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Homogeneous Kähler manifolds of non-positive Ricci curvature

By

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Introduction.

Let M be a homogeneous Kähler manifold. It is known that the Ricci curvature of M is negative if and only if M is biholomorphic to a homogenous bounded domain ([9] or [11]). Hano and Kobayashi [6] constructed a canonical fibering of a homogeneous complex manifold with an invariant volume element. In general, we know only that the fiber is a homogeneous complex submanifold and that the base space is a homogeneous symplectic manifold. In this paper, we shall prove that if the Ricci curvature of the homogeneous Kähler manifold M is non-positive, then the fiber is flat with respect to the induced Kähler metric and the base space admits a natural complex structure so that the canonical projection is holomorphic. Moreover, we can show that the base space is biholomorphic to a homogeneous bounded domain. Thus we obtain

Main Theorem. *Every homogeneons Kähler manifold of non-positive Ricci curvature is a holomorphic fiber space over a homogeneons bounded domain and each fiber is a flat homogeneous Kähler manifold with the induced Kähler metric.*

To prove Main Theorem, we use the fact that every homogeneous Kähler manifold of non-negative Ricci curvature is a product of a flat homogeneous Kähler manifold and a compact simply connected homogeneous Kähler manifold. This is essentially proved by Cheeger and Gromoll [3] for a simply connected homogeneous Kähler manifold. We state in Appendix the proof for general case.

Throughout this paper, for a Kähler manifold M, $\mathrm{Aut}(M)$ means the group of all holomorphic isometries of M and $\mathrm{Aut}^0(M)$ denotes its identity component. We denote by $\boldsymbol{R}^n$ (resp. $\boldsymbol{C}^n$) the n-dimensional real (resp. complex) euclidean space.

§ 1. Preliminaries.

Let $M=G/K$ be a homogeneous Kähler manifold of a connected Lie group G by a closed subgroup K. Let us denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. We also denote by π the projection of G onto G/K. There

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corresponds to the invariant complex structure J of M, an endomorphism j of $\mathfrak{g}$ satisfying

$$(1.1)\qquad \begin{aligned} &j\mathfrak{k}\subset\mathfrak{k}, \qquad j^2X\equiv -X \qquad (\mathrm{mod}\,\mathfrak{k})\\ &\mathrm{Ad}\,k\circ jX\equiv j\circ\mathrm{Ad}\,kX \qquad (\mathrm{mod}\,\mathfrak{k})\\ &[jX, jY]\equiv[X, Y]+j[jX, Y]+j[X, jY] \qquad (\mathrm{mod}\,\mathfrak{k})\\ &\pi_*\circ jX_e=J\circ\pi_*X_e, \end{aligned}$$

for $X, Y\in\mathfrak{g}$ and $k\in K$, where e denotes the unit element of G.

For any $X\in\mathfrak{g}$, $\mathrm{ad}\,jX-j\circ\mathrm{ad}\,X$ leaves $\mathfrak{k}$ invariant and therefore it induces an endomorphism of $\mathfrak{g}/\mathfrak{k}$. According to Koszul [8], we define a linear form $\psi_{G/K}$ on $\mathfrak{g}$ by

$$\psi_{G/K}(X)=\mathrm{Tr}_{\mathfrak{g}/\mathfrak{k}}(\mathrm{ad}\,jX-j\circ\mathrm{ad}\,X) \qquad \text{for} \quad X\in\mathfrak{g}.$$

We call $\psi_{G/K}$ the Koszul form of G/K. We then have ([8])

$$(1.2)\qquad \begin{aligned} &\psi_{G/K}([jX, jY])=\psi_{G/K}([X, Y])\\ &\psi_{G/K}(\mathrm{Ad}\,kX)=\psi_{G/K}(X) \end{aligned}$$

where $X, Y\in\mathfrak{g}$ and $k\in K$. Moreover let us denote by R the Ricci curvature form of M. Then we have

$$(1.3)\qquad \psi_{G/K}([X, Y])=-2R(\pi_*X_e, \pi_*Y_e) \qquad \text{for} \quad X, Y\in\mathfrak{g}.$$

In particular, $\psi_{G/K}([jX, X])\geqq 0$ for any $X\in\mathfrak{g}$ if and only if M is of non-positive Ricci curvature.

A real vector space W endowed with a complex structure j and an skew-symmetric bilinear form Ω is called *symplectic* if the following conditions are satisfied:

$$\Omega(jw, jw')=\Omega(w, w') \qquad \text{for} \quad w, w'\in W,$$

$$\Omega(jw, w)>0 \qquad \text{if} \quad w\neq 0.$$

Let $Sp(W)$ denote the group of all linear transformations f of W satisfying $\Omega(fw, fw')=\Omega(w, w')$ $(w, w'\in W)$ and denote by $K(W)$ the subgroup defined by $K(W)=\{f\in Sp(W)\,;\,j\circ f=f\circ j\}$. The Lie algebra $\mathfrak{sp}(W)$ of $Sp(W)$ consists of all linear endomorphisms f of W satisfying $\Omega(fw, w')+\Omega(w, fw')=0$ and the Lie algebra $\mathfrak{k}(W)$ of $K(W)$ consists of all $f\in\mathfrak{sp}(W)$ such that $f\circ j=j\circ f$. The homogeneous space $Sp(W)/K(W)$ admits an $Sp(W)$-invariant complex structure which corresponds to the endomorphism I of $\mathfrak{sp}(W)$ given by

$$I(f)=\frac{1}{2}(j\circ f-f\circ j) \qquad \text{for} \quad f\in\mathfrak{sp}(W).$$

With respect to this complex structure, $Sp(W)/K(W)$ is biholomorphic to a symmetric bounded domain (cf. [7]).

