J. Math. Soc. Japan
Vol. 57, No. 2, 2005

Homogenization for stochastic partial differential equations derived from nonlinear filterings with feedback

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(Received Sept. 3, 2003)
(Revised Jun. 14, 2004)

Abstract. We discuss the homogenization of stochastic partial differential equations whose coefficients are rapidly oscillating and are perturbed by a diffusion process. Such class of equations appear in nonlinear filtering problems with feedback. We specify the constant coefficients of the limit equation. The constants are essentially different from the case where the coefficients do not contain perturbed factors.

1. Introduction.

In this paper, we deal with the following stochastic partial differential equations (SPDEs) with small parameter $\varepsilon > 0$

$$\begin{cases} dp^{\varepsilon}(t,x) = L^{\varepsilon}(t)\, p^{\varepsilon}(t,x)\, dt + M^{\varepsilon}(t)\, p^{\varepsilon}(t,x)\, dY_t, \quad 0 \le t \le T\,, \\ p^{\varepsilon}(0,x) = p_0(x) \in L^2(\boldsymbol{R}^d), \end{cases} \tag{1.1}$$

where $Y = (Y_t)_{t\in[0,T]}$ is an n-dimensional standard Brownian motion, and $L^{\varepsilon} = L^{\varepsilon}(t)$ and $M^{\varepsilon} = M^{\varepsilon}(t) = (M_1^{\varepsilon}(t), \cdots, M_n^{\varepsilon}(t))$ represent the linear differential operator and the multiplicative operator acting on a function on $\boldsymbol{R}^d$ defined by

$$L^{\varepsilon}(t) = \nabla_{x_i}(a^{ij}(x/\varepsilon, Z_t^{\varepsilon}/\varepsilon)\, \nabla_{x_j}\cdot\,), \quad M_k^{\varepsilon}(t) = h_k(x/\varepsilon, Z_t^{\varepsilon}/\varepsilon)\ \cdot\ .$$

Note that ∇_{x_i} $(i = 1, \dots, d)$ are the partial derivatives with respect to x_i and that we use the summation convention throughout the paper. The symbol $Z^{\varepsilon} = (Z_t^{\varepsilon})_{t\in[0,T]}$ stands for a solution to the following stochastic differential equation (SDE) on $\boldsymbol{R}^n$

$$\begin{cases} dZ_t^{\varepsilon} = f(Z_t^{\varepsilon}/\varepsilon)\, dt + Q\, dY_t, \quad 0 \le t \le T\,, \\ Z_0^{\varepsilon} = z \in \boldsymbol{R}^n\,, \end{cases} \tag{1.2}$$

where $Q = (Q^{kl})$ denotes an $(n \times n)$-matrix. All coefficients $a = (a^{ij}(x,z))$, $h = (h_k(x,z))$ and $f = (f^k(z))$ are assumed to be periodic with period 1 in all components.

Our aim is to show that as ε goes to zero the family of solutions to (1.1) converges in law to the solution of an SPDE having both spatially and temporally homogeneous coefficients. It turns

2000 *Mathematics Subject Classification.* Primary 60H15; Secondary 35R60, 93E11.

Key Words and Phrases. homogenization, stochastic partial differential equations, Zakai equations, nonlinear filtering.

out that the limit equation satisfies the SPDE

$$\begin{cases} dp^0(t,x) = L^0 p^0(t,x)\,dt + M^0 p^0(t,x)\,dY_t, \quad 0 \le t \le T\,, \\ p^0(0,x) = p_0(x) \in L^2(\boldsymbol{R}^d)\,, \end{cases} \tag{1.3}$$

where

$$L^0 = \overline{c}^{ij}\, \nabla_{x_i} \nabla_{x_j} \,\cdot\, - \overline{g}^i\, \nabla_{x_i}\,\cdot\,, \quad M_k^0 = \overline{h}_k\,\cdot\,\,, \tag{1.4}$$

and the constants $\overline{c} = (\overline{c}^{ij})$, $\overline{g} = (\overline{g}^i)$ $(i,j = 1,\dots,d)$ and $\overline{h} = (\overline{h}_k)$ $(k = 1,\dots,n)$ are characterized by

$$\overline{c}^{ij} = \langle\langle (\delta^i_{i'} + \nabla_{x_{i'}} \chi^i)\, a^{i'j'}\, (\delta^j_{j'} + \nabla_{x_{j'}} \chi^j) \rangle\rangle + \langle\langle \nabla_{z_k} \chi^i A^{kl}\, \nabla_{z_l} \chi^j \rangle\rangle\,,$$

$$\overline{g}^i = \langle\langle h_k\, Q^{kl}\, \nabla_{z_l} \chi^i \rangle\rangle\,, \qquad \overline{h}_k = \left(\int_{\boldsymbol{T}^n} \left(\int_{\boldsymbol{T}^d} h_k(x,z)\,dx \right)^2 dz \right)^{1/2},$$

with the notation $\langle\langle \cdot \rangle\rangle := \int_{\boldsymbol{T}^d \times \boldsymbol{T}^n} \cdot\, dxdz$, where $\boldsymbol{T}^d$ and $\boldsymbol{T}^n$ represent the d-dimensional and n-dimensional unit torus respectively, and ∇_{z_k} $(k = 1,\dots,n)$ denote the partial derivatives with respect to z_k. The symbols (δ^i_j) $(i,j = 1,\dots,d)$ and (A^{kl}) $(k,l = 1,\dots,n)$ stand for Kronecker's delta and the $(n \times n)$-matrix defined by $A = QQ^*/2$ respectively, and we denote by $\chi^m = (\chi^m(x,z))$ $(m = 1,\dots,d)$ periodic functions with period 1 in all components which satisfy $\langle\langle \chi^m \rangle\rangle = 0$ and the following auxiliary partial differential equations (PDEs) on $\boldsymbol{R}^d \times \boldsymbol{R}^n$

$$\nabla_{x_i}(a^{ij}(x,z)\, \nabla_{x_j} \chi^m(x,z)) + A^{kl}\, \nabla_{z_k} \nabla_{z_l} \chi^m(x,z) + (\nabla_{x_i} a^{im})(x,z) = 0\,. \tag{1.5}$$

The limit equation (1.3) does not depend on f.

