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## Homogenization of almost periodic monotone operators

by

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**ABSTRACT.** — We determine some sufficient conditions for the G-convergence of sequences of quasi-linear monotone operators, together with an asymptotic formula for the G-limit. We then prove a homogenization theorem for quasiperiodic monotone operators and, eventually, extend this result to general almost periodic monotone operators using an approximation result and a closure lemma.

*Key words* : Homogenization, G-convergence, almost periodic functions, monotone operators, quasi-linear equations.

**RÉSUMÉ.** — Nous déterminons quelques conditions suffisantes pour la G-convergence de suites d'opérateurs monotones quasi linéaires, avec une

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formule asymptotique pour la G-limite. Ensuite nous démontrons un théorème d'homogénéisation pour les opérateurs monotones quasi-périodiques et, enfin, nous étendons ce résultat aux opérateurs monotones presque périodiques en utilisant un résultat d'approximation.

## INTRODUCTION

In this paper we consider a class of quasi-linear operators  $\mathcal{A} : H_0^{1,p}(\Omega) \rightarrow H^{-1,q}(\Omega)$  of the form

$$\mathcal{A}u = -\operatorname{div}(a(x, Du)),$$

where  $\Omega$  is a bounded open subset of  $\mathbf{R}^n$ ,  $1 < p < +\infty$ ,  $1/p + 1/q = 1$ , and the function  $a : \Omega \times \mathbf{R}^n \rightarrow \mathbf{R}^n$  satisfies suitable measurability, continuity, and monotonicity assumptions. By  $M_\Omega$  we denote the set of such functions  $a$ . In order to study the behaviour of boundary value problems of the type

$$\begin{cases} -\operatorname{div}(a(x, Du)) = f & \text{on } \Omega, \\ u \in H_0^{1,p}(\Omega) \end{cases}$$

under perturbations of the function  $a \in M_\Omega$ , a notion of G-convergence has been introduced in [16]. Its definition and main properties are recalled in Section 1.

The main purpose of Section 2 is to determine some conditions on a sequence of functions  $(a_h)$  in  $M_\Omega$  which imply G-convergence. It turns out that one of them is simply the strong convergence in  $(L^1(\Omega))^n$  of the sequence  $(a_h(\cdot, \xi))$ , for every  $\xi \in \mathbf{R}^n$  (see Theorem 2.1). A necessary and sufficient condition involves the limit behaviour, as  $h$  tends to  $+\infty$ , of the integrals

$$\int_A a_h(x, Dv_h + \xi) dx$$

for every  $\xi \in \mathbf{R}^n$  and for every  $A$  in a suitable family of open sets, where the functions  $v_h$  are the solutions to the Dirichlet boundary value problems

$$\begin{cases} -\operatorname{div}(a_h(x, Dv_h + \xi)) = 0 & \text{on } A, \\ v_h \in H_0^{1,p}(A) \end{cases}$$

(see Theorem 2.3). The proof of this result relies on a representation formula for functions  $a \in M_\Omega$  given in Theorem 2.2.

Section 3 is concerned with the homogenization of operators defined by functions of the class  $M_{\Omega}$ ; *i.e.*, the G-convergence of sequences of functions  $(a_h)$  of the form

$$a_h(x, \xi) = a\left(\frac{x}{\varepsilon_h}, \xi\right),$$

where  $(\varepsilon_h)$  is a sequence of positive real numbers converging to 0. We suppose that the function  $a: \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^n$  satisfies the usual measurability, continuity, and monotonicity assumptions, and a condition of quasiperiodicity with respect to  $x$  (*see* Definition 3.1). By using the results obtained in Section 2, we prove that there exists a monotone operator  $b: \mathbf{R}^n \rightarrow \mathbf{R}^n$  such that, for every  $f \in H^{-1, q}(\Omega)$ , the solutions  $u_h$  and the momenta  $a\left(\frac{x}{\varepsilon_h}, Du_h\right)$  of the Dirichlet boundary value problems

$$\begin{cases} -\operatorname{div}\left(a\left(\frac{x}{\varepsilon_h}, Du_h\right)\right) = f & \text{on } \Omega, \\ u_h \in H_0^{1, p}(\Omega) \end{cases}$$

converge, as  $(\varepsilon_h)$  tends to 0, to the solution  $u$  and the momentum  $b(Du)$  of the homogenized problem

$$\begin{cases} -\operatorname{div}(b(Du)) = f & \text{on } \Omega, \\ u \in H_0^{1, p}(\Omega). \end{cases}$$

Theorem 3.4 gives also an asymptotic formula for the function  $b$ ; its proof generalizes a construction used in [36].