Lemma 1.1. *Let $p, q\in\mathfrak{sp}(W)$. Assume that*

$$p\circ j-j\circ p-q-j\circ q\circ j=0.$$

Then $\mathrm{Tr}_W j\circ[p, q]\leqq 0$ *and the equality holds if and only if* $p, q\in\mathfrak{k}(W)$.

Proof. From the condition, we have

$$\begin{aligned}\mathrm{Tr}_W j\circ[p, q]&=\mathrm{Tr}_W j\circ p\circ q-\mathrm{Tr}_W p\circ j\circ q\\&=-\mathrm{Tr}_W q^2-\mathrm{Tr}_W (j\circ q)^2\\&=-\frac{1}{2}\mathrm{Tr}_W (q\circ j-j\circ q)^2.\end{aligned}$$

We set $s=q\circ j-j\circ q$. Let B denote the positive definite symmetric bilinear form on W given by $B(w, w')=\Omega(jw, w')$. We then have $B(sw, w')=B(w, sw')$. Therefore $\mathrm{Tr}_W s^2\geqq 0$ and the equality holds if and only if $s=0$. This implies $q\in\mathfrak{k}(W)$ and from the condition, $p\in\mathfrak{k}(W)$. q. e. d.

Cheeger and Gromoll [3] showed that every connected complete riemannian manifold M of non-negative Ricci curvature is isometric to $\boldsymbol{R}^m\times M'$, where M' does not contain any line. Moreover if M is homogeneous, then M' is compact. Clearly, if M is simply connected, then in the de Rham decomposition of M, $\boldsymbol{R}^m$ is the flat factor and M' coincides with the product of the irreducible non-flat factors. Therefore for a simply connected homogeneous Kähler manifold, we already know the following

Theorem 1.2. *Every homogeneous Kähler manifold of non-negative Ricci curvature is holomorphically isometric to a product of a flat homogeneous Kähler manifold and a compact simply connected homogeneous Kähler manifold.*

We can show this theorem for general case by a simple observation about the action of the fundamental group (see, Appendix).

As an immediate consequence of Theorem 1.2, we have

Corollary 1.3.*) *Every homogeneous Kähler manifold of vanishing Ricci curvature is flat.*

§ 2. Hano-Kobayashi fiberings.

Let $M=G/K$ be the homogeneous Kähler manifold and let R be the Ricci form of M. Hano and Kobayashi constructed in [6] a fibering of a homogeneous complex manifold with an invariant volume element. Applying their result to our case, there exists a unique closed subgroup L of G having the following properties:

(a) L contains K and L/K is connected.

Consider G/K as a fiber bundle over G/L with fiber L/K. Then

*) More generally, Alekseevskii and Kimel'fel'd [1] stated that every homogeneous riemannian manifold of vanishing Ricci tensor is flat.

(b) The restriction of R to each fiber is idetically zero.

(c) There exists a non-degenerate anti-symmetric bilinear form σ on G/L such that $\Phi^*\sigma=R$, where Φ denotes the projection of G/K onto G/L.

Let $\mathfrak{l}$ be the Lie algebra of L. From the properties above, it follows

$$\mathfrak{l}=\{X\in\mathfrak{g}\,;\,R(\pi_*X_e,\,\pi_*Y_e)=0 \text{ for any } Y\in\mathfrak{g}\}. \tag{2.1}$$

In particular, $\mathfrak{l}$ is j-invariant and hence L/K is a complex submanifold. But in general, G/L may not admit a G-invariant complex structure. The notations being as above, we shall prove the following

Proposition 2.1. *Assume that the Ricci curvature of the homogeneous Kähler manifold G/K is non-positive. Then in the Hano-Kobayashi fibering of G/K, we have*

(1) *The fiber L/K is a flat homogeneous Kähler manifold with respect to the induced Kähler metric.*

(2) *The base space G/L admits a G-invariant complex structure such that the projection $\Phi: G/K\to G/L$ is holomorphic. With respect to this complex structure, $-\sigma$ gives a G-invariant Kähler form on G/L.*

Let us define a closed subgroup $\hat{L}$ of G by

$$\hat{L}=\{g\in G\,;\,\psi_{G/K}(\mathrm{Ad}\,gX)=\psi_{G/K}(X) \text{ for any } X\in\mathfrak{g}\}.$$

From (1.2), (1.3) and (2.1), $\hat{L}$ contains K and the Lie algebra of $\hat{L}$ coincides with $\mathfrak{l}$.

Lemm 2.2. *$\hat{L}$ contains L.*

Proof. Let L^0 be the identity component of L. Then $L^0\subset\hat{L}$. Since L/K is connected, L^0 acts transitively on L/K. Now our assertion follows from

$$L/K=L^0/L^0\cap K\subset\hat{L}\cap L/K\subset L/K. \qquad \text{q. e. d.}$$

We now assume that the Ricci curvature of G/K is non-positive. Let us set $W=\mathfrak{g}/\mathfrak{l}$. The endomorphism j induces a complex structure of W in a natural manner which will be denoted by the same letter j and the form $\psi_{G/K}([X,\,Y])$ $(X,\,Y\in\mathfrak{g})$ gives a non-degenerate anti-symmetric bilinear form ω on W. Then $(W,\,j,\,\omega)$ is a symplectic space. For any $g\in\hat{L}$, $\mathrm{Ad}\,g$ leaves $\mathfrak{l}$ invariant and therefore induces a linear transformation $\tau(g)$ of W. From the definition of $\hat{L}$, the correspondence: $g\to\tau(g)$ is a homomorphism of $\hat{L}$ into $Sp(W)$. Clearly $\tau(K)\subset K(W)$. Since $L\subset\hat{L}$ by Lemma 2.2, we get a mapping η of L/K to $Sp(W)/K(W)$ by setting $\eta(gK)=\tau(g)K(W)$ for $gK\in L/K$.

