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HOMOGENIZATION FOR TIME-DEPENDENT TWO-DIMENSIONAL INCOMPRESSIBLE GAUSSIAN FLOWS[1]

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Dedicated to the memory of Sergei Kozlov

We consider the diffusive scaling limit for the transport of a passive scalar in a two-dimensional time-dependent incompressible Gaussian velocity field and in the presence of molecular diffusivity. We prove that homogenization holds in this limiting regime and we derive some simple properties of the effective diffusivity tensor.

1. Introduction and results. The study of the large time behavior of the solutions of the advection–diffusion equation

$$\frac{\partial C(t,x)}{\partial t} = \kappa\,\Delta C(t,x) + (\mathbf{v}(t,x)\nabla)U(t,x), \tag{1}$$

when $\mathbf{v}(t,x)$ is an incompressible random vector field which is stationary in time and homogeneous in space, has attracted the attention of many applied mathematicians in the last decades. This equation governs the time evolution of concentrations of passive tracers in a fluid. It is used to study the salinity or the temperature of the ocean, for example. This equation is also used to model the spread of pollutants at the surface of the ocean. For all these reasons, it has been the object of active research both experimently and theoretically (see [1], [2], [4], [6], [10], [14] and [19], for example). A number of new tools have been developed to understand its rich dynamical scaling behavior, but so far all the published works have focused on either time-independent flows (see, for example, [5], [13] or [21]) or very special time-dependent flows (see for example [3] for a remarkable analysis of shear flows). In this paper, we study the diffusive scaling limit for time-dependent flows.

Let us describe the specific model we work with. We consider the advection–diffusion equation (1) on the two-dimensional plane $\mathbb{R}^2$. We restrict ourselves to the two-dimensional case for the sake of simplicity. Our analysis (and the results that we prove) can be carried out in the general case of finite-dimensional divergence free Gaussian velocity fields with finitely Fourier modes. We assume that the medium is incompressible in the sense

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that the velocity field is divergence free. Because the dimension is 2, there exists a scalar function $\phi(t, x)$, called the stream function, which satisfies $\mathbf{v}(t, x) = \Delta^{\perp} \phi(t, x)$ or in other words:

$$\mathbf{v}(t, x) = \begin{bmatrix} \dfrac{\partial \phi(t, x)}{\partial x_2} \\ -\dfrac{\partial \phi(t, x)}{\partial x_1} \end{bmatrix}. \tag{2}$$

In the general d-dimensional case, the stream function ϕ has to be replaced by d scalar functions and the technicalities of the proof are more involved, though the methods remain the same. We shall assume that the stream function is of the form

$$\phi(t, x) = \sum_{i=1}^{n} \left[a_i(t)\cos(k_i x) + b_i(t)\sin(k_i x) \right],$$

where $k_1, \dots, k_n$ are n points (Fourier modes) in $\mathbb{R}^2$ and where

$$\{a_1, b_1, \dots, a_n, b_n\}$$

are mutually independent scalar Ornstein–Ulhenbeck processes. The form of this specific model of velocity field is essentially imposed by the conditions of (1) incompressibility, (2) Markov property in time and (3) finite spectral support. This last assumption was introduced in [9] as a way to find a reasonable approximation of an isotropic incompressible Gaussian velocity field which would be amenable to numerical simulations. See also [6] for more on the numerical simulations of the transport properties of general Gaussian velocity fields with Kolmogorov spectra. We assume that

$$\begin{aligned} da_i(t) &= -\alpha_i a_i(t)\, dt + \sigma_i\, dz_i^a(t), \\ db_i(t) &= -\alpha_i b_i(t)\, dt + \sigma_i\, dz_i^b(t), \end{aligned} \tag{3}$$

where the α_i and σ_i are positive constants and where the $z_1^a(t)$, $z_1^b(t), \dots, z_n^a(t)$ and $z_n^b(t)$ are $2n$ independent standard scalar Wiener processes. As usual, we denote by $\|k_i\|$ the Euclidean norm of the vector k_i. We shall use the notation $\mathbb{P}$ for the probability measure of the space on which the random velocity field is defined and denote by $\mathbb{E}$ the corresponding expectation.

We are interested in the long-time large-scale behavior of the solutions of the advection–diffusion equation (1). We know that under the rescaling of the diffusion approximation limit, this solution converges in distribution to the solution of a stochastic partial differential equation and the moments converge to the solution of a deterministic heat equation. See, for example, [7]. We work here under a different rescaling regime, but we still expect that the asymptotics of the first moment of the solution, in other words the limit

$$\lim_{N\to\infty} \mathbb{E}\left\{C\left(N^2 t, \frac{x}{N}\right)\right\}, \tag{4}$$

which we would like to compute for fixed t and x, be governed by a heat equation. For each realization of the velocity field, the microscopic motion leading to (1) is given by the stochastic differential equation

$$\begin{aligned} dX_t &= \sqrt{2\kappa}\, dW(t) + \mathbf{v}(t, X_t)\, dt, \\ X_0 &= x, \end{aligned} \tag{5}$$

where $\{W(t);\ t \geq 0\}$ is a standard Wiener process (with values in $\mathbb{R}^2$). We shall assume that this Wiener process is defined on the same probability space as the velocity field and that it is independent of the Wiener processes $z_i^a(t)$ and $z_i^b(t)$ appearing in the definition of the velocity field. The limit (4) can be best understood by means of the behavior of the rescaled microscopic motion given by the process

$$Y_N(t) = \frac{1}{N} X_{N^2 t}$$

when N is large. The main result of the paper is the following:

THEOREM 1.1. *The process $\{Y_N(t);\ t \geq 0\}$ converges in distribution to a two-dimensional Brownian motion $\{B(t);\ t \geq 0\}$ with covariance matrix $D(\kappa)$ depending only upon the diffusivity constant κ.*

As an immediate consequence of this result we have the following corollary.