In Section 4 we extend the homogenization result of Section 3 to almost periodic operators (in the sense of Besicovitch; *see* Definition 4.1) defined by functions of the class  $M_{\mathbf{R}^n}$  using an approximation result (Lemma 4.4) and a closure lemma (Lemma 4.3).

The notion of G-convergence for second order linear elliptic operators was studied by E. De Giorgi and S. Spagnolo in the symmetric case (*see* [40], [41], [42], [21]), and then extended to the non-symmetric case by F. Murat and L. Tartar under the name of H-convergence (*see* [43], [44], and [34]). A further extension to higher order linear elliptic operators can be found in [47] together with an extensive bibliography on this subject. Results for the quasi-linear case are given, among others, in [46], [37], [23], [22] and [16].

For the related problems in homogenization theory under periodicity hypotheses on  $a(\cdot, \xi)$ , we refer to the books [2], [39], and [1], which contain a wide bibliography on this topic. Homogenization results for quasi-linear operators are obtained in [4], [5], [6], [25], [26], [17], while the almost periodic case for linear equations is studied in [28] and [36]. A corrector result for quasi-linear periodic equations has been obtained in

[18], whereas an analogous theorem for the almost periodic case will appear in [10].

From another point of view, the homogenization of a class of variational integrals of the form

$$F_h(u) = \int_{\Omega} f\left(\frac{x}{\varepsilon_h}, Du\right) dx,$$

which is related to the homogenization of the operators  $-\operatorname{div}\left(\partial_{\xi} f\left(\frac{x}{\varepsilon_h}, \xi\right)\right)$ , has been studied in [32], [15], and [7], using the techniques of  $\Gamma$ -convergence introduced by E. De Giorgi. Homogenization results for variational integrals under almost periodicity assumptions have been proven in [8], [9], [11].

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1. NOTATIONS AND PRELIMINARY RESULTS

Let  $p$  be a real constant,  $1 < p < +\infty$ , and let  $q$  be its dual exponent,  $1/p + 1/q = 1$ . The Euclidean norm and the scalar product in  $\mathbf{R}^n$  are denoted by  $|\cdot|$  and  $(\cdot, \cdot)$ , respectively.

DEFINITION 1.1. - Given four constants  $\alpha, \beta, c_1$ , and  $c_2$ , such that  $c_1 > 0, c_2 > 0, 0 \leq \alpha \leq 1 \wedge (p-1), p \vee 2 \leq \beta < +\infty$ , we denote by  $M(\alpha, \beta, c_1, c_2)$  the class of all functions  $a: \mathbf{R}^n \rightarrow \mathbf{R}^n$  which fulfill the following conditions:

- (i)  $|a(0)| \leq c_1$ ;
- (ii)  $a$  satisfies the following inequalities of equicontinuity and strict monotonicity:

$$|a(\xi_1) - a(\xi_2)| \leq c_1 (1 + |\xi_1| + |\xi_2|)^{p-1-\alpha} |\xi_1 - \xi_2|^\alpha \tag{1.1}$$

$$(a(\xi_1) - a(\xi_2), \xi_1 - \xi_2) \geq c_2 (1 + |\xi_1| + |\xi_2|)^{p-\beta} |\xi_1 - \xi_2|^\beta \tag{1.2}$$

for every  $\xi_1, \xi_2 \in \mathbf{R}^n$ .

For every open subset  $\mathcal{O}$  of  $\mathbf{R}^n$ , by  $M_{\mathcal{O}}(\alpha, \beta, c_1, c_2)$  we denote the class of all functions  $a: \mathcal{O} \times \mathbf{R}^n \rightarrow \mathbf{R}^n$  which satisfy the following conditions:

- (iii) for a. e.  $x \in \mathcal{O}, |a(x, \cdot)| \in M(\alpha, \beta, c_1, c_2)$ ;
- (iv) for every  $\xi \in \mathbf{R}^n, a(\cdot, \xi)$  is Lebesgue measurable.