Lemma 2.3. *The mapping η is holomorphic.*

Proof. From (1.1), we have for any $X\in\mathfrak{l}$,

$$\tau(jX)\circ j=j\circ\tau(jX)+\tau(X)+j\circ\tau(X)\circ j, \tag{2.2}$$

where we also denote by τ the induced homomorphism of $\mathfrak{l}$ to $\mathfrak{sp}(W)$. This is equivalent to $I(\tau(X))\equiv\tau(jX)$ (mod $\mathfrak{k}(W)$), proving that η is holomorphic. q. e. d.

We are now in a position to prove Proposition 2.1. For any $X\in\mathfrak{l}$, we have

$$0=\psi_{G/K}([jX, X])=\psi_{L/K}([jX, X])+\mathrm{Tr}_W\,\tau(j[jX, X])-\mathrm{Tr}_W\,j\circ\tau([jX, X])$$
$$=\psi_{L/K}([jX, X])-\mathrm{Tr}_W\,j\circ[\tau(jX), \tau(X)].$$

From (2.2) and Lemma 1.1, we have $\psi_{L/K}([jX, X])\leqq 0$. Therefore the Ricci curvature of L/K is non-negative. Hence by Theorem 1.2, L/K is biholomorphic to a product of a compact simply connected homogeneous Kähler manifold and a flat homogeneous Kähler manifold. Therefore $\eta(L/K)$ is a single point because η is holomorphic and $Sp(W)/K(W)$ is biholomorphic to a homogeneous bounded domain. As a consequence, we have $\tau(L)\subset K(W)$. This imples

(2.3) $\qquad \mathrm{Ad}\,g\circ jX\equiv j\circ\mathrm{Ad}\,gX \quad (\mathrm{mod}\,\mathfrak{l})$ for any $X\in\mathfrak{g}$ and $g\in L$

and by Lemma 1.1 again we have $\mathrm{Tr}_W\,j\circ[\tau(jX), \tau(X)]=0$, whence

(2.4) $\qquad \psi_{L/K}([jX, X])=0 \quad$ for any $X\in\mathfrak{l}$.

The equation (2.3) means that G/L admits a G-invariant complex structure such that the projection of G/K onto G/L is holomorphic. Clearly $-\sigma$ is a G-invariant Kähler form of G/L with respect to this complex structure. By (2.4), we know that the Ricci curvature of L/K is identically zero. It follows from Corollary 1.3 that L/K is flat, completing the proof of Proposition 2.1.

§ 3. Remarks.

Let M be a homogeneous complex manifold. If the universal covering space of M is biholomorphic to a homogeneous bounded domain, then M itself is biholomorphic to a homogeneous bounded domain (see, [7] or [10]). Similarly, if M is a homogeneous Kähler manifold whose universal covering space is compact simply connected, then M itself is compact simply connected because there exists a compact semi-simple subgroup of $\mathrm{Aut}(M)$ acting on M transitively ([2]). We now prove

Proposition 3.1. *Let M be a homogeneous Kähler manifold. Assume that its universal covering space $\tilde{M}$ is biholomorphic to a product of a homogeneous bounded domain M_1 and a compact simply connected homogeneous complex manifold M_2. Then M itself is biholomorphic to $M_1\times M_2$.*

Let $G=\mathrm{Aut}^0(M)$ and let K be the isotropy subgroup of G. Let $\tilde{G}$ be the universal covering group of G and $\tilde{K}=\phi^{-1}(K)$, ϕ denoting the projection of $\tilde{G}$ onto G. Let $\tilde{K}^0$ be the identity component of $\tilde{K}$. Then $\tilde{M}=\tilde{G}/\tilde{K}^0$ and $M=\tilde{G}/\tilde{K}$. Let $\pi_1(M)$ be the fundamental group of M. Then $\pi_1(M)\cong\tilde{K}/\tilde{K}^0$ and every element of $\pi_1(M)$ represented by $k\in\tilde{K}$ acts on $\tilde{M}$ holomorphically and isometrically in the following way

(3.1) $$\tilde{M}=\tilde{G}/\tilde{K}^0\ni g\tilde{K}^0 \longrightarrow gk\tilde{K}^0\in\tilde{G}/\tilde{K}^0.$$

Since M_2 is compact, every holomorphic transformation f of $\tilde{M}$ induces an homlomorphic transformation f_1 of M_1 such that $f_1\circ p_1=p_1\circ f_1$, where p_1 denotes the projection of $\tilde{M}$ onto M_1. Then there exists a connected closed subgroup $\tilde{A}$ of $\tilde{G}$ containing $\tilde{K}^0$ such that $M_1=\tilde{G}/\tilde{A}$ and $M_2=\tilde{A}/\tilde{K}^0$. Let $k\in\tilde{K}$. We denote by θ_k the holomorphic transformation of $\tilde{M}$ defined by (3.1). Then $p_1\circ\theta_k(\tilde{A}/\tilde{K}^0)$ is a single point. This means that for any $a\in\tilde{A}$, $ak\tilde{A}=k\tilde{A}$. Therefore for any $k\in\tilde{K}$, we have

