J. Austral. Math. Soc. (*Series A*) **46** (1989), 343–355

HOMOGENEOUS MODELS AND ALMOST DECIDABILITY

TERRY MILLAR

(Received 1 February 1987; revised 6 January 1988)

Communicated by C. J. Ash

Abstract

Countable homogeneous models are 'simple' objects from a model theoretic point of view. From a recursion theoretic point of view they can be complex. For instance the elementary theory of such a model might be undecidable, or the set of complete types might be recursively complex. Unfortunately even if neither of these conditions holds, such a model still can be undecidable. This paper investigates countable homogeneous models with respect to a weaker notion of decidability called *almost decidable*. It is shown that for theories that have only countably many type spectra, any countable homogeneous model of such a theory that has a Σ_2 type spectrum is almost decidable.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 03 C 57.

This paper continues an investigation of the notion of "almost decidability," introduced in [1], that approximates decidability with respect to countable structures. The *type spectrum* of a model $\mathfrak{A}$ is the set of complete types realized by the model. A *type spectrum* of a theory is a type spectrum of one of its models. We restrict our attention to theories whose complete types are all recursive. The main result of this paper is that if T is a decidable complete theory with only countably many type spectra, then every countable homogeneous model of T with a Σ_2^0 type spectrum is almost decidable. It was shown in [2] that if T has countably many complete types and there is a decidable model realizing all the recursive types of the theory, then every countable homogeneous model of the theory with a Σ_2^0 type spectrum is

The research for this paper was partially supported by ARGS grant #8215113 and NSF grant #DMS8501521.

decidable. In [3] there is an example of a complete decidable theory with an undecidable countable homogeneous model whose type spectrum is Σ_1^0. In [4] there is an example of a complete decidable theory with only countably many type spectra, such that the theory has an undecidable countable homogeneous model.

A tree Tr will be a subset of $2^{<\omega}$ that is closed under predecessors. For elements f, g in $2^{<\omega} \cup w^{\omega}$, define $f \leq g$ just if $\forall i < lh(f)[f(i) = g(i)]$. Thus if Tr is a tree and $f \leq g \in \mathrm{Tr}$, then $f \in \mathrm{Tr}$. A function $f \in 2^{\omega}$ is a *branch* of Tr just if every proper initial segment of f under $\leq$ is in Tr. Let $\{\pi_i | i < \omega\}$ be an effective enumeration of all formulas of L, and let P_{ω} be the set of all finite subsets of $\{\pi_i | i < \omega\}$. Let $\{\mu_i | i < \omega\}$ be a standard enumeration of all partial recursive functions $\mu\colon \omega \to \omega$. We will assume that we have fixed recursive numberings of such objects as ω^n, etc. Let $\mathrm{TySp}(\mathfrak{A})$ be the type spectrum of $\mathfrak{A}$. Let $\theta(x/y)$ denote the formula obtained from θ by replacing all free occurrences of 'x' in θ by 'y', attending to bound variable changes as necessary. Let $\exists\theta[\overline{x}]\#$ denote the formula obtained from θ by adjoining, as a prefix, a block of existential quantifiers to θ. The variables in the block are those free variables occurring in θ that are *not* in $\overline{x}$. Let $\exists\theta[\overline{x}]*$ denote the formula obtained from $\exists\theta[\overline{x}]\#$ by replacing all free occurrences of $\overline{x}(i)$ in θ by x_i, $i < lh(\overline{x})$. This modification simply renumbers the variables in $\overline{x}$, starting with '0' and ending with '$lh(\overline{x}) - 1$'. Let $\theta[/\overline{x}]$ denote the formula obtained from θ by replacing all free occurrences of x_i in θ by $\overline{x}(i)$, $i < lh(\overline{x})$. This modification allows a renumbering of the variables $x_i \to \overline{x}(i)$, $i < lh(\overline{x})$. A set of formulas will sometimes be identified with its conjunction. Most other notation not defined is common.

Now define F to be a *model tree* (for the language L) just if

(1) F is a function with domain $2^{<\omega}$ and range a subset of P_{ω},

(2) $\delta \leq \alpha \to F_{\delta} \subseteq F_{\alpha}$, and

(3) $\forall f \in 2^{\omega}[\bigcup_{\alpha < f} F(\alpha)$ is the elementary diagram of an L-structure].

If F is a model tree and $g \in 2^{\omega}$, then we shall use $\mathfrak{A}_g$ to denote the L-structure corresponding to g, as in Condition (3). An L-structure $\mathfrak{A}$ is *almost decidable* just if there is a recursive model tree F such that for all but countably many $g \in 2^{\omega}$, $\mathfrak{A}_g \cong \mathfrak{A}$.

THEOREM. *If T is a complete theory with only countably many type spectra, then every countable homogeneous model $\mathfrak{A}$ of T whose type spectrum is Σ_2^0 is almost decidable.*

Proof. We will construct a recursive model tree for $\mathfrak{A}$. Fix $S_1 \in \Sigma_2^0$ such that

$$\forall\Gamma \in \mathrm{TySp}(\mathfrak{A})\exists^\omega n \in S_1[(n) \text{ is the characteristic function of } \Gamma]$$
$$\text{and } \forall n \in S_1\exists\Gamma \in \mathrm{TySp}(\mathfrak{A})[(n) \text{ is the characteristic function of } \Gamma].$$