It is easy to see that (iii) implies that there exist constants  $c_3 > 0, c_4 > 0$ , such that

$$|a(x, \xi)| \leq c_3 (1 + |\xi|)^{p-1}, \tag{1.3}$$

$$|\xi|^p \leq c_4 (1 + (a(x, \xi), \xi)) \tag{1.4}$$

for a.e.  $x \in \mathcal{O}$ , for every  $\xi \in \mathbf{R}^n$ . The proof of (1.3) is trivial. As for the proof of (1.4), we just observe that by Young's inequality we have

$$|\xi|^p \leq \frac{p}{\beta} (1 + |\xi|)^{p-\beta} |\xi|^\beta + \frac{\beta-p}{\beta} (1 + |\xi|)^p.$$

In the case  $\mathcal{O} = \mathbf{R}^n$ , we simply use the notation  $M_{\mathbf{R}^n}$  for  $M_{\mathbf{R}^n}(\alpha, \beta, c_1, c_2)$ .

Let us fix from now on a bounded open subset  $\Omega$  of  $\mathbf{R}^n$ . Given  $a \in M_\Omega(\alpha, \beta, c_1, c_2)$ , it can be proved that for every  $f \in H^{-1,q}(\Omega)$  there exists a unique solution  $u \in H_0^{1,p}(\Omega)$  to the following Dirichlet boundary value problem

$$\left. \begin{aligned} -\operatorname{div}(a(x, Du)) &= f \quad \text{on } \Omega, \\ u &\in H_0^{1,p}(\Omega). \end{aligned} \right\} \tag{1.5}$$

For a proof we refer, for instance, to [27], Chapter III, Corollary 1.8, or to [31], Chapter 2, Theorem 2.1. The solution to (1.5) satisfies a Meyers' regularity estimate (see [33]) that will be needed in the sequel in the particular case where  $\Omega$  is a cube, as stated in the following theorem.

**THEOREM 1.2.** — *Let  $Q$  be a cube in  $\mathbf{R}^n$  and let  $w \in H_0^{1,p}(Q)$  be the weak solution to the equation*

$$\left. \begin{aligned} -\operatorname{div}(a(x, Dw)) &= 0 \quad \text{on } Q, \\ w &\in H_0^{1,p}(Q). \end{aligned} \right\}$$

*Then there exists  $\eta > 0$  such that  $w \in H^{1,p+\eta}(Q)$  and*

$$\|Dw\|_{(\mathbb{L}^{p+\eta}(Q))^n} \leq C(Q) \|Dw\|_{(\mathbb{L}^p(Q))^n}. \tag{1.6}$$

*The constant  $\eta$  depends only on  $c_1, c_2, n, p$ , while  $C(Q)$  depends in addition on  $Q$ . Moreover, a simple rescaling argument shows that we can take*

$$C(tQ) = t^{-n/r} C(Q)$$

*for all  $t > 0$ , where  $\frac{1}{r} + \frac{1}{q} + \frac{1}{p+\eta} = 1$ .*

In order to study the behaviour of problem (1.5) under perturbations of the function  $a$  we make use of the following notion of G-convergence.

**DEFINITION 1.3.** — We say that a sequence  $(a_h)$  in  $M_\Omega(\alpha, \beta, c_1, c_2)$  G-converges to  $a \in M_\Omega(\alpha, \beta, c_1, c_2)$  if, for every  $f \in H^{-1,q}(\Omega)$  and for every sequence  $(f_h)$  converging to  $f$  strongly in  $H^{-1,q}(\Omega)$ , the solutions  $u_h$  to

the equations

$$\left. \begin{aligned} -\operatorname{div}(a_h(x, D u_h)) &= f_h \quad \text{on } \Omega, \\ u_h &\in H_0^{1,p}(\Omega), \end{aligned} \right\} \tag{1.7}$$

satisfy the following conditions:

$$u_h \rightarrow u \quad \text{weakly in } H_0^{1,p}(\Omega), \tag{1.8}$$

$$a_h(x, D u_h) \rightarrow a(x, D u) \quad \text{weakly in } (L^q(\Omega))^n, \tag{1.9}$$

where  $u$  is the solution to the equation

$$\left. \begin{aligned} -\operatorname{div}(a(x, D u)) &= f \quad \text{on } \Omega, \\ u &\in H_0^{1,p}(\Omega). \end{aligned} \right\} \tag{1.10}$$