(3.2) $$k\tilde{A}k^{-1}=\tilde{A}.$$

Let $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{a}$ be the Lie algebras of $\tilde{G}$, $\tilde{K}$, and $\tilde{A}$ respectively and let j be the endomorphism of $\mathfrak{g}$ corresponding to the complex structure of G/K. We have for any $k\in\tilde{A}$ and by (1.1) for any $k\in\tilde{K}$

(3.3) $$\mathrm{Ad}\,k\circ jX\equiv j\circ\mathrm{Ad}\,kX \pmod{\mathfrak{a}} \quad \text{for all } X\in\mathfrak{g}.$$

Lemma 3.2. *$\tilde{K}$ is contained in $\tilde{A}$.*

Proof. Let $\mathfrak{g}^c$ and $\mathfrak{a}^c$ be the complexifications of $\mathfrak{a}$ and $\mathfrak{g}$ respectively and let $\tilde{A}'$ be the subgroup of $\tilde{G}$ consisting of all k of $\tilde{G}$ satisfying (3.2) and (3.3). We set $\mathfrak{g}_-=\mathfrak{a}^c+\{X+\sqrt{-1}jX\,;X\in\mathfrak{g}\}$. Let us denote by $\mathfrak{n}(\mathfrak{g}_-)$ the normalizer of $\mathfrak{g}_-$ in $\mathfrak{g}^c$. Then the Lie algebra of $\tilde{A}'$ coincides with $\mathfrak{n}(\mathfrak{g}_-)\cap\mathfrak{g}$. On the other hand, since $\tilde{G}/\tilde{A}$ is a homogeneous bounded domain, its canonical hermitian form is positive definite. Therefore by a result of Hano [5], $\mathfrak{a}=\mathfrak{n}(\mathfrak{g}_-)\cap\mathfrak{g}$. Since $\tilde{A}\subset\tilde{A}'$, $\tilde{G}/\tilde{A}$ is a covering space of $\tilde{G}/\tilde{A}'$. Clearly there exists a $\tilde{G}$-invariant complex structure on $\tilde{G}/\tilde{A}'$ such that the projection of $\tilde{G}/\tilde{A}$ onto $\tilde{G}/\tilde{A}'$ is holomorphic. Consequently, $\tilde{G}/\tilde{A}=\tilde{G}/\tilde{A}'$ and hence $\tilde{A}'=\tilde{A}$. q. e. d.

Consider the homogeneous space $\tilde{A}/\tilde{K}$. As a complex submanifold of $\tilde{G}/\tilde{K}$, it has an $\tilde{A}$-invariant Kähler structure. Since $M_2=\tilde{A}/\tilde{K}^0$ is a covering space of $\tilde{A}/\tilde{K}$, we can conclude that $\tilde{A}/\tilde{K}$ is simply connected and hence $\tilde{K}$ is connected. Therefore $\tilde{M}=M$, proving Proposition 3.1.

Corollary 3.3. *Let G/K be a homogeneous Kähler manifold. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K and let j be the endomorphism of $\mathfrak{g}$ corresponding to the invariant complex structure of G/K. Assume that G acts almost effectively on G/K and assume that there exists a linear form ψ on $\mathfrak{g}$ satisfying*

$$\psi([\mathfrak{k},\mathfrak{g}])=0,$$
$$\psi([jX,jY])=\psi([X,Y]) \quad \text{for } X, Y\in\mathfrak{g},$$
$$\psi([jX,X])>0 \quad \text{if } X\notin\mathfrak{k}.$$

Then G/K is biholomorphic to a product of a homogeneous bounded domain and a compact simply connected homogeneous complex manifold.

Proof. From the conditions, the system $(\mathfrak{g}, \mathfrak{k}, j, \psi)$ is an effective j-algebra.

Therefore by [11], the universal covering space of G/K is biholomorphic to a product of a homogeneous bounded domain and a compact simply connected homogeneous complex manifold. Now Corollary 3.3 follows from Proposition 3.1.
q. e. d.

§4. The structure of G/L.

Let G/K be the homogeneous Kähler manifold of non-positive Ricci curvature and let L be the connected subgroup of G as before. We keep the notations in §2. From now on, we assume that the action of G on G/K is effective.

Lemma 4.1*). *There exists an ideal* $\mathfrak{l}'$ *of* $\mathfrak{l}$ *such that*

$$\mathfrak{l}=\mathfrak{l}'+\mathfrak{k} \qquad (\textit{semi-direct}).$$

Proof. Let $\mathfrak{k}(\mathfrak{l})$ be the largest ideal of $\mathfrak{l}$ contained in $\mathfrak{k}$. Since L/K admits an L-invariant metric, we have $\mathfrak{k}(\mathfrak{l})=\{X\in\mathfrak{k}\,;\,[X, Y]\in\mathfrak{k}$ for any $Y\in\mathfrak{g}\}$. By Lemma 1.2 of [10], there exists an ideal $\mathfrak{h}$ of $\mathfrak{l}$ such that

$$\mathfrak{l}=\mathfrak{h}+\mathfrak{k}(\mathfrak{l}) \qquad (\text{direct sum of ideals}).$$

