and $P = (p_1, \ldots, p_n) \in \mathbb{C}^n$. The involution τ of $\mathfrak{su}(n,1)$ given by $\tau(X) = -\bar{X}^T$ defines the Cartan decomposition $\mathfrak{su}(n,1) = \mathfrak{k} + \mathfrak{p}$, where

$$\mathfrak{k} = \left\{ \begin{pmatrix} Z & 0 \\ 0 & ic \end{pmatrix} : \operatorname{tr} Z + ic = 0 \right\} \cong \mathfrak{s}(\mathfrak{u}(n) \oplus \mathfrak{u}(1)), \quad \mathfrak{p} = \left\{ \begin{pmatrix} 0 & P^T \\ \bar{P} & 0 \end{pmatrix} \right\}.$$

The element A_0 of $\mathfrak{p}$ defined by $P = (0, \ldots, 0, 1)$ generates a maximal $\mathbb{R}$-diagonalizable subalgebra $\mathfrak{a}$ of $\mathfrak{su}(n,1)$. Let f_0 be the linear functional on $\mathfrak{a}$ given by $f_0(A_0) = 1$. If $n > 1$, the set of roots of $(\mathfrak{su}(n,1), \mathfrak{a})$ is $\Sigma = \{ \pm f_0, \pm 2f_0 \}$, the set $\Pi = \{ f_0 \}$ is a system of simple roots, and the corresponding positive root system is $\Sigma^+ = \{ f_0, 2f_0 \}$. If $n = 1$, $\Sigma = \{ \pm 2f_0 \}$, and $\Pi = \Sigma^+ = \{ 2f_0 \}$.

Let E_{ij} be the matrix in $\mathfrak{gl}(n, \mathbb{C})$ such that the entry at the i-th row and the j-th column is 1 and the other entries are all zero. The root vector spaces are

$$\begin{aligned} \mathfrak{g}_{f_0} &= \langle Z_j, Z'_j : 1 \leqslant j \leqslant n-1 \rangle \text{ (if } n > 1), \quad \mathfrak{g}_{2f_0} = \langle U \rangle, \\ \mathfrak{g}_{-f_0} &= \langle W_j, W'_j : 1 \leqslant j \leqslant n-1 \rangle \text{ (if } n > 1), \quad \mathfrak{g}_{-2f_0} = \langle V \rangle, \end{aligned}$$

where

$$\begin{aligned} Z_j &= E_{jn} - E_{j,n+1} - E_{nj} - E_{n+1,j}, \quad Z'_j = i(E_{jn} - E_{j,n+1} + E_{nj} + E_{n+1,j}), \\ W_j &= E_{jn} + E_{j,n+1} - E_{nj} + E_{n+1,j}, \quad W'_j = i(E_{jn} + E_{j,n+1} + E_{nj} - E_{n+1,j}), \\ U &= i(E_{nn} - E_{n,n+1} + E_{n+1,n} - E_{n+1,n+1}), \\ V &= i(E_{nn} + E_{n,n+1} - E_{n+1,n} - E_{n+1,n+1}). \end{aligned}$$

If $n > 2$, the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ is $Z_{\mathfrak{k}}(\mathfrak{a}) = \langle C_r, F_{jk}, H_{jk} : r, j, k = 1, \ldots, n-1, j < k \rangle \cong \mathfrak{u}(n-1)$, where

$$C_r = 2iE_{rr} - iE_{nn} - iE_{n+1,n+1}, \quad F_{jk} = E_{jk} - E_{kj}, \quad H_{jk} = i(E_{jk} + E_{kj})$$

and $\mathfrak{su}(n,1) = (Z_{\mathfrak{k}}(\mathfrak{a}) + \mathfrak{a}) + \sum_{f \in \Sigma} \mathfrak{g}_f$ is the restricted-root space decomposition. We also have the Iwasawa decomposition $\mathfrak{su}(n,1) = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$, where $\mathfrak{n} = \mathfrak{g}_{f_0} + \mathfrak{g}_{2f_0} = \langle U, Z_j, Z'_j : 1 \leqslant j \leqslant n-1 \rangle$.

If $n = 2$, we put $C = C_1 = \operatorname{diag}(2i, -i, -i)$, $Z = Z_1$, $Z' = Z'_1$, and in this case C generates $Z_{\mathfrak{k}}(\mathfrak{a})$, and $\mathfrak{a} + \mathfrak{n} = \langle A_0, U, Z, Z' \rangle$. If $n = 1$, $Z_{\mathfrak{k}}(\mathfrak{a}) = 0$, we have the restricted-root space decomposition $\mathfrak{su}(1,1) = \mathfrak{a} + (\mathfrak{g}_{2f_0} + \mathfrak{g}_{-2f_0}) = \langle A_0 \rangle + \langle U, V \rangle$, and the solvable part in the Iwasawa decomposition is $\mathfrak{a} + \mathfrak{n} = \langle A_0, U \rangle$.

By using the Cartan decomposition $\mathfrak{su}(n,1) = \mathfrak{k} + \mathfrak{p}$, we express each element $X \in \mathfrak{su}(n,1)$ as the sum $X = X_{\mathfrak{k}} + X_{\mathfrak{p}}$ ($X_{\mathfrak{k}} \in \mathfrak{k}$, $X_{\mathfrak{p}} \in \mathfrak{p}$). In particular, we have

$$\begin{aligned} U_{\mathfrak{k}} &= i(E_{nn} - E_{n+1,n+1}), & U_{\mathfrak{p}} &= i(E_{n+1,n} - E_{n,n+1}), \\ (Z_j)_{\mathfrak{k}} &= E_{jn} - E_{nj}, & (Z_j)_{\mathfrak{p}} &= -(E_{n+1,j} + E_{j,n+1}), \\ (Z'_j)_{\mathfrak{k}} &= i(E_{jn} + E_{nj}), & (Z'_j)_{\mathfrak{p}} &= i(E_{n+1,j} - E_{j,n+1}). \end{aligned}$$

From the basis $\{ A_0, U, Z_j, Z'_j : 1 \leqslant j \leqslant n-1 \}$ of $\mathfrak{a} + \mathfrak{n}$ and the generators of $Z_{\mathfrak{k}}(\mathfrak{a})$ above, we get the basis $\{ C_r, F_{jk}, H_{jk}, U_{\mathfrak{k}}, (Z_r)_{\mathfrak{k}}, (Z'_r)_{\mathfrak{k}} : r, j, k = 1, \ldots, n-1, j < k \}$ of $\mathfrak{k}$, and the basis $\{ A_0, U_{\mathfrak{p}}, (Z_j)_{\mathfrak{p}}, (Z'_j)_{\mathfrak{p}} : 1 \leqslant j \leqslant n-1 \}$ of $\mathfrak{p}$. Notice that, if $n = 1$, $\mathfrak{k} = \langle U_{\mathfrak{k}} \rangle$ and $\mathfrak{p} = \langle A_0, U_{\mathfrak{p}} \rangle$, and if $n = 2$ we have $\mathfrak{k} = \langle C, U_{\mathfrak{k}}, Z_{\mathfrak{k}}, Z'_{\mathfrak{k}} \rangle$, and $\mathfrak{p} = \langle A, U_{\mathfrak{p}}, Z_{\mathfrak{p}}, Z'_{\mathfrak{p}} \rangle$. We also decompose $\mathfrak{k} = \mathfrak{k}' + \mathfrak{c}$, where $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}] = \langle C_r - U_{\mathfrak{k}}, F_{jk}, H_{jk}, (Z_r)_{\mathfrak{k}}, (Z'_r)_{\mathfrak{k}} : r, j, k = 1, \ldots, n-1, j < k \rangle \cong \mathfrak{su}(n)$, and $\mathfrak{c}$ is the center of $\mathfrak{k}$, which is generated by the element $E_J = \frac{1}{2n+1}(C_1 + \cdots + C_{n-1} + (n+1)U_{\mathfrak{k}})$ such that $\mathrm{ad}_{E_J} : \mathfrak{p} \to \mathfrak{p}$ defines the complex structure on $\mathbb{C}\mathrm{H}(n)$. By the isomorphisms $\mathfrak{p} \cong \mathfrak{su}(n,1)/\,\mathfrak{k} \cong \mathfrak{a} + \mathfrak{n}$, we obtain the complex structure J acting on $\mathfrak{a} + \mathfrak{n}$ as follows.