The study of homogenization for PDEs has been largely developed for the last two decades, and numerous publications can be found at present. The books [**3**], [**9**] give us large numbers of results obtained before the nineties with an extensive bibliography. The former book treats the homogenization of linear, second-order PDEs with periodic coefficients by two different approaches, that is, analytic and probabilistic (see also [**12**], [**14**] and references therein). The latter one is concerned with the homogenization on stationary random fields (we refer to [**5**], [**10**] for more recent results). The papers [**4**], [**15**] deal with another sort of homogenization in random environment; they consider second order PDEs whose coefficients are periodic function of the space variable, and perturbed by an ergodic diffusion process.

On the other hand, few studies are found on the homogenization problem of SPDEs. The literature [**1**] consider the homogenization of the SPDE having the operators

$$L^\varepsilon = \nabla_{x_i}(a^{ij}(x/\varepsilon) \nabla_{x_j} \cdot) - \nabla_{x_i}(g^i(x/\varepsilon)\,\cdot)\,, \quad M_k^\varepsilon = h_k^\varepsilon(x)\,\cdot\,\,, \tag{1.6}$$

under the assumption of pointwise convergence : $\lim_{\varepsilon\downarrow 0} h_k^\varepsilon(x) = h_k(x)$ for all $k = 1,\dots,n$. However, this assumption is rather strong since it forbids an oscillatory behavior of h_k^ε written as

$h_k^\varepsilon(x) = h_k(x/\varepsilon)$ by periodic functions h_k. Motivated by this problem, our previous paper [**8**] deals with the case where h_k^ε allows such oscillation by taking, in place of (1.6), the operators

$$L^\varepsilon = a^{ij}(x/\varepsilon)\nabla_{x_i}\nabla_{x_j}\,\cdot\, + \varepsilon^{-1} b^i(x/\varepsilon)\nabla_{x_i}\,\cdot\,, \quad (M_k^\varepsilon u)(x) = B_k(x, x/\varepsilon, u(x))\,,$$

and studies its homogenization.

The principal interest of the present paper is to know how L^ε and M^ε are homogenized when we add random factors in the coefficients. In fact, we get different limit operators from that obtained in [**8**] because of the presence of Z^ε. Besides, contrary to [**8**], the limit operator L^0 is determined not only by L^ε but also by M^ε since the constants $\overline{g}^i$ contain the functions h_k in their integrand. Remark that this term does not appear in the case where the coefficients do not depend on Z^ε. The reason why L^0 does not depend on f will be revealed at the end of Section 4. Roughly speaking, the constant $\overline{g}^i$ should involve intrinsically the term of the form $\langle\langle f^l \nabla_{z_l} \chi^i \rangle\rangle$, but it can be shown that this term is equal to zero by the particularity of χ^m.

Finally, we point out that the SPDEs (1.1) often appear in certain nonlinear filtering problems. Take $f = 0$, $Q = I$ and σ such that $\sigma\sigma^* = 2a$, and consider the following nonlinear filtering problem with feedback terms

$$\begin{cases} dX_t^\varepsilon = \varepsilon^{-1} b(X_t^\varepsilon/\varepsilon, Y_t/\varepsilon)\, dt + \sigma(X_t^\varepsilon/\varepsilon, Y_t/\varepsilon)\, dW_t^\varepsilon, \quad X_0^\varepsilon = \xi, \\ Y_t = \int_0^t h(X_s^\varepsilon/\varepsilon, Y_s/\varepsilon)\, ds + \hat{W}_t^\varepsilon, \end{cases}$$

where $W^\varepsilon = (W_t^\varepsilon)$ and $\hat{W}^\varepsilon = (\hat{W}_t^\varepsilon)$ are mutually independent standard Brownian motions with respect to the probability measure P^ε defined by

$$\left.\frac{dP^\varepsilon}{dP}\right|_{\mathscr{F}_t^\varepsilon} = \exp\left(\int_0^t h(X_s^\varepsilon/\varepsilon, Y_s/\varepsilon)\, dY_s - \frac{1}{2} \sum_{k=1}^n \int_0^t |h_k(X_s^\varepsilon/\varepsilon, Y_s/\varepsilon)|^2\, ds \right),$$

with $\mathscr{F}_t^\varepsilon = \sigma(W_s^\varepsilon, Y_s \,|\, s \le t)$. Note that in this case $Z^\varepsilon = (Z_t^\varepsilon)$ is identical with $Y = (Y_t)$. Then, the solution of (1.1) appears in the following representation formula for the optimal filter

$$E^{P^\varepsilon}[\,\psi(X_t^\varepsilon) \,|\, \sigma(Y_s; s \le t)\,] = \frac{\int_{\mathbf{R}^d} \psi(x)\, p^\varepsilon(t,x)\, dx}{\int_{\mathbf{R}^d} p^\varepsilon(t,x)\, dx}\,, \qquad P\text{-a.s.}$$

We refer to [**2**] for more information about physical and engineering aspects of the homogenization in nonlinear filtering problems.