The universal covering space of L/K is a complex euclidean space $\boldsymbol{C}^n$. Let $\mathfrak{e}(\boldsymbol{C}^n)$ denote the Lie algebra of $\mathrm{Aut}(\boldsymbol{C}^n)$. It is easy to see that every semi-simple subalgebra of $\mathfrak{e}(\boldsymbol{C}^n)$ is compact. Let H be the connected subgroup of $\mathrm{Aut}(\boldsymbol{C}^n)$ corresponding to $\mathfrak{h}$ and let $\bar{H}$ be the closure of H in $\mathrm{Aut}(\boldsymbol{C}^n)$. Let H_0 and $\bar{H}_0$ be the isotropy subgroups of H and $\bar{H}$ respectively. Since $\bar{H}/\bar{H}_0=H/H_0=\boldsymbol{C}^n$, $\bar{H}_0$ is a maximal compact subgroup of $\bar{H}$. Let $\mathfrak{h}=\mathfrak{r}+\mathfrak{s}$ be a Levi-decomposition of $\mathfrak{h}$, where $\mathfrak{r}$ is the radical of $\mathfrak{h}$ and $\mathfrak{s}$ is a semi-simple subalgebra. Let S be the connected subgroup corresponding to $\mathfrak{s}$. Since S is compact, there exists $g\in\bar{H}$, such that $\mathrm{Ad}\,gS\subset\bar{H}_0$. Since H is a normal subgroup of $\bar{H}$, we have $\mathrm{Ad}\,gS\subset H_0\cap H=H_0$. Therefore we get $\mathrm{Ad}\,g\mathfrak{s}\subset\mathfrak{h}\cap\mathfrak{k}$. Thus we may assume $\mathfrak{s}\subset\mathfrak{h}\cap\mathfrak{k}$. Then $\mathfrak{h}=\mathfrak{r}+\mathfrak{h}\cap\mathfrak{k}$. Noting that $[\mathfrak{r}, \mathfrak{r}]\cap(\mathfrak{h}\cap\mathfrak{k})=0$, we can find an $\mathrm{ad}(\mathfrak{h}\cap\mathfrak{k})$-invariant subspace $\mathfrak{c}$ of $\mathfrak{r}$ satisfying

$$\mathfrak{r}=[\mathfrak{r}, \mathfrak{r}]+(\mathfrak{r}\cap\mathfrak{k})+\mathfrak{c} \qquad (\text{vector space direct sum}).$$

If we set $\mathfrak{l}'=[\mathfrak{r}, \mathfrak{r}]+\mathfrak{c}$, then $\mathfrak{l}'$ satisfies the desired properties. q. e. d.

Let Ψ be the Kähler form of G/K and let $\rho=\pi^*\Psi$. Then ρ is a left invariant skew-symmetric bilinear form on G and hence it may be regarded as a skew-symmetric bilinear form on $\mathfrak{g}$.

Let $\mathfrak{l}'$ be as in Lemma 4.1. We may assume that $j\mathfrak{l}'=\mathfrak{l}'$. By a result of Dorfmeister [4], $\mathfrak{l}'$ is decomposed as

$$\mathfrak{l}'=\mathfrak{l}_0+\mathfrak{l}_1,$$

where $\mathfrak{l}_0$ is a j-invariant abelian ideal of $\mathfrak{l}'$ given by $\mathfrak{l}_0=[\mathfrak{l}', \mathfrak{l}']$ and $\mathfrak{l}_1$ is a j-

*) The author was informed of this fact from J. Dorfmeister.

invariant subalgebra defined by $\mathfrak{l}_1=\{X\in\mathfrak{l}' ; \rho(X, Y)=0$ for any $Y\in\mathfrak{l}_0\}$. We note that both $\mathfrak{l}_0$ and $\mathfrak{l}_1$ is invariant under ad$\mathfrak{k}$. For any $X\in\mathfrak{l}_1$, we denote by D_X the semi-simple part of ad$X$. Then $\{D_X ; X\in\mathfrak{l}_1\}$ is a commuting family of derivations of $\mathfrak{g}$. Since ad$X$ is nilpotent on $\mathfrak{l}_1+\mathfrak{k}$, we have

$$D_X(\mathfrak{l}_1+\mathfrak{k})=0. \tag{4.1}$$

We also know from [4] that $D_X|_{\mathfrak{l}_0}$ has only purely imaginary eigenvalues and

$$\begin{aligned} &D_X\circ jY=j\circ D_XY \quad \text{for any} \quad Y\in\mathfrak{l}', \\ &\rho(D_XY, Z)+\rho(Y, D_XZ)=0 \qquad \text{for any} \quad Y, Z\in\mathfrak{l}'. \end{aligned} \tag{4.2}$$

Let us denote by τ the linear isotropy representation of $\mathfrak{l}$ on $\mathfrak{g}/\mathfrak{l}$. We already know that τ is a unitary representation. Therefore $\tau(X)$ is semi-simple and its eigenvalues are purely imaginary. Furthermore

$$\tau(\mathfrak{l}_0)=0, \tag{4.3}$$

because $\mathfrak{l}'$ is solvable and $\mathfrak{l}_0=[\mathfrak{l}', \mathfrak{l}']$. Now it is clear that D_X has only purely imaginary eigenvalues and

$$D_XY\equiv[X, Y] \quad (\mathrm{mod}\,\mathfrak{l}) \quad \text{for any } Y\in\mathfrak{g}. \tag{4.4}$$