*Remark 1.4.* – It can be proved that this definition of G-convergence is independent of the boundary condition. More precisely, if  $\varphi \in H^{1,p}(\Omega)$ , if the sequence  $(a_h) \in M_\Omega(\alpha, \beta, c_1, c_2)$  G-converges to  $a \in M_\Omega(\alpha, \beta, c_1, c_2)$ ,  $(f_h)$  converges to  $f$  strongly in  $H^{-1,q}(\Omega)$ , and  $u_h$  are the solutions in  $H^{1,p}(\Omega)$  to the equations

$$\left. \begin{aligned} -\operatorname{div}(a_h(x, D u_h)) &= f_h \quad \text{on } \Omega, \\ u_h - \varphi &\in H_0^{1,p}(\Omega), \end{aligned} \right\}$$

then

$$u_h \rightarrow u \quad \text{weakly in } H^{1,p}(\Omega),$$

$$a_h(x, D u_h) \rightarrow a(x, D u) \quad \text{weakly in } (L^q(\Omega))^n,$$

where  $u$  is the solution to the equation

$$\left. \begin{aligned} -\operatorname{div}(a(x, D u)) &= f \quad \text{on } \Omega, \\ u - \varphi &\in H_0^{1,p}(\Omega). \end{aligned} \right\}$$

A proof of this fact can be found, for instance, in [16], Theorem 3.8.

The next two theorems concern a localization property and a compactness result for G-convergence. Their proofs can be deduced from Theorem 6.1 and Theorem 4.1 in [16] respectively, by using Theorem 7.9 and Corollaries 7.10-7.12 therein.

Let  $\Omega'$  be an open subset of  $\Omega$ . For  $a \in M_\Omega(\alpha, \beta, c_1, c_2)$  we denote by  $a'$  the function of  $M_{\Omega'}(\alpha, \beta, c_1, c_2)$  defined by  $a' = a|_{\Omega' \times \mathbb{R}^n}$ . Then the following localization property holds.

**THEOREM 1.5.** – *Let  $(a_h)$  be a sequence in  $M_\Omega(\alpha, \beta, c_1, c_2)$  which G-converges to  $a$  in  $M_\Omega(\alpha, \beta, c_1, c_2)$ . Then  $(a'_h)$  G-converges to  $a'$  in  $M_{\Omega'}(\alpha, \beta, c_1, c_2)$ .*

**THEOREM 1.6.** – *Let  $(a_h)$  be a sequence in  $M_\Omega(\alpha, \beta, c_1, c_2)$ . Then there exist suitable positive constants  $c'_1, c'_2$  and a subsequence  $(a_{\sigma(h)})$  of  $(a_h)$  which G-converges to a function  $a$  of the class  $M_\Omega\left(\frac{\alpha}{p-\alpha}, \beta, c'_1, c'_2\right)$ .*

In order to simplify the notation, the classes  $M_\Omega(\alpha, \beta, c_1, c_2)$  and  $M_\Omega\left(\frac{\alpha}{p-\alpha}, \beta, c'_1, c'_2\right)$  given by Theorem 1.6 will be denoted, from now on, by  $M_\Omega$  and  $M'_\Omega$  respectively.

Finally, we recall a lemma of compensated compactness type (see [35], [45]) which will be used in the sequel. For its proof see, for example, Lemma 3.4 in [16].

LEMMA 1.7. — *Let  $(u_h)$  be a sequence converging to  $u$  weakly in  $H^{1,p}(\Omega)$ . Let  $(g_h)$  be a sequence converging to  $g$  weakly in  $(L^q(\Omega))^n$  with  $(\operatorname{div} g_h)$  converging to  $\operatorname{div} g$  strongly in  $H^{-1,q}(\Omega)$ . Then*

$$\int_\Omega (g_h, D u_h) \varphi \, dx \rightarrow \int_\Omega (g, D u) \varphi \, dx$$

for every  $\varphi \in \mathcal{C}_0^\infty(\Omega)$ .

## 2. SUFFICIENT CONDITIONS FOR THE G-CONVERGENCE AND REPRESENTATION FORMULA FOR MONOTONE OPERATORS

In this section we investigate some conditions on a sequence of