Homogeneous Ricci soliton hypersurfaces in the complex hyperbolic spaces

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Submanifold Theory in Symmetric Spaces
AND Lie Theory in Finite and Infinite Dimensions

Introduction (1/2)

Our interests are

- **geometry of homogeneous submanifolds(i.e., orbits) in symmetric spaces of noncompact type, and**
- **left-invariant geometric structures on Lie groups.**

Today's talk is about

- **Ricci soliton Lie hypersurfaces in $\mathbb{C}\mathrm{H}^n$.**
  - **Ricci soliton: a generalization of Einstein manifolds,**
  - **Lie hypersurf.: a kind of homog. hypersurf., (they can be isometric to with some Lie gr. with left-invariant metric)**

Introduction (2/2)

▼ Main result is

- the classification of Ricci soliton Lie hypersurf. in $\mathbb{C}\mathrm{H}^n$.
- More precisely,

Thm. (Hashinaga & K. & Tamaru, in preparation)

$M \subset \mathbb{C}\mathrm{H}^n$ is a Ricci soliton Lie hypersurf. $\Leftrightarrow$

- if $n > 2$, $M \cong$ a horosphere,
- if $n = 2$, $M \cong$ a horosphere or the homog. minimal ruled hypersurf.

Here, "$\cong$" means "be isometrically congruent to."

Ricci soliton

Def:

Def.

(M, g) is a <u>Ricci soliton</u> $:\Leftrightarrow$

$$\exists c \in \mathbb{R}, X \in \mathfrak{X}(M) \text{ s.t. } \mathrm{ric}_g = cg + \mathfrak{L}_X g,$$

- ric_g is the Ricci tensor of (M, g),
- $\mathfrak{L}_X$ is the Lie derivative.

Remark:

- Einstein $\Rightarrow$ Ricci soliton.
- Homogeneous / left-invariant Ricci solitons provide interesting examples of Ricci solitons.

Lie hypersurface

Def:

- G/K : a symm. sp. of noncpt. type,
- $G = KAN$: the Iwasawa decomp. of G,
- $S := AN$: the solvable part of the Iwasawa decomp.

Def.

Lie hypersurface := the orbit $S'.p$ $(S \supset S' : \mathrm{codim}\, 1)$.

Remark:

- in this case, the action $S' \curvearrowright G/K$
  - is of cohomogeneity 1, and
  - has no singular orbits.
- Lie hypersurf. $S'.p$ is isometric to Lie gr.

Motivation and Background

▼ **Backgroud:**

- We consider the actions $S' \curvearrowright G/K$ for some $S' \subset S$.
- They provide good examples of homog. submfd. in G/K.

▼ **Recent studies show that:**

Fact

- $N.p$ is a Ricci soliton (Lauret, 2001),
- S_Φ : the solvable part of a parabolic subgr. of $G \Rightarrow$ $(S_\Phi).p$ is Einstein (Tamaru, 2011, K.-Tamaru, to appear)

▼ **In this talk:**

- we consider the case when $S \supset S'$: $\mathrm{codim}\, 1$.

We introduce a Lie alg. $\mathfrak{s}$:

- **which is the solvable part of the Iwasawa decomp. of**

$$\mathbb{C}\mathrm{H}^n = \mathrm{SU}(1,n)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n)),$$

- **to construct Lie hypersurf. in terms of Lie alg.**

Lie alg. $\mathfrak{a}$, $\mathfrak{n}$, and $\mathfrak{s}$:

- $\mathfrak{a} = \mathrm{span}\{A_0\}$, $\mathfrak{n} = \mathrm{span}\{X_1, Y_1, \ldots, X_{n-1}, Y_{n-1}, Z_0\}$.
- $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n} = \mathrm{span}\{A_0, X_1, Y_1, \ldots, X_{n-1}, Y_{n-1}, Z_0\}$ **s.t.**

$$[A_0, X_i] = (1/2)X_i, [A_0, Y_i] = (1/2)Y_i,$$

$$[A_0, Z_0] = Z_0, [X_i, Y_i] = Z_0.$$

- $\langle , \rangle_0$ **: inner product(metric) on $\mathfrak{s}$ s.t. the above is o.n.b. then, $(\mathfrak{s}, \langle , \rangle_0)$ is isometric to $\mathbb{C}\mathrm{H}^n$**

▼ $\mathrm{codim}\,1$ **Lie subalg.** $\mathfrak{s}(\theta)$, **and Lie subgr.** $S(\theta)$ **:**

Def.