$$JA_0 = -U, \quad JU = A_0, \quad JZ_r = Z'_r, \quad JZ'_r = -Z_r. \tag{3.2}$$

We consider the scalar product $\langle\, ,\rangle$ on $\mathfrak{a} + \mathfrak{n}$ defined by the isomorphism $\mathfrak{a} + \mathfrak{n} \cong \mathfrak{p}$ and $\frac{1}{4(n+1)} B_{|\,\mathfrak{p} \times \mathfrak{p}}$. Then $(\mathfrak{a} + \mathfrak{n}, \langle\ ,\ \rangle, J)$ is a Hermitian vector space, and the basis $\{ A_0, U, Z_r, Z'_r : 1 \leqslant r \leqslant n-1 \}$ of $\mathfrak{a} + \mathfrak{n}$ is orthonormal. We consider the solvable factor AN (with Lie algebra $\mathfrak{a} + \mathfrak{n}$) of the Iwasawa decomposition of $SU(n,1)$ with the invariant metric g and almost complex structure J defined by $\langle\, ,\rangle$ and J, respectively.

The Lie brackets of the elements of the basis of $\mathfrak{a} + \mathfrak{n}$ are given by

$$\begin{gathered}
[A_0, U] = 2U, \quad [A_0, Z_j] = Z_j, \quad [A_0, Z'_j] = Z'_j, \quad [Z_j, Z'_r] = -\delta_{jr} 2U, \\
[U, Z_j] = [U, Z'_j] = [Z_j, Z_r] = [Z'_j, Z'_r] = 0.
\end{gathered}$$

The Levi-Civita connection ∇ is given by $2g(\nabla_X Y, Z) = g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y)$ for all $X, Y, Z \in \mathfrak{a} + \mathfrak{n}$. So, the covariant derivatives between generators of $\mathfrak{a} + \mathfrak{n}$ are given by

$$\begin{gathered}
\nabla_{A_0} A_0 = \nabla_{A_0} U = \nabla_{A_0} Z_r = \nabla_{A_0} Z'_r = 0, \\
\nabla_U A_0 = -2U, \quad \nabla_U U = 2A_0, \quad \nabla_U Z_r = Z'_r, \quad \nabla_U Z'_r = -Z_r, \\
\nabla_{Z_j} A_0 = -Z_j, \quad \nabla_{Z_j} U = Z'_j, \quad \nabla_{Z_j} Z_r = \delta_{jr} A_0, \quad \nabla_{Z_j} Z'_r = -\delta_{jr} U, \\
\nabla_{Z'_j} A_0 = -Z'_j, \quad \nabla_{Z'_j} U = -Z_j, \quad \nabla_{Z'_j} Z_r = \delta_{jr} U, \quad \nabla_{Z'_j} Z'_r = \delta_{jr} A_0.
\end{gathered} \tag{3.3}$$

The components of the curvature tensor field R are given by

$$\begin{array}{llll}
R_{A_0 U} A_0 = -4U, & R_{A_0 U} U = 4A_0, & R_{A_0 U} Z_r = 2Z'_r, & R_{A_0 U} Z'_r = -2Z_r, \\
R_{A_0 Z_j} A_0 = -Z_j, & R_{A_0 Z_j} U = Z'_j, & R_{A_0 Z_j} Z_r = \delta_{jr} A_0, & R_{A_0 Z_j} Z'_r = -\delta_{jr} U, \\
R_{A_0 Z'_j} A_0 = -Z'_j, & R_{A_0 Z'_j} U = -Z_j, & R_{A_0 Z'_j} Z_r = \delta_{jr} U, & R_{A_0 Z'_j} Z'_r = \delta_{jr} A_0, \\
R_{U Z_j} A_0 = -Z'_j, & R_{U Z_j} A_0 = -Z_j, & R_{U Z_j} Z_r = \delta_{jr} U, & R_{U Z_j} Z'_r = \delta_{jr} A_0, \\
R_{U Z'_j} A_0 = Z_j, & R_{U Z'_j} U = -Z'_j, & R_{U Z'_j} Z_r = -\delta_{jr} A_0, & R_{U Z'_j} Z'_r = \delta_{jr} U,
\end{array}$$

$$\begin{array}{lll}
R_{Z_k Z_j} A_0 = R_{Z_k Z_j} U = 0, & R_{Z_j Z'_r} A_0 = 2\delta_{jr} U, & R_{Z_j Z'_r} U = -2\delta_{jr} A_0, \\
R_{Z_k Z_j} Z_r = \delta_{jr} Z_k - \delta_{kr} Z_j, & R_{Z_k Z_j} Z'_r = \delta_{jr} Z'_k - \delta_{kr} Z'_j, & R_{Z'_k Z'_j} = R_{Z_k Z_j}, \\
R_{Z_j Z'_j} Z_r = -2(1 + \delta_{jr} Z'_r), & R_{Z_j Z'_j} Z'_r = 2(1 + \delta_{jr}) Z_r & \\
R_{Z_k Z'_j} Z_r = -\delta_{jr} Z'_k - \delta_{kr} Z'_j, & R_{Z_k Z'_j} Z_r = \delta_{jr} Z_k - \delta_{kr} Z_j, & (k \neq j).
\end{array}$$

In particular we see that the invariant metric on AN has constant holomorphic sectional curvature -4.

3.2 Homogeneous Kähler structures on $\mathbb{C}\mathrm{H}(n) \equiv AN$

We will determine the homogeneous Kähler structures on $\mathbb{C}\mathrm{H}(n) \equiv AN$ in terms of the basis of left-invariant forms $\alpha, \beta, \gamma^j, \gamma'^j$, $1 \leqslant j \leqslant n-1$, dual to A_0, U, Z_j, Z'_j. If S is a homogeneous Riemannian structure on AN and $\widetilde{\nabla} = \nabla - S$, the condition $\widetilde{\nabla} g = 0$ in (2.1) is equivalent to $S_{XYZ} + S_{XZY} = 0$ for all $X, Y, Z \in \mathfrak{a} + \mathfrak{n}$. Moreover, $\widetilde{\nabla} R = 0$ is equivalent to the condition

$$\left(\nabla_X R\right)_{Y_1Y_2Y_3Y_4} = -R_{S_XY_1Y_2Y_3Y_4} - R_{Y_1S_XY_2Y_3Y_4} - R_{Y_1Y_2S_XY_3Y_4} - R_{Y_1Y_2Y_3S_XY_4},$$

for all $Y_1, Y_1, Y_3, Y_4 \in \mathfrak{a} + \mathfrak{n}$. Replacing (Y_1, Y_2, Y_3, Y_4) by (A_0, U, A_0, Z_j), (A_0, U, A_0, Z'_j), (A_0, U, Z_k, Z_j), and (A_0, U, Z_k, Z'_j), one gets that $S_{XUZ_j} = S_{XA_0Z'_j}$, $S_{XUZ'_j} = -S_{XA_0Z_j}$, $S_{XZ_kZ'_j} = -S_{XZ'_kZ_j}$, and $S_{XZ_kZ_j} = S_{XZ'_kZ'_j}$, respectively. It is easy to see that the condition $\widetilde{\nabla} R = 0$ holds if and only if the last four Eqs. are satisfied for all $X \in \mathfrak{a} + \mathfrak{n}$. These Eqs. also show (see (3.2)) that the condition $S \cdot J = 0$ of homogeneous Kähler structure (see Proposition 2.2) is fulfilled. We put

$$\omega(X) = S_{XA_0U}, \quad \sigma^j(X) = S_{XA_0Z_j} = -S_{XUZ'_j}, \quad \tau^j(X) = S_{XA_0Z'_j} = S_{XUZ_j}, \tag{3.4}$$

$$\theta^{kj}(X) = S_{XZ_kZ'_j} = S_{XZ_jZ'_k}, \quad \psi^{kj}(X) = S_{XZ_kZ_j} = S_{XZ'_kZ'_j}. \tag{3.5}$$