This paper is organized as follows. In the next section, we state our main result after giving standing assumptions. In Section 3, we prove tightness of the family of solutions to (1.1) on an appropriate function space. Section 4 is devoted to the identification of the limit measure.

2. Assumptions and main result.

Throughout this paper, we make the following standing assumptions.

ASSUMPTION 2.1.

(1) $a \in C^2(\boldsymbol{R}^{d+n};\boldsymbol{R}^d \otimes \boldsymbol{R}^d)$, $h \in C^1(\boldsymbol{R}^{d+n};\boldsymbol{R}^n)$, and $f \in C^1(\boldsymbol{R}^n;\boldsymbol{R}^n)$ are periodic functions with period 1 in all components.

(2) $a = (a^{ij}(x,z))$ is symmetric and strictly elliptic, that is, $a^{ij}(x,z) = a^{ji}(x,z)$ and there exists $\alpha > 0$ such that $\alpha|\xi|^2_{\boldsymbol{R}^d} \le (a(x,z)\,\xi,\xi)_{\boldsymbol{R}^d} \le \alpha^{-1}|\xi|^2_{\boldsymbol{R}^d}$ for all $\xi \in \boldsymbol{R}^d$ and $(x,z) \in \boldsymbol{R}^{d+n}$.

(3) $A = (A^{kl})$ is positive definite.

Let $H = L^2(\boldsymbol{R}^d)$ be the Hilbert space with inner product $(u,v)_H := \int_{\boldsymbol{R}^d} u(x)v(x)\,dx$ and norm $|u|_H := \sqrt{(u,u)_H}$, and let $V = H^1(\boldsymbol{R}^d)$ be the Sobolev space of order 1 with norm $|u|_V := \sqrt{|u|^2_H + \sum_{i=1}^d |\nabla_{x_i} u|^2_H}$. We denote by H' and V' the dual spaces of H and V respectively. Then, under the identification $H = H'$ by the Riesz representation theorem, we have the inclusions $V \hookrightarrow H \hookrightarrow V'$ that are dense and continuous.

The conditions (1) and (2) of Assumption 2.1 ensure the existence and uniqueness of solution to (1.1) (see Theorem 1.3 of [**13**]).

THEOREM 2.1. *Let $(\Omega,\mathscr{F},P;\mathscr{F}_t,Y_t)$ be a filtered probability space with a standard $(\mathscr{F}_t)$-Brownian motion $Y = (Y_t)$, and let $Z^\varepsilon = (Z^\varepsilon_t)$ be a solution to* (1.2). *Then, there exists an $(\mathscr{F}_t)$-progressively measurable process $p^\varepsilon = (p^\varepsilon_t) \in L^2(\Omega \times [0,T];V)$ such that*

$$(p^\varepsilon_t,v)_H = (p_0,v)_H + \int_0^t {}_{V'}\langle L^\varepsilon(s)\,p^\varepsilon_s,v\rangle_V\,ds + \int_0^t (M^\varepsilon(s)\,p^\varepsilon_s,v)_H\,dY_s\,, \qquad \forall v \in V\,, \tag{2.1}$$

for almost all $(t,\omega) \in [0,T] \times \Omega$, where ${}_{V'}\langle\,\cdot\,,\,\cdot\,\rangle_V$ stands for the duality product between V and V'. Such process is unique in the following sense

$$P\big(\,p^\varepsilon_t = q^\varepsilon_t \;\text{ in }\; V',\; \forall t \in [0,T]\big) = 1$$

for all p^ε and q^ε satisfying (2.1). *Moreover, the solution p^ε satisfies the energy equality*

$$\begin{aligned}|p^\varepsilon_t|^2_H = |p_0|^2_H &+ 2\int_0^t {}_{V'}\langle L^\varepsilon(s)\,p^\varepsilon_s,p^\varepsilon_s\rangle_V\,ds \\ &+ 2\int_0^t (M^\varepsilon(s)\,p^\varepsilon_s,p^\varepsilon_s)_H\,dY_s + \sum_{k=1}^n \int_0^t |M^\varepsilon_k(s)\,p^\varepsilon_s|^2_H\,ds\,, \qquad P\text{-a.s.}\end{aligned} \tag{2.2}$$

Now we are in position to state our main result. We denote by $E = \{\,v \mid ve^{-\lambda\theta} \in V'\}$ the weighted Sobolev space with norm $|v|_E := |ve^{-\lambda\theta}|_{V'}$, where $\lambda > 0$ is a fixed number and $\theta = \theta(\cdot)$ is a smooth and strictly positive function on $\boldsymbol{R}^d$ such that $\theta(x) = |x|$ for all $|x| \ge 1$. Let $S = C(0,T;E)$ be the set of continuous functions taking their values in E equipped with the uniform topology, and let Π^ε and Π^0 be the laws on S of the solutions to (1.1) and (1.3) respectively. Remark that (1.3) has a unique solution since $\bar{c} = (\bar{c}^{ij})$ is positive definite. Moreover, we can show the uniqueness in law of (1.3) on S by using the uniqueness theorem of Yamada-Watanabe type (see p. 89 of [**11**]).

Our main result is the following.

THEOREM 2.2. *The family of probability measures* $\{\Pi^\varepsilon; \varepsilon > 0\}$ *converges to* Π^0 *as* ε *goes to zero.*

3. Tightness.

In this section, we prove the tightness of $\{\Pi^\varepsilon; \varepsilon > 0\}$. To begin with, we check the following uniform estimate.