Also fix recursive S such that $\forall n[n \in S_1 \leftrightarrow \exists x\forall y S(x, y, n)]$. Part of the difficulty during the construction is that we will be unsure of which $\{n\}$ actually represent types in $\mathrm{TySp}(\mathfrak{A})$. Typically we will wish to determine membership in the represented type. Therefore, we will establish a convention. During the construction we will never be attempting to determine membership according to $\{n\}$ alone. Instead, this will be balanced against an attempt to find a y such that $\neg S(r, y, n)$ for an associated r. If all membership questions of interest are answered before such a y is found, fine. Otherwise we will entirely abandon the pair $\langle n, r\rangle$. The point is that if $\forall y S(r, y, n)$, then in fact $\{n\}$ does represent a type in $\mathrm{TySp}(\mathfrak{A})$. Even if n does belong to S_1 and r is not a witness, there will be other attempts made to verify n in such a way that the construction succeeds. Denote by Γ_m the potential type of $\mathrm{TySp}(\mathfrak{A})$ with witness m. If at some stage a y has been found such that $\neg S(r, y, n)$, where $m = \langle n, r\rangle$, then we will say Γ_m has *died* by that stage, otherwise it is *promising*. So we will now assume that if a membership question is to be determined of $T_{\langle n,r\rangle}$, then simultaneously a search is initiated to find a y such that $\neg S(r, y, n)$.

The construction will proceed by induction on $2^{<\omega}$. The objective is to define $\Sigma_\delta \in P_\omega$ for each $\delta \in 2^{<\omega}$. At stage $2s + 1$ of the construction, either π_s or $\neg\pi_s$ will be put into Σ_δ for each $\delta \in 2^{s+1}$. Finitely many other formulas may also be put into Σ_δ. Fix an effective enumeration $\{\overline{x}_i | i < \omega\}$ of $\{x_j | j < \omega\}^{<\omega}$. There are three primary strategies that will be implemented during the construction. The first is to ensure that $\mathrm{TySp}(\mathfrak{A}) = \mathrm{TySp}(\mathfrak{A}_g)$ for each $g \in 2^\omega$. The second strategy is to ensure that in every model of the tree each tuple of elements realizing a type in $\mathrm{TySp}(\mathfrak{A})$ has extensions realizing the consistent extensions of the type in $\mathrm{TySp}(\mathfrak{A})$. This will give at least selective homogeneity. The third strategy is to attempt to have models in the tree realize in common only those types that are realized in $\mathfrak{A}$. This strategy will be implemented by attempting to ensure that every finite tuple of elements realizes a type that is principal over some finite set of elements in the employment of strategies one and two. We will now phrase these strategies as requirements.

P_n: Γ_n dies, or is realized in every model of the tree.

$Q_{\langle n,m,a\rangle}$: either (1) Γ_n dies; or (2) Γ_m dies; or (3) $\neg[\Gamma_n \subseteq \Gamma_m]$; or (4) for every model of the tree, either (i) $\overline{x}_a$ does not realize Γ_n or (ii) there is a j such that $\overline{x}_a{}^\frown\overline{x}_j$ realizes Γ_m.

Each of these first two kinds of requirements will be active or terminated for a given node of $2^{<\omega}$ at any given stage of the construction. Termination is permanent and is precipitated by either the death of the associated type or the fact that for the Q requirements, (1), (2), (3) or (4)(i) obtains. Condition (3) will hold at stage s just if (1) or (2) holds or else the first s formulas in Γ_n are not in Γ_m. Condition (4)(i) will hold at stage s for all $f \in 2^\omega$ such that $\delta \leq f$ just if $\Sigma_\delta \cup \Gamma_n(\overline{x}_a)$ is inconsistent. A death will cause the termination of the requirement for the entire tree. A failure of $\overline{x}_a$ to realize the appropriate type in a subtree of the tree will cause termination of the associated Q requirement in that subtree. Let $P(v,t,\alpha)$ be the tuple associated with the requirement P_v at node α at stage t. If P_v is terminated by stage t, then $P(v,t,\alpha) = \langle\ \rangle$, the empty sequence. Similarly, let $Q(v,t,\alpha)$ be the tuple $\overline{x}_a{}^\frown\overline{x}_j$ associated with requirement Q_v at stage t at node α if Q_v is active at stage t at node α, and otherwise $Q(v,t,\alpha) = \langle\ \rangle$.

Another major difficulty is amalgamating the types associated with the different P and Q requirements in a consistent way. The obvious candidates for such amalgamations are the types in $\mathrm{Sp}(\mathfrak{A})$. Unfortunately types are infinite objects. Therefore containment of one type in another type of greater arity can not be determined in an effective manner. However, since the types are all recursive, if one type is inconsistent with another then we can find this out at some finite stage of the construction. Our strategy in that case will be to abandon the candidate constants of the offending type associated with the requirement of lowest priority. We will assign a new candidate to the realization of that type, and simultaneously search for a new type that amalgamates the previous types. We will treat this attempt to amalgamate types as a requirement.

R: all the types associated with active P, Q, and R requirements of higher priority are amalgamated by some type in $\mathrm{tySP}(\mathfrak{A})$.

In order to be more specific about the R requirements, we must establish the priorities of the various requirements. The priorities of the P, Q, and R requirements are listed below:

$$P_n : 5n;$$
$$R_{n,0} : 5n+1;$$
$$Q_n : 5n+2;$$
$$R_{n,1} : 5n+3.$$

$\mathrm{Amal}(n,i,t,\alpha)$ will be either undefined or the index of the type associated with $R_{n,i}$ at stage t in the neighborhood determined by $\alpha, i = 0, 1$. Define

$R(n, i, t, \alpha), i = 0, 1$, inductively on n:

$$R(0, 0, t, \alpha) =_{df} P(0, t, \alpha);$$
$$R(n, 1, t, \alpha) =_{df} R(n, 0, t, \alpha)\hat{}Q(n, t, \alpha);$$
$$R(n+1, 0, t, \alpha) =_{df} R(n, 1, t, \alpha)\hat{}P(n, t, \alpha).$$

The final set of requirements, $\{S_n | n < \omega\}$, will control the attempt to make the type spectra of the different models in the model tree as nearly disjoint as possible. For each n, requirements S_n will have priority $5n + 4$.

$$S_{\langle\alpha,i,j\rangle}: \forall f, g \in 2^\omega[\text{if } \alpha\hat{}\langle 0\rangle < f \text{ and } \alpha\hat{}\langle 1\rangle < g \text{ then}$$
$$\{\exists\theta[\overline{x}_i] * | \exists\beta < f[\theta \in F(\beta)]\} \neq \{\exists\theta[\overline{x}_j] * | \exists\beta < g[\theta \in F(\beta)]\}].$$

Not all of these requirements will be met, but for all but countably many branches of the tree a failure on the branch will be caused by a realization of a type in TySp($\mathfrak{A}$), which of course should not be avoided.