COROLLARY 1.1. *Assume that for each integer N the scalar function $C_N(t, x)$ solves the parabolic problem*

$$\begin{aligned} \frac{\partial C_N(t,x)}{\partial t} &= \kappa\, \Delta C_N(t,x) + N\big(\mathbf{v}(N^2 t, x/N)\Delta\big) C_N(t,x) \\ C_N(0,x) &= f(x), \end{aligned} \tag{6}$$

where the initial condition $f(x)$ is smooth and has compact support. Then the function $\mathbb{E}\{C_N(t, x)\}$ converges locally in the L^2 sense to the solution $C(t, x)$ of the (deterministic) parabolic equation

$$\begin{aligned} \frac{\partial C(t,x)}{\partial t} &= \nabla \cdot \big(D(\kappa)\nabla C(t,x)\big), \\ C(0,x) &= f(x). \end{aligned} \tag{7}$$

Theorem 1.1 is more general than other homogenization results for Gaussian fields (such as [2], [3] or [20], for example) because of the time dependence of the velocity field. After the completion of this paper, we received a preprint [16] addressing the same homogenization problem for incompressible time-dependent Gaussian velocity fields. Even though the results are very similar in nature, the proofs are very different. Indeed, we derive the time decorrelation estimates needed in the proof from the ergodic properties of the Lagrangian velocity while the authors of [16] use the assumption of compact support for the time correlation of the Eulerian velocity field. Moreover, our proof is not restricted to the case $\kappa = 0$.

Our proof of Theorem 1.1 relies on the ergodic properties of the (microscopic) motion in the random velocity field as observed in a frame moving with X_t. Such a Lagrangian velocity is a stationary Markov process. This fact has been known for a long time by workers in homogenization theory. The difficulty is usually in proving that this process is ergodic enough for the additive functionals of interest to satisfy a central limit theorem. This is usually proved by decomposing the additive functional into a martingale part to which a form of the central limit theorem for martingales can be applied, and a remainder part which is shown to be negligible in the limit. See, for example, [21], [14], [19] or [18]. Our proof follows these time-honored lines. Its main thrust is the rewriting of the Lagrangian velocity as a simple function of a Markov process for which we can give a complete analysis. In particular we identify the infinitesimal generator, we show that it has a unique invariant measure and we derive its ergodic properties by providing an exponential bound for the large-time behavior of the semigroup. This crucial estimate is obtained by comparison with the Ornstein–Ulhenbeck semigroup. This comparison argument is important because our Markov process is not symmetric and the standard arguments (see, e.g., [15]) cannot be used directly.

The paper is organized as follows. The next section is devoted to the analysis of the velocity field in a Lagrangian frame. We establish the Markov property, we identify the infinitesimal generator and we derive the ergodic properties which are needed in the proof of the results stated above. Section 3 contains the proof of these results. Section 4 is devoted to the analysis of the dependence of the effective diffusivity matrix $D(\kappa)$ upon the diffusivity constant κ.

2. The velocity field in a Lagrangian frame. We first fix the initial point $x \in \mathbb{R}^2$ and we denote by X_t the solution of the equation of motion (5). We then define the Lagrangian velocity field as

$$\mathbf{v}(t, z) = \tilde{\mathbf{v}}(t, X_t + z). \tag{8}$$

for $z \in \mathbb{R}^2$. Obviously

$$\tilde{\mathbf{v}}(t, z) = \nabla^{\perp} \tilde{\phi}(t, z)$$

with

$$\begin{aligned}\tilde{\phi}(t, z) &= \phi(t, X_t + z)\\ &= \sum_{i=1}^{n} \left[\tilde{a}_i(t)\cos(k_i z) + \tilde{b}_i(t)\sin(k_i z)\right],\end{aligned}$$

where

$$\tilde{a}_i(t) = a_i(t)\cos(k_i X_t) + b_i(t)\sin(k_i X_t),$$
$$\tilde{b}_i(t) = -a_i(t)\sin(k_i X_t) + b_i(t)\cos(k_i X_t),$$

for $i = 1, \dots, n$. Notice that

$$\mathbf{v}(t, X_t) = \tilde{\mathbf{v}}(t, 0) = \sum_{i=1}^{n} k_i^{\perp} \tilde{b}_i(t), \tag{9}$$

where we use the notation $k_i^{\perp} = [k_i^2 - k_i^1]^{\mathsf{T}}$. In the same way that the distribution of the original velocity field $\mathbf{v}(t, x)$ is determined by the $2n$-dimensional Ornstein–Ulhenbeck process

$$v(t) = (a_1(t), b_i(t), \dots, a_n(t), b_n(t)),$$

the distribution of the Lagrangian velocity field is determined by the process

$$\tilde{v}(t) = \big(\tilde{a}_1(t), \tilde{b}_i(t), \dots, \tilde{a}_n(t), \tilde{b}_n(t)\big).$$