Let B be the closure of the automorphism group of $\mathfrak{g}$ generated by $\{D_X ; X\in\mathfrak{l}_1\}$. Then B is a compact abelian group. We set

$$\tilde{\rho}(Y, Z)=\int_B \rho(bX, bY)db \qquad \text{for} \quad Y, Z\in\mathfrak{g},$$

where db is the normalized Haar measure of B. Using (4.1) and (4.2), we can see

$$\begin{aligned} &d\tilde{\rho}=0, \qquad \tilde{\rho}(\mathfrak{k}, \mathfrak{g})=0, \\ &\tilde{\rho}(Y, Z)=\rho(Y, Z) \qquad \text{for} \quad Y, Z\in\mathfrak{l}, \\ &\tilde{\rho}(D_XY, Z)+\tilde{\rho}(Y, D_XZ)=0 \qquad \text{for} \quad X\in\mathfrak{l}_1 \quad \text{and} \quad Y, Z\in\mathfrak{g}. \end{aligned} \tag{4.5}$$

Let us set

$$\mathfrak{t}=\{X\in\mathfrak{g} ; \tilde{\rho}(X, Y)=0 \quad \text{for any } Y\in\mathfrak{l}\}.$$

Lemma 4.2. (1) $\mathfrak{g}=\mathfrak{t}+\mathfrak{l}$ *and* $\mathfrak{l}\cap\mathfrak{t}=\mathfrak{k}$.
(2) $\mathfrak{t}$ *is a subalgebra.*

Proof. Assertion (1) follows from (4.5) and the definition of $\mathfrak{t}$.

By (4.3), $[\mathfrak{l}_0, \mathfrak{g}]\subset\mathfrak{l}$. Therefore using $d\tilde{\rho}=0$, we have $\tilde{\rho}([\mathfrak{t}, \mathfrak{t}], \mathfrak{l}_0)\subset\tilde{\rho}([\mathfrak{l}_0, \mathfrak{t}], \mathfrak{t})=0$. Moreover using (4.4) and (4.5), we have for any $X\in\mathfrak{l}_1$ and $Y, Z\in\mathfrak{t}$

$$\begin{aligned} \tilde{\rho}(X, [Y, Z])&=\tilde{\rho}([X, Y], Z)+\tilde{\rho}(Y, [X, Z]) \\ &=\tilde{\rho}(D_XY, Z)+\tilde{\rho}(Y, D_XZ) \\ &=0, \end{aligned}$$

proving $\tilde{\rho}(\mathfrak{l}_1, [\mathfrak{t}, \mathfrak{t}])=0$. Hence we get $[\mathfrak{t}, \mathfrak{t}]\subset\mathfrak{t}$. q. e. d.

Let $\mathfrak{k}(\mathfrak{t})$ be the largest ideal of $\mathfrak{t}$ contained in $\mathfrak{k}$. By a result of [10] there exists an ideal $\mathfrak{t}'$ of $\mathfrak{t}$ such that $\mathfrak{t}=\mathfrak{t}'+\mathfrak{k}(\mathfrak{t})$ (direct sum of ideals). We then have $\mathfrak{g}=\mathfrak{t}'+\mathfrak{l}$. Let T' be the connected subgroup of G corresponding to $\mathfrak{t}'$. Then T' acts on G/L transitively and almost effectively. There corresponds to the invariant complex structure of G/L, an endomorphism j' of $\mathfrak{t}'$ such that $j'X\equiv jX$ (mod $\mathfrak{l}$) for $X\in\mathfrak{t}'$. Then $(\mathfrak{t}', \mathfrak{t}'\cap\mathfrak{k}, j', \psi_{G/K})$ is an effective j-algebra. Thus by Corollary 3.3, we have

Lemma 4.3. *The base space G/L is biholomorphic to the product of a homogeneous bounded domain M_1 and a compact simply connected homogeneous complex manifold M_2.*

As before there exists a connected closed subgroup A of G containing L such that $M_1=G/A$ and $M_2=A/L$. Then A/K is a Kähler submanifold of G/K and the fibering: $A/K\to A/L$ is nothing but the Hano-Kobayashi fibering of A/K.

Lemma 4.4. *The Ricci curvature of A/K is non-positive.*

Proof. Let $\mathfrak{a}$ be the Lie algebra of A and let $X\in\mathfrak{a}$. We then have

$$\psi_{G/K}([jX, X])=\psi_{G/A}([jX, X])+\psi_{A/K}([jX, X]).$$

Since $\psi_{G/K}([jX, X])\geqq 0$ and $\psi_{G/A}([jX, X])=0$, we have $\psi_{A/K}([jX, X])\geqq 0$.

q. e. d.

§ 5. Proof of Main Theorem.