We have $\theta^{kj} = \theta^{jk}$ and $\psi^{kj} = -\psi^{jk}$. Now, we must determine the conditions for the 1-forms ω, σ^j, τ^j, θ^{kj} and σ^{kj} under which the condition $\widetilde{\nabla} S = 0$ in (2.1) is satisfied.

By (3.3), (3.4) and (3.5), the connection $\widetilde{\nabla} = \nabla - S$ is given by

$$\begin{aligned}
\widetilde{\nabla}_X A_0 &= -(2\beta + \omega)(X)U - \textstyle\sum_j (\gamma^j + \sigma^j)(X)Z_j - \sum_j (\gamma'^j + \tau^j)(X)Z'_j, \\
\widetilde{\nabla}_X U &= (2\beta + \omega)(X)A_0 - \textstyle\sum_j (\gamma'^j + \tau^j)(X)Z_j + \sum_j (\gamma^j + \sigma^j)(X)Z'_j, \\
\widetilde{\nabla}_X Z_j &= (\gamma^j + \sigma^j)(X)A_0 + (\gamma'^j + \tau^j)(X)U + (\beta - \theta^j)(X)Z'_j \\
&\quad + \textstyle\sum_{k\neq j} (\psi^{kj}(X)Z_k - \theta^{kj}(X)Z'_k), \\
\widetilde{\nabla}_X Z'_j &= (\gamma'^j + \tau^j)(X)A_0 - (\gamma^j + \sigma^j)(X)U + (\theta^j - \beta)(X)Z_j \\
&\quad + \textstyle\sum_{k\neq j} (\theta^{kj}(X)Z_k - \psi^{kj}(X)Z'_k).
\end{aligned}$$

Now, replacing (V_1, V_2) in the Eq. $\big(\widetilde{\nabla}_X S\big)(W, V_1, V_2) = 0$ by (A_0, U), (A_0, Z_j), (A_0, Z'_j), (Z_k, Z_j) and (Z_k, Z'_j), respectively, we obtain that the condition $\widetilde{\nabla} S = 0$ is equivalent to the following conditions:

$$\begin{aligned}
\widetilde{\nabla}\omega &= 2\textstyle\sum_j \big((\gamma^j + \sigma^j) \otimes \tau^j - (\gamma'^j + \tau^j) \otimes \sigma^j\big), \\
\widetilde{\nabla}\sigma^j &= -(\beta + \omega + \theta^j) \otimes \tau^j + (\gamma'^j + \tau^j) \otimes (\omega + \theta^j) \\
&\quad + \textstyle\sum_{k\neq j} \big(\psi^{kj} \otimes \sigma^k - \theta^{kj} \otimes \tau^k + (\gamma'^k + \tau^k) \otimes \theta^{kj} - (\gamma^k + \sigma^k) \otimes \psi^{kj}\big), \\
\widetilde{\nabla}\tau^j &= (\beta + \omega + \theta^j) \otimes \sigma^j - (\gamma^j + \sigma^j) \otimes (\omega + \theta^j) \\
&\quad + \textstyle\sum_{k\neq j} \big(\theta^{kj} \otimes \sigma^k + \psi^{kj} \otimes \tau^k - (\gamma^k + \sigma^k) \otimes \theta^{kj} - (\gamma'^k + \tau^k) \otimes \psi^{kj}\big), \\
\widetilde{\nabla}\theta^{kj} &= (\gamma^j + \sigma^j) \otimes \tau^k + (\gamma^k + \tau^k) \otimes \tau^j - (\gamma'^j + \tau^j) \otimes \sigma^k - (\gamma'^k + \tau^k) \otimes \sigma^j \\
&\quad + \textstyle\sum_l \psi^{lk} \wedge \theta^{jl} + \sum_l \theta^{lk} \wedge \psi^{jl}, \\
\widetilde{\nabla}\psi^{kj} &= (\gamma^k + \sigma^k) \otimes \sigma^j - (\gamma^j + \sigma^j) \otimes \sigma^k - (\gamma'^k + \tau^k) \otimes \tau^j - (\gamma'^j + \tau^j) \otimes \tau^k \\
&\quad + \textstyle\sum_l \theta^{lk} \wedge \theta^{jl} - \sum \psi^{lk} \wedge \psi^{jl},
\end{aligned} \tag{3.6}$$

where $\theta^j = \theta^{jj}$. Thus, from (3.4) and (3.5), we have

Theorem 3.1. *All the homogeneous Kähler structures on* $\mathbb{C}\mathrm{H}(n) \equiv AN$ *are given by*

$$S = \omega \otimes (\alpha \wedge \beta) + \sum_{j=1}^{n-1} \left(\sigma^j \otimes (\alpha \wedge \gamma^j - \beta \wedge \gamma'^j) + \tau^j \otimes (\alpha \wedge \gamma'^j + \beta \wedge \gamma^j) + \theta^{jj} \otimes (\gamma^j \wedge \gamma'^j) \right)$$
$$+ \sum_{1 \leqslant k < j \leqslant n-1} \left(\psi^{kj} \otimes (\gamma^k \wedge \gamma^j + \gamma'^k \wedge \gamma'^j) + \theta^{kj} \otimes (\gamma^k \wedge \gamma'^j + \gamma^j \wedge \gamma'^k) \right),$$

where ω, σ^j, τ^j, θ^{kj}, ψ^{kj}, $(1 \leqslant k, j \leqslant n-1)$, *are* 1*-forms on* AN *satisfying* $\theta^{jk} = \theta^{kj}$, $\psi^{jk} = -\psi^{kj}$ *and the Eqs.* (3.6).

If $n = 2$, we put $\gamma = \gamma^1$, $\gamma' = \gamma'^1$, so that $\{\alpha, \beta, \gamma, \gamma'\}$ is the basis of left-invariant forms on $AN = \mathbb{C}\mathrm{H}(2)$ dual to $\{A_0, U, Z, Z'\}$, and we have

Corollary 3.2. *All the homogeneous Kähler structures on the complex hyperbolic plane* $\mathbb{C}\mathrm{H}(2) \equiv AN$ *are given by*

$$S = \omega \otimes (\alpha \wedge \beta) + \sigma \otimes (\alpha \wedge \gamma - \beta \wedge \gamma') + \tau \otimes (\alpha \wedge \gamma' + \beta \wedge \gamma) + \theta \otimes (\gamma \wedge \gamma'),$$

where ω, σ, τ *and* θ *are* 1*-forms on* AN *satisfying*

$$\begin{aligned} \widetilde{\nabla}\omega &= 2(\gamma + \sigma) \otimes \tau - 2(\gamma' + \tau) \otimes \sigma = \widetilde{\nabla}\theta, \\ \widetilde{\nabla}\sigma &= -(\beta + \omega + \theta) \otimes \gamma + (\gamma' + \tau) \otimes (\omega + \theta), \\ \widetilde{\nabla}\tau &= (\beta + \omega + \theta) \otimes \sigma - (\gamma + \sigma) \otimes (\omega + \theta). \end{aligned}$$

If $n = 1$, $\{\alpha, \beta\}$ is the basis of 1-invariant forms on the 2-dimensional solvable Lie group $AN = \mathbb{C}\mathrm{H}(1)$ dual to the basis $\{A_0, U\}$ of $\mathfrak{a} + \mathfrak{n}$, and we have

Corollary 3.3. *All the homogeneous Kähler structures on the complex hyperbolic line (or real hyperbolic plane)* $\mathbb{C}\mathrm{H}(1) \equiv AN$ *are given by* $S = \omega \otimes (\alpha \wedge \beta)$, *where* ω *is a* 1*-form on* AN *satisfying* $\widetilde{\nabla}\omega = 0$.

