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Homogenization and Global Responses of Inhomogeneous Spherical Nonlinear Elastic Shells

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Abstract. Homogenization of radially inhomogeneous spherical nonlinear elastic shells subject to internal pressure is studied. The equivalent homogeneous material is defined in such a way that it gives rise to exactly the same global response to the pressure load as that of the inhomogeneous shell. For a shell with general strain–energy function and inhomogeniety, the strain–energy function of the equivalent homogeneous material is determined explicitly. The resulting formula is used to study layered composite shells. The equivalent homogeneous material for an infinitely fine layered composite shell is examined, and is found to give not only the same global response, but also the same average stress field as the composite shell does.

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1. Introduction

Homogenization theory has played a fundamental role in the development of mathematical models for describing the constitutive response of composite materials. The basic idea is to identify a homogeneous material that is equivalent, in a certain sense, to a given inhomogeneous material. This idea assumes great importance from both the theoretical and practical perspective. For this reason, considerable research effort has been expended in developing homogenization models during the last four decades. A fair sample of the works, which is by no means exhaustive, includes those by Eshelby [2], Hill [6, 7], Christensen [1], Willis [29, 30], Hashin [4], Francfort and Murat [3], Milton and Kohn [12], Kohn [9], Nemat-Nasser and Hori [14], Ponte Castañeda [18], Suquet [27], Ponte Castañeda and Suquet [19], Milton [11], and Sahimi [22, 23]. While it is plausible that a composite material may be replaced, with regard to its response characteristics, by a homogeneous material in a length scale much larger than

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that of the inhomogeneity, a complete and rigorous treatment of such a mathematical equivalence does not appear available.

To be precisely defined, such an equivalent homogeneous material must be understood as the limiting behavior of a sequence of composite materials with increasing finer microstructures of common characteristics. This idea is expounded in the review article by Kohn [9], that summarizes the theoretical development of mathematical modeling of composite materials up to 1988. For composites of linear elastic materials, an attempt is made to precisely define effective moduli, i.e., the components of the elasticity tensor of the equivalent homogeneous material. To this end, a sequence of composites involving a parameter ϵ is introduced. For periodic composites and random composites, ϵ represents the length scale of the microstructure. The characteristic functions of this sequence of composites are generated by re-scaling a basic characteristic function through ϵ. For a given load, the elastostatic solutions of this sequence of composites give a sequence of displacement fields u^ϵ. The effective moduli are defined as those that produce, under the same load, a displacement field that is the limit of u^ϵ as $\epsilon \to 0$.

This definition is certainly precise and physically well motivated. The existence of effective moduli so defined, however, has not been fully established. Questions arise as to whether the limit of u^ϵ exists for a given load, and if the limit does exist, whether there exist elastic moduli that produce the limiting displacement field for the given load. Of course, even after the existence of such effective moduli is established for *all* possible loads, there remains the question whether the effective moduli are the same for all the loads. The concept of effective moduli would not be useful if the latter uniqueness issue is not resolved. To our knowledge, these issues are at best partially addressed for some special composite structures. For composites of nonlinear elastic materials, the problems of the existence and uniqueness of equivalent homogeneous materials are even more challenging. Among few others, a rigorous analysis is given by Müller [13] for the integral of a scalar function which is spatially periodical with a periodicity parameter ϵ. In the physical context, the scalar function in question can be the strain–energy function of a periodically inhomogeneous elastic material. It is shown that there exist a "homogenized" energy function, whose integral is the limit of the integral of the inhomogeneous energy function as ϵ approaches 0. In this cited work, the displacement field is taken to be independent of the value of ϵ, which is not the case for solutions of the elastic composites with varying inhomogeneity.

The above definition of the effective moduli through limit processes does not seem to lead to a practical way of finding the effective moduli. For this reason, researchers have developed various "operation-based" definitions. A detailed review of some important developments can be found in the paper by Ponte Castañeda and Suquet [19]. A common approach is to use a representative volume element and the spatial averages of the field variables over the volume

element. For an elastic material, it has been shown that if the representative volume element is subjected to a homogeneous (affine) displacement boundary condition, then the derivative of the average strain–energy function with respect to the average strain equals the average stress. The average strain–energy function so obtained is then taken as the effective strain–energy function for the composite. The effective moduli can be obtained by taking the second order derivative of the effective strain–energy function.

This approach, intuitively reasonable, of replacing a complicated structure by a simple one allows great facility in developing computational models for the effective properties of composites. It avoids all together the difficulty associated with proving the existence of an equivalent homogeneous material. However, the issue of how "effective" the effective properties are, in terms of describing the behavior of real composites, becomes rather elusive. A representative volume element is clearly an idealization. To what extent this idealization is valid is rarely quantified. For a given composite structure, a number of different representative volume elements are often proposed, which lead to different models to predict effective properties. While much effort has been devoted to comparing the different models, little has been done to compare a model with the real composite. Furthermore, the limitation of using homogeneous boundary conditions has not been fully addressed. Indeed, spatially the stress and strain can vary rapidly in a composite material even when it is subjected to a homogeneous boundary condition.

As much progress has been made in deriving effective properties of the linear elastic composite materials, relatively less has been done for large deformations of composite materials with nonlinear constitutive functions. Many concepts developed for small deformation deformations of linear composites need be re-examined. For example, the volume average of a stress (or strain) measure may not correspond to the boundary data measured in experiment. Hill [8] addressed this issue, arguing that macro-variables defined in terms of surface data are *not* necessary volume averages of their microscopic counterparts. Indeed, he shows that the only stress whose macroscopic value always coincides with its volume average is that formed from the contravariant components of Kirchhoff stress. Hill's analysis does not invoke an overall constitutive function that relates the macroscopic measures of stress and deformation. However, the theoretic development in that direction is clearly implied.

Such an overall constitutive function, if it exists, would play a central role in the analysis of nonlinear composites. In this light, considerable research efforts have been devoted to studying its properties, assuming that it does exist. One approach, in parallel to that for linear composites, has been to define a macroscopic energy in terms of the macroscopic deformation, with the presumption that the derivative of the macroscopic energy would give the macroscopic stress. Among others, Ogden [15] studied the properties of the macroscopic energy using the extremum principles, and established upper and

lower bounds on this energy. Various improved bounds [17, 21, 28] have been subsequently derived. It is noted that a bound on the energy function does not in general lead to a bound on the stress itself.

As summarized in Ponte Castañeda and Suquet's review article [19], most research efforts on nonlinear composites are concentrated on plasticity and creep deformation of polycrystals with special forms (such as power law) for the constitutive functions. It is usually assumed that the effective properties are described by the constitutive functions of the same type as those of the constituents. Often small deformation is assumed despite that the material behavior is nonlinear. The analysis for large deformations of nonlinear elastic composites is lacking to a large extent. In a series of papers, Saravanan and Rajagopal [24 – 26] have shown that if the same types of constitutive functions are taken for the equivalent homogeneous material and the composite, then using the equivalence of energy or the equivalence of moduli is geometry dependent for a special deformation, and is deformation dependent for a special geometry. Thus, different values of the equivalent moduli can be obtained when correlations are made with respect to different experiments.

Of course, the merit of the existing theories for composite materials should not be understated. Among other things, they overcome the difficulty associated with the direct stress analysis of a real composite structure. Ultimately, this is the very reason for the development of these theories – to provide mathematical models to predict the macroscopic behavior of composite materials. The importance of verifying that a model does indeed give reasonably accurate description of the global response of a composite material to the loads is not to be ignored. In this regard, the recent work of Lahellec, Mazerolle and Michel [10] of comparing the theoretical prediction with the finite-element solutions and the experimental measurements should be highly appreciated. Of course, a numerical solution, or an experiment, is for a particular material and geometry, and therefore cannot lead to the general verification of a model, as effectively as an analytic solution can.