Definition. *$S_{\langle\alpha,i,j\rangle}$ allows θ through with respect to $\{\Sigma_\delta | \delta \in 2^{s+1}\}$* just if $\forall\beta, \gamma \in 2^{s+1}$ [where $n = \langle\alpha, i, j\rangle$, $t = 2s + 1$, $a = \text{Amal}(n, 1, t, \gamma)$, and $b = \text{Amal}(n, 1, t, \beta)$] *if*

(1) $\alpha\hat{}\langle 0\rangle \leq \gamma, \alpha\hat{}\langle 1\rangle \leq \beta$,
(2) $gG_a(R(n, 1, s, \gamma)) \cup \exists\Sigma_\gamma[\overline{x}_i]\# \cup \exists\Sigma_\beta[\overline{x}_j] * [/\overline{x}_i]$ is consistent, and
(3) $\Gamma_b(R(n, 1, s, \beta)) \cup \exists\Sigma_\beta[\overline{x}_j]\# \cup \exists\Sigma_\gamma[\overline{x}_i] * [/\overline{x}_j]$ is consistent,

then for π equal to each of

$$\exists(\Sigma_\gamma \cup \{\theta^k\})[\overline{x}_i]*,$$
$$\exists(\Sigma_\beta \cup \{\theta^k\})[\overline{x}_j]*, \qquad k < 2,$$

(1) if $\Gamma_a(R(n, 1, s, \gamma)) \cup \Sigma_\gamma \cup \{\pi[/\overline{x}_i]\}$ is consistent then

$$\Gamma_a(R(n, 1, s, \gamma)) \vdash \Sigma_\gamma \to \pi[/\overline{x}_i],$$

and

(2) if $\Gamma_b(R(n, 1, s, \beta)) \cup \Sigma_\beta \cup \{\pi[/\overline{x}_j]\}$ consistent then

$$\Gamma_b(R(n, 1, s, \beta)) \vdash (\Sigma_\beta \to \pi[/\overline{x}_j]).$$

Definition. *θ resolves $S_{\langle\alpha,i,j\rangle}$ past $\langle\gamma, \beta\rangle(1h(\gamma) = 1h(\beta) = s + 1)$ with respect to $\{\Sigma_\delta | \delta \in 2^{s+1}\}$* [let n, a, and b be as above] just if

(1) $\alpha\hat{}\langle 0\rangle \leq \gamma$ and $\alpha\hat{}\langle 1\rangle \leq \beta$,
(2) for π as above and for some $v < 2$
   (i) $\Gamma_a(R(n, 1, s, \gamma)) \cup \Sigma_\gamma \cup \{\pi^v[/\overline{x}_i]\}$ is consistent, and
   (ii) $\Gamma_b(R(n, 1, s, \beta)) \cup \Sigma_\beta \cup \{\neg\pi^v[/\overline{x}_j]\}$ is consistent.

LEMMA 1. *If*

(1) *for each* $\delta \in 2^{s+1}$, $T \cup \Sigma_\delta$ *is consistent,*
and

(2) $S_{\langle\alpha,i,j\rangle}$ *does not allow* θ *through* s *with respect to* $\{\Sigma_\delta | \delta \in 2^{s+1}\}$, *then there are* γ, β *such that* θ *resolves* $S_{\langle\alpha,i,j\rangle}$ *past* $\langle\gamma, \beta\rangle$ *with respect to* $\{\Sigma_\delta | \delta \in 2^{s+1}\}$.

PROOF. This is just chasing definitions, so we have the details to the reader.

For a, π, v, γ, and β as above, $\langle\pi^v, \neg\pi^v\rangle$ is called the *resolvent* for $S_{\langle\alpha,i,j\rangle}$ past $\langle\gamma, \beta\rangle$ with respect to $\{\Sigma_\delta | \delta \in 2^{s+1}\}$.

DEFINITION. The requirements are *variable through* t *at stage* $p+1$ *with respect to* $\{C_\delta | \delta \in s+1\}$ just if for all $m \leq t, \delta \in 2^{s+1}$, and $i < 2$, where $\sigma = \delta|_s$ and $a = \mathrm{Amal}(m, i, p, \sigma)$ then

(1) if $m = 4v + 2i + 1$ then $\Gamma_a(R(m, i, p, \sigma)) \cup \Sigma_\sigma \cup C_\delta$ is consistent,

(2) if $m = 4v$ and P_m is active then $\Gamma_m(P(m, p, \sigma)) \cup \Sigma_\sigma \cup C_j$ is consistent, and

(3) if $m = 4v+2$ and Q_m is active, where $m = \langle n, v, b\rangle$, then $\Gamma_v(Q(m, p, \sigma)) \cup \Sigma_\sigma \cup C_\delta$ is consistent.

By our convention of using initial sequences of variables, the arity of a type Γ_n is the least i for which $(x_{i-1} = x_{i-1})$ does not belong to Γ_n. This can always be determined for any type Γ_n that is not deceased. Therefore we will assume that the arity of any type is known.