The sample paths of this process are continuous by construction. Because of the homogeneity in space of the original velocity field, for each $i = 1, \dots, n$, the scaling parameter and the standard deviation of the processes $a_i(t)$ and $b_i(t)$ have to be the same and consequently the distribution of the Ornstein–Ulhenbeck process $\{(a_i(t), b_i(t)) \colon t \geq 0\}$ in $\mathbb{R}^2$ is invariant under the rotations of the plane. Since the couple $(\tilde{a}_i(t), \tilde{b}_i(t))$ is merely a rotation of the couple $(a_i(t), b_i(t))$, they have the same norms. Also, for $i \neq j$, the processes $\{(\tilde{a}_i(t), \tilde{b}_i(t)) \colon t \geq 0\}$ and $\{(\tilde{a}_j(t), \tilde{b}_j(t)) \colon t \geq 0\}$ are dependent even though the processes $\{(a_i(t), b_i(t)) \colon t \geq 0\}$ and $\{(a_j(t), b_j(t)) \colon t \geq 0\}$ are independent. The process $\{v(t);\ t \geq 0\}$ is a diffusion process whose ergodic properties are well known. In particular, its (unique) invariant measure is the Gaussian measure

$$\begin{aligned} &\nu(da_1\, db_1 \cdots da_n\, db_n) \\ &\quad = \frac{\alpha_1 \cdots \alpha_n}{\pi^n \sigma_1^2 \cdots \sigma_n^2} \exp\left[\frac{-\sum_{i=1}^{n} \alpha_i\big(a_i^2 + b_i^2\big)}{\sigma_i^2}\right] da_1\, db_1 \cdots da_n\, db_n. \end{aligned} \tag{10}$$

We shall prove that ν is also the distribution of $\tilde{v}(t)$ [if the process $v(t)$ is started with its invariant distribution]. The first important result of this section concerns the Markov property of the process $\{\tilde{v}(t);\ t \geq 0\}$. It is contained in the statement of the following theorem where we also identify the infinitesimal generator.

THEOREM 2.1. *The $\mathbb{R}^{2n}$-valued stochastic process $\tilde{v}(T)$ is the $2n$-dimensional diffusion process with infinitesimal generator*

$$\mathscr{L}_\kappa = \mathscr{L}_0 + \kappa \mathscr{L}', \tag{11}$$

where

$$\mathscr{L}_0 = \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2\left(\frac{\partial^2}{\partial\tilde{a}_i^2} + \frac{\partial^2}{\partial\tilde{b}_i^2}\right) - \sum_{i=1}^{n}\alpha_i\left(\tilde{a}_i\frac{\partial}{\partial\tilde{a}_i} + \tilde{b}_i\frac{\partial}{\partial\tilde{b}_i}\right) + \sum_{i,j=1}^{n}(k_i\times k_j)\tilde{b}_j\left(\tilde{b}_i\frac{\partial}{\partial\tilde{a}_i} - \tilde{a}_i\frac{\partial}{\partial\tilde{b}_i}\right) \tag{12}$$

and

$$\mathscr{L}' = \sum_{i,j=1}^{n}\left(\tilde{b}_i\tilde{b}_j\frac{\partial^2}{\partial\tilde{a}_i\,\partial\tilde{a}_j} - 2\tilde{b}_i\tilde{a}_j\frac{\partial^2}{\partial\tilde{a}_i\,\partial\tilde{b}_j} + \tilde{a}_i\tilde{a}_j\frac{\partial^2}{\partial\tilde{b}_i\,\partial\tilde{b}_j}\right)(k_ik_j) - \sum_{i=1}^{n}\left(\tilde{a}_i\frac{\partial}{\partial\tilde{a}_i} + \tilde{b}_i\frac{\partial}{\partial\tilde{b}_i}\right)\|k_i\|^2. \tag{13}$$

We use the notation $(k_i\times k_j) = k_i^1k_j^2 - k_i^2k_j^1$ *for the cross product of the two-dimensional vectors* k_i *and* k_j.

REMARKS. Observing the velocity field in a Lagrangian frame is not a new idea. See, for example, [17] and [18]. It has been routinely used in recent works on homogenization even though attention has been restricted to stationarity (i.e., time independent) velocity fields.

The observation that the velocity field in a Lagrangian frame still has the Markov property will be crucial to us, not merely for its own sake, but because we are able to use this fact to derive useful ergodic properties and precise estimates (by comparison arguments involving simpler processes).

The quadratic form of the coefficients of the second order part of the operator $\mathscr{L}'$ is nonnegative definite and consequently the operator $\mathscr{L}_\kappa$ is strongly elliptic.

The spectral properties of the generator $\mathscr{L}_0$ corresponding to the case $\kappa = 0$ play a crucial role in the proof given in [8] of the positivity of the Lyapunov exponent of the Jacobian flow.