LEMMA 3.1. *Let* $p^\varepsilon = (p_t^\varepsilon)$ *be a solution to* (1.1). *Then, there exists a constant* $C > 0$ *such that*

$$\sup_{\varepsilon>0} E\Big[\sup_{0\le t\le T} |p_t^\varepsilon|_H^4\Big] + \sup_{\varepsilon>0} E\left[\left(\int_0^T |p_t^\varepsilon|_V^2\,dt\right)^2\right] \le C\,|p_0|_H^4\,. \tag{3.1}$$

PROOF. By applying Ito's formula to the semi-martingale $|p_t^\varepsilon|_H^2$, which satisfies (2.2), we can see

$$\begin{aligned} &|p_t^\varepsilon|_H^4 + 4\alpha\int_0^t |p_s^\varepsilon|_H^2\,|p_s^\varepsilon|_V^2\,ds \\ &\quad\le |p_0|_H^4 + C\int_0^t |p_s^\varepsilon|_H^4\,ds + 4\int_0^t |p_s^\varepsilon|_H^2\,(M^\varepsilon(s)\,p_s^\varepsilon, p_s^\varepsilon)_H\,dY_s\,, \quad P\text{-a.s.}, \end{aligned} \tag{3.2}$$

for some constant $C > 0$. Here and in the following, we denote by $C > 0$ different constants independent of $\varepsilon > 0$. Making use of Burkholder's inequality, we can estimate the stochastic integral part as

$$4E\left[\sup_{0\le t\le T}\left|\int_0^t |p_s^\varepsilon|_H^2 (M^\varepsilon(s)\,p_s^\varepsilon, p_s^\varepsilon)_H\,dY_s\right|\right] \le \frac{1}{2}E\Big[\sup_{0\le t\le T}|p_t^\varepsilon|_H^4\Big] + CE\left[\int_0^T |p_t^\varepsilon|_H^4\,dt\right].$$

Thus, by considering the expectation of both sides of (3.2) after taking the supremum in $t \in [0,T]$, we obtain by Gronwall's lemma that $\sup_{\varepsilon>0} E[\sup_{0\le t\le T}|p_t^\varepsilon|_H^4] \le C\,|p_0|_H^4$.

On the other hand, the energy equality (2.2) yields

$$\left(2\alpha\int_0^T |p_t^\varepsilon|_V^2\,dt\right)^2 \le 4\,|p_0|_H^4 + C\sup_{0\le t\le T}|p_t^\varepsilon|_H^4 + 8\left(\int_0^T (M^\varepsilon(t)p_t^\varepsilon, p_t^\varepsilon)_H\,dY_t\right)^2,$$

form which we obtain $\sup_{\varepsilon>0} E[(\int_0^T |p_t^\varepsilon|_V^2\,dt)^2] \le C\,|p_0|_H^4$. Hence we have completed the proof. $\square$

Let us denote by $C_c^\infty(\boldsymbol{R}^d)$ the set of smooth functions on $\boldsymbol{R}^d$ with compact supports. The following lemma gives the equicontinuity of $\{(p^\varepsilon,\psi)_H; \varepsilon > 0\}$ for every $\psi \in C_c^\infty(\boldsymbol{R}^d)$.

LEMMA 3.2. *For each* $\psi \in C_c^\infty(\boldsymbol{R}^d)$, *there exists a constant* $C > 0$ *such that*

$$\sup_{\varepsilon>0} E\left[\,\left|(p_t^\varepsilon,\psi)_H - (p_s^\varepsilon,\psi)_H\right|^4\right] \le C\,|p_0|_H^4\,|t-s|^2\,, \quad 0 \le \forall s < \forall t \le T\,. \tag{3.3}$$

PROOF. Remark first that there exists a constant $C > 0$ such that L^ε and M^ε satisfy ${}_{V'}\langle L^\varepsilon(r)p_r^\varepsilon, \psi\rangle_V \le C\,|p_r^\varepsilon|_V\,|\psi|_V$ and $(M_k^\varepsilon(r)p_r^\varepsilon, \psi)_H \le C\,|p_r^\varepsilon|_H\,|\psi|_H$ for all $r \in [0,T]$. Thus, we can easily show

$$E\left[\left|(p_t^\varepsilon - p_s^\varepsilon, \psi)_H\right|^4\right] \le C\,|t-s|^2 E\left[\left(\left(\int_0^T |p_r^\varepsilon|_V^2\,dr\right)^2\right]\right. + C\,|t-s|^2 E\left[\sup_{0\le r\le T} |p_r^\varepsilon|_H^4\right],$$

which implies (3.3) by virtue of (3.1). □

PROPOSITION 3.1. *The family of probability measures* $\{\Pi^\varepsilon;\varepsilon > 0\}$ *is tight in* S.

PROOF. By the estimate (3.1) and the compactness of the injection $H \hookrightarrow E$ (cf. Lemma 9.21 of [**6**]), we have only to check the tightness of the family of real valued processes $\{(p^\varepsilon, \psi);\varepsilon > 0\}$ for each $\psi \in C_c^\infty(\boldsymbol{R}^d)$ (see for example [**7**]). But in view of (3.1), (3.3) and Kolmogorov's tightness criterion, we can conclude that $\{(p^\varepsilon, \psi);\varepsilon > 0\}$ is a tight family in $C([0,T];\boldsymbol{R})$. Hence, we get the desired result. □

4. Identification of the limit measure.

By Proposition 3.1, we can extract a subsequence of $\{\Pi^\varepsilon;\varepsilon > 0\}$ having a limit $\overline{\Pi}$. Hereafter we fix such converging subsequence arbitrarily and denote it by $\{\Pi^\varepsilon;\varepsilon > 0\}$ again to avoid heavy notation. The goal of this section is to prove $\overline{\Pi} = \Pi^0$ as probability measures on S. For this purpose, we adopt martingale formulation for infinite dimensional diffusion processes following the notation in [**11**].