The construction.

Stage 0. $\Sigma_\delta =_{df} 0$, for $\delta \in 2^{<\omega}$.

Stage $2s$. This stage just provides the usual Henkin witnesses to existential statements. So assume that in some consistent, uniformly effective fashion $\pi(x/x_k)$ has been put into Σ_α for some 'new' x_k if $\exists x\pi \in \Sigma_\alpha, \alpha \in 2^{s+1}$.

Stage $p + 1 = 2s + 1$. This stage consists of many steps. We will execute a procedure repeatedly until a certain condition obtains. The procedure takes and sets various parameters. Initialilze the parameters controlling the diagrams by defining, for $\delta \in 2^s, i < 2, C_{\delta^\frown\langle i\rangle} =_{df} \Sigma_\delta$. Now execute Procedure on $\langle\{C_\delta | i < 2, \delta \in 2^s\}, s, 0\rangle$. After the procedure call terminates, let $\langle\{C_\delta | \delta \in 2^{s+1}\}, t, z\rangle$ be the final values of the parameters. If the requirements are viable through t at stage $p+1$ with respect to $\{C_\delta | \delta \in 2^{s+1}\}$, then $\Sigma_\delta =_{df} C_\delta \cup \Sigma_\sigma$ for $\delta \in 2^{s+1}$, where $\sigma = \delta|_s$. Otherwise, choose $\{D_\delta | \delta \in 2^{s+1}\}, D_\delta \subseteq \{\pi_s, \neg\pi_s\}$ to maximize $u \leq s$ such that the requirements are viable through u at stage $p+1$ with respect to $\{D_\delta | \delta \in 2^{s+1}\}$ (choose π_s whenever both maximizes u). In this case, $\Sigma_\delta =_{df} D_\delta \cup \Sigma_\sigma$ for $\delta \in 2^{s+1}$, where $\sigma = \delta|_s$.

https://doi.org/10.1017/S1446788700030834 Published online by Cambridge University Press

Procedure $(\{C_\delta|\delta \in 2^{s+1}\}, t, z)$: if for every $\delta \in 2^{s+1}, \pi_s^k \in C_\delta$ for some $k = 0$ or 1, then the procedure ends. Otherwise fix the lexicographically least $\mu \in 2^{s+1}$ such that π_s^k is not in C_μ for any $k < 2$.

Execute Function on arguments $\langle \mu, \pi_s, \mu, \pi_s, t, z\rangle$, returning, let us say, $\langle \alpha, \theta, \beta, \psi, r, z\rangle$. If $r < t$ then we say that $S_{\langle\delta,i,j\rangle}$ *has acted*, where $S_{\langle\delta,i,j\rangle}$ is the requirement of priority $5r+4$. Now execute Procedure on $\langle\{C'_\sigma|\sigma \in 2^{s+1}\}, t, z+1\rangle$, where the C'_σ are defined as follows:

if $r = t$, then

$$C'_\sigma =_{df} \begin{cases} C_\alpha \cup \{\theta\}, & \sigma = \alpha, \\ C_\beta \cup \{\psi\}, & \sigma = \beta, \\ C_\sigma, & \text{otherwise;} \end{cases}$$

if $r < t, \sigma \in 2^{s+1}$ such that $\delta^\frown\langle 0\rangle \le \sigma$ and

$$\Gamma_{\mathrm{Amal}(r-1,1,p,\delta)}(R(r-1,1,p,\delta)) \vdash (\exists C_\sigma[\overline{x}_i]\# \leftrightarrow \exists C_\alpha[\overline{x}_i]\#),$$

then $C'_\sigma =_{df} C_\sigma \cup \{\theta\}$;

if $\delta^\frown\langle 1\rangle \le \sigma$ and

$$\Gamma_{\mathrm{Amal}(r-1,1,p,\delta)}(R(r-1,1,p,\delta)) \vdash (\exists C_\sigma[\overline{x}_j]\# \leftrightarrow \exists C_\beta[\overline{x}_j]\#),$$

then $C'_\sigma =_{df} C_\sigma \cup \{\psi\}$;

and for all other $\sigma \in 2^{s+1}$,

$$C'_\sigma =_{df} C_\sigma.$$

Function $(\delta, \theta, \gamma, \psi, r, z)$: determine first if there is an $S_{\langle\alpha,i,j\rangle}$ of priority greater than $5r+4$ that does not allow θ through s with respect to $\{C_\mu|\mu \in 2^{s+1}\}$. If there is no such requirement, then return $\langle\delta, \theta, \beta, \psi, r, z\rangle$. Otherwise fix the $S_{\langle\alpha,i,j\rangle}$ of highest priority $r' < r$ that does not allow θ. By Lemma 1 fix the lexicographically least $\beta, \lambda \in 2^{s+1}$ such that θ resolves $S_{\langle\alpha,i,j\rangle}$ past $\langle\beta,\lambda\rangle$ with respect to $\{C_\mu|\mu \in 2^{s+1}\}$. Let $\langle\pi, \neg\pi\rangle$ be the corresponding resolvent. Now execute Function on $\langle\lambda, \neg\pi, \beta, \pi, r', z\rangle$, and return whatever is returned.

Before the stage is over, we must tidy up. Determine the greatest $u \le s$ for which the requirements are viable through u at stage $p+1$ with respect to 0. For all $m \le u$, $i, j < 2$ and $\delta \in 2^s$

(1) if $m = 5v$ then $P(m, p+1, \delta^\frown\langle j\rangle) =_{df} P(m, p, \delta)$;

(2) if $m = 5v+2$ then $Q(m, [+1, \delta^\frown\langle j\rangle) =_{df} Q(m, p+1, \delta)$;

(3) if $m = 5v+2i+1$ then $\mathrm{Amal}(m, i, p+1, \delta^\frown\langle j\rangle) =_{df} \mathrm{Amal}(m, i, p, \delta)$.