While such a verification is very difficult for most composites, there are composite structures for which analytic solutions of the equilibrium equations are available. This makes it possible to compare the exact solution with the solution of the homogenization model, or to evaluate the assumptions made in the model. One such structure, a radially inhomogeneous spherical shell under internal pressure, is studied in this work. The analytic solution can be obtained for general incompressible isotropic elastic materials with arbitrary radial inhomogeneity. This enables us to define the equivalent homogeneous material as one which gives precisely the same global response as the inhomogeneous shell does. In particular, the inhomogeneous shell and the equivalent homogeneous shell cannot be distinguished by the experiment that measures the boundary displacements and the boundary tractions. Although this is not the definition adopted by most researchers, it is our belief that a good micromechanics-based definition

should be at least consistent with the definition based on equivalent global response of the inhomogeneous and the homogenized bodies. After all, a theory, however sophisticated, would be of little value if it gives grossly wrong predictions with respect to the global response of a composite.

In Section 2, the boundary value problem of a spherical shell subject to uniform internal pressure is formulated and solved. The shell is composed of a radially inhomogeneous, incompressible, isotropic, elastic material. The form of the strain–energy function, as well as the radial inhomogeniety, are left arbitrary. Nonlinear elasticity theory is used to find the exact solution for large deformations of the shell. The global response of the shell, i.e., the relationship between the displacement and the pressure is obtained.

The equivalent homogeneous material, defined as one that gives exactly the same global response as the inhomogeneous shell, is studied in Section 3. We do not assume that the strain–energy function for the equivalent homogeneous material has a particular form as related to that of the inhomogeneous shell. It is to be determined purely by the equivalence requirement. By solving an integro-differential equation, we explicitly establish the existence of the equivalent homogeneous material. The strain–energy function of this material, which is unique in terms of its restriction to spherically symmetric deformations, is expressed in terms of the strain–energy function of the inhomogeneous material.

A spherical shell composed of two layers of neo-Hookean materials is studied in Section 4. The strain–energy function of the equivalent homogeneous material is found in series form. A closed-form expression for the strain–energy function does not appear possible. However, by an asymptotic analysis, we show that the equivalent homogeneous material is *not* neo-Hookean. It is worth noting that a parallel can be drawn between the present solution and the solution of Hashin [5] for a Mooney–Rivlin hollow spherical elastic shell under uniform tractions on the outer surface. He stated that the exact solution of this boundary value problem defines the effective stress–strain relation of the equivalent homogeneous material. The focus of his work was not on characterizing the constitutive function of the equivalent homogeneous material for spherical shells of general elastic materials. Nevertheless, it can be readily shown from his solution that the equivalent homogeneous material is not of the Mooney–Rivlin type.

In the concluding Section 5 of the present paper, a much more complicated composite shell is studied, that is composed of an arbitrary number of layers of two homogeneous, incompressible, isotropic, elastic materials with strain–energy functions of general form. The local volume fraction of each component is assumed to be constant along the radius. Although the expression for the strain–energy function of the equivalent homogeneous material, involving a double series, is hopelessly complicated, we find that this expression reduces to a very simple form when passing to the limit as the number of the layers tends to infinity and the thickness of each layer tends to zero. The effective strain–energy function for this composite of infinitely fine microstructures actually follows a

simple mixture rule. Finally, the average strain and the average stress for this infinitely fine structure are studied. It is shown that they agree with the strain and stress fields of the equivalent homogeneous shell.

2. Response of an Inhomogeneous Spherical Shell to Internal Pressure

Consider a spherical shell which occupies, in a reference configuration, the region

$$\Omega \equiv \{(R, \Theta, \Phi) : A \leq R \leq B, 0 \leq \Theta \leq \pi, 0 \leq \Phi \leq 2\pi\},$$

where R, Θ and Φ are spherical polar coordinates, and A and B are the inner and outer radii of the undeformed shell.

The shell is assumed to be radially inhomogeneous. Subject to a load of uniform internal pressure p, the shell undergoes a deformation of inflation. The global response of the shell is defined by the relationship between p and the boundary displacements of the shell. We assume that the deformation of the shell is spherically symmetric under the uniform internal pressure as given by

$$r = r(R), \ \theta = \Theta, \ \phi = \Phi, \tag{1}$$

where r, θ and ϕ denote the spherical coordinates of the material particle (R, Θ, Φ) after deformation, and $r(R)$ is a C^1 function. The physical components of the deformation gradient $\mathbf{F}$ for the spherical deformation (1) are

$$\mathbf{F} = \begin{pmatrix} r'(R) & 0 & 0 \\ 0 & r(R)/R & 0 \\ 0 & 0 & r(R)/R \end{pmatrix}. \tag{2}$$

The deformation gradient has the polar decomposition

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR},$$

where $\mathbf{R}$ is the rotation tensor, and $\mathbf{U}$ and $\mathbf{V}$ are, respectively, the right and left stretch tensors. The principal stretches λ_1, λ_2 and λ_3 are the eigenvalues of $\mathbf{U}$ or $\mathbf{V}$. For the spherical deformation (1), we have

$$\lambda_1 = r'(R), \ \lambda_2 = \lambda_3 = \frac{r(R)}{R}. \tag{3}$$

The material of the shell is assumed to be incompressible, that is, it can undergo only isochoric deformations and thus we require that

$$\det\mathbf{F} = \frac{r^2 r'(R)}{R^2} = \lambda_1\lambda_2\lambda_3 = 1. \tag{4}$$

Integrating equation (4) yields

$$r(R) = \left(R^3 - A^3 + a^3\right)^{\frac{1}{3}}, \tag{5}$$

where a is the inner radius of the deformed shell. Equation (5) shows that a spherically symmetric deformation of the form (1) is completely determined once the displacement of the inner surface is given. Hence, if two spherical shells have the same global response, i.e., if they have the same relationship between p and the boundary displacements, then they have the same relationship between the internal pressure p and the entire displacement field.

The material of the inhomogeneous shell is further assumed to be isotropic and hyperelastic in the reference configuration. Such a material possesses a strain–energy function of the form (see, for example, [16])

$$W = \hat{W}(\mathbf{F}, R) = W(\lambda_1, \lambda_2, \lambda_3, R). \tag{6}$$

The reduced strain–energy function W is symmetric in the principal stretches:

$$W(\lambda_1, \lambda_2, \lambda_3, R) = W(\lambda_2, \lambda_1, \lambda_3, R) = W(\lambda_1, \lambda_3, \lambda_2, R). \tag{7}$$

The Piola–Kirchhoff stress tensor $\mathbf{S}$ is given by

$$\mathbf{S} = -\hat{p}\ \mathbf{F}^{-T} + \frac{\partial \hat{W}}{\partial \mathbf{F}},$$

where $\hat{p}$ is the indeterminate part of the stress due to the incompressibility constraint. The Cauchy stress tensor $\mathbf{T}$ is related to the Piola–Kirchhoff stress tensor by

$$\mathbf{T} = \mathbf{S}\mathbf{F}^T$$

For a deformation gradient of form (2), the physical components of Piola–Kirchhoff stress are given by [16]

$$\mathbf{S} = \begin{pmatrix} S_R & 0 & 0 \\ 0 & S_\Theta & 0 \\ 0 & 0 & S_\Phi \end{pmatrix},$$

where

$$S_R = -\hat{p}\lambda_1^{-1} + W_1, S_\Theta = S_\Phi = -\hat{p}\lambda_2^{-1} + W_2, W_i \equiv \frac{\partial W}{\partial \lambda_i}(\lambda_1, \lambda_2, \lambda_2, R). \tag{8}$$