For $\theta \in [0, \pi/2]$,

- $\mathfrak{s}(\theta) := \mathfrak{s} \ominus \mathrm{span}\{\cos(\theta) X_1 + \sin(\theta) A_0\}$
- $S(\theta)$: the connected Lie subgr. of S with Lie alg. $\mathfrak{s}(\theta)$

- Note that
  - $S(\theta)$ is a simply-connected Lie gr.
  - $\mathfrak{s}(\pi/2) = \mathfrak{n}$ $(S(\pi/2) = N)$.

Lie hypersurfaces in $\mathbb{C}\mathrm{H}^n$ (3/3)

Congruency class of Lie hypersurf. in $\mathbb{C}\mathrm{H}^n$:

Thm. (Berndt, 1998)

$\{$Lie hypersurf. in $\mathbb{C}\mathrm{H}^n\}/_{\cong} = \{S(\theta).o \mid \theta \in [0, \pi/2]\}$.

- that is, every Lie hypersurf. in $\mathbb{C}\mathrm{H}^n \cong$

  (1) $\theta = \pi/2$;

  $S(\pi/2).o = N.o$: a horosphere,

  (2) $\theta = 0$;

  $S(0).o$: the homog. minimal ruled hypersurf.,

  (3) $\theta \in (0, \pi/2)$;

  $S(\theta).o \cong S(0).p$: the equidistant hypersurf. of $S(0).o$.

Main result (1/2)

- **Recall that**
  - (M, g) **is a** **Ricci soliton**
    $:\Leftrightarrow \exists c \in \mathbb{R}, X \in \mathfrak{X}(M)$ **s.t.** $\mathrm{ric}_g = cg + \mathfrak{L}_X g$.
  - **Einstein** $\Rightarrow$ **Ricci soliton.**
- **It is known that**
  - $\nexists$ **hypersurf. in** $\mathbb{C}\mathrm{H}^n$ **s.t. Einstein, and**
  - **a horosphere** $(= S(\pi/2).o = N.o)$ **is a Ricci soliton.**

Main result (2/2)

- **Main result:**

Thm. (Hashinaga & K. & Tamaru, in preparation)

Lie hypersurf. $S(\theta).o$ is a Ricci soliton $\Leftrightarrow$

- if $n > 2$, $\theta = \pi/2$,
- if $n = 2$, $\theta = 0, \pi/2$.

- **Recall:**
  - $S(\pi/2).o$: a horosphere.
  - $S(0).o$: the homog. minimal ruled hypersurf.

Proof of main result (1/2)

Algebraic Ricci soliton:

- G : simply-connected Lie gr. , $(G, g) \leftrightarrow (\mathfrak{g}, \langle , \rangle)$.
- (G, g) is an <u>algebraic Ricci soliton</u>

  $:\Leftrightarrow \exists c \in \mathbb{R}, \exists D \in \mathrm{Der}(\mathfrak{g})$ s.t. $\mathrm{Ric}_{\langle , \rangle} = c \cdot \mathrm{id} + D$.

It is a key theorem of our proof that :

Thm. (Lauret, 2011)

For (G, g) a simply-connected Lie gr.,

- algebraic Ricci soliton $\Rightarrow$ Ricci soliton, and
- if G is completely solvable (i.e., solvable and the all eigenvalues of $\mathrm{ad}\, X$ are real),

  Ricci soliton $\Rightarrow$ algebraic Ricci soliton.

Proof of main result (2/2)

Sketch of proof:

- Identify Lie hypersurf. $S(\theta).o \leftrightarrow$ Lie gr. $S(\theta)$.
- Check that

Lem.

$S(\theta)$ is completely solvable.

- and, $S(\theta)$ is a Ricci soliton $\Leftrightarrow$ an alg. Ricci soliton.
- Recall that $\mathrm{Ric}_{\langle,\rangle} = c \cdot \mathrm{id} + D$ $(c \in \mathbb{R}, D \in \mathrm{Der})$.
- $\mathrm{Ric}_{\langle,\rangle}$ of $\mathfrak{s}(\theta)$ has been calculated (Hamada - Hoshikawa - Tamaru, 2012).
- We have only to calculate Der of $\mathfrak{s}(\theta)$.

Conclusion and further problems

Conclusion: in this talk

- **We study "Ricci soliton Lie hypersurf. in $\mathbb{C}\mathrm{H}^n$."**

Further problems:

- **Examine geometry of the orbits of**
  - **the actions $S' \curvearrowright G/K$ for some $S' \subset S$,**
  - **for example, some polar actions on $\mathbb{C}\mathrm{H}^n$.**

Thank you for your attention!