Remark 3.4. If $S = \omega \otimes (\alpha \wedge \beta)$ is a homogeneous Kähler structure on $\mathbb{C}\mathrm{H}(1)$, and $\omega = \lambda\alpha + \mu\beta$, where λ and μ are functions on $\mathbb{C}\mathrm{H}(1)$, the condition $\widetilde{\nabla}\omega = 0$ together with the structure Eq. $[A_0, U] = 2U$ gives $\lambda = \mu = 0$ or $\lambda^2 + \mu^2 = 4$, and we have that there are infinite homogeneous Kähler structures on $\mathbb{C}\mathrm{H}(1)$. However (see [26], Th. 4.4), up to isomorphism, there are only two homogeneous structures on the real hyperbolic plane: one of them is $S = 0$ ($\lambda = \mu = 0$), and the other, which is given by $S_X Y = g(X, Y)\xi_0 - g(\xi_0, Y)X$, with $\xi_0 = 2A_0$ (for $X, Y \in \mathfrak{a} + \mathfrak{n} = \langle A_0, U \rangle$), corresponds to $S = \omega \otimes (\alpha \wedge \beta)$, with $\omega = -2\beta$ ($\lambda = 0$, $\mu = -2$).

Remark 3.5. For each $n > 0$, $S = 0$ is a homogeneous Kähler structure on $\mathbb{C}\mathrm{H}(n) \equiv AN$, the corresponding canonical connection is $\widetilde{\nabla} = \nabla$, its holonomy algebra is $\mathfrak{k} \cong \mathfrak{s}(\mathfrak{u}(n) \oplus \mathfrak{u}(1))$, the associated reductive decomposition is the Cartan decomposition $\mathfrak{su}(n, 1) = \mathfrak{k} + \mathfrak{p}$, and it gives the description of $\mathbb{C}\mathrm{H}(n)$ as symmetric space $\mathbb{C}\mathrm{H}(n) = SU(n, 1)/S(U(n) \times U(1))$.

Now, our purpose is to obtain nontrivial homogeneous Kähler structures on $\mathbb{C}\mathrm{H}(n)$, $n \geqslant 2$, their associated reductive decompositions, and the corresponding descriptions as homogeneous Kähler spaces.

We will seek for solutions for which $\sigma^j = -\gamma^j$, $\tau^j = -\gamma'^j$. In this case, we have

$$\begin{aligned} \widetilde{\nabla}\gamma^j &= (\beta - \theta^j) \otimes \gamma'^j + \textstyle\sum_{k\neq j}(\psi^{kj} \otimes \gamma^k - \theta^{kj} \otimes \gamma'^k), \\ \widetilde{\nabla}\gamma'^j &= (\theta^j - \beta) \otimes \gamma^j + \textstyle\sum_{k\neq j}(\theta^{kj} \otimes \gamma^k + \psi^{kj} \otimes \gamma'^k). \end{aligned}$$

(Obviously, the last summands on the right hand-side in each one of the two Eqs. above do not appear if $n = 2$.) By the second and third Eqs. in (3.6), we must have $\omega = -2\beta$, which also satisfies the first Eqs. in (3.6), because $\widetilde{\nabla}\beta = (2\beta + \omega) \otimes \alpha - \sum_j(\gamma'^j + \tau^j) \otimes \gamma^j + \sum_j(\gamma^j + \sigma^j) \otimes \gamma'^j = 0$. If $n = 2$, by Corollary 3.2, we only have to determine θ such that $\widetilde{\nabla}\theta = 0$. If we put $\theta = a\,\alpha + b\,\beta + c\,\gamma + c'\,\gamma'$, by using also the structure Eqs. of $\mathfrak{a} + \mathfrak{n} = \langle\, A_0, U, Z, Z' \,\rangle$, we obtain that $c = c' = 0$ and a and b are constant. For $n > 2$ we put $\theta^j = \theta^{jj} = a_j\alpha + b_j\beta$, $\theta^{kj} = c_{kj}\alpha$, $\psi^{kj} = p_{kj}\alpha$, $(k \neq j)$, with $a_j, b_j, c_{kj}, p_{kj} \in \mathbb{R}$. Then, if $\sigma^j = -\gamma^j$, $\tau^j = -\gamma'^j$ and $\omega = -2\beta$, Eqs. (3.6) are satisfied if and only if one has

$$p_{kj}(b_k - b_j) = c_{kj}(b_k - b_j) = 0.$$

Consequently, we get

Proposition 3.6. *For $n > 2$, the space $\mathbb{C}\mathrm{H}(n)$ admits the multi-parametric family of homogeneous Kähler structures $S = S^{a_j,b_j,c_{kj},p_{kj}}$ given in terms of the generators of $\mathfrak{a} + \mathfrak{n}$ by the following table.*

Table I	A_0	U	Z_j	Z'_j
S_{A_0}	0	0	$a_j Z'_j + \sum_{l\neq j}(p_{jl}Z_l + c_{jl}Z'_l)$	$-a_j Z_j + \sum_{l\neq j}(p_{jl}Z'_l - c_{jl}Z_l)$
S_U	$-2U$	$2A_0$	$b_j Z'_j$	$-b_j Z_j$
S_{Z_k}	$-Z_k$	Z'_k	$\delta_{kj}A_0$	$-\delta_{kj}U$
$S_{Z'_k}$	$-Z'_k$	$-Z_k$	$\delta_{kj}U$	$\delta_{kj}A_0$

The complex hyperbolic plane $\mathbb{C}\mathrm{H}(2)$ admits the two-parametric family of homogeneous Kähler structures $S = S^{a,b}$ given in terms of the generators of $\mathfrak{a} + \mathfrak{n}$ by the following table.

Table II	A_0	U	Z	Z'
S_{A_0}	0	0	aZ'	$-aZ$
S_U	$-2U$	$2A_0$	bZ'	$-bZ$
S_Z	$-Z$	Z'	A_0	$-U$
$S_{Z'}$	$-Z'$	$-Z$	U	A_0

If $S = S^{a_j,b_j,c_{kj},p_{kj}}$, with respect to the basis $\{\, A_0, U, Z_j, Z'_j \,\}$ of $\mathfrak{a} + \mathfrak{n}$, the connection $\widetilde{\nabla} = \nabla - S$ is given by

$$\begin{aligned} \widetilde{\nabla}_{A_0} Z_j &= -a_j Z'_j - \textstyle\sum_{l\neq j}(p_{jl}Z_l + c_{jl}Z'_l), \quad \widetilde{\nabla}_U Z_j = (1 - b_j)Z'_j, \\ \widetilde{\nabla}_{A_0} Z'_j &= a_j Z_j - \sum_{l\neq j}(p_{jl}Z'_l - c_{jl}Z_l), \quad \widetilde{\nabla}_U Z'_j = (b_j - 1)Z_j, \end{aligned}$$

with the rest vanishing. Hence, the components of the curvature tensor field are $\widetilde{R}_{A_0 U} = -\widetilde{R}_{Z_k Z'_k} = 2\sum_j (1-b_j)(Z'_j \otimes \gamma^j - Z_j \otimes \gamma'^j)$, and the rest zero.