In the absence of body forces, the equations of equilibrium take the form

$$\text{Div}\ \mathbf{S} = \mathbf{0}, \tag{9}$$

where the divergence operator is with respect to the reference configuration. For the spherical deformation (1), the components of $\mathbf{F}$ are functions of R alone. Equation (9) then implies that $\hat{p}$ is also a function of R alone, and so are the components of $\mathbf{S}$. The only non-trivial component of equation (9) is

$$\frac{dS_R}{dR} + \frac{2}{R}(S_R - S_\Theta) = 0. \tag{10}$$

The boundary conditions are

$$S_R(A) = -p\lambda_1^{-1}(A),\ S_R(B) = 0. \tag{11}$$

It follows from equations (11), (10), (4), (3) and (8) that

$$\begin{aligned} p &= \lambda_1(B)S_R(B) - \lambda_1(A)S_R(A) \\ &= \int_A^B \frac{d}{dR}(\lambda_1 S_R)dR \\ &= \int_A^B \left[\frac{d\lambda_1}{dR}S_R - \frac{2\lambda_1}{R}(S_R - S_\Theta)\right]dR \\ &= \int_A^B \frac{2\lambda_1}{R\lambda_2}(\lambda_2 S_\Theta - \lambda_1 S_R)dR \\ &= \int_A^B \frac{2\lambda_1}{R\lambda_2}(\lambda_2 W_2 - \lambda_1 W_1)dR \end{aligned} \tag{12}$$

Equation (12) gives, with aid of equations (3) and (5), the relation between p and a, and thus defines the global response of the spherical shell. This equation can be written in a more explicit form in terms of the circumferential stretch

$$\lambda \equiv \frac{(R^3 - A^3 + a^3)^{\frac{1}{3}}}{R}. \tag{13}$$

We then have

$$\lambda_1 = \lambda^{-2}, \lambda_2 = \lambda_3 = \lambda. \tag{14}$$

The restriction of the strain–energy function W on the principal stretches satisfying equation (14) is defined by

$$\tilde{W}(\lambda, R) \equiv W\left(\lambda^{-2}, \lambda, \lambda, R\right). \tag{15}$$

Differentiating equation (15) with respect to λ and using equation (7), we find that

$$\frac{\partial \tilde{W}}{\partial \lambda}(\lambda, R) = -2\lambda^{-3}W_1 + 2W_2. \tag{16}$$

On substituting equation (16) into equation (12) and using equations (14) and (13), we find that

$$p = p(a) \equiv \int_A^B \frac{1}{R\lambda^2} \frac{\partial \tilde{W}(\lambda, R)}{\partial \lambda}\bigg|_{\lambda=\left(1+\frac{a^3-A^3}{R^3}\right)^{1/3}} dR. \tag{17}$$

Equation (17) shows how the internal pressure p depends on the deformed inner radius a. Since the deformation in the entire shell is completely determined by a through equation (5), function $p(a)$ defined by equation (17) represents the response of the spherical shell to the pressure load. Once the material of the shell (radially inhomogeneous, isotropic, incompressible) is given, the response function of the shell can be computed through equation (17). The task at hand is to determine whether there exists a homogeneous, isotropic, incompressible, elastic material, so that a spherical shell composed of such a material would have exactly the same response function $p(a)$.

For later reference, the stress components are derived here. Substitution of equation (8) into equation (10) gives

$$\frac{d\hat{p}}{dR} = \lambda_1 \frac{dW_1}{dR} + \frac{2\lambda_1}{R}(W_1 - W_2). \tag{18}$$

On integrating equation (18) and using the boundary conditions (11), we find that

$$\hat{p} = p + \lambda_1 W_1 + \int_A^R \frac{2\lambda_1}{R\lambda_2}(\lambda_1 W_1 - \lambda_2 W_2) dR. \tag{19}$$

Finally, substituting equation (19) back into equation (8), and using equations (16) and (17), we arrive at

$$S_R = -\lambda^2 \int_R^B \frac{1}{R\lambda^2} \frac{\partial \tilde{W}(\lambda, R)}{\partial \lambda} dR, \; S_\Theta = \frac{S_R}{\lambda^3} + \frac{1}{2} \frac{\partial \tilde{W}(\lambda, R)}{\partial \lambda}, \tag{20}$$

where λ is given by equation (13).

3. Globally Equivalent Homogeneous Shell

For the radially inhomogeneous spherical shell discussed in the preceding section, a response function $p(a)$ can be computed through equation (17). A homogeneous spherical shell (of the same geometry) is said to be globally equivalent to the inhomogeneous shell if it has exactly the same response function $p(a)$. We emphasize here that the global equivalence of spherical shells defined here is different from the local equivalence of material elements introduced in the

literature of composite materials through the average stresses and average strains (see, for example, [1]). However, the notion of global equivalence is introduced on the premise that if a homogeneous material that is locally equivalent to the given inhomogeneous material does exist, then two spherical shells composed of these two materials should also be globally equivalent. In other words, the globally equivalent homogeneous materials should contain *all* candidates for the locally equivalent materials.

In this work, we wish to find a globally equivalent material that is homogeneous, incompressible, isotropic and hyperelastic. This class of materials is chosen as the simplest possibility and is expected to render the desired global equivalence only for the particular load (internal pressure) under consideration. For more general loads, a globally equivalent homogeneous material, if it exists, is likely to be anisotropic. For the present problem, we denote the strain–energy function of the equivalent homogeneous, incompressible, isotropic and hyperelastic material by

$$U = U(\lambda_1, \lambda_2, \lambda_3),$$

where λ_i are again the principal stretches. Compared with the strain–energy function (6) for the inhomogeneous material, the strain–energy function $U(\lambda_1, \lambda_2, \lambda_3)$ has no dependence on the radial coordinate R. The response function of the homogeneous shell is determined in the same way as that presented in the preceding section for the inhomogeneous shell. We thus adopt the same notation for the analysis of the homogeneous shell, except for the obvious substitution of U for W. In analogy to equation (15), we define

$$\tilde{U}(\lambda) \equiv U\left(\lambda^{-2}, \lambda, \lambda\right), \tag{21}$$

where λ is again given by equation (13) with R, A and a now being used for the homogeneous shell. The response function for the homogeneous shell is then given by equation (17) with all appropriate replacements of the variables.

The determination of a globally equivalent homogeneous material hence amounts to finding a function $\widetilde{U}(\lambda)$ that satisfies

$$\int_A^B \frac{\tilde{U}'(\lambda)}{R\lambda^2}\bigg|_{\lambda=\left(1+\frac{a^3-A^3}{R^3}\right)^{1/3}} dR = \int_A^B \frac{1}{R\lambda^2}\frac{\partial \tilde{W}(\lambda, R)}{\partial \lambda}\bigg|_{\lambda=\left(1+\frac{a^3-A^3}{R^3}\right)^{1/3}} dR, \tag{22}$$

for all $a \in [A, \infty)$. To solve this integro-differential equation, we first make a change of variables (13) on the left-hand side of equation (22), yielding

$$\int_{\frac{a}{A}}^{\left(1+\frac{a^3-A^3}{B^3}\right)^{1/3}} \frac{\tilde{U}'(\lambda)}{1-\lambda^3} d\lambda = \int_A^B \frac{1}{R\lambda^2}\frac{\partial \tilde{W}(\lambda, R)}{\partial \lambda}\bigg|_{\lambda=\left(1+\frac{a^3-A^3}{R^3}\right)^{1/3}} dR. \tag{23}$$