PROOF. According to the Stroock–Varadhan theory [25], we use the martingale formulation of diffusion processes. To prove the desired result, it is sufficient to check that, for any smooth function F with compact support in $\mathbb{R}^{2n}$,

$$M^F(t) = F(\tilde{v}(t)) - F(\tilde{v}(0)) - \int_0^t[\mathscr{L}_\kappa F](\tilde{v}(s))\,ds \tag{14}$$

is a martingale. In order to do this, we rewrite

$$\begin{aligned} F(\tilde{v}(t)) &= F\big(\tilde{a}_1(t),\tilde{b}_1(t),\dots,\tilde{a}_n(t),\tilde{b}_n(t)\big) \\ &= \tilde{F}\big(a_1(t),b_1(t),\dots,a_n(t),b_n(t),X_t\big) \end{aligned}$$

as a function of the original Ornstein–Ulhenbeck processes $a_1(t), b_1(t),\dots,$ $a_n(t), b_n(t)$ and the tracer motion X_t [recall formula (2) and the definition (5) of X_t]. We then apply Itô's formula. Using the obvious fact that

$$\begin{aligned} d[a_i, b_j]_t &= 0, \\ d[a_i, a_j]_t &= d[b_i, b_j]_t = \sigma_i^2 \delta_{i,j}, \\ d[X, a_i]_t &= d[X, b_i]_t = 0, \\ d[X, X]_t &= 2\kappa\, dt \end{aligned}$$

and taking into account equations (3) and (5), a simple calculation yields

$$\begin{aligned} M_F(t) = \sum_{i=1}^{n} \int_0^t \bigg[& \frac{\partial F}{\partial \tilde{a}_i}(\tilde{v}(s))\big(\cos(X_s)\, dz_i^a(s) + \sin(X_t)\, dz_i^b(s)\big) \\ & + \frac{\partial F}{\partial \tilde{b}_i}(\tilde{v}(s))\big(-\sin(X_s)\, dz_i^a(s) + \cos(X_t)\, dz_i^b(s)\big)\bigg] \\ & + \sqrt{2\kappa} \sum_{i=1}^{n} \int_0^t \left(\tilde{b}_i \frac{\partial F}{\partial \tilde{a}_i} - \tilde{a}_i \frac{\partial F}{\partial \tilde{b}_i} \right)(k_i dW(t)), \end{aligned} \tag{15}$$

which is obviously a martingale. This completes our proof. □

Next we identify the adjoints of the components of the infinitesimal generator $\mathscr{L}_\kappa$. The ergodic properties of the process $\tilde{v}(t)$ will follow. Recall that the adjoints of the operators $\mathscr{L}_0$ and $\mathscr{L}'$ are the operators $\mathscr{L}_0^*$ and $\mathscr{L}'^*$, satisfying

$$\int_{\mathbb{R}^{2n}} G \mathscr{L}_0 F\, d\nu = \int_{\mathbb{R}^{2n}} F \mathscr{L}_0^* G\, d\nu \tag{16}$$

and

$$\int_{\mathbb{R}^{2n}} G \mathscr{L}' F\, d\nu = \int_{\mathbb{R}^{2n}} F \mathscr{L}'^* G\, d\nu \tag{17}$$

for all real-valued smooth functions F and G with compact supports in $\mathbb{R}^{2n}$.

PROPOSITION 2.1. *The adjoint operators* $\mathscr{L}_0^*$ *and* $\mathscr{L}'^*$ *are given by*

$$\begin{aligned} \mathscr{L}_0^* = {} & \frac{1}{2} \sum_{i=1}^{n} \sigma_i^2 \left(\frac{\partial^2}{\partial \tilde{a}_i^2} + \frac{\partial^2}{\partial \tilde{b}_i^2} \right) - \sum_{i=1}^{n} \alpha_i \left(\tilde{a}_i \frac{\partial}{\partial \tilde{a}_i} + \tilde{b}_i \frac{\partial}{\partial \tilde{b}_i} \right) \\ & - \sum_{i,j=1}^{n} (k_i \times k_j) \tilde{b} j \left(\tilde{b}_i \frac{\partial}{\partial \tilde{a}_i} - \tilde{a}_i \frac{\partial}{\partial \tilde{b}_i} \right) \end{aligned} \tag{18}$$

and $\mathscr{L}'^* = \mathscr{L}'$.

The proof is a straightforward application of the formula of integration by parts.

REMARK. The above characterization implies that ν is an invariant measure for the diffusion process $\tilde{v}(t)$ and that the symmetrization $\mathscr{L}_0^{(s)} = (\mathscr{L}_0 +$

$\mathscr{L}_0^*)/2$ of $\mathscr{L}_0$ is the generator of the $2n$-dimensional Ornstein–Ulhenbeck process $v(t)$ comprising the original coefficients $a_i(t)$ and $b_i(t)$. This remark is of crucial importance because of the spectral gap (and the ensuing ergodic properties) of the latter. Similarly, we shall use the notations $\mathscr{L}_\kappa^{(s)}$ for the symmetrizations of the operator $\mathscr{L}_\kappa$.

The quadratic form $\mathscr{D}_0$ corresponding to the symmetric operator $\mathscr{L}_0^{(s)}$ is

$$\begin{aligned}\mathscr{D}_0(F) &= \int_{\mathbb{R}^{2n}} F(-\mathscr{L}_0^{(s)}F)\,d\nu\\ &= \int_{\mathbb{R}^{2n}} F(-\mathscr{L}_0 F)\,d\nu\\ &= \frac{1}{2}\int_{\mathbb{R}^{2n}} \sum_{i=1}^{n} \sigma_i^2\left[\left(\frac{\partial F}{\partial \tilde{a}_i}\right)^2 + \left(\frac{\partial F}{\partial \tilde{b}_i}\right)^2\right] d\nu\end{aligned}$$

for any real-valued smooth functions F with compact support. The quadratic form corresponding to $\mathscr{L}'^{(s)}$ is given by

$$\begin{aligned}\mathscr{D}'(F) &= \int_{\mathbb{R}^{2n}} F(-\mathscr{L}'F)\,d\nu\\ &= \int_{\mathbb{R}^{2n}} \left\| \sum_{i=1}^{n}\left(\tilde{b}_i\frac{\partial F}{\partial \tilde{a}_i} - \tilde{a}_i\frac{\partial F}{\partial \tilde{b}_i}\right)k_i\right\|^2 d\nu.\end{aligned}$$