Let $X = (X_t)_{t\in[0,T]}$ and $\mathscr{X} = (\mathscr{X}_t)_{t\in[0,T]}$ be the canonical process and the canonical filtration on S, respectively. For $\phi \in C_c^\infty(\boldsymbol{R})$, $\psi \in C_c^\infty(\boldsymbol{R}^d)$ and $t \in [0,T]$, we define the functional $\Gamma_t = \Gamma_t^{\phi,\psi} : S \longrightarrow \boldsymbol{R}$ by

$$\Gamma_t^{\phi,\psi}(w) := \phi((w_t,\psi)_H) - \phi((w_0,\psi)_H) - \int_0^t \phi'((w_r,\psi)_H)\,(w_r,(L^0)^*\psi)_H\,dr$$
$$- \frac{1}{2}\sum_{k=1}^n \int_0^t \phi''((w_r,\psi)_H)\,(w_r,\overline{h}_k\psi)_H^2\,dr,$$

where $(L^0)^*$ denotes the adjoint operator of L^0. Then, from the uniqueness in law of the limit equation (1.3), we can show the uniqueness of probability measures under which $(\Gamma_t^{\phi,\psi})$ is a $(\mathscr{X}_t)$-martingale for every $\phi \in C_c^\infty(\boldsymbol{R})$ and $\psi \in C_c^\infty(\boldsymbol{R}^d)$ (see p. 76 of [**11**]). Thus, we have only to check $E^{\overline{\Pi}}[\Psi_s(\Gamma_t^{\phi,\psi} - \Gamma_s^{\phi,\psi})] = 0$ for arbitrarily fixed $0 \le s < t \le T$ and bounded $(\mathscr{X}_s)$-measurable functional Ψ_s, where $E^{\overline{\Pi}}$ stands for the expectation with respect to the probability measure $\overline{\Pi}$.

Now, we set $\psi^\varepsilon(x,z) = \psi(x) + \varepsilon\,\chi^m(x/\varepsilon, z/\varepsilon)\,\psi_{x_m}(x)$, where $\psi_{x_m} := \partial\psi/\partial x_m$. Recall that $\chi^m = \chi^m(x,z)$ $(m = 1,\ldots,d)$ are bounded and periodic solutions to (1.5) that belong to $C^2(\boldsymbol{R}^{d+n})$. Then, by Ito's formula, we have

$$\int_0^t \phi'((p_r^\varepsilon, \psi^\varepsilon(\,\cdot\,,Z_r^\varepsilon))_H)\,(p_r^\varepsilon,(\mathscr{M}^\varepsilon\psi^\varepsilon)(\,\cdot\,,Z_r^\varepsilon))_H\,dY_r \tag{4.1}$$

$$
\begin{aligned}
&= \phi((p_t^\varepsilon, \psi^\varepsilon(\cdot, Z_t^\varepsilon))_H) - \phi((p_0, \psi^\varepsilon(\cdot, z))_H) \\
&\quad - \int_0^t \phi'((p_r^\varepsilon, \psi^\varepsilon(\cdot, Z_r^\varepsilon))_H)\,(p_r^\varepsilon, (\mathscr{L}^\varepsilon \psi^\varepsilon)(\cdot, Z_r^\varepsilon))_H\, dr \\
&\quad - \frac{1}{2}\sum_{k=1}^n \int_0^t \phi''((p_r^\varepsilon, \psi^\varepsilon(\cdot, Z_r^\varepsilon))_H)\,(p_r^\varepsilon, (\mathscr{M}_k^\varepsilon \psi^\varepsilon)(\cdot, Z_r^\varepsilon))_H^2\, dr,
\end{aligned}
$$

where

$$
\begin{aligned}
(\mathscr{L}^\varepsilon \psi^\varepsilon)(x,z) &= \{(c^{ij}(x/\varepsilon, z/\varepsilon) + d^{ij}(x/\varepsilon, z/\varepsilon)\}\, \psi_{x_i x_j}(x) \\
&\quad + g^i(x/\varepsilon, z/\varepsilon)\, \psi_{x_i}(x) + \varepsilon\,(a^{ij}\chi^m)(x/\varepsilon, z/\varepsilon)\, \psi_{x_i x_j x_m}(x), \\
(\mathscr{M}_k^\varepsilon \psi^\varepsilon)(x,z) &= Q^{kl}\,(\nabla_{z_l}\chi^m)(x/\varepsilon, z/\varepsilon)\, \psi_{x_m}(x) \\
&\quad + h_k(x/\varepsilon, z/\varepsilon)\, \psi(x) + \varepsilon\,(h_k\chi^m)(x/\varepsilon, z/\varepsilon)\, \psi_{x_m}(x),
\end{aligned}
$$

and $c^{ij} = (c^{ij}(x,z))$, $d^{ij} = (d^{ij}(x,z))$ and $g^i = (g^i(x,z))$ are defined by

$$
c^{ij} = a^{ij} + a^{im}(\nabla_{x_m}\chi^j), \quad d^{ij} = \nabla_{x_m}(a^{mj}\chi^i), \quad g^i = (h_k Q^{kl} + f^l)(\nabla_{z_l}\chi^i).
$$

PROPOSITION 4.1. *Let us denote the left-hand side of the equality* (4.1) *by* Λ_t^ε. *Then,*

$$
\lim_{\varepsilon\downarrow 0} E[\Psi_s(p^\varepsilon)(\Gamma_t(p^\varepsilon) - \Gamma_s(p^\varepsilon) - \Lambda_t^\varepsilon + \Lambda_s^\varepsilon)] = 0. \tag{4.2}
$$