Introducing the circumferential stretch Λ of the inner surface through

$$\Lambda \equiv \frac{a}{A},$$

we can rewrite equation (23) as

$$\int_{\left(1+\frac{A^3\Lambda^3-A^3}{B^3}\right)^{1/3}}^{\Lambda} \frac{\tilde{U}'(\lambda)}{\lambda^3-1}d\lambda = \int_A^B \frac{1}{R\lambda^2}\frac{\partial \tilde{W}(\lambda,R)}{\partial\lambda}\bigg|_{\lambda=\left(1+\frac{A^3\Lambda^3-A^3}{R^3}\right)^{1/3}} dR. \tag{24}$$

By the isotropy, the function U possesses the same symmetry as that described by equation (7). It implies that

$$\tilde{U}'(1) = 0.$$

Also, function U is assumed to be of C^2, so we can define a function

$$f(\Lambda) \equiv \int_1^{\Lambda} \frac{\tilde{U}'(\lambda)}{\lambda^3-1}d\lambda. \tag{25}$$

It then follows that equation (24) can be further rewritten as a recurrence equation

$$f(\Lambda) = f\left(\left(1+\frac{A^3\Lambda^3-A^3}{B^3}\right)^{1/3}\right) + \int_A^B \frac{1}{R\lambda^2}\frac{\partial \tilde{W}(\lambda,R)}{\partial\lambda}\bigg|_{\lambda=\left(1+\frac{A^3\Lambda^3-A^3}{R^3}\right)^{1/3}} dR, \tag{26}$$

which holds for all $\Lambda \in [1, \infty)$.

On repeatedly applying the recurrence equation (26), and using equation (25) and the fact that $\lim_{m\to\infty}(A^{3m}\Lambda^3 - A^{3m})/B^{3m} = 0$, we arrive at

$$\begin{aligned} f(\Lambda) &= f\left(\left[1+\frac{A^{3m}\left(\Lambda^3-1\right)}{B^{3m}}\right]^{1/3}\right) + \sum_{n=1}^{m}\int_A^B \frac{1}{R\lambda^2}\frac{\partial \tilde{W}(\lambda,R)}{\partial\lambda}\bigg|_{\lambda=\left[1+\frac{A^{3n}\left(\Lambda^3-1\right)}{B^{3n-3}R^3}\right]^{1/3}} dR \\ &= \sum_{n=1}^{\infty}\int_A^B \frac{1}{R\lambda^2}\frac{\partial \tilde{W}(\lambda,R)}{\partial\lambda}\bigg|_{\lambda=\left[1+\frac{A^{3n}\left(\Lambda^3-1\right)}{B^{3n-3}R^3}\right]^{1/3}} dR. \end{aligned} \tag{27}$$

The last series converges uniformly. By differentiating equation (27), we obtain

$$\frac{\tilde{U}'(\Lambda)}{\Lambda^3-1}=\sum_{n=1}^{\infty}\int_A^B\frac{\partial}{\partial\Lambda}\left\{\frac{1}{R\lambda^2}\frac{\partial\tilde{W}(\lambda,R)}{\partial\lambda}\bigg|_{\lambda=\left[1+\frac{A^{3n}(\Lambda^3-1)}{B^{3n-3}R^3}\right]^{1/3}}\right\}dR.$$

This equation can be rearranged and integrated to give the restricted strain–energy function for the equivalent homogeneous material:

$$\begin{aligned}\tilde{U}(\Lambda)&=\sum_{n=1}^{\infty}\int_A^B\int_1^{\Lambda}(\Gamma^3-1)\frac{\partial}{\partial\Gamma}\left\{\frac{1}{R\lambda^2}\frac{\partial\tilde{W}(\lambda,R)}{\partial\lambda}\bigg|_{\lambda=\left[1+\frac{A^{3n}(\Gamma^3-1)}{B^{3n-3}R^3}\right]^{1/3}}\right\}d\Gamma dR\\&=\sum_{n=1}^{\infty}\int_A^B\left\{\frac{\Lambda^3-1}{R\lambda^2}\frac{\partial\tilde{W}(\lambda,R)}{\partial\lambda}\bigg|_{\lambda=\left[1+\frac{A^{3n}(\Lambda^3-1)}{B^{3n-3}R^3}\right]^{1/3}}\right.\\&\quad\left.-\int_1^{\Lambda}\frac{3\Lambda^2}{R\lambda^2}\frac{\partial\tilde{W}(\lambda,R)}{\partial\lambda}\bigg|_{\lambda=\left[1+\frac{A^{3n}(\Lambda^3-1)}{B^{3n-3}R^3}\right]^{1/3}}d\Lambda\right\}dR\\&=\sum_{n=1}^{\infty}\int_A^B\frac{\Lambda^3-1}{R}\left[\frac{1}{\lambda^2}\frac{\partial\tilde{W}(\lambda,R)}{\partial\lambda}-\frac{3\tilde{W}(\lambda,R)}{\lambda^3-1}\right]_{\lambda=\left[1+\frac{A^{3n}(\Lambda^3-1)}{B^{3n-3}R^3}\right]^{1/3}}dR.\end{aligned}\tag{28}$$

Here, it has been assumed that

$$\tilde{U}(1)=0,\tilde{W}(1,R)=0.\tag{29}$$

For any given inhomogeneous strain–energy function $\tilde{W}(\lambda,R)$, not only does equation (28) demonstrate the existence of an equivalent homogeneous material, it also renders a unique expression for $\tilde{U}(\lambda)$. This, of course, does not imply the uniqueness of the strain–energy function $U(\lambda_1,\ \lambda_2,\ \lambda_3)$ for the equivalent homogeneous material, as $\tilde{U}(\lambda)$ restricts the value of $U(\lambda_1,\ \lambda_2,\ \lambda_3)$ through equation (21) only on the 1-dimensional set ${\lambda_1}^{-1/2}=\lambda_2=\lambda_3$ in the domain of U. In general, we do not expect to obtain a closed-form expression for $\tilde{U}(\lambda)$. Numerical computation of the integral and summation may be necessary.

Equation (28) can be written in the following alternate form, which is more convenient to use for further computation:

$$\begin{aligned}\tilde{U}(\Lambda)=\sum_{n=1}^{\infty}\int_A^B(\Lambda^3-1)&\left\{\frac{1}{\lambda^3-1}\frac{\partial\tilde{W}(\lambda,R)}{\partial R}\bigg|_{\lambda=\left(1+\frac{A^{3n}(\Lambda^3-1)}{B^{3n-3}R^3}\right)^{1/3}}\right.\\&\left.-\frac{d}{dR}\left[\frac{\tilde{W}(\lambda,R)}{\lambda^3-1}\bigg|_{\lambda=\left(1+\frac{A^{3n}(\Lambda^3-1)}{B^{3n-3}R^3}\right)^{1/3}}\right]\right\}dR.\end{aligned}\tag{30}$$

For the special case where $\tilde{W}$ is independent of R, which corresponds to a homogeneous material, equation (30) leads to, as expected,

$$\begin{aligned}\tilde{U}(\Lambda) &= \sum_{n=1}^{\infty}(\Lambda^3-1)\left[\left.\frac{\tilde{W}(\lambda)}{\lambda^3-1}\right|_{\lambda=\left(1+\frac{A^{3n-3}(\Lambda^3-1)}{B^{3n-3}}\right)^{1/3}}-\left.\frac{\tilde{W}(\lambda)}{\lambda^3-1}\right|_{\lambda=\left(1+\frac{A^{3n}(\Lambda^3-1)}{B^{3n}}\right)^{1/3}}\right]\\ &= (\Lambda^3-1)\left[\frac{\tilde{W}(\Lambda)}{\Lambda^3-1}-\lim_{\lambda\to 1}\frac{\tilde{W}(\lambda)}{\lambda^3-1}\right]\\ &= \tilde{W}(\Lambda),\end{aligned}$$

where use has been made of equation (29) and of the equation $\lim_{\lambda\to 1} d\tilde{W}(\lambda)/d\lambda = 0$, as implied by equation (7).