For all m such that $u \le m \le s$, and for $i, j < 2$ and $\delta \in 2^s$

(1) if $m = 5v$ and P_m is active, then let $P(m, p+1, \delta^\frown\langle j\rangle) =_{df} \overline{x}_a \wedge \overline{x}_k$, where k is the least integer such that

(i) $\overline{x}_k$ has the same arity as Γ_m,

and

(ii) no element of $\overline{x}_k$ occurs in any formula of $\Sigma_{\delta^\frown\langle j\rangle}$;

(2) if $m = 5v$ and P_m is terminated, then $P(m, p+1, \delta^\frown\langle j\rangle) =_{df} \langle\ \rangle$;

(3) if $m = 5v+2$ and Q_m is active, then let $Q(m, p+1, \delta^\frown\langle j\rangle) =_{df} \overline{x}_a{}^\frown\overline{x}_k$, where $m = \langle n, c, a\rangle$ and k is the least integer such that

(i) $\overline{x}_a{}^\frown\overline{x}_k$ has the same arity as Γ_c, and

(ii) no element of $\overline{x}_k$ occurs in any formula of $\Sigma_{\delta^\frown\langle j\rangle}$;

(4) if $m = 5v+2$ and Q_m is terminated, then $Q(m, p+1, \delta^\frown\langle j\rangle) =_{df} \langle\ \rangle$;

(5) if $m = 5v + 2i + 1$ then $\mathrm{Amal}(m, i, p+1, \delta^\frown\langle j\rangle) =_{df} k$, where k is the least integer such that

(i) $k > \mathrm{Amal}(m, i, p, \delta)$ (if the latter is defined),

(ii) Γ_k has the same arity as $R(m, i, p+1, \delta^\frown\langle j\rangle)$.

This ends the stage and the construction.

LEMMA 2. *Each Function call in the construction terminates.*

PROOF. This is immediate, since the fifth argument on successive calls of Function from Function is strictly decreasing and bounded below by 0.

LEMMA 3. *Each Procedure call in the construction terminates.*

PROOF. It is enough to see that only finitely often at stages $2s+1$ can a given $\delta \in 2^{s+1}$ be the lexicographically least such that π_s^k does not belong to C_δ for either $k = 0$ or $k = 1$. So fix such a δ. Notice that, once δ is least such, then until π_s^k shows up in C_δ for some $k = 0$ or $k = 1$, Procedure continues to select δ on each of its subsequent calls from within Procedure. But why is the condition eventually satisfied? The only way it is not automatically satisfied as if Procedure calls Function on $\langle \delta, -, \delta, -, -, -\rangle$ which then returns $\langle \gamma, -, \beta, -, -, -\rangle$ for the sake of $S_{\langle \alpha, i, j\rangle}$. But then by the change in C_γ and C_β at that point in the construction, it can not happen that a subsequent Function call on $\langle \delta, -, \delta, -, -, -\rangle$ returns $\langle \gamma, -, \beta, -, -, -\rangle$ for the sake of $S_{\langle \alpha, i, j\rangle}$. This is because once the changes are made in C_γ and C_β, γ, β do not satisfy (2) and (3) in the definition of "allows". Therefore Function will be called from Procedure for the sake of δ at most $(s)(2^{2s})$ times.

LEMMA 4. $\forall f \in 2^\omega[\{\theta | \exists \alpha \leq f[\theta \in \Sigma_\alpha]\}$ *is the elementary diagram of a model of* T].

PROOF. This is routine to check. Notice that for each s, and $\alpha \leq f, \alpha \in 2^{s+1}, \pi_s^k \in \Sigma_\alpha$ for one of $k < 2$. Consistency is proved by induction and depends on the construction and the definition of 'allows' and 'resolves'. The details are left to the reader.

https://doi.org/10.1017/S1446788700030834 Published online by Cambridge University Press

DEFINITION. *$S_{\langle\delta,i,j\rangle}$ is clear through $\langle\alpha,\beta\rangle$ by s with respect to* $\{C_\sigma | \sigma \in 2^{s+1}\}$ just if $\forall\gamma,\lambda \in 2^{s+1}$ [where $\langle\delta,i,j\rangle = n, p = 2s, a = \mathrm{Amal}(n,1,p,\alpha)$, and $b = \mathrm{Amal}(n,1,p,\lambda)$],

if $\alpha \le \gamma$ and $\beta \le \lambda$ *then* either

$$\Gamma_a(R(n,1,p,\gamma) \cup \{\exists C_\gamma[\overline{x}_i]\#, \exists C_\lambda[\overline{x}_j] * (\overline{x}_i)\}$$

is not consistent, or

$$\Gamma_b(R(n,1,p,\lambda) \cup \{\exists C_{i\lambda}[\overline{x}_j]\#, \exists C_\gamma[\overline{x}_i] * (/\overline{x}_j)\}$$

is not consistent. If $C_\sigma = \Sigma_\sigma$, $\sigma \in 2^{s+1}$, then we just say that $S_{\langle\delta,i,j\rangle}$ *is clear through* $\langle\alpha,\beta\rangle$ *by stage* s.