Let G/K be a homogeneous Kähler manifold of non-positive Ricci curvature and let G/L be the base space in the Hano-Kobayashi fibering of G/K. For the proof of Main Theorem, it remains to prove that G/L is biholomorphic to a homogeneous bounded domain. To do this, by virtue of Lemmas 4.3 and 4.4, it is sufficient to prove the following

Proposition 5.1. *Let G/K be a homogeneous Kähler manifold of non-positive Ricci curvature. Consider the Hano-Kobayashi fibering: $G/K\to G/L$. Assume that G/L is compact simply connected. We then have $G=L$.*

It is sufficient to prove this proposition assuming that G acts effectively on G/K. Let $\mathfrak{l}_0$, $\mathfrak{l}_1$, D_X, $\mathfrak{t}$ and $\mathfrak{t}'$ be as § 4. Since $\mathfrak{t}'$ is the Lie algebra of a transitive subgroup of Aut(G/L) and since G/L is a compact simply connected homogeneous Kähler manifold, we know that $\mathfrak{t}'$ is semi-simple. It follows that $\mathfrak{t}$ is reductive and hence it is decomposed as

$$\mathfrak{t}=\mathfrak{c}+\mathfrak{s},$$

where $\mathfrak{c}$ denotes the center of $\mathfrak{t}$ and $\mathfrak{s}$ is the semi-simple part of $\mathfrak{t}$. Note that $\mathfrak{c}$ is contained in $\mathfrak{k}$. It is well known that there exists $Z_0\in\mathfrak{s}$ such that

$$\mathfrak{k}=\{X\in\mathfrak{t}\,;\,[X, Z_0]=0\}. \tag{5.1}$$

Clearly, D_X leaves $\mathfrak{t}$ invariant and hence it induces a derivation of $\mathfrak{s}$. Therefore for every $X \in \mathfrak{l}_1$, there exists s_X of $\mathfrak{s}$ such that

$$D_X Y = [s_X, Y] \quad \text{for any} \quad Y \in \mathfrak{s}. \tag{5.2}$$

Since $D_X Z_0 = 0$ by (4.1), we know $s_X \in \mathfrak{k}$ from (5.1) and (5.2). Let $\mathfrak{l}(\mathfrak{g})$ be the largest ideal of $\mathfrak{g}$ contained in $\mathfrak{l}$. By (4.4) and (5.2), we have $X - s_X \in \mathfrak{l}(\mathfrak{g})$. From (4.3), we also know that $\mathfrak{l}_0$ is contained in $\mathfrak{l}(\mathfrak{g})$. Therefore we get

$$\mathfrak{l} = \mathfrak{l}(\mathfrak{g}) + \mathfrak{k}. \tag{5.3}$$

Hence we may assume $j\mathfrak{l}(\mathfrak{g}) \subset \mathfrak{l}(\mathfrak{g})$. Let $\mathfrak{g}'$ be the subspace given by

$$\mathfrak{g}' = \{X \in \mathfrak{g}\,;\ \rho(X, Y) = 0 \quad \text{for any } Y \in \mathfrak{l}\}.$$

Using (5.3) and the fact that $\mathfrak{l}(\mathfrak{g})$ is a j-invariant ideal of $\mathfrak{g}$, we can see that $\mathfrak{g}'$ is a j-invariant subalgebra and satisfies

$$\mathfrak{g} = \mathfrak{g}' + \mathfrak{l} \quad \text{and} \quad \mathfrak{g}' \cap \mathfrak{l} = \mathfrak{k}.$$

By the same arguments as for $\mathfrak{t}$, we can show that $\mathfrak{g}'$ is reductive.

Lemma 5.2. *For every $X \in \mathfrak{g}'$,*

$$\mathrm{Tr}_{\mathfrak{l}/\mathfrak{k}}(\mathrm{ad}\, j[jX, X] - j \circ \mathrm{ad}\,[jX, X]) = 0.$$

Proof. Let us set $V = \mathfrak{l}/\mathfrak{k}$. Since $\mathfrak{g}/\mathfrak{g}' \cong \mathfrak{l}/\mathfrak{k}$, the linear isotropy representation of $\mathfrak{g}'$ induces a representation γ on V. The form ρ induces a skew-symmetric bilinear form ω on V. Then (V, j, ω) is a symplectic space. By the definition of $\mathfrak{g}'$, γ is a symplectic representation. Let G' be the connected subgroup of G corresponding to $\mathfrak{g}'$. Then we obtain a holomorphic mapping of $G'/G' \cap K$ to $Sp(V)/K(V)$. Since $G'/G' \cap K$ is biholomorphic to G/L and since G/L is compact, the image of $G'/G' \cap K$ is a single point. This means that $\gamma(X)$ commutes with j for any $X \in \mathfrak{g}'$. As a result

$$\mathrm{Tr}_V\, j \circ [\gamma(jX), \gamma(X)] = 0.$$

It follows that

$$\begin{aligned} &\mathrm{Tr}_{\mathfrak{l}/\mathfrak{k}}(\mathrm{ad}\, j[jX, X] - j \circ \mathrm{ad}\,[jX, X]) \\ &\quad = \mathrm{Tr}_V(\gamma(j[jX, X]) - j \circ [\gamma(jX), \gamma(X)]) = 0. \end{aligned}$$ q. e. d.

Let $X \in \mathfrak{g}'$. We then have from Lemma 5.2,

$$\begin{aligned} \psi_{G/K}([jX, X]) &= \psi_{G/L}([jX, X]) + \mathrm{Tr}_{\mathfrak{l}/\mathfrak{k}}(\mathrm{ad}\, j[jX, X] - j \circ \mathrm{ad}\,[jX, X]) \\ &= \psi_{G/L}([jX, X]). \end{aligned}$$

Since G/L is compact and simply connected, we have from [8], $\psi_{G/L}([jX, X]) \leqq 0$ and the equality holds if and only if $X \in \mathfrak{l}$. From the assumption $\psi_{G/K}([jX, X]) \geqq 0$ we then have $X \in \mathfrak{g}' \cap \mathfrak{l} = \mathfrak{k}$. Thus we get $\mathfrak{g}' = \mathfrak{k}$, proving Proposition 5.1.

We have proved the following theorem and completed the proof of Main Theorem.