For an inhomogeneous shell, the expression of the strain–energy function $\tilde{U}$ for the equivalent homogeneous material may not be obtained in closed form. In the following sections, we will examine two composite shells, one with two layers of neo-Hookean materials, the other with two arbitrary elastic materials in infinitely fine structures.

4. A Two-layer Neo-Hookean Shell

In this section, we study an inhomogeneous neo-Hookean shell. The main conclusion to be drawn from this example is that the equivalent homogeneous shell is *not* composed of a neo-Hookean material. This demonstrates that the practice of seeking equivalent homogeneous materials in the same class of constitutive functions is not warranted.

Consider an inhomogeneous isotropic neo-Hookean shell for which

$$W(\lambda_1,\lambda_2,\lambda_3,R) = \begin{cases} \frac{1}{2}\mu_1\left(\lambda_1^2+\lambda_2^2+\lambda_3^2-3\right) & \text{when } A<R<C,\\ \frac{1}{2}\mu_2\left(\lambda_1^2+\lambda_2^2+\lambda_3^2-3\right) & \text{when } C<R<B,\end{cases}$$

where $C \in (A, B)$ is the interface between the two layers of neo-Hookean materials, and μ_1 and μ_2 are positive material constants that correspond to the shear moduli of these two materials for infinitesimal deformations. By equation (15), we have

$$\tilde{W}(\lambda,R) = \begin{cases} \frac{1}{2}\mu_1(2\lambda^2+\lambda^{-4}-3) & \text{when } A<R<C,\\ \frac{1}{2}\mu_2(2\lambda^2+\lambda^{-4}-3) & \text{when } C<R<B.\end{cases} \tag{31}$$

The strain–energy function of the globally equivalent homogeneous material is obtained by substituting equation (31) into equation (30):

$$\tilde{U}(\Lambda) = \sum_{n=1}^{\infty} (\Lambda^3 - 1) \left[\frac{\mu_1 (2\lambda_A^2 + \lambda_A^{-4} - 3)}{2(\lambda_A^3 - 1)} - \frac{\mu_1 (2\lambda_C^2 + \lambda_C^{-4} - 3)}{2(\lambda_C^3 - 1)} + \frac{\mu_2 (2\lambda_C^2 + \lambda_C^{-4} - 3)}{2(\lambda_C^3 - 1)} - \frac{\mu_2 (2\lambda_B^2 + \lambda_B^{-4} - 3)}{2(\lambda_B^3 - 1)} \right], \tag{32}$$

where

$$\lambda_R \equiv \left[1 + \frac{A^{3n}(\Lambda^3 - 1)}{B^{3n-3} R^3} \right]^{1/3}, \tag{33}$$

for $R = A$, B or C.

It is an easy matter to check that if $\mu_2 = \mu_1$ or if $C = B$, equation (32) gives

$$\tilde{U}(\lambda) = \mu_1 (2\lambda^2 + \lambda^{-4} - 3).$$

This is the expected result when the two-layer shell reduces to a homogeneous shell. For a true two-layer shell, there seems to be no closed form expression for equation (32). Nevertheless, it will be shown that in general $\tilde{U}$ is *not* of the following form for a neo-Hookean material

$$\tilde{U}_{NH}(\lambda) = \mu (2\lambda^2 + \lambda^{-4} - 3), \tag{34}$$

μ being a constant. This leads to the conclusion that the equivalent homogeneous material of a composite of neo-Hookean materials is not neo-Hookean. To demonstrate this, we take the following values of the geometry and material constants

$$A = 1,\ B = 3^{1/3},\ C = 2^{1/3},\ \mu_1 = 1,\ \mu_2 = 2. \tag{35}$$

Substituting equation (35) into equation (32), computing the derivatives of $\tilde{U}$ at $\Lambda = 1$, we find, after a tedious but straightforward calculation, that

$$\tilde{U}(1) = 0, \tilde{U}'(1) = 0, \tilde{U}''(1) = 15, \tilde{U}'''(1) = -\frac{501}{8}.$$

Hence, for small deformations, we have the following Taylor series expansion for $\tilde{U}(\lambda)$:

$$\tilde{U}(\lambda) = \frac{15}{2}(\lambda - 1)^2 - \frac{167}{16}(\lambda - 1)^3 + \ldots \tag{36}$$

On the other hand, the Taylor series expansion for a neo-Hookean material (equation (34)) is

$$\tilde{U}_{NH}(\lambda) = 6\mu(\lambda - 1)^2 - 10\mu(\lambda - 1)^3 + \ldots \tag{37}$$

Clearly, equations (36) and (37) are not compatible.

This result reveals a limitation of the notion of effective elastic moduli which are invariably introduced in the composite material literature as the material constants of a homogeneous material that is equivalent to a given composite material. This notion is based on the assumption that both the composite material and the equivalent homogeneous material have constitutive functions of the same form, i.e., for both the stress tensor is a linear function of the strain tensor. While this may be the case for composites of linear elastic materials, no such claim can be made of the form of the constitutive function of the homogeneous material that is equivalent to a given composite of nonlinear elastic materials. In the above example, the material of the composite shell is characterized by a simple constitutive function with two material constants μ_1 and μ_2, but the globally equivalent homogeneous material appears to have a complicated constitutive function.

It may seem that the two-layer neo-Hookean shell considered in this section is one of the simplest composites of nonlinear elastic materials. Yet the equivalent homogeneous material assumes a rather complex description. The task of finding equivalent homogeneous materials of composites of general nonlinear elastic materials with more complicated micro-structures appears prohibitively difficult. It is thus particularly surprising that the homogeneous material equivalent to a composite shell of two *arbitrary* nonlinear elastic materials with infinitely fine structure actually has a very simple constitutive function, as is to be demonstrated in the next section.

5. Composite Shells of Infinitely Fine Structure

Consider a $2m$-layer composite shell of two arbitrary isotropic, incompressible, nonlinear elastic materials. The strain–energy function of this composite material is given by

$$W(\lambda_1,\lambda_2,\lambda_3,R)=\begin{cases} W^{(1)}(\lambda_1,\lambda_2,\lambda_3) & \text{when } A_i<R<C_i,\\ W^{(2)}(\lambda_1,\lambda_2,\lambda_3) & \text{when } C_i<R<B_i,\ i=1,2,\ldots,m,\end{cases} \tag{38}$$

where $W^{(1)}$ and $W^{(2)}$ are the strain–energy functions of the two materials of the composite, and

$$A_i\equiv\left[\frac{i-1}{m}\left(B^3-A^3\right)+A^3\right]^{1/3},$$
$$C_i\equiv\left[\frac{i+c-1}{m}\left(B^3-A^3\right)+A^3\right]^{1/3},\ B_i\equiv\left[\frac{i}{m}\left(B^3-A^3\right)+A^3\right]^{1/3}, \tag{39}$$

A and B being again the inner and outer radii of the spherical shell, and c the volume fraction of material 1. Here the radii of various layers have been so chosen that the volume fraction of material 1 remains at a constant value c throughout the thickness of the shell. Obviously, the two-layer neo-Hookean shell considered in the previous section is a special case where m=1 and $W^{(k)}(\lambda_1,\lambda_2,\lambda_3)=1/2\ \mu_k\ (\lambda_1^2+\lambda_2^2+\lambda_3^2-3)$, $k=1,2$.