If $b_j = 1$ for all $j = 1, \ldots, n-1$, the holonomy algebra of $\widetilde{\nabla}$ is trivial and the reductive decompositions associated to the homogeneous Kähler structures given in Proposition 3.6 are given by $\widetilde{\mathfrak{g}}^{a_j, c_{kj}, p_{kj}} = \{0\} + (\mathfrak{a} + \mathfrak{n})$ with nonvanishing brackets, by (2.3), given by

$$\begin{aligned} &[A_0, U] = 2U, \quad [A_0, Z_j] = Z_j + a_j Z'_j + \textstyle\sum_{l \neq j} (p_{jl} Z_l + c_{jl} Z'_l), \\ &[A_0, Z'_j] = -a_j Z_j + Z'_j + \textstyle\sum_{l \neq j} (p_{jl} Z'_l + c_{jl} Z_l), \quad [Z_j, Z'_j] = -2U. \end{aligned} \tag{3.7}$$

On the other hand, the element $\hat{A}_0 = \lambda_1 C_1 + \cdots + \lambda_{n-1} C_{n-1} + \sum_{j<l} (c_{jl} H_{jl} - p_{jl} F_{jl}) + A_0$ of $\mathfrak{su}(n,1)$ generates a subspace $\mathfrak{e}^{\lambda_j, c_{kj}, p_{kj}}$ of $Z_{\mathfrak{k}}(\mathfrak{a}) + \mathfrak{a}$, and the structure Eqs. of the Lie subalgebra $\mathfrak{e}^{\lambda_j, c_{kj}, p_{kj}} + \mathfrak{n}$ of $\mathfrak{su}(n,1)$ are

$$\begin{aligned} &[\hat{A}_0, U] = 2U, \quad [\hat{A}_0, Z_j] = Z_j + (3\lambda_j + \textstyle\sum_{l \neq j} \lambda_l) Z'_j + \sum_{l \neq j} (p_{jl} Z_l + c_{jl} Z'_l), \\ &[\hat{A}_0, Z'_j] = -(3\lambda_j + \textstyle\sum_{l \neq j} \lambda_l) Z_j + Z'_j + \sum_{l \neq j} (p_{jl} Z'_l + c_{jl} Z_l), \quad [Z_j, Z'_j] = -2U, \end{aligned} \tag{3.8}$$

with the rest vanishing. From (3.7) and (3.8), it follows that $\widetilde{\mathfrak{g}}^{a_j, c_{kj}, p_{kj}}$ is isomorphic to $\mathfrak{e}^{\lambda_j, c_{kj}, p_{kj}} + \mathfrak{n}$.

Now, for the structure $S = S^{a_j, b_j, c_{kj}, p_{kj}}$ in Table I, suppose that $b_j \neq 1$ for some $j = 1, \ldots, n-1$. Then, $\rho = \widetilde{R}_{A_0 U} = -\widetilde{R}_{Z_k Z'_k} = 2\sum_j (1-b_j)(Z'_j \otimes \gamma^j - Z_j \otimes \gamma'^j)$ generates the holonomy algebra $\widetilde{\mathfrak{h}}^{a_j, b_j, c_{kj}, p_{kj}}$ of $\widetilde{\nabla} = \nabla - S$, and the reductive decomposition associated to S is $\widetilde{\mathfrak{g}}^{a_j, b_j, c_{kj}, p_{kj}} = \widetilde{\mathfrak{h}}^{a_j, b_j, c_{kj}, p_{kj}} + (\mathfrak{a} + \mathfrak{n}) = \langle \rho, A_0, U, Z_j, Z'_j \rangle$ with structure Eqs., by (2.3), given by

$$\begin{aligned} &[\rho, A_0] = [\rho, U] = 0, \quad [\rho, Z_j] = 2(1-b_j) Z'_j, \quad [\rho, Z'_j] = 2(b_j - 1) Z_j, \\ &[A_0, U] = \rho + 2U, \quad [A_0, Z_j] = Z_j + a_j Z'_j + \textstyle\sum_{l \neq j} (p_{jl} Z_l + c_{jl} Z'_l), \\ &[A_0, Z'_j] = -a_j Z_j + Z'_j + \textstyle\sum_{l \neq j} (p_{jl} Z'_l + c_{jl} Z_l), \\ &[U, Z_j] = (b_j - 1) Z'_j, \quad [U, Z'_j] = (1-b_j) Z_j, \quad [Z_k, Z'_j] = -\delta_{kj} (\rho + 2U). \end{aligned} \tag{3.9}$$

If $\mathfrak{u} \cong \mathfrak{u}(1)$ is the subspace of $Z_{\mathfrak{k}}(\mathfrak{a})$ generated by $C = C_1 + \cdots + C_{n-1}$, it is easy to see that the Lie algebra $\widetilde{\mathfrak{g}}^{a_j, b_j, c_{kj}, p_{kj}}$ is isomorphic to the Lie subalgebra $\mathfrak{u} + \mathfrak{e}^{\lambda_j, c_{kj}, p_{kj}} + \mathfrak{n} = \langle C, \hat{A}_0, U, Z_j, Z'_j \rangle$ of $\mathfrak{su}(n,1)$. We deduce

Theorem 3.7. *Let $S = S^{a_j, b_j, c_{kj}, p_{kj}}$ be the homogeneous Kähler structure on $\mathbb{C}\mathrm{H}(n)$, $n > 2$, given by Table I, and let $\mathfrak{e}^{\lambda_j, c_{kj}, p_{kj}}$ be the subspace of $Z_{\mathfrak{k}}(\mathfrak{a}) + \mathfrak{a}$ generated by*

$$\hat{A}_0 = \textstyle\sum_j \lambda_j C_j + \sum_{1 \leqslant j < l \leqslant n-1} (c_{jl} H_{jl} - p_{jl} F_{jl}) + A_0, \qquad \left(\lambda_j = \frac{n a_j - \sum_{l \neq j} a_l}{2n+2}\right),$$

and $\mathfrak{u} = \langle C_1 + \cdots + C_{n-1} \rangle$. If $b_j = 1$ for all $j = 1, \ldots, n-1$, the corresponding group of isometries is the connected subgroup $E^{\lambda_j, c_{kj}, p_{kj}} N$ of $SU(n,1)$ whose lie algebra is $\mathfrak{e}^{\lambda_j, c_{kj}, p_{kj}} + \mathfrak{n}$. If $b_j \neq 1$ for some $j = 1, \ldots, n-1$, the corresponding group of isometries is the connected subgroup $U(1) E^{\lambda_j, c_{kj}, p_{kj}} N$ of $SU(n,1)$ whose Lie algebra is $\mathfrak{u} + \mathfrak{e}^{\lambda_j, c_{kj}, p_{kj}} + \mathfrak{n}$.