Therefore the quadratic form $\mathscr{D}_\kappa = \mathscr{D}_0 + \kappa\mathscr{D}'$ of the symmetric operator $\mathscr{L}_\kappa^{(s)}$ satisfies

$$\mathscr{D}_\kappa(F) \geq \mathscr{D}_0(F).$$

The operator $-\mathscr{L}_0^{(s)}$ is a nonnegative self-adjoint operator on the Hilbert space $L^2(\mathbb{R}^{2n}, d\nu)$ and its spectrum is the set of numbers $p_1\alpha_1 + \cdots + p_n\alpha_n$ for nonnegative integers $p_1, \dots, p_n$. In particular, this implies that

$$\inf_{\{F:\, \int F\,d\nu = 0,\, \int F^2\,d\nu = 1\}} \mathscr{D}_0(F) = \alpha$$

if we set $\alpha = \min_{i=1,\dots,n} \alpha_i$. Consequently

$$\inf_{\{F:\, \int F\,d\nu = 0,\, \int F^2\,d\nu = 1\}} \mathscr{D}_\kappa(F) \geq \alpha. \tag{19}$$

PROPOSITION 2.2. *Let us denote by* $\langle\cdot,\cdot\rangle$ *the inner product and by* $\|\cdot\|$ *the norm of the Hilbert space* $L^2(\mathbb{R}^{2n}, d\nu)$. *If* $F \in L^2(\mathbb{R}^{2n}, d\nu)$ *is real and such that* $\int F\,d\nu = 0$, *then*

$$\|\exp(t\mathscr{L}_\kappa)F\| \leq \exp(-\alpha t)\|F\|. \tag{20}$$

PROOF. Let us set

$$g(t) = \|F_t\|^2 \quad \text{with } F_t = \exp(t\mathscr{L}_\kappa)F.$$

Then a direct calculation yields

$$\begin{aligned} \frac{dg(t)}{dt} &= \langle F_t, \mathscr{L}_\kappa F_t \rangle + \langle \mathscr{L}_\kappa F_t, F_t \rangle \\ &= -2\langle F_t, -\mathscr{L}_\kappa^{(s)} F_t \rangle \\ &= -2\mathscr{D}_\kappa(F) \end{aligned} \tag{21}$$

and since $\int F_t \, d\nu = 0$, it follows from (19) that

$$\mathscr{D}_\kappa(F_t) \geq \alpha \langle F_t, F_t \rangle \tag{22}$$

and (21) and (19) imply that

$$\frac{dg(t)}{dt} \leq -2\alpha g(t), \tag{23}$$

from which the inequality $g(t) \leq g(0)e^{-2\alpha t}$ follows. This completes the proof. □

The following corollary is an immediate consequence of the above estimation.

COROLLARY 2.1. *If F is an arbitrary function of $L^2(\mathbb{R}^{2n}, d\nu)$ which satisfies $\int F \, d\nu = 0$, then there exists a unique $G \in L^2(\mathbb{R}^{2n}, d\nu)$ such that*

$$\int G \, dv = 0 \quad \textit{and} \quad \mathscr{L}_\kappa G = F.$$

PROOF. We first notice that the integral

$$G = \int_0^\infty \exp(t\mathscr{L}_\kappa) F \, dt$$

converges in $L^2(\mathbb{R}^{2n}, d\nu)$; because of the above estimation, the function G so defined satisfies $\int G \, d\nu = 0$ as an immediate consequence of Proposition 2.1. By construction we have $\mathscr{L}_\kappa G = F$ and uniqueness follows from the following argument. If $G \in L^2(\mathbb{R}^{2n}, d\nu)$ is such that

$$\int G \, d\nu = 0 \quad \text{and} \quad \mathscr{L}_\kappa G = 0,$$

then $\mathscr{D}_\kappa G = 0$ and consequently $\mathscr{D}_0 G = 0$ by (2) from which it follows that $G = 0$. □

3. Proof of the main result. Recalling formula (9) giving the Lagrangian velocity $\mathbf{v}(t, X_t) = \tilde{v}(t, 0)$, one sees that the equation of motion (5) can be rewritten in the form

$$X_t = x + \sqrt{2\kappa}\, W(t) + \int_0^t A\tilde{v}(s) \, ds \tag{24}$$

for some constant deterministic $2 \times (2n)$ matrix A. The first part of the proof is concerned with the tightness of the sequence $\{(N^{-1}X_{N^2 t})_{t \geq 0};\ N = 1, 2, \dots\}$ of processes. We use Kolmogorov's criterion. We shall prove that for each