It follows from equation (15) that

$$\tilde{W}(\lambda, R) = \begin{cases} \tilde{W}^{(1)}(\lambda) & \text{when } A_i < R < C_i, \\ \tilde{W}^{(2)}(\lambda) & \text{when } C_i < R < B_i, \ i = 1, 2, \ldots, m, \end{cases} \tag{40}$$

where

$$\tilde{W}^{(k)}(\lambda) \equiv W^{(k)}\left(\lambda^{-2}, \lambda, \lambda\right), \ k = 1, 2. \tag{41}$$

By substituting equation (40) into equation (30), we obtain the strain–energy function of the homogeneous material that is globally equivalent to the composite shell:

$$\tilde{U}(\Lambda) = \sum_{n=1}^{\infty} \sum_{i=1}^{m} \left(\Lambda^3 - 1\right) \left[\frac{\tilde{W}^{(1)}(\lambda_{A_i})}{\lambda_{A_i}^3 - 1} - \frac{\tilde{W}^{(1)}(\lambda_{C_i})}{\lambda_{C_i}^3 - 1} + \frac{\tilde{W}^{(2)}(\lambda_{C_i})}{\lambda_{C_i}^3 - 1} - \frac{\tilde{W}^{(2)}(\lambda_{B_i})}{\lambda_{B_i}^3 - 1} \right], \tag{42}$$

where λ_R is again given for various values of R by equation (33).

A closed form expression for $\tilde{U}$ as given by equation (42) is not available for a finite m. However, the limit of $\tilde{U}$ as m tends to infinity can be obtained. Physically, this corresponds to a composite shell of infinitely fine structure, and thus provides a reasonable approximation for the composites of which the characteristic length of the microstructure (layer thickness) is sufficiently small.

To this end, we introduce variables

$$x \equiv \frac{i}{m}, \ \epsilon \equiv \frac{1}{m}. \tag{43}$$

Then by equations (33) and (39), the circumferential stretches appearing in equation (42) can be written as

$$\begin{aligned} \lambda_{A_i} &= \left\{ 1 + \frac{A^{3n}\left(\Lambda^3 - 1\right)}{B^{3n-3}[(x - \epsilon)(B^3 - A^3) + A^3]} \right\}^{1/3}, \\ \lambda_{C_i} &= \left\{ 1 + \frac{A^{3n}\left(\Lambda^3 - 1\right)}{B^{3n-3}[(x + (c-1)\epsilon)(B^3 - A^3) + A^3]} \right\}^{1/3}, \\ \lambda_{B_i} &= \left\{ 1 + \frac{A^{3n}\left(\Lambda^3 - 1\right)}{B^{3n-3}[x(B^3 - A^3) + A^3]} \right\}^{1/3}. \end{aligned} \tag{44}$$

This allows us to write the expression under the summation signs in equation (42) as a function of x and ϵ:

$$g(x, \epsilon) \equiv \left(\Lambda^3 - 1\right) \left[\frac{\tilde{W}^{(1)}(\lambda_{A_i})}{\lambda_{A_i}^3 - 1} - \frac{\tilde{W}^{(1)}(\lambda_{C_i})}{\lambda_{C_i}^3 - 1} + \frac{\tilde{W}^{(2)}(\lambda_{C_i})}{\lambda_{C_i}^3 - 1} - \frac{\tilde{W}^{(2)}(\lambda_{B_i})}{\lambda_{B_i}^3 - 1} \right]. \tag{45}$$

Since

$$\lambda_{A_i}|_{\epsilon=0} = \lambda_{C_i}|_{\epsilon=0} = \lambda_{B_i},$$

we have $g(x, 0) = 0$. By the definition of integral, it can be shown that

$$\lim_{m\to\infty}\sum_{i=1}^{m} g\left(\frac{i}{m},\frac{1}{m}\right) = \int_0^1 \frac{\partial g}{\partial \epsilon}(x,0)dx. \tag{46}$$

The proof of equation (46) is given in Appendix. By equation (44), we have

$$\left.\frac{d\lambda_{A_i}}{d\epsilon}\right|_{\epsilon=0} = \frac{\left(\lambda_{A_i}^3-1\right)(B^3-A^3)}{3\lambda_{A_i}^2[x(B^3-A^3)+A^3]},$$

$$\left.\frac{d\lambda_{C_i}}{d\epsilon}\right|_{\epsilon=0} = \frac{(1-c)\left(\lambda_{C_i}^3-1\right)(B^3-A^3)}{3\lambda_{C_i}^2[x(B^3-A^3)+A^3]}, \left.\frac{d\lambda_{B_i}}{d\epsilon}\right|_{\epsilon=0} = 0. \tag{47}$$

It then follows from equations (42), (43), (45–47) that

$$\begin{aligned}
\lim_{m\to\infty}\tilde{U}(\Lambda) &= \sum_{n=1}^{\infty}\int_0^1 (\Lambda^3-1)\left\{\frac{\left(\lambda_{A_i}^3-1\right)(B^3-A^3)}{3\lambda_{A_i}^2[x(B^3-A^3)+A^3]}\frac{d}{d\lambda_{A_i}}\frac{\tilde{W}^{(1)}(\lambda_{A_i})}{\lambda_{A_i}^3-1}\right.\\
&\quad -\frac{(1-c)\left(\lambda_{C_i}^3-1\right)(B^3-A^3)}{3\lambda_{C_i}^2[x(B^3-A^3)+A^3]}\frac{d}{d\lambda_{C_i}}\frac{\tilde{W}^{(1)}(\lambda_{C_i})}{\lambda_{C_i}^3-1}\\
&\quad \left.+\frac{(1-c)\left(\lambda_{C_i}^3-1\right)(B^3-A^3)}{3\lambda_{C_i}^2[x(B^3-A^3)+A^3]}\frac{d}{d\lambda_{C_i}}\frac{\tilde{W}^{(2)}(\lambda_{C_i})}{\lambda_{C_i}^3-1}\right\}_{\epsilon=0} dx\\
&= \sum_{n=1}^{\infty}\int_0^1 \frac{(\Lambda^3-1)(\lambda^3-1)(B^3-A^3)}{3\lambda^2[x(B^3-A^3)+A^3]}\left[c\frac{d}{d\lambda}\frac{\tilde{W}^{(1)}(\lambda)}{\lambda^3-1}+(1-c)\frac{d}{d\lambda}\frac{\tilde{W}^{(2)}(\lambda)}{\lambda^3-1}\right]_{\lambda=\lambda_{B_i}} dx\\
&= \sum_{n=1}^{\infty}\int_0^1 -(\Lambda^3-1)\left[c\frac{d}{d\lambda}\frac{\tilde{W}^{(1)}(\lambda)}{\lambda^3-1}+(1-c)\frac{d}{d\lambda}\frac{\tilde{W}^{(2)}(\lambda)}{\lambda^3-1}\right]_{\lambda=\lambda_{B_i}}\frac{d\lambda_{B_i}}{dx}dx\\
&= \sum_{n=1}^{\infty}(\Lambda^3-1)\left\{\left[\frac{c\tilde{W}^{(1)}(\lambda)}{\lambda^3-1}+\frac{(1-c)\tilde{W}^{(2)}(\lambda)}{\lambda^3-1}\right]_{\lambda=\lambda_{B_i},\ x=0}\right.\\
&\quad \left.-\left[\frac{c\tilde{W}^{(1)}(\lambda)}{\lambda^3-1}+\frac{(1-c)\tilde{W}^{(2)}(\lambda)}{\lambda^3-1}\right]_{\lambda=\lambda_{B_i},x=1}\right\}\\
&= \sum_{n=1}^{\infty}(\Lambda^3-1)\left\{\left.\frac{c\tilde{W}^{(1)}(\lambda)+(1-c)\tilde{W}^{(2)}(\lambda)}{\lambda^3-1}\right|_{\lambda=\left[1+\frac{A^{3n-3}(\Lambda^3-1)}{B^{3n-3}}\right]^{1/3}}\right.\\
&\quad \left.-\left.\frac{c\tilde{W}^{(1)}(\lambda)+(1-c)\tilde{W}^{(2)}(\lambda)}{\lambda^3-1}\right|_{\lambda=\left[1+\frac{A^{3n}(\Lambda^3-1)}{B^{3n}}\right]^{1/3}}\right\}\\
&= (\Lambda^3-1)\left[\left.\frac{c\tilde{W}^{(1)}(\lambda)+(1-c)\tilde{W}^{(2)}(\lambda)}{\lambda^3-1}\right|_{\lambda=\Lambda}-\lim_{\lambda\to 1}\frac{c\tilde{W}^{(1)}(\lambda)+(1-c)\tilde{W}^{(2)}(\lambda)}{\lambda^3-1}\right]\\
&= c\tilde{W}^{(1)}(\Lambda)+(1-c)\tilde{W}^{(2)}(\Lambda).
\end{aligned} \tag{48}$$