Before proving this proposition, we point out here that the convergence (4.2) implies $E^{\overline{\Pi}}[\Psi_s(\Gamma_t - \Gamma_s)] = 0$, and in consequence $\overline{\Pi} = \Pi^0$. This claim can be verified as follows. For $N > 0$, we define $\theta_N : S \longrightarrow S$ by $\theta_N(w)(t) := |w_t|_E^{-1} \min\{|w_t|_E, N\}\, w_t$, and set $\Gamma_t^N(w) = \Gamma_t(\theta_N(w))$. Clearly, $\theta_N(p^\varepsilon) = p^\varepsilon$ on the event $\{\sup_{0\le t\le T} |p_t^\varepsilon|_E \le N\}$. Thus, taking into account that

$$
\sup_{N>0}\sup_{0\le t\le T} E[|\Gamma_t^N(p^\varepsilon)|^2] \le C\left(1 + E\left[\sup_{0\le t\le T} |p_t^\varepsilon|_H^4\right]\right),
$$

we can show that for all $\varepsilon > 0$ and $N > 0$,

$$
\begin{aligned}
&\left| E[\Psi_s(p^\varepsilon)(\Gamma_t(p^\varepsilon) - \Gamma_s(p^\varepsilon))] - E[\Psi_s(p^\varepsilon)(\Gamma_t^N(p^\varepsilon) - \Gamma_s^N(p^\varepsilon))] \right|^2 \\
&\quad \le CP(\{\sup_{0\le t\le T} |p_t^\varepsilon|_E > N\}) \times \left(1 + E\left[\sup_{0\le t\le T} |p_t^\varepsilon|_H^4\right]\right) \le \frac{C}{N^2}.
\end{aligned}
$$

Therefore, by using (4.2) and the martingale property of (Λ_t^ε), we obtain

$$
\overline{\lim_{\varepsilon\downarrow 0}} \left| E[\Psi_s(p^\varepsilon)(\Gamma_t^N(p^\varepsilon) - \Gamma_s^N(p^\varepsilon))] \right| \le \frac{C}{N}.
$$

Furthermore, if Ψ_s is continuous on S, then the left-hand side of the above inequality is equal to $|E^{\overline{\Pi}}[\Psi_s(\Gamma_t^N-\Gamma_s^N)]|$ since Γ_t^N is bounded and continuous on S and Π^ε converges to $\overline{\Pi}$. Thus, in consideration of the fact $\sup_{0\le t\le T}E^{\overline{\Pi}}[|\Gamma_t|]<\infty$, we obtain $E^{\overline{\Pi}}[\Psi_s(\Gamma_t-\Gamma_s)]=0$ by letting $N\to\infty$. This equality is also valid for all bounded $(\mathscr{X}_s)$-measurable functional Ψ_s by approximation.

Hence it remains to prove Proposition 4.1.

PROOF OF PROPOSITION 4.1. We set $\Gamma_t(p^\varepsilon)-\Gamma_s(p^\varepsilon)-\Lambda_t^\varepsilon+\Lambda_s^\varepsilon:=\Phi_1^\varepsilon+\Phi_2^\varepsilon+\Phi_3^\varepsilon+\Phi_4^\varepsilon$, where

$$\Phi_1^\varepsilon=\phi((p_t^\varepsilon,\psi)_H)-\phi((p_t^\varepsilon,\psi^\varepsilon(\,\cdot\,,Z_t^\varepsilon))_H)-\{\phi((p_s^\varepsilon,\psi)_H)-\phi((p_s^\varepsilon,\psi^\varepsilon(\,\cdot\,,Z_s^\varepsilon))_H)\},$$

$$\begin{aligned}\Phi_2^\varepsilon=&\int_s^t\{\phi'((p_r^\varepsilon,\psi^\varepsilon(\,\cdot\,,Z_r^\varepsilon))_H)-\phi'((p_r^\varepsilon,\psi)_H)\}(p_r^\varepsilon,(\mathscr{L}^\varepsilon\psi^\varepsilon)(\,\cdot\,,Z_r^\varepsilon))_H\,dr,\\&+\frac{1}{2}\sum_{k=1}^n\int_s^t\{\phi''((p_r^\varepsilon,\psi^\varepsilon(\,\cdot\,,Z_r^\varepsilon))_H)-\phi''((p_r^\varepsilon,\psi)_H)\}(p_r^\varepsilon,(\mathscr{M}_k^\varepsilon\psi^\varepsilon)(\,\cdot\,,Z_r^\varepsilon))_H^2\,dr,\end{aligned}$$

$$\Phi_3^\varepsilon=\int_s^t\phi'((p_r^\varepsilon,\psi)_H)(p_r^\varepsilon,(\mathscr{L}^\varepsilon\psi^\varepsilon-(L^0)^*\psi)(\,\cdot\,,Z_r^\varepsilon))_H\,dr,$$

$$\Phi_4^\varepsilon=\frac{1}{2}\sum_{k=1}^n\int_s^t\phi''((p_r^\varepsilon,\psi)_H)\{(p_r^\varepsilon,(\mathscr{M}_k^\varepsilon\psi^\varepsilon)(\,\cdot\,,Z_r^\varepsilon))_H^2-(p_r^\varepsilon,\overline{h}_k\psi)_H^2\}\,dr.$$

We prove $\lim_{\varepsilon\downarrow0}E[\Psi_s(p^\varepsilon)\Phi_i^\varepsilon]=0$ one by one for each $i=1,\dots,4$.