If $S^{a,b}$ is the homogeneous Kähler structure on the complex hyperbolic plane $\mathbb{C}\mathrm{H}(2)$ *given by Table II,* $\mathfrak{e}^{\lambda} = \langle \hat{A}_0 \rangle$*, where* $\hat{A}_0 = \lambda C + A_0$*,* $(\lambda = a/3)$*, and* $\mathfrak{u} = \langle C \rangle$*, then the corresponding group of isometries is* (i) *the subgroup* $E^{\lambda}N$ *of* $SU(2,1)$ *generated by the Lie subalgebra* $\mathfrak{e}^{\lambda} + \mathfrak{n}$ *of* $\mathfrak{su}(2,1)$*, if* $b = 1$*;* (ii) *the subgroup* $U(1)E^{\lambda}N$ *of* $SU(2,1)$ *generated by* $\mathfrak{u} + \mathfrak{e}^{\lambda} + \mathfrak{n}$*, if* $b \neq 1$*.*

Remark 3.8. Each structure $S^{a_j,b_j,c_{kj},p_{kj}}$, with $b_j = 1$ for all j, is also characterized by the fact that $\widetilde{\nabla} = \nabla - S^{a_j,b_j,c_{kj},p_{kj}}$ is the canonical connection for the Lie group $E^{\lambda_j,c_{kj},p_{kj}}N$, which is the connection for which every left-invariant vector field on $E^{\lambda_j,c_{kj},p_{kj}}N$ is parallel. Each one of these groups acts simply transitively on $\mathbb{C}\mathrm{H}(n)$ and it provides a description of $\mathbb{C}\mathrm{H}(n)$ as a homogeneous space. If all the parameters a_j, c_{kj}, p_{kj} are zero, then $\mathfrak{e}^{\lambda_j,c_{kj},p_{kj}} = \mathfrak{a}$, and we get the usual description as a solvable Lie group $\mathbb{C}\mathrm{H}(n) = AN$. In this case, the corresponding homogeneous structure is given by $S_X Y = \nabla_X Y$ for all $X, Y \in \mathfrak{a} + \mathfrak{n}$. If $b_j \neq 1$ for some $j = 1, \ldots, n-1$, we get the descriptions as homogeneous space $\mathbb{C}\mathrm{H}(n) = U(1)E^{\lambda_j,c_{kj},p_{kj}}N/U(1)$.

3.3 Principal line bundle over $\mathbb{C}\mathrm{H}(n)$

By (3.2), the fundamental 2-form of the Kähler structure (J, g) of $\mathbb{C}\mathrm{H}(n) \equiv AN$ is given by $\Omega = \alpha \wedge \beta - \sum_{j=1}^{n-1} \gamma^j \wedge \gamma'^j = -\frac{1}{2}\, d\beta$, where $\{\, \alpha, \beta, \gamma^j, \gamma'^j : 1 \leqslant j \leqslant n-1 \,\}$ is the basis of left-invariant 1-forms on AN dual to the basis $\{\, A_0, U, Z_j, Z'_j \,\}$ of $\mathfrak{a} + \mathfrak{n}$. We consider the principal line bundle $\pi \colon \bar{M} \to \mathbb{C}\mathrm{H}(n)$, and identify the bundle space $\bar{M}$ with $AN \times \mathbb{R}$ and π with the projection on AN. The fundamental vector field ξ is identified with $\frac{d}{dt}$, and the 1-form $\eta = dt - \pi^*\beta$ is also regarded as a connection form on the bundle. If φ and $\bar{g}$ are given by (2.7), then $(\varphi, \xi, \eta, \bar{g})$ is a Sasakian structure on $\bar{M}$.

By (a) of Proposition 2.5, each homogeneous Kähler structure $S^{a_j,b_j,c_{kj},p_{kj}}$ on $\mathbb{C}\mathrm{H}(n)$ given in Theorem 3.7 defines a homogeneous Sasakian structure $\bar{S}^{a_j,b_j,c_{kj},p_{kj}}$ on $\bar{M}$ which gives a description of $\bar{M}$ as either the connected subgroup $E^{\lambda_j,c_{kj},p_{kj}}N \times \mathbb{R}$ of $SU(n,1) \times \mathbb{R}$ (if $b_j = 1$ for all $j = 1, \ldots, n-1$), or as the homogeneous space $(U(1)E^{\lambda_j,c_{kj},p_{kj}}N \times \mathbb{R})/U(1)$.

On the other hand, from (b) of Proposition 2.5, it follows

Proposition 3.9. *The bundle space* $\bar{M}$ *of the line bundle* $\pi \colon \bar{M} \to \mathbb{C}\mathrm{H}(n)$ *admits the family of homogeneous Sasakian structures* $\{S^t : t \in \mathbb{R}\}$ *given, in terms of the horizontal lifts of the generators of* $\mathfrak{a} + \mathfrak{n}$ *and the fundamental vector field* ξ*, by the following table.*

Table III	A_0^H	U^H	Z_j^H	$Z_j'^H$	ξ
$S^t_{A_0^H}$	0	$-\xi$	0	0	U^H
$S^t_{U^H}$	ξ	0	0	0	$-A^H$
$S^t_{Z_k^H}$	0	0	0	$\delta_{kj}\xi$	$-Z_k'^H$
$S^t_{Z_k'^H}$	0	0	$-\delta_{kj}\xi$	0	Z_k^H
S^t_{ξ}	tU^H	$-tA^H$	$-tZ_j'^H$	tZ_j^H	0

Remark 3.10. For each $p \in \bar{M}$, if $c_{12}(S^t)_p$ is the map from the tangent space $T_p(\bar{M})$ to its dual given by $c_{12}(S^t)_p(\tilde{X}) = \sum_{i=1}^{2n+1} S_{e_i e_i \tilde{X}}$, where $\{e_i\}$ is an orthonormal basis

of $T_p(\bar{M})$, then $c_{12}(S^t)_p$ vanishes for every $t \in \mathbb{R}$. According to Tricerri-Vanhecke's classification of homogeneous Riemannian structures in [26], each S^t is of type $\mathcal{T}_2 \oplus \mathcal{T}_3$. Moreover, if $t=-1$, we have $S_{\tilde{X}}\tilde{Y} + S_{\tilde{Y}}\tilde{X} = 0$, then S^{-1} is of type $\mathcal{T}_3$, which means that $\bar{M}$ is a naturally reductive Riemannian space. If $t=2$, then each cyclic sum $\mathfrak{S}_{\tilde{X}\tilde{Y}\tilde{Z}} S_{\tilde{X}\tilde{Y}\tilde{Z}}$ vanishes, and hence $\bar{M}$ is of type $\mathcal{T}_2$, which may be also expressed by saying that $\bar{M}$ is a cotorsionless manifold (see [13]).

We will construct the reductive decomposition $\tilde{\mathfrak{g}}_t = \tilde{\mathfrak{h}}_t + \bar{\mathfrak{m}}$ associated to each homogeneous Sasakian structure S^t, where $\bar{\mathfrak{m}} = T_o(\bar{M})$, with $o \in \bar{M}$, is generated by $\tilde{A} = (A_0^H)_o$, $\tilde{U} = (U^H)_o$, $\tilde{Z}_j = (Z_j^H)_o$, $\tilde{Z}'_j = (Z'_j)_o^H$, $\bar{\xi} = \xi_o$, $(1 \leqslant j \leqslant n-1)$, and $\tilde{\mathfrak{h}}_t$ is the holonomy algebra of the connection $\widetilde{D}^t = D - S^t$. Each connection $\widetilde{D}^t$ is given by

Table IV	A_0^H	U^H	Z_j^H	$Z_j'^H$	ξ
$\widetilde{D}^t_{A_0^H}$	0	0	0	0	0
$\widetilde{D}^t_{U^H}$	$-2U^H$	$2A_0^H$	$Z_j'^H$	$-Z_j^H$	0
$\widetilde{D}^t_{Z_k^H}$	$-Z_k^H$	$Z_k'^H$	$\delta_{kj}A_0^H$	$-\delta_{kj}U^H$	0
$\widetilde{D}^t_{Z_k'^H}$	$-Z_k'^H$	$-Z_k^H$	$\delta_{kj}U^H$	$\delta_{kj}A_0^H$	0
$\widetilde{D}^t_{\xi}$	$(1-t)U^H$	$(t-1)A^H$	$(t-1)Z_j'^H$	$(1-t)Z_j^H$	0