The extremely simple end result of equation (48) states that the strain–energy function of the equivalent homogeneous material for the composite of infinitely fine structure follows the simple rule of mixture. This is not to be normally expected when the strain and stress fields are inhomogeneous in the composite. It is likely that such a relation holds only for special geometry and loading conditions of which the present case is a particular example. It requires further research to identify other boundary value problems for which the equivalent homogeneous strain–energy functions have a simple form, or just a form that is practically manageable.

This work is concluded by an analysis to draw possible connections between the homogeneous material defined by the global response and the homogeneous material defined through the average stress and the average strain, a practice commonly followed in the literature of composites. In particular, we shall demonstrate that for the composite with infinitely fine structures, the average stress and the average deformation do agree with those of the homogeneous material we have found through the global response of the composite.

It is clear from equation (5) that the spherically symmetric deformation studied in this work is independent of the form of the strain–energy function, due to the incompressibility. Hence, the deformation fields in the composite shell and in the homogeneous shell are trivially identical. On the other hand, the comparison of the stress fields is much less straightforward.

For the given $2m$-layer composite shell under discussion, the strain–energy function is given by equations (38 – 41). The stress components at any point in the composite shell are given by equation (20). Here we shall present the analysis for the radial stress S_R. The same conclusion holds for the circumferential stress S_Θ.

Let $R \in [A, B]$ be fixed. There is a unique integer j such that

$$j < 2 + \frac{m(R^3 - A^3)}{B^3 - A^3} \leq j + 1.$$

It follows from equation (39) that

$$A_{j-1} < R \leq A_j,$$

i.e., A_j is the closest material 1 point to R from above. Substituting equation (40) into equation (20) and applying the mean-value theorem, we find that

$$\begin{aligned} S_R(R) &= -\lambda^2 \int_R^B \frac{\tilde{W}_\lambda(\lambda, R)}{R\lambda^2} dR \\ &= -\lambda^2 \int_R^{A_j} \frac{\tilde{W}_\lambda(\lambda, R)}{R\lambda^2} dR - \lambda^2 \sum_{i=j}^{m} \int_{A_i}^{B_i} \frac{\tilde{W}_\lambda(\lambda, R)}{R\lambda^2} dR \end{aligned}$$

$$= -\lambda^2 \int_R^{A_j} \frac{\tilde{W}_\lambda(\lambda, R)}{R\lambda^2} dR - \lambda^2 \sum_{i=j}^{m} \left[\int_{A_i}^{C_i} \frac{\tilde{W}_\lambda^{(1)}(\lambda)}{R\lambda^2} dR + \int_{C_i}^{B_i} \frac{\tilde{W}_\lambda^{(2)}(\lambda)}{R\lambda^2} dR \right]$$

$$= -\lambda^2 \int_R^{A_j} \frac{\tilde{W}_\lambda(\lambda, R)}{R\lambda^2} dR$$

$$-\lambda^2 \sum_{i=j}^{m} \left[(C_i - A_i) \frac{\tilde{W}_\lambda^{(1)}\left(\hat{\lambda}_i\right)}{\hat{R}_i \hat{\lambda}_i^2} + (B_i - C_i) \frac{\tilde{W}_\lambda^{(2)}\left(\check{\lambda}_i\right)}{\check{R}_i \check{\lambda}_i^2} dR \right]$$

$$= -\lambda^2 \int_R^{A_j} \frac{\tilde{W}_\lambda(\lambda, R)}{R\lambda^2} dR - \lambda^2 \sum_{i=j}^{m} (B_i - A_i)$$

$$\times \left[\frac{(C_i - A_i) \tilde{W}_\lambda^{(1)}\left(\hat{\lambda}_i\right)}{(B_i - A_i) \hat{R}_i \hat{\lambda}_i^2} + \frac{(B_i - C_i) \tilde{W}_\lambda^{(2)}\left(\check{\lambda}_i\right)}{(B_i - A_i) \check{R}_i \check{\lambda}_i^2} \right], \tag{49}$$

where $\hat{R}_i \in (A_i, C_i)$, $\check{R}_i \in (C_i, B_i)$, and

$$\hat{\lambda}_i \equiv \frac{\left(\hat{R}_i - A^3 + a^3\right)^{\frac{1}{3}}}{\hat{R}_i}, \quad \check{\lambda}_i \equiv \frac{\left(\check{R}_i - A^3 + a^3\right)^{\frac{1}{3}}}{\check{R}_i}.$$

The last summation in equation (49) corresponds to a partition of the interval $[A_j, B]$. Hence, it represents an approximation of an integral over $[A_j, B]$. This approximation becomes exact as $m \to \infty$. By equation (39), we have

$$\lim_{m\to\infty} \frac{C_i - A_i}{B_i - A_i} = c, \quad \lim_{m\to\infty} \frac{B_i - C_i}{B_i - A_i} = 1 - c.$$

Hence, the integrand in question is

$$\frac{c \tilde{W}_\lambda^{(1)}(\lambda) + (1 - c) \tilde{W}_\lambda^{(2)}(\lambda)}{R\lambda^2}.$$

Moreover, the last integral in equation (49) tends to zero as $m \to \infty$. We thus arrive at

$$\lim_{m\to\infty} S_R(R) = -\lambda^2 \int_R^B \frac{c \tilde{W}_\lambda^{(1)}(\lambda) + (1 - c) \tilde{W}_\lambda^{(2)}(\lambda)}{R\lambda^2} dR, \tag{50}$$

where λ is given by equation (13).

The average radial stress $\bar{S}_R(R)$ is defined by the volume average of $S_R(R)$ over a representative volume element. Since $S_R(R)$ given by equation (50) for the composite shell of infinitely fine structure is a continuous function of R, the limit of $\bar{S}_R(R)$ as the representative volume element approaches zero is the same as $S_R(R)$ given in equation (50). On the other hand, the radial stress in the equivalent homogeneous shell is obtained by substituting the strain–energy function (48) into equation (18). This results in exactly the same expression on the right-hand side of equation (50). It can thus be concluded that the radial stress in the homogeneous shell that is globally equivalent to the composite shell of infinitely fine structure is identical to the average stress in the composite shell.