LEMMA 5. *For all* $f \in 2^\omega$, *for all* $m < \omega$ *the following limits exist:*
(i) $\mathrm{Lim}_s P(m,s,f|_s)$;
(ii) $\mathrm{Lim}_s R(m,0,s,f|_s)$;
(iii) $\mathrm{Lim}_s \mathrm{Amal}(m,0,s,f|_s)$;
(iv) $\mathrm{Lim}_s Q(m,s,f|_s)$;
(v) $\mathrm{Lim}_s R(m,1,s,f|_s)$;
(vi) $\mathrm{Lim}_s \mathrm{Amal}(m,1,s,f|_s)$.
And
(vii) S_m *acts only finitely often.*

PROOF. The proof is by induction on m and for all $f \in 2^\omega$ simultaneously. So fix such an f and assume that the lemma is true for all $n < m$. If $m = 0$ or Γ_m ever dies, then by the construction it is easy to see that $\mathrm{Lim}_s P(m,s,f|_s) = P(m,0,\langle\ \rangle)$ or $\langle\ \rangle$ and that (ii) and (iii) also hold. So assume that $m > 0$ and Γ_a never dies, and let $a = \mathrm{Lim}_s \mathrm{Amal}(m-1,1,s,f|_s)$ (by the induction hypothesis). Also, fix a stage s_0 such that all the limits for smaller $n < m$ for (i)–(vi) have been permanently achieved, and such that no S_n acts after stage s_0. Since Γ_a never dies, $\Gamma_m \in \mathrm{TySp}(\mathfrak{A})$. Also, we have that $\Gamma_a \in \mathrm{TySp}(\mathfrak{A})$. Therefore there is some type $\Sigma \in \mathrm{TySp}(\mathfrak{A})$ such that $\Gamma_a(\overline{z}) \cup \Gamma_m(\overline{w}) \subseteq \Sigma(\overline{z}\frown\overline{w})$. By the construction, it follows that if $\mathrm{Amal}(m,0,t,f|_t)$ is ever defined as a witness for Σ for some $t > s_0$, then $\mathrm{Lim}_s \mathrm{Amal}(m,0,s,f|_s) = \mathrm{Amal}(m,0,t,f|_t)$. But since each type in $\mathrm{TySp}(\mathfrak{A})$ has infinitely many representatives, it follows from the construction that the only way such a witness for Σ could be avoided as if $\mathrm{Lim}_s \mathrm{Amal}(m,0,s,f|_s)$ existed and were finite. But then clearly, from the construction, (i) and (ii) also hold. In either case we have the lemma for (i)–(iii).

Next let $m = \langle u,v,a\rangle$. If Γ_u or Γ_v ever die, or it is not the case that $\Gamma_u \subseteq \Gamma_v$, or $\overline{x}_a$ does not realize Γ_u, then from the construction it is clear that $\mathrm{Lim}_s Q(m, s, f|_s) = \langle\ \rangle$ and that (v)–(vi) also holds. So assume otherwise and fix s_0 as above, and let $a = \mathrm{Lim}_s\mathrm{Amal}(m, 0, s, f|_s)$. Again we have that $\Gamma_v, \Gamma_a \in \mathrm{TySp}(\mathfrak{A})$, and thus there is a $\Sigma \in \mathrm{TySp}(\mathfrak{A})$ such that $\Gamma_a(\overline{z}) \cup \Gamma_v(\overline{w}) \subseteq \Sigma(\overline{z}^\frown\overline{w})$. Now the argument is just as before.

Finally we must show that (vii) holds. Let $m = \langle\delta, i, j\rangle$. By the induction hypothesis, our proof of (i)–(vi), and the compactness of 2^ω, fix an r such that for any $f \in 2^\omega$ the various limits in (i)–(v) for $n \leq m$ have been permanently achieved. Also pick r large enough that no S_n acts after stage r, for $n < m$. It is now sufficient to show that if $S_{\langle\delta,i,j\rangle}$ acts at stage $2s+1 > 2r+1$, then each time it does so it becomes clear through some $\langle\alpha, \beta\rangle$ by s such that it was not clear through $\langle\alpha, \beta\rangle$ by $s-1$, where $\alpha, \beta \in 2^r$; this is because there are less than 2^{2r} such $\langle\alpha, \beta\rangle$. Therefore fix a stage $2t+1 > 2r+1$ where $S_{\langle\delta,i,j\rangle}$ acted. Let $\{C_\sigma | \sigma \in 2^{t+1}\}$ be the value of the Cs just before Function returns, let us say $\langle\gamma, \pi, \lambda, \neg\pi, v\rangle$, to Procedure at stage $2t+1$, where the priority of $S_{\langle\delta,i,j\rangle}$ is $v < t$. So in particular $S_{\langle\delta,i,j\rangle}$ is not clear through $\langle\alpha, \beta\rangle$ by t with respect to $\{C_\sigma | \sigma \in 2^{t+1}\}$, where $\alpha \leq \gamma$ and $\beta \leq \lambda$, $\alpha, \beta \in 2^r$. Now by the construction we are done as long as for all $\gamma', \lambda' \in 2^{t+1}$ such that $\alpha \leq \gamma'$ and $\beta \leq \lambda'$ (where $a = \mathrm{Lim}_s\mathrm{Amal}(m, 1, s, \alpha), \overline{y} = \mathrm{Lim}_s R(m, 1, s, \alpha)$, $b = \mathrm{Lim}_s\mathrm{Amal}(m, 1, s, \beta)$, and $\overline{z} = \mathrm{Lim}_s R(m, 1, s, \beta)$),

$$\Gamma_a(\overline{y}) \vdash (\exists\sigma_\gamma[\overline{x}_i]\# \leftrightarrow \exists\sigma_{\gamma'}[\overline{x}_i]\#)$$

and

$$\Gamma_b(\overline{z}) \vdash (\exists\sigma_\lambda[\overline{x}_j]\# \leftrightarrow \exists\sigma_{\lambda'}[\overline{x}_j]\#).$$