Theorem 5.3. *Let G/K be a homogeneous Kähler manifold of non-positive Ricci curvature. Then there exists a closed subgroup L contatning K such that*

(a) *L/K is a flat homogeneous Kähler submanifold of G/K.*

(b) *G/L admits a G-invariant complex structure with respect to which G/L is biholomorphic to a homogeneous bounded domain and the canonical projection of G/K onto G/L is holomorphic.*

Appendix.

We will give here the proof of Theorem 1.2. We first show the following

Lemma. *Let M be a compact simply connected homogeneous Kähler manifold and let f be an element of* $\mathrm{Aut}(M)$. *Assume that there exists a connected subgroup G of* $\mathrm{Aut}(M)$ *such that*

(a) *G acts transitively on M,*

(b) *f commutes with each element of G.*

Then f is the identity transformation of M.

Proof. We first note that the Lie algebra of a subgroup of $\mathrm{Aut}(M)$ acting transitively is compact and semi-simple. Define a compact subgroup C by

$$C=\{h\in \mathrm{Aut}(M)\,;\, gh=hg \quad \text{for any } g\in G\}.$$

and put $G'=CG$. Both G' and G are compact semi-simple subgroups. We denote by $\mathfrak{g}'$, $\mathfrak{g}$, and $\mathfrak{c}$ the Lie algebras of G', G, and C respctively. We have $\mathfrak{g}'=\mathfrak{g}+\mathfrak{c}$ (direct sum of ideals). Let K' and K be the isotropy subgroups of G' and G at a point of M. We denote by $\mathfrak{k}'$ and $\mathfrak{k}$ the corresponding Lie algebras. It is well known that there exists $Z\in\mathfrak{g}'$ such that $\mathfrak{k}'=\{X\in\mathfrak{g}'\,;\,[X, Z]=0\}$. We then have $\mathfrak{k}'=\mathfrak{k}+\mathfrak{k}'\cap\mathfrak{c}$. Since $\dim\mathfrak{g}'/\mathfrak{k}'=\dim\mathfrak{g}/\mathfrak{k}$, we get $\dim\mathfrak{c}=\dim\mathfrak{c}\cap\mathfrak{k}'$. This means that $\mathfrak{c}$ is contained in $\mathfrak{k}'$ and hence $\mathfrak{c}=0$. It follows that G is the identity component of G' and C is a finite group. Moreover C is a normal subgroup of G'. In fact, let $g\in G'$ and $a\in C$. Since $gGg^{-1}=G$, we have for any $h\in G$, $gag^{-1}h=gag^{-1}hgg^{-1}=gg^{-1}hgag^{-1}=hgag^{-1}$, proving $gag^{-1}\in C$. It follows that CK' is a compact subgroup of G' and $M=G'/K'$ is a covering space of G'/CK'. The homogeneous space G'/CK' admits naturally a G'-invariant Kähler structure so that the projection: $G'/K'\to G'/CK'$ is holomorphic and isometric. Now G'/CK' is a homogeneous Kähler manifold on which a connected semi-simple Lie group acts transitively, holomorphically and isometrically. Therefore by Borel [2], G'/CK' is simply connected and hence we get $K'=CK'$. This means $C\subset K'$ and hence $C=\{e\}$, because G' acts effectively on M. q. e. d.

We now prove Theorem 1.2. Let M be a homogeneous Kähler manifold of non-negative Ricci curvature and let $G=\mathrm{Aut}^0(M)$. Then $M=G/K$, K being the isotropy subgroup. Denote by $\tilde{G}$ the universal covering group of G and by ϕ the projection of $\tilde{G}$ onto G. Let $\tilde{K}=\phi^{-1}(K)$ and let $\tilde{K}^0$ be the identity component of $\tilde{K}$. Then $\tilde{M}=\tilde{G}/\tilde{K}^0$ is the universal covering space of M and it has a natural $\tilde{G}$-invariant Kähler structure so that the projection is holomorphic and isometric. We already know from [3] that $\tilde{M}$ is isomorphic to $\boldsymbol{C}^n \times M'$, where M' is a compact simply connected homogeneous Kähler manifold. Let $\pi_1(M)$ be the fundamental group of M. Let f be an element of $\pi_1(M)$. We express the action of f on $\tilde{M}$ as

$$f(z, w)=(f_0(z, w), f'(z, w)),$$

where $z \in \boldsymbol{C}^n$ and $w \in M'$. Since M' is compact, f_0 does not depend on w. For the proof of Theorem 1.2, it is sufficient to show that $f'(z, w)=w$ for any z and w. We fix a point $z \in \boldsymbol{C}^n$. Define a map $f'_z: M' \to M'$ by $f'_z(w)=f'(z, w)$. We can easily see that f'_z is an element of $\mathrm{Aut}(M')$. Since $\mathrm{Aut}^0(\tilde{M})=\mathrm{Aut}^0(\boldsymbol{C}^n) \times \mathrm{Aut}^0(M')$, the group $\tilde{G}$ acts $\boldsymbol{C}^n$ and M' in a natural manner. Let $\tilde{H}$ be the isotropy subgroup of $\tilde{G}$ at the point z. We then have $\tilde{G}/\tilde{H}=\boldsymbol{C}^n$ and $\tilde{H}/\tilde{K}^0=M'$. In view of (3.1), we can easily see that f'_z commutes with the action of $\tilde{H}$ on M'. Therefore from Lemma, we have $f'_z=1$, proving Theorem 1.2.

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