A similar analysis for the circumferential stress S_Θ has been carried out. For the sake of brevity, the details are omitted. One remark is that unlike the radial stress, the circumferential stress is discontinuous across the interface between two layers, due to the presence of the last term $\partial\tilde{W}/\partial\lambda$ in equation (20). As a result, the limit of $S_\Theta(R)$ as $m \to \infty$ does not exist, in contrast to equation (50). However, the limit of the average circumferential stress over a fixed representative volume element exists as $m \to \infty$. Furthermore, this limit tends, as the representative volume element approaches zero, to

$$-\frac{1}{\lambda}\int_R^B \frac{c\tilde{W}_\lambda^{(1)}(\lambda)+(1-c)\tilde{W}_\lambda^{(2)}(\lambda)}{R\lambda^2}\,dR+\frac{1}{2}\left[c\,\tilde{W}_\lambda^{(1)}(\lambda)+(1-c)\,\tilde{W}_\lambda^{(2)}(\lambda)\right],$$

which is precisely the circumferential stress in the globally equivalent homogeneous shell. Therefore, not only do the composite shell and the globally equivalent homogeneous shell have exactly the same relationship between the boundary deformation and the boundary load, they also have the same average stress and average deformation when the composite has infinitely fine structure.

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Appendix: Proposition. *If* $g \in C^2([0,1]\times[0,1];\mathbb{R})$ *satisfies* $g(x, 0) = 0$*, then equation* (46) *holds.*

Proof. By the mean-value theorem, we have

$$g\left(\frac{i}{m},\frac{1}{m}\right)=g\left(\frac{i}{m},0\right)+\frac{1}{m}\frac{\partial g}{\partial \epsilon}\left(\frac{i}{m},0\right)+\frac{1}{2m^2}\frac{\partial^2 g}{\partial \epsilon^2}\left(\frac{i}{m},\frac{\theta}{m}\right),$$

where $\theta \in (0, 1)$. Since g is of C^2, there exists $M > 0$, such that

$$\left|\frac{\partial^2 g}{\partial \epsilon^2}(x,\epsilon)\right| < M.$$

By the definition of integral, we find that

$$\begin{aligned}
&\left|\lim_{m\to\infty}\sum_{i=1}^{m} g\left(\frac{i}{m},\frac{1}{m}\right)-\int_0^1 \frac{\partial g}{\partial \epsilon}(x,0)dx\right| \\
&\quad=\left|\lim_{m\to\infty}\sum_{i=1}^{m} g\left(\frac{i}{m},\frac{1}{m}\right)-\lim_{m\to\infty}\sum_{i=1}^{m}\frac{\partial g}{\partial \epsilon}\left(\frac{i}{m},0\right)\frac{1}{m}\right| \\
&\quad=\left|\lim_{m\to\infty}\sum_{i=1}^{m}\frac{1}{2m^2}\frac{\partial^2 g}{\partial \epsilon^2}\left(\frac{i}{m},\frac{\theta}{m}\right)\right| \\
&\quad\le \lim_{m\to\infty}\frac{M}{2m}=0.
\end{aligned}$$

□

References

1. R.M. Christensen, *Mechanics of Composite Materials*. Wiley Interscience (1979).
2. J.D. Eshelby, Elastic inclusions and inhomogeneities. In: I.N. Sneddon and R. Hill (eds.), *Progress in Solid Mechanics*, Vol. 2. North-Holland, Amsterdam (1961) pp. 89–140.
3. G. Franfort and F. Murat, Homogenization and optimal bounds in linear elasticity. *Arch. Rational Mech. Anal.* **94** (1986) 307–334.
4. Z. Hashin, Analysis of composite materials – A survey. *J. Appl. Mech.* **50** (1983) 481–505.
5. Z. Hashin, Large isotropic elastic deformation of composites and porous media. *Int. J. Solids Struct.* **21** (1985) 711–720.
6. R. Hill, Elastic properties of reinforced solids: Some theoretical principles. *J. Mech. Phys. Solids* **11** (1963) 357–372.
7. R. Hill, The essential structure of constitutive laws for metal composites and polycrystals. *J. Mech. Phys. Solids* **15** (1967) 79–95.
8. R. Hill, On constitutive macro-variables for heterogeneous solids at finite strain. *Proc. R. Soc. Lond. A* **326** (1972) 131–147.

9. R.V. Kohn, Recent progress in the mathematical modelling of composite materials. In: G. Sih et al. (eds.), *Composite Material Response: Constitutive Relations and Damage Mechanisms*. Elsevier (1988) pp. 155–176.
10. L. Lahellec, F. Mazerolle and J.C. Michel, Second-order estimate of the macroscopic behavior of periodic hyperelastic composites: Theory and experimental validation. *J. Mech. Phys. Solids* **52** (2004) 27–49.
11. G.W. Milton, *The theory of composites.* Cambridge University Press (2002).
12. G.W. Milton and R.V. Kohn, Variational bounds of the effective moduli of anisotropic composites. *J. Mech. Phys. Solids* **36** (1988) 597–629.
13. S. Müller, Homogenization of nonconvex integral functions and cellular elastic materials. *Arch. Rational Mech. Anal.* **99** (1987) 189–211.
14. S. Nemat-Nasser and M. Hori, *Micromechanics: Overall Properties of Heterogeneous Solids.* Elsevier, Amsterdam (1993).
15. R.W. Ogden, Extremum principles in non-linear elasticity and their application to composites – I Theory. *Int. J. Solids Struct.* **14** (1978) 265–282.
16. R.W. Ogden, *Non-linear elastic deformations*. Dover Publications (1997).
17. P. Ponte Castañeda, The overall constitutive behaviour of nonlinear elastic composites. *Proc. R. Soc. Lind. A* **422** (1989) 147–171.
18. P. Ponte Castañeda, Nonlinear composite materials: Effective constitutive behavior and microstructure evolution. In: P. Suquet (ed.), *Continuum Micromechanics*, CISM Courses and Lecture Notes No. 377. Springer (1997) pp. 131–195.
19. P. Ponte Castañeda and P. Suquet, Nonlinear composites. *Adv. Appl. Mech.* **34** (1998) 171–302.
20. P. Ponte Castañeda and E. Tiberio, A second-order homogenization method in finite elasticity and applications to black-filled elastomers. *J. Mech. Phys. Solids* **48** (2000) 1389–1411.
21. P. Ponte Castañeda and J.R. Willis, On the overall properties of nonlinearly viscous composites. *Proc. R. Soc. Lond. A* **416** (1988) 217–244.
22. M. Sahimi, *Heterogeneous materials I, linear transport and optical properties*. Springer (2003a).
23. M. Sahimi, *Heterogeneous materials II, nonlinear and breakdown properties and atomistic modeling*. Springer (2003b).
24. U. Saravanan and K.R. Rajagopal, On the role of inhomogeneities in the deformation of elastic bodies. *Math. Mech. Solids* **8** (2003) 349–376.
25. U. Saravanan and K.R. Rajagopal, A comparison of the response of isotropic inhomogeneous elastic cylindrical and spherical shells and their homogenized counterparts. *J. Elast.* (2004), in press.
26. U. Saravanan and K.R. Rajagopal, Inflation, extension, torsion and shearing of an inhomogeneous compressible right circular annular cylinder. *Math. Mech. Solids* (2005), in press.
27. P. Suquet, Effective properties of nonlinear composites. In: P. Suquet (ed.), *Continuum Micromechanics*, CISM Courses and Lecture Notes No. 377. Springer (1997) pp. 197–264.
28. D.R.S. Talbot and J.R. Willis, Variational principles for inhomogeneous non-linear media. *J. Appl. Math.* **35** (1985) 39–54.
29. J.R. Willis, Variational and related methods for the overall properties of composite materials. In: C.-S. Yih (ed.), *Advances in Applied Mechanics,* Vol. 21 (1981) 2–78.
30. J.R. Willis, Elasticity theory of composites. In: H.G. Hopkins and M.J. Sewell (eds.), *Mechanics of Solids*. Pergamon, Oxford (1982) pp. 653–686.