$T > 0$ there exists a constant $C_T > 0$ such that

$$\mathbb{E}\left\{\left\|\frac{1}{N}X_{N^2t} - \frac{1}{N}X_{N^2s}\right\|^4\right\} \le C_T|t-s|^2 \tag{25}$$

for all $0 \le s,\ t \le T$. Obviously, it follows from (24) and the properties of the increments of a Wiener process that it is enough to prove

$$\frac{1}{N^4}\mathbb{E}\left\{\left\|\int_{N^2s}^{N^2t} A\tilde{v}(u)\,du\right\|^4\right\} \le C_T|t-s|^2 \tag{26}$$

for all $0 \le s,\ t \le T$. Using the stationarity of the process $\tilde{v}$ it is enough to prove that one has

$$\sup_{t\ge 0}\mathbb{E}\left\{\left\|\frac{1}{\sqrt{t}}\int_0^t A\tilde{v}(u)\,du\right\|^4\right\} \le C \tag{27}$$

for some constant C. Obviously, the estimate (27) will hold if we can prove that

$$\sup_{0\le t\le\infty}\ \mathbb{E}\left\{\exp\left(\varepsilon t^{-1/2}\int_0^t A_i\tilde{v}(u)\,du\right)\right\} < \infty \tag{28}$$

where $\varepsilon = \pm 1$ and where $i = 1, 2$. We use the notation A_i for the ith row of the matrix A. Coming back to (28) and using the Feynman–Kac formula, we can rewrite the expectation in an operator form as

$$\begin{aligned}\mathbb{E}\left\{\exp\left(\varepsilon t^{-1/2}\int_0^t A_i\tilde{v}(u)\,du\right)\right\} &= \left\langle \exp\left(t\left(\mathscr{L}_\kappa + t^{-1/2}F\right)\right)\mathbf{1}, \mathbf{1}\right\rangle \\ &\le \left\|\exp\left(t\left(\mathscr{L}_\kappa + t^{-1/2}F\right)\right)\mathbf{1}\right\|,\end{aligned} \tag{29}$$

where we use the notation F for the linear function $\tilde{v} \hookrightarrow F(\tilde{v}) = \varepsilon A_i\tilde{v}$ and $\mathbf{1}$ for the constant function identically equal to 1 and the norm in the rightmost expression is the norm of the Hilbert space $L^2(\mathbb{R}^{2n}, d\nu)$. Using the same argument as in the proof of Proposition 2.2, one proves easily that

$$\left\|\exp\left(t\left(\mathscr{L}_n + t^{-1/2}F\right)\right)\mathbf{1}\right\| \le \exp(-\lambda_t t)$$

if we denote by λ_t the infimum of the spectrum of the self-adjoint operator $-\mathscr{L}_0^{(s)} - t^{-1/2}F$. Using Jensen's inequality for the time integral in the exponential, one sees that the expectation in (28) remains bounded as $t \searrow 0$. Consequently, the proof of the tightness will be complete if we can prove that $\lambda_t = O(1/t)$ when $t \to \infty$. Since the operator of multiplication by the function $t^{-1/2}F$ is a $-\mathscr{L}_0^{(s)}$-bounded perturbation in the sense of the lemma on page 17 of [22], one can use the perturbation theory for the eigenvalues of the discrete spectrum of self-adjoint operators to conclude. Indeed, for t large enough, the infimum of he spectrum of the operator $-\mathscr{L}_0^{(s)} - t^{-1/2}F$ is an isolated eigenvalue λ_t. Moreover this eigenvalue is an analytic function of $t^{-1/2}$ for $t^{-1/2}$ in a neighborhood of 0. Finally we have the expansion

$$\lambda_t = \lambda_\infty + a_0 t^{-1/2} + a_1 t^{-1} + o(t^{-1}),$$

where λ_∞ denotes the infimum of the spectrum of the unperturbed operator $-\mathscr{L}_0^{(s)}$. Obviously $\lambda_\infty = 0$. Consequently, the proof of the tightness will be complete if we can prove that $a_0 = 0$. But this fact is immediate if we use the explicit formula

$$a_0 = \langle \mathbf{1}, F\mathbf{1} \rangle = \int F(\tilde{v})\, d\nu(\tilde{v})$$

for a_0. This formula is derived in detail in the finite-dimensional case in [22], pages 6 and 7, but its proof applies as well to the present situation of a regular analytic family.

The second part of the proof consists of proving that there is only one possible limit point. This will prove that the sequence of processes $\{(N^{-1}X_{N^2 t})_{t \ge 0};\ N = 1, 2, \dots\}$ is indeed convergent. For every $l \in \mathbb{R}^2$, one has

$$lX_t = lx + \sqrt{2\kappa}\int_0^t d(l'W(s)) + \int_0^t F_l(\tilde{v}(s))\, ds, \tag{30}$$

where we use the notation F_l for the linear function

$$\tilde{v} = \left(\tilde{a}_1, \tilde{b}_1, \dots, \tilde{a}_n, \tilde{b}_n\right) \hookrightarrow F_l(\tilde{v}) = lA\tilde{v} = \sum_{i=1}^{n} (k_i \times l)\tilde{b}_i.$$

The proof relies on an approximation of the bounded variation term in the right-hand side of equation (30) by a martingale. Let $U_{\kappa, l}$ be the unique mean zero solution of the equation $\mathscr{L}_\kappa U_{\kappa, l} = F_l$. Then this bounded variation term can be written in the form

$$\int_0^t F_l(\tilde{v}(s))\, ds = U_{\kappa, l}(\tilde{v}(t)) - U_{\kappa, l}(\tilde{v}(0)) - M_t^{U_{\kappa, l}}.$$

For typographical reasons we shall use the notation M_t for the martingale $M_t^{U_{\kappa, l}}$. Henceforth:

$$\frac{1}{N} lX_{N^2 t} = \frac{1}{N} lx + \frac{1}{N} U_{\kappa, l}(\tilde{v}(N^2 t)) - \frac{1}{N} U_{\kappa, l}(\tilde{v}(0)) - \frac{1}{N} M_{N^2 t} + \frac{\sqrt{2\kappa}}{N}\int_0^{N^2 t} d(lW(s)).$$