Assume that this is false in order to obtain a contradiction. Fix the least s, $r < s \leq t$, such that there are $\gamma, \gamma', \lambda, \lambda' \in 2^{s+1}$, $\alpha \leq \gamma, \gamma'$ and $\beta \leq \lambda, \lambda'$ such that the above fails. Now fix the least z during stage $2s+1$ such that Procedure on some $\langle\{C_\delta | \delta \in 2^{s+1}\}, t, z\rangle$ satisfying

$$\Gamma_a(\overline{y}) \vdash (\exists C_\gamma[\overline{x}_i]\# \leftrightarrow \exists C_{\gamma'}[\overline{x}_i]\#)$$

and

$$\Gamma_b(\overline{z}) \vdash (\exists C_\lambda[\overline{x}_j]\# \leftrightarrow \exists C_{\lambda'}[\overline{x}_j]\#)$$

returns $\langle\{D_\delta | \delta \in 2^{s+1}\}, u, z+1\rangle$ such that one of the following fails:

$$\Gamma_a(\overline{y}) \vdash (\exists D_\gamma[\overline{x}_i]\# \leftrightarrow \exists D_{\gamma'}[\overline{x}_i]\#),$$
$$\Gamma_b(\overline{z}) \vdash (\exists D_\lambda[\overline{x}_j]\# \leftarrow \exists D_{\lambda'}[\overline{x}_j]\#).$$

Therefore $u < s$ and let us assume the first one fails; the other is argued similarly. Let $\langle\xi, \theta, \chi, \psi, u, z+1\rangle$ be the last Function return and $S_{\langle\sigma,k,1\rangle}$ be the requirement of priority $5u+4$. Notice that by our choice of r, $u \geq v$. Also it must be that $\sigma^\frown\langle p\rangle \leq \gamma$ or $\sigma^\frown\langle p\rangle \leq \gamma'$ for some $p < 2$. Let us assume

$\sigma^\frown\langle 0\rangle \le \gamma$; all the other cases are argued similarly. It is easy to check then that

$$\Gamma_a(\overline{y}) \cup \{\exists C_\gamma[\overline{x}_i]\#, \exists(C_\gamma \cup \{\theta\})[\overline{x}_i]\#\}$$

is consistent and

$$\Gamma_a(\overline{y}) \cup \{\exists C_\gamma[\overline{x}_i]\#, \neg(\exists(C_\gamma \cup \{\theta\})[\overline{x}_i]\#)\}$$

is consistent. Therefore, since $s \le t$ and thus $R_{\langle\delta,i,j\rangle}$ is not cleared through $\langle\alpha,\beta\rangle$ by s with respect to $\{C_\sigma | \sigma \in 2^{s+1}\}$, $R_{\langle\delta,i,j\rangle}$ does not allow θ through s with respect to $\{C_\sigma | \sigma \in 2^{s+1}\}$. But then by the definition of Function in the construction, it must be that $u = v$ and so $S_{\langle\sigma,k,1\rangle}$ is actually $S_{\langle\delta,i,j\rangle}$. However, since by our assumptions

$$\Gamma_a(\overline{y}) \vdash (\exists C_\gamma[\overline{x}_i]\# \leftrightarrow \exists C_{\gamma'}[\overline{x}_i]\#),$$

then by the construction the same remains true after the redefinition, so this is a contradiction. This completes the proof of the lemma.

LEMMA 6. *For every $S_{\langle\delta,i,j\rangle}$ there is an s such that $\forall\alpha,\beta \in 2^{s+1}$,*

(I) $\forall f \in S^\omega \forall t > s$ *if* $\alpha \le f$ *then* $\mathrm{Amal}(m,1,t,f|_t) = \mathrm{Amal}(m,1,s,f|_s)$ *and* $R(m,1,t,f|_t) = R(m,1,s,f|_s)$; *and*

(II) *if*

(1) $\delta^\frown\langle 0\rangle \le \alpha$ [*let* $\mathrm{Amal}(m,1,s,\alpha) = a$ *and* $R(m,1,s,\alpha) = \overline{y}$],

(2) $\delta^\frown\langle 1\rangle \le \beta$ [*let* $\mathrm{Amal}(m,1,s,\beta) = b$ *and* $R(m,1,s,\beta) = \overline{z}$],

(3) $\Gamma_a(\overline{y}) \cup \Sigma_\alpha \cup \{\exists\Sigma_\beta[\overline{x}_j] * (/\overline{x}_i)\}$ *is consistent,*

and

(4) $\Gamma_b(\overline{z}) \cup \Sigma_\beta \cup \{\exists\Sigma_\alpha[\overline{x}_i] * (/\overline{x}_j)\}$ *is consistent,*

then

$$\Sigma_\alpha[\overline{x}_i]\# \text{ is a complete formula in the theory } \Gamma_a(\overline{y}).$$

PROOF. Let v be the priority of $S_{\langle\delta,i,j\rangle}$ and fix an s such that no requirement of priority $u \le v$ acts after stage $2s$, and all limits associated with requirements of higher priority have permanently achieved their final value; this can be done by Lemma 5. It is enough to show that this s works for the lemma. Assume not, in order to obtain a contradiction. Fix $\alpha,\beta \in 2^{s+1}$ so that (1)–(4) above are true, but for which the conclusion fails. Since $\exists\Sigma_\alpha[\overline{x}_i]\#$ is not a complete formula in $\Gamma_a(\overline{y})$, there is an $r \ge s$ such that $\Gamma_a(\overline{y}) \cup \Sigma_\alpha \cup \{\exists(\Sigma_\alpha \cup \{\pi_r\})[\overline{x}_i]\#\}$ and $\Gamma_a(\overline{y}) \cup \Sigma_\alpha \cup \{\neg\exists(\Sigma_\alpha \cup \{\pi_r\})[\overline{x}_i]\#\}$ are both consistent. Thus $S_{\langle\delta,i,j\rangle}$ does not allow π_r through s with respect to $\{\Sigma_\sigma | \sigma \in 2^{s+1}\}$. By the construction, it must be that by the end of stage $2r+1$, $S_{\langle\delta,i,j\rangle}$ does allow π_r through r with respect to $\{\Sigma_\sigma | \sigma \in 2^{r+1}\}$. Therefore fix the greatest t such that $S_{\langle\delta,i,j\rangle}$ does allow π_r through t with respect to $\{\Sigma_\sigma | \sigma \in 2^{t+1}\}$. Fix the least z during stage $2t+1$ such that $S_{\langle\delta,i,j\rangle}$ does