Note first that ψ^ε satisfies $|\psi^\varepsilon(\,\cdot\,,z)-\psi|_H\le\varepsilon|\chi^m|_{L^\infty}|\psi_{x_m}|_H$ for all $z\in\boldsymbol{R}^n$ by definition, where $|\cdot|_{L^\infty}$ stands for the L^∞-norm. Thus, we get $|E[\Psi_s(p^\varepsilon)\Phi_1^\varepsilon]|\le\varepsilon CE[\sup_{0\le t\le T}|p_t^\varepsilon|_H]$, which implies $\lim_{\varepsilon\downarrow0}E[\Psi_s(p^\varepsilon)\Phi_1^\varepsilon]=0$. Similarly, we can easily show that

$$\big|E[\Psi_s(p^\varepsilon)\Phi_2^\varepsilon]\big|\le\varepsilon C\Big(E[\sup_{0\le t\le T}|p_t^\varepsilon|_H^2]+E[\sup_{0\le t\le T}|p_t^\varepsilon|_H^3]\Big)\xrightarrow[\varepsilon\downarrow0]{}0$$

since $|\mathscr{L}^\varepsilon\psi^\varepsilon(\,\cdot\,,z)|_H$ and $|\mathscr{M}_k^\varepsilon\psi^\varepsilon(\,\cdot\,,z)|_H$ are bounded, uniformly in $\varepsilon>0$ and $z\in\boldsymbol{R}^n$.

In order to prove $\lim_{\varepsilon\downarrow0}E[\Psi_s(p^\varepsilon)\Phi_3^\varepsilon]=0$, we prepare the following lemma.

LEMMA 4.1. *Let $\alpha\in C^1(\boldsymbol{R}^{d+n})$ be a periodic function with period 1 in all components such that $\langle\langle\alpha\rangle\rangle=0$. Then, for every $\varphi\in C_c^\infty(\boldsymbol{R}^d)$, we have*

$$\lim_{\varepsilon\downarrow0}E\left[\Psi_s(p^\varepsilon)\int_s^t\phi'((p_r^\varepsilon,\psi)_H)(p_r^\varepsilon,\alpha(\,\cdot\,/\varepsilon,Z_r^\varepsilon/\varepsilon)\,\varphi)_H\,dr\right]=0.\tag{4.3}$$

PROOF. For each $z\in\boldsymbol{R}^n$, we set $\tilde\alpha(z):=\int_{\boldsymbol{T}^d}\alpha(x,z)\,dx$, and consider the PDE on $\boldsymbol{R}^d$

$$\Delta_x\eta(\,\cdot\,,z)=\alpha(\,\cdot\,,z)-\tilde\alpha(z),\tag{4.4}$$

where Δ_x stands for the Laplace operator with respect to $x=(x_1,\dots,x_d)$. Recall that (4.4) admits a unique periodic solution $\eta(\,\cdot\,,z)\in C^2(\boldsymbol{R}^d)$ with period 1 in all components such that $\tilde\eta(z):=\int_{\boldsymbol{T}^d}\eta(x,z)\,dx=0$. Then, by the integration by parts formula,

$$(p_r^\varepsilon, \alpha(\cdot/\varepsilon, Z_r^\varepsilon/\varepsilon)\,\varphi)_H = (p_r^\varepsilon, \varphi)_H\, \tilde{\alpha}(Z_r^\varepsilon/\varepsilon) - \varepsilon \sum_{i=1}^{d} (\nabla_{x_i} p_r^\varepsilon, (\nabla_{x_i}\eta)(\cdot/\varepsilon, Z_r^\varepsilon/\varepsilon)\,\varphi)_H$$
$$- \varepsilon \sum_{i=1}^{d} (p_r^\varepsilon, (\nabla_{x_i}\eta)(\cdot/\varepsilon, Z_r^\varepsilon/\varepsilon)\,\nabla_{x_i}\varphi)_H\,.$$

Thus, it is sufficient for (4.3) to prove

$$\lim_{\varepsilon\downarrow 0} E\left[\Psi_s(p^\varepsilon)\int_s^t \phi'((p_r^\varepsilon,\psi)_H)(p_r^\varepsilon,\varphi)_H\,\tilde{\alpha}(Z_r^\varepsilon/\varepsilon)\,dr\right] = 0\,. \tag{4.5}$$

Let us take the N-partition $(s,t] := \bigcup_{i=1}^N (s_i, s_{i+1}]$, where $s_i = s + N^{-1}(t-s)(i-1)$. Then,

$$\left| E\left[\Psi_s(p^\varepsilon)\int_s^t \phi'((p_r^\varepsilon,\psi)_H)(p_r^\varepsilon,\varphi)_H\,\tilde{\alpha}(Z_r^\varepsilon/\varepsilon)\,dr\right]\right|$$
$$\le C\sum_{i=1}^N \int_{s_i}^{s_{i+1}} E[\,|(p_r^\varepsilon - p_{s_i}^\varepsilon,\psi)_H|\,|p_r^\varepsilon|_H + |(p_r^\varepsilon - p_{s_i}^\varepsilon,\varphi)_H|\,]\,dr$$
$$+ \sum_{i=1}^N \left| E\left[\Psi_s(p^\varepsilon)\phi'((p_{s_i}^\varepsilon,\psi)_H)(p_{s_i}^\varepsilon,\varphi)_H \int_{s_i}^{s_{i+1}} \tilde{\alpha}(Z_r^\varepsilon/\varepsilon)\,dr\right]\right| =: I_1^\varepsilon + I_2^\varepsilon\,.$$