The sum of the first three terms converges to zero for fixed t since

$$\limsup_{N \to \infty} \mathbb{E}\left\{\left[\frac{1}{N} U_{\kappa, l}(\tilde{v}(N^2 t))\right]^2\right\} = \limsup_{N \to \infty} \frac{1}{N^2}\int_{\mathbb{R}^{2n}} (U_{\kappa, l})^2\, d\nu = 0.$$

More generally, this convergence is in the sense of the convergence of the finite-dimensional marginals. Since we have already proved that the family of processes in the left-hand side is tight, the result will follow from the control and the identification of the limit of the continuous local martingale

$$M_t^{(N)} = \frac{1}{N} M_{N^2 t} - \frac{\sqrt{2\kappa}}{N}\int_0^{N^2 t} d(lW(s))$$

by means of the central limit theorem for martingales (see, e.g., [11]). To use this result it is enough to prove the convergence of the quadratic variation of this continuous local martingale. The quadratic variation of M_t is given by the "opérateur carré du champ." More precisely,

$$[M, M]_t = \int_0^t \left[\mathscr{L}_\kappa U_{\kappa,l}^2 - 2U_{\kappa,l}\mathscr{L}_\kappa U_{\kappa,l}\right](\tilde{v}(s))\,ds,$$

from which we get

$$\begin{aligned}
&[M^{(N)}, M^{(N)}]\\
&\quad = \frac{1}{N^2}\int_0^{N^2 t}\Bigg[\sum_{i=1}^n \sigma_i^2\left(\left|\frac{\partial U_{\kappa,l}}{\partial \tilde{a}_i}(\tilde{v}(s))\right|^2 + \left|\frac{\partial U_{\kappa,l}}{\partial \tilde{b}_i}(\tilde{v}(s))\right|^2\right)\\
&\qquad\qquad + 2\kappa\left(\left\|\left[l - \sum_{i=1}^n\left(\tilde{b}_i\frac{\partial U_{\kappa,l}}{\partial \tilde{a}_i} - \tilde{a}_i\frac{\partial U_{\kappa,l}}{\partial \tilde{b}_i}\right)k_i\right\|(\tilde{v}(s))\right)^2\Bigg]\,ds.
\end{aligned}$$

The ergodic theorem for an additive functional of the ergodic Markov process (see, e.g., [23]) implies that the right-hand side converges, for each fixed t, to $(l'D(\kappa)l)t$ where the effective diffusivity matrix $D(\kappa)$ is, by its quadratic form,

$$\begin{aligned}
lD(\kappa)l &= \int_{\mathbb{R}^{2n}}\sum_{i=1}^n \sigma_i^2\left[\left(\frac{\partial U_{\kappa,l}}{\partial \tilde{a}_i}\right)^2 + \left(\frac{\partial U_{\kappa,l}}{\partial \tilde{b}_i}\right)^2\right]d\nu\\
&\quad + 2\kappa\int_{\mathbb{R}^{2n}}\left(\left[l_1 - \sum_{i=1}^n\left(\tilde{b}_i\frac{\partial U_{\kappa,l}}{\partial \tilde{a}_i} - \tilde{a}_i\frac{\partial U_{\kappa,l}}{\partial \tilde{b}_i}\right)k_i^1\right]^2\right.\\
&\qquad\qquad \left. + \left[l_2 - \sum_{i=1}^n\left(\tilde{b}_i\frac{\partial U_{\kappa,l}}{\partial \tilde{a}_i} - \tilde{a}_i\frac{\partial U_{\kappa,l}}{\partial \tilde{b}_i}\right)k_i^2\right]^2\right)d\nu
\end{aligned} \tag{31}$$

for any l in $\mathbb{R}^2$. Consequently, the martingale convergence theorem implies that $(1/N)l \cdot X_{N^2 t}$ converges weakly to a process of Brownian motion with variance–covariance matrix $D(\kappa)$. □

The method, used in our proof to approximate the additive functional of an ergodic Markov process by a martingale, has been proven to be very useful in other contexts, for example in the study of the long-time behavior of a random dynamics (see, e.g., [13], [15] or [24]).

4. Effective diffusivity. In this section, we shall derive some qualitative properties of the effective diffusivity $D(\kappa)$ introduced above. It is not hard to see that this quantity is well defined when $\kappa = 0$. In fact,

$$lD(0)l = \int_{\mathbb{R}^{2n}} \sum_{i=1}^{n} \sigma_i^2 \left[\left(\frac{\partial U_{0,l}}{\partial \tilde{a}_i} \right)^2 + \left(\frac{\partial U_{0,l}}{\partial \tilde{b}_i} \right)^2 \right] d\nu,$$

where $U_{0,l}$ is the unique mean 0 function in $L^2(\mathbb{R}^{2n}, d\nu)$ satisfying

$$\mathscr{L}_0 U_{0,l} = F_l.$$

The dependence of the effective diffusivity upon the molecular diffusivity κ is highly nonlinear and describing this dependence is a very interesting and very difficult question in general (see, nevertheless, [3] and [13] for results in the case of time-independent velocity fields). In the next proposition we attempt to give some qualitative information on this dependence.

PROPOSITION 4.1. *If the matrix* $D(\kappa)$ *is defined as in* (31), *then* (i) $\lim_{\kappa \downarrow 0} D(\kappa) = D(0)$ *and* (ii) $D(\kappa) > 2\kappa I$ *in the sense that* $l'D(0)l > 2\kappa l'l$, *for all nonzero vectors* $l \in \mathbb{R}^2$.