https://doi.org/10.1017/S1446788700030834 Published online by Cambridge University Press

not allow π_r through t with respect to $\{C_\sigma | \sigma \in 2^{t+1}\}$, but $S_{\langle\delta,i,j\rangle}$ does allow π_r through t with respect to $\{D_\sigma | \sigma \in 2^{t+1}\}$, where Procedure returns $\langle\{D_\delta | \delta \in 2^{t+1}\}, w, z+1\rangle$ from input $\langle\{C_\delta | \delta \in 2^{t+1}\}, v, z\rangle$. Then it is not difficult to check that some requirement of priority $u \leq v$ must have acted at stage $2t+1$. Since $t > s$ we have the desired contradiction.

Now we complete the proof of the theorem. Define $F(\alpha) =_{df} \Sigma_\alpha$ for $\alpha \in 2^{<\omega}$. By Lemma 5 and the construction it follows that F is a recursive model tree. Now suppose that $f, g \in 2^\omega$. We claim that every type realized in both $\mathfrak{A}_f$ and $\mathfrak{A}_g$ is also realized in $\mathfrak{A}$. For suppose that $\overline{y} \in |\mathfrak{A}_f|^n$ and $\overline{z} \in |\mathfrak{A}_g|^n$ realize the same type, for some n. Fix the corresponding $\overline{x}_i$ and $\overline{x}_j$ respectively. Let $\delta \in 2^{<\omega}$ be such that for some $k < 2, \delta^\frown\langle k\rangle \leq f$ and $\delta^\frown\langle 1-k\rangle \leq g$, and let $m = \langle\delta, i, j\rangle$ (assume without loss that $i = 0$). If $\overline{y}$ and $\overline{z}$ were to realize the same type, then, letting $a = \mathrm{Lim}_t\mathrm{Amal}(m, 1, t, f|_t), \overline{w} = \mathrm{Lim}_t R(m, 1, t, f|_t), b = \mathrm{Lim}_t\mathrm{Amal}(m, 1, t, g|_t)$ and $\overline{u} = \mathrm{Lim}_t R(m, 1, t, g|_t)$, we would have that

$$\Gamma_a(\overline{w}) \cup \{\exists\theta[\overline{x}_i]\# | \forall\alpha \leq f[\theta \in \Sigma_\alpha]\} \cup \{\exists\theta[\overline{x}_i] * (/\overline{x}_j) | \forall\beta \leq g[\theta \in \Sigma_\beta]\}$$

and

$$\Gamma_b(\overline{u}) \cup \{\exists\theta[\overline{x}_j]\# | \forall\beta \leq g[\theta \in \Sigma_\beta]\} \cup \{\exists\theta[\overline{x}_j] * (/\overline{x}_i) | \forall\alpha \leq f[\theta \in \Sigma_\alpha]\}$$

would be consistent. From this and Lemma 9, there would be an $\alpha \leq f$ such that $\{\exists\theta[\overline{x}_i]\# | \exists\beta \leq \alpha[\theta \in \Sigma_\beta]\}$ generates a principal type in $\Gamma_a(\overline{w})$. But then since Γ_a is realized in $\mathfrak{A}$, the type realized by $\overline{y}$ in $\mathfrak{A}_f$ must also be realized in $\mathfrak{A}$. This establishes the claim.

Thus, since T is assumed to have only countably many complete types, at most countably many $\mathfrak{A}_f$ realize a type that is not realized in $\mathfrak{A}$. But then that says that all but at most countably many of the $\mathfrak{A}_f$ satisfy $\mathrm{TySp}(\mathfrak{A}_f) \subseteq \mathrm{TySp}(\mathfrak{A})$. By Lemma 8(i)–(iii) and the construction it follows that for every $f \in 2^\omega$, $\mathrm{TySp}(\mathfrak{A}) \subseteq \mathrm{TySp}(\mathfrak{A}_f)$. Therefore, for all but countably many $f \in 2^\omega$, $\mathrm{TySp}(\mathfrak{A}_f) = \mathrm{TySp}(\mathfrak{A})$. Finally, by Lemma 8(iv)–(vi) and the construction, it follows that except for countably many $f \in 2^\omega$, $\mathfrak{A}_f$ is homogeneous. So, except for countably many $f \in 2^\omega$, $\mathfrak{A}_f$ and $\mathfrak{A}$ are isomorphic. This concludes the proof of the theorem.

References

[1] T. Millar, 'Prime models and almost decidability', *J. Symbolic Logic* **51** (1986), 412–420.

[2] T. Millar, 'Type structure complexity and decidability', *Trans. Amer. Math. Soc.* **271** (1981), 73–81.

https://doi.org/10.1017/S1446788700030834 Published online by Cambridge University Press

[3] T. Millar, 'Homogeneous models and decidability', *Pacific J. Math.* **91** (1980), 407–418.
[4] T. Millar, 'Decidability and the number of countable models', *Ann. Pure Appl. Logic* **27** (1984), 137–153.

Department of Mathematics
University of Wisconsin
Madison, Wisconsin 53706
U.S.A.