By (3.3) and Hölder's inequality,

$$I_1^\varepsilon \le C\left\{1 + \left(E\left[\sup_{0\le t\le T}|p_t^\varepsilon|_H^{4/3}\right]\right)^{3/4}\right\}\sum_{i=1}^N \left(\int_{s_i}^{s_{i+1}}(r-s_i)^2\,dr\right)^{1/4}(s_{i+1}-s_i)^{3/4} \le \frac{C}{\sqrt{N}}\,. \tag{4.6}$$

On the other hand, by using $\lim_{\varepsilon\downarrow 0} E[(\int_{s_i}^{s_{i+1}}\tilde{\alpha}(Z_r^\varepsilon/\varepsilon)\,dr)^2\,|\,\mathscr{F}_{s_i}] = 0$ (see p. 400 of [**3**]), we can verify that for each fixed $N > 0$,

$$I_2^\varepsilon \le C\sqrt{E\left[\sup_{0\le t\le T}|p_t^\varepsilon|_H^2\right]}\sum_{i=1}^N \sqrt{E\left[\left(\int_{s_i}^{s_{i+1}}\tilde{\alpha}(Z_r^\varepsilon/\varepsilon)\,dr\right)^2\right]} \xrightarrow[\varepsilon\downarrow 0]{} 0\,.$$

Thus, in combination with (4.6), we obtain (4.5). □

Now we return to the proof of $\lim_{\varepsilon\downarrow 0} E[\Psi_s(p^\varepsilon)\Phi_3^\varepsilon] = 0$. First, we check $\langle\langle c^{ij}\rangle\rangle = \bar{c}^{ij}$, $\langle\langle d^{ij}\rangle\rangle = 0$ and $\langle\langle g^i\rangle\rangle = \bar{g}^i$. By the integration by parts formula, we can see $\langle\langle d^{ij}\rangle\rangle = 0$ and

$$\bar{c}^{ij} - \langle\langle c^{ij}\rangle\rangle = -\langle\langle \chi^i\{\nabla_{x_{i'}}(a^{i'j'}\nabla_{x_{j'}}\chi^j) + A^{kl}\,\nabla_{z_k}\nabla_{z_l}\chi^j + \nabla_{x_{i'}}a^{i'j}\}\rangle\rangle = 0\,.$$

The equality $\langle\langle g^i\rangle\rangle = \bar{g}^i$ can be seen as follows. For $z \in \boldsymbol{R}^n$ and $m = 1,\ldots,d$, we define $\tilde{\chi}^m(z) := \int_{\boldsymbol{T}^d}\chi^m(x,z)\,dx$. Then, in view of (1.5), $\tilde{\chi}^m$ satisfy $A^{kl}\nabla_{z_k}\nabla_{z_l}\tilde{\chi}^m(z) = 0$. Since (A^{kl}) is positive definite and $\tilde{\chi}^m$ are periodic, the strong maximum principle implies that $\tilde{\chi}^m$ are constant functions; in particular, $\nabla_{z_l}\tilde{\chi}^m(z) = 0$ for all $z \in \boldsymbol{R}^n$. Therefore $\langle\langle f^l\,\nabla_{z_l}\chi^m\rangle\rangle =$

$\int_{\boldsymbol{T}^n} f(z)\,\nabla_{z_l}\tilde{\chi}^m(z)\,dz=0$, and we obtain $\langle\langle g^i\rangle\rangle=\overline{g}^i$. Hence, applying Lemma 4.1 to $\alpha=c^{ij}-\overline{c}^{ij}$, $\alpha=d^{ij}$ and $\alpha=g^i-\overline{g}^i$, we get $\lim_{\varepsilon\downarrow 0}E[\Psi_s(p^\varepsilon)\Phi_3^\varepsilon]=0$.

Finally, we shall show $\lim_{\varepsilon\downarrow 0}E[\Psi_s(p^\varepsilon)\Phi_4^\varepsilon]=0$. Recall that $Q^{kl}\nabla_{z_l}\tilde{\chi}^m(z)=0$ for all $z\in\boldsymbol{R}^n$. Then, by taking $\alpha(x,z)=Q^{kl}\nabla_{z_l}\chi(x,z)$ and $\alpha(x,z)=h_k(x,z)-\tilde{h}_k(z)$, where $\tilde{h}_k(z):=\int_{\boldsymbol{T}^d}h_k(x,z)\,dx$, we can show similarly to the proof of Lemma 4.1 that

$$\begin{aligned}\big|E[\Psi_s(p^\varepsilon)\Phi_4^\varepsilon]\big| &\le \sum_{k=1}^n\left|E\left[\Psi_s(p^\varepsilon)\int_s^t\phi''((p_r^\varepsilon,\psi)_H)(p_r^\varepsilon,\psi)_H^2\{(\tilde{h}_k(Z_r^\varepsilon/\varepsilon))^2-|\overline{h}_k|^2\}\,dr\right]\right|\\ &\quad+(\varepsilon+\varepsilon^2)C\left\{E\left[\sup_{0\le t\le T}|p_t^\varepsilon|_H^4\right]+E\left[\left(\int_0^T|p_t^\varepsilon|_V^2\,dt\right)^2\right]\right\}.\end{aligned}$$

Since the first term of the right-hand side converges to zero by the same argument as in the proof of (4.5), we obtain $\lim_{\varepsilon\downarrow 0}E[\Psi_s(p^\varepsilon)\Phi_4^\varepsilon]=0$, and the proof of Proposition 4.1 has been completed.

ACKNOWLEDGMENT. The author would like to express his sincere thanks to the referee. He pointed out the errors in the first version of this paper.

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