REMARK. The claim in part (i) gives the leading order of the behavior of $D(\kappa)$ for small κ. However, the analysis of the higher-order terms of a possible expansion is an interesting and completely open problem.

The lower bound given in part (ii) confirms rigorously (at least for our model) the well-known principle of diffusivity enchancement for incompressible flows. There is an extensive literature on this problem for incompressible flows and especially for stationary flows. See [12] and references therein for more on this phenomenon.

PROOF. Equation (31) gives

$$\begin{aligned} l'D(\kappa)l = {} & \int_{\mathbb{R}^{2n}} \sum_{i=1}^{n} \sigma_i^2 \left[\left(\frac{\partial U_{\kappa,l}}{\partial \tilde{a}_i} \right)^2 + \left(\frac{\partial U_{\kappa,l}}{\partial \tilde{b}_i} \right)^2 \right] d\nu \\ & + 2\kappa \int_{\mathbb{R}^{2n}} \left\{ \left[\left(\sum_{i=1}^{n} \left(\tilde{b}_i \frac{\partial U_{\kappa,l}}{\partial \tilde{a}_i} - \tilde{a}_i \frac{\partial U_{\kappa,l}}{\partial \tilde{b}_i} \right) k_i^1 \right] ^2 \right. \\ & \left. + \left[\left(\sum_{i=1}^{n} \left(\tilde{b}_i \frac{\partial U_{\kappa,l}}{\partial \tilde{a}_i} - \tilde{a}_i \frac{\partial U_{\kappa,l}}{\partial \tilde{b}_i} \right) k_i^2 \right]^2 \right\} d\nu \\ & 2\kappa l'l - 4\kappa \int_{\mathbb{R}^{2n}} \sum_{i=1}^{n} \left(\tilde{b}_i \frac{\partial U_{\kappa,l}}{\partial \tilde{a}_i} - \tilde{a}_i \frac{\partial U_{\kappa,l}}{\partial \tilde{b}_i} \right) (k_i l)\, d\nu \\ = {} & A_1 + A_2 + A_3 - A_4. \end{aligned}$$

We use integration by parts to compute the last term. We obtain

$$\begin{aligned} A_4 &= 4\kappa \int_{\mathbb{R}^{2n}} \sum_{i=1}^{n} \left(\tilde{a}_i \tilde{b}_i U_{\kappa,l} - \tilde{a}_i \tilde{b}_i U_{\kappa,l} \right) d\nu \\ &= 0. \end{aligned}$$

This implies that

$$l'D(\kappa)l = \int_{\mathbb{R}^{2n}} \sum_{i=1}^{n} \sigma_i^2 \left[\left(\frac{\partial U_{\kappa,l}}{\partial \tilde{a}_i} \right)^2 + \left(\frac{\partial U_{\kappa,l}}{\partial \tilde{b}_i} \right)^2 \right] d\nu$$

$$+ 2\kappa \int_{\mathbb{R}^{2n}} \left\{ \left[\left(\sum_{i=1}^{n} \left(\tilde{b}_i \frac{\partial U_{\kappa,l}}{\partial \tilde{a}_i} - \tilde{a}_i \frac{\partial U_{\kappa,l}}{\partial \tilde{b}_i} \right) k_i^1 \right]^2 \right.$$

$$\left. + \left[\left(\sum_{i=1}^{n} \left(\tilde{b}_i \frac{\partial U_{\kappa,l}}{\partial \tilde{a}_i} - \tilde{a}_i \frac{\partial U_{\kappa,l}}{\partial \tilde{b}_i} \right) k_i^2 \right]^2 \right\} d\nu + 2\kappa l' l$$

$$> 2\kappa l' l.$$

In the above, we used the fact that $U_{\kappa,l}$ is not identically 0. This completes the proof of part (ii).

Multiplying both sides of the definition $\mathscr{L}_\kappa U_{\kappa,l} = F_l$ by $U_{\kappa,l}$ and integrating with respect to ν,

$$\mathscr{D}_\kappa(U_{\kappa,l}) = -\int_{\mathbb{R}^{2n}} U_{\kappa,l} F_l \, d\nu \le C_l \|U_{\kappa,l}\|, \tag{32}$$

where C_l is a constant which depends only upon l and the Fourier modes k_i. On the other hand, from (19) we have

$$D(U_{\kappa,l}) \ge \alpha \|U_{\kappa,l}\|^2$$

since $U_{\kappa,l}$ has mean zero. Hence, $\|U_{\kappa,l}\| \le C_l/\alpha$, and together with (32), this implies that

$$D(U_{\kappa,l}) \le C_1^2/\alpha.$$

Thus the set of $U_{\kappa,l}$'s parameterized by κ is compact in the $L^2(\nu)$ because of the well-known H^1 embedding theorem. A standard argument from the L^2 theory of elliptic equations gives that $U_{\kappa,l}$ converges to $U_{0,l}$ in the sense of the D_0 norm as defined in Section 2. Hence, on account of the equality in (32), the second term A_2 tends to 0 as κ goes to 0 and the convergence of the energies

$$\lim_{\kappa \downarrow 0} \mathscr{D}_0(U_{\kappa,l}) = \mathscr{D}_0(U_{0,l}),$$

from which we conclude that

$$\lim_{\kappa \downarrow 0} D(\kappa) = D(0).$$

This completes the proof. □

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