Let $\widetilde{R}^t$ be the curvature of $\widetilde{D}^t$, and let $\{\bar{\alpha}, \bar{\beta}, \bar{\gamma}^j, \bar{\gamma}'^j, \bar{\eta}\}$ be the basis dual to the basis $\{\tilde{A}, \tilde{U}, \tilde{Z}_j, \tilde{Z}'_j, \bar{\xi}\}$ of $\bar{\mathfrak{m}}$. The holonomy algebra $\tilde{\mathfrak{h}}_t$ of $\widetilde{D}^t$ is generated by the curvature operators ρ_0, ρ_r, φ_r, ψ_r, σ_{jk}, τ_{jk} $(r,j,k = 1,\dots,n-1,\ j<k)$, given by

$$\begin{aligned}
\rho_0 &= \widetilde{R}^t_{\tilde{A}\tilde{U}} = 2(t-3)(\bar{\alpha}\otimes\tilde{U} - \bar{\beta}\otimes\tilde{A}) + 2(2-t)\textstyle\sum_{j=1}^{n-1}(\bar{\gamma}^j\otimes\tilde{Z}'_j - \bar{\gamma}'^j\otimes\tilde{Z}_j),\\
\rho_r &= \widetilde{R}^t_{\tilde{Z}_r\tilde{Z}'_r} = 2(2-t)(\bar{\alpha}\otimes\tilde{U} - \bar{\beta}\otimes\tilde{A}) + 2(t-3)(\bar{\gamma}^r\otimes\tilde{Z}'_r - \bar{\gamma}'^r\otimes\tilde{Z}_r)\\
&\qquad\qquad + 2(t-2)\textstyle\sum_{j\neq r}(\bar{\gamma}^j\otimes\tilde{Z}'_j - \bar{\gamma}'^j\otimes\tilde{Z}_j),\\
\varphi_r &= \widetilde{R}^t_{\tilde{A}\tilde{Z}_r} = -\widetilde{R}^t_{\tilde{U}\tilde{Z}'_r} = -\bar{\alpha}\otimes\tilde{Z}_r + \bar{\beta}\otimes\tilde{Z}'_r + \bar{\gamma}^r\otimes\tilde{A} - \bar{\gamma}'^r\otimes\tilde{U},\\
\psi_r &= \widetilde{R}^t_{\tilde{U}\tilde{Z}_r} = \widetilde{R}^t_{\tilde{A}\tilde{Z}'_r} = -\bar{\alpha}\otimes\tilde{Z}'_r - \bar{\beta}\otimes\tilde{Z}_r + \bar{\gamma}^r\otimes\tilde{U} + \bar{\gamma}'^r\otimes\tilde{A},\\
\sigma_{jk} &= \widetilde{R}^t_{\tilde{Z}_j\tilde{Z}_k} = \widetilde{R}^t_{\tilde{Z}'_j\tilde{Z}'_k} = -\bar{\gamma}^j\otimes\tilde{Z}_k - \bar{\gamma}'^j\otimes\tilde{Z}'_k + \bar{\gamma}^k\otimes\tilde{Z}_j + \bar{\gamma}'^k\otimes\tilde{Z}'_j,\\
\tau_{jk} &= \widetilde{R}^t_{\tilde{Z}_j\tilde{Z}'_k} = \widetilde{R}^t_{\tilde{Z}_k\tilde{Z}'_j} = -\bar{\gamma}^j\otimes\tilde{Z}'_k + \bar{\gamma}'^j\otimes\tilde{Z}_k - \bar{\gamma}^k\otimes\tilde{Z}'_j + \bar{\gamma}'^k\otimes\tilde{Z}_j.
\end{aligned}$$

(If $n=2$, the operators σ_{jk} and τ_{jk} do not appear, that is $\tilde{\mathfrak{h}}_t = \langle \rho_0, \rho_1, \varphi_1, \psi_1\rangle$, and if $n=1$, $\tilde{\mathfrak{h}}_t$ is generated by $\rho_0 = \widetilde{R}^t_{\tilde{A}\tilde{U}} = 2(t-3)(\bar{\alpha}\otimes\tilde{U} - \bar{\beta}\otimes\tilde{A})$.) The Lie structure of $\tilde{\mathfrak{g}}_t = \tilde{\mathfrak{h}}_t + \bar{\mathfrak{m}}$ is defined by the Eqs. (2.3). If $t \neq (2n+1)/n$, the subalgebra $\tilde{\mathfrak{h}}_t$ is isomorphic to the Lie algebra $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(n) + \mathfrak{u}(1)) \cong \mathfrak{u}(n)$ in § 3.1, via the map $h\colon \tilde{\mathfrak{h}}_t \to \mathfrak{k}$ given by $h(\rho_0) = 2U_{\mathfrak{k}}$, $h(\rho_r) = -(C_r + U_{\mathfrak{k}})$, $h(\varphi_r) = (Z_r)_{\mathfrak{k}}$, $h(\psi_r) = (Z'_r)_{\mathfrak{k}}$, $h(\sigma_{jk}) = F_{jk}$, $h(\tau_{jk}) = -H_{jk}$. If we put $\hat{\rho}_0 = \frac{1}{2}(\rho_0 - 2\bar{\xi})$, $\hat{\rho}_r = -\frac{1}{2}\rho_0 - \rho_r - \bar{\xi}$, then $\widehat{\mathfrak{su}}(n,1) = \langle \hat{\rho}_0, \hat{\rho}_r, \varphi_r, \psi_r, \sigma_{jk}, \tau_{jk}, \tilde{A}, \tilde{U}, \tilde{Z}_r, \tilde{Z}'_r : r,j,k = 1,\dots,n-1, j<k\rangle$ is an ideal of $\tilde{\mathfrak{g}}_t$, and the map h extends to a Lie algebra isomorphism $\tilde{h}\colon \widehat{\mathfrak{su}}(n,1) \to \mathfrak{su}(n,1) = \mathfrak{k} + \mathfrak{p}$, given by $\tilde{h}(\hat{\rho}_0) = U_{\mathfrak{k}}$, $\tilde{h}(\hat{\rho}_r) = C_r$, $\tilde{h}(\varphi_r) = (Z_r)_{\mathfrak{k}}$, $\tilde{h}(\psi_r) = (Z'_r)_{\mathfrak{k}}$, $\tilde{h}(\sigma_{jk}) = F_{jk}$, $\tilde{h}(\tau_{jk}) = -H_{jk}$,

$\tilde{h}(\tilde{A}) = A_0$, $\tilde{h}(\tilde{U}) = U_\mathfrak{p}$, $\tilde{h}(\tilde{Z}_r) = (Z_r)_\mathfrak{p}$, $\tilde{h}(\tilde{Z}'_r) = (Z'_r)_\mathfrak{p}$. Moreover, $\tilde{\mathfrak{g}}_t$ is the semidirect product of $\widehat{\mathfrak{su}}(n,1)$ and the line generated by $\bar{\xi}$ under the homomorphism $\delta_t\colon \langle\bar{\xi}\rangle \to \mathrm{Der}(\widehat{\mathfrak{su}}(n,1))$, given by $\delta_t(\bar{\xi})(\tilde{A}) = (t-1)\tilde{U}$, $\delta_t(\bar{\xi})(\tilde{U}) = (1-t)\tilde{A}$, $\delta_t(\bar{\xi})(\tilde{Z}_r) = (1-t)\tilde{Z}'_r$, $\delta_t(\bar{\xi})(\tilde{Z}'_r) = (t-1)\tilde{Z}_r$, and $\delta_t(\bar{\xi})(\langle\,\hat{\rho}_0, \hat{\rho}_r, \varphi_r, \psi_r, \sigma_{jk}, \tau_{jk}\,\rangle) = 0$. So, we have

Proposition 3.11. *The reductive decomposition associated to the homogeneous Sasakian structure S^t, $t \neq (2n+1)/n$, on the total space of the line bundle $\bar{M} \to \mathbb{C}\mathrm{H}(n)$ is $\tilde{\mathfrak{g}}_t = \tilde{\mathfrak{h}}_t + \bar{\mathfrak{m}}$, where $\tilde{\mathfrak{h}}_t \cong \mathfrak{s}(\mathfrak{u}(n) + \mathfrak{u}(1)) \cong \mathfrak{u}(n) \subset \mathfrak{su}(n,1)$, and $\bar{\mathfrak{m}} = \mathfrak{p} + \langle\bar{\xi}\rangle = \langle\, A_0, U_\mathfrak{p}, (Z_r)_\mathfrak{p}, (Z'_r)_\mathfrak{p}, \bar{\xi} \,:\, 1 \leqslant r \leqslant n-1\,\rangle$. Moreover, $\tilde{\mathfrak{g}}_t$ is the semidirect product $\tilde{\mathfrak{g}}_t = \langle\bar{\xi}\rangle \ltimes_{\delta_t} \mathfrak{su}(n,1)$, where $\delta_t(\bar{\xi})(A_0) = (t-1)U_\mathfrak{p}$, $\delta_t(\bar{\xi})(U_\mathfrak{p}) = (1-t)A_0$, $\delta_t(\bar{\xi})((Z_r)_\mathfrak{p}) = (1-t)(Z'_r)_\mathfrak{p}$, $\delta_t(\bar{\xi})((Z'_r)_\mathfrak{p}) = (t-1)(Z_r)_\mathfrak{p}$, and $\delta_t(\bar{\xi})(\tilde{\mathfrak{h}}_t) = 0$.*

If $n \geqslant 2$ and $t = (2n+1)/n$, then it is easy to see that $\rho_0 = \rho_1 + \cdots + \rho_{n-1}$, and we put $\tilde{\rho}_r = \frac{1}{2}(\rho_0 + \rho_r)$, $1 \leqslant r \leqslant n-1$. In this case, $\tilde{\mathfrak{g}}_{\frac{2n+1}{n}} = \tilde{\mathfrak{h}}_{\frac{2n+1}{n}} + \bar{\mathfrak{m}}$ coincides with the reductive decomposition $\mathfrak{su}(n,1) = \mathfrak{k}' + \mathfrak{m}'$, where $\mathfrak{k}' = [\mathfrak{k},\mathfrak{k}] \cong \mathfrak{su}(n)$, and $\mathfrak{m}' = \mathfrak{p} + \langle\mathfrak{c}\rangle$, being $\mathfrak{c}$ the center of $\mathfrak{k}$, which is generated by the element E_J such that $\mathrm{ad}_{E_J}\colon \mathfrak{p} \to \mathfrak{p}$ defines the complex structure of $\mathbb{C}\mathrm{H}(n)$. In fact, we have the isomorphism $f\colon \tilde{\mathfrak{g}}_{\frac{2n+1}{n}} \to \mathfrak{su}(n,1)$ given by $f(\tilde{\rho}_r) = \frac{1}{2}(U_\mathfrak{k} - C_r)$, $f(\varphi_r) = (Z_r)_\mathfrak{k}$, $f(\psi_r) = (Z'_r)_\mathfrak{k}$, $f(\sigma_{jk}) = F_{jk}$, $f(\tau_{jk}) = -H_{jk}$, $f(\tilde{A}) = A_0$, $f(\tilde{U}) = U_\mathfrak{p}$, $f(\tilde{Z}_r) = (Z_r)_\mathfrak{p}$, $f(\tilde{Z}'_r) = (Z'_r)_\mathfrak{p}$, and $f(\bar{\xi}) = -\frac{n+1}{n}E_J = -\frac{1}{2n}(C_1 + \cdots + C_{n-1} + (n+1)U_\mathfrak{k})$, and, in particular, $f(\tilde{\mathfrak{h}}_{\frac{2n+1}{n}}) = \mathfrak{k}'$ and $f(\bar{\mathfrak{m}}) = \mathfrak{m}'$. If $n = 1$ and $t = 3$, then $\rho_0 = 0$. In this case, $\tilde{\mathfrak{h}}_3 = 0$, $\mathfrak{k}' = [\mathfrak{k},\mathfrak{k}] = 0$, $\mathfrak{c} = \langle E_J\rangle$, $E_J = \frac{1}{2}U_\mathfrak{k}$, $\tilde{\mathfrak{g}}_3 = \{0\} + \bar{\mathfrak{m}}$ is the reductive decomposition $\mathfrak{su}(1,1) = \{0\} + \mathfrak{m}'$, where $\bar{\mathfrak{m}} = \langle\tilde{A}, \tilde{U}, \bar{\xi}\rangle$, $\mathfrak{m}' = \langle A_0, U_\mathfrak{p}, U_\mathfrak{k}\rangle$, and $f\colon \tilde{\mathfrak{g}}_3 \to \mathfrak{su}(1,1)$ such that $f(\tilde{A}) = A_0$, $f(\tilde{U}) = U_\mathfrak{p}$, $f(\bar{\xi}) = -U_\mathfrak{k}$. Hence, we have obtained

Proposition 3.12. *The reductive decomposition associated to the homogeneous Sasakian structure S^t, with $t = (2n+1)/n$, on the total space of the line bundle $\bar{M} \to \mathbb{C}\mathrm{H}(n)$ is $\mathfrak{su}(n,1) = \mathfrak{k}' + \mathfrak{m}'$, where $\mathfrak{k}' = [\mathfrak{k},\mathfrak{k}] \cong \mathfrak{su}(n)$, and $\mathfrak{m}' = \mathfrak{p} + \mathfrak{c}$, $\mathfrak{c} = \langle E_J\rangle$ being the center of $\mathfrak{k}$.*

Remark 3.13. The reductive decomposition $\mathfrak{su}(n,1) = \mathfrak{k}' + \mathfrak{m}'$ associated to the homogeneous Sasakian structure S^t, with $t = \frac{2n+1}{n}$, provides the description of $\bar{M}$ as the homogeneous space $\widetilde{SU}(n,1)/K'$, where $\widetilde{SU}(n,1)$ is the universal covering of $SU(n,1)$, and $K' \cong SU(n)$ is the connected subgroup of $\widetilde{SU}(n,1)$ whose Lie algebra is $\mathfrak{k}' \cong \mathfrak{su}(n)$. (In particular, if $n = 1$, $\bar{M}$ is the universal covering space of $Sl(2,\mathbb{R})$.) These spaces appear in the classification by Jiménez and Kowalski [16] of complete simply connected φ-symmetric Sasakian manifolds, and they are also Sasakian space forms (they have constant φ-sectional curvature -7). Notice that for a Sasakian manifold, the condition of being a locally symmetric space is too strong, because in this case it is a space of constant curvature (Okumura [23]). For this reason, Takahashi [25] introduced φ-symmetric spaces in Sasakian geometry as generalizations of Sasakian space forms. They are also analogues of Hermitian symmetric spaces. A φ-symmetric space is a complete connected regular Sasakian manifold $\bar{M}$ that fibers over a Hermitian symmetric space M so that the geodesic involutions of M lift to involutive automorphisms of the Sasakian structure on $\bar{M}$. Moreover, each complete simply connected φ-symmetric space is a naturally reductive homogeneous space (Blair and Vanhecke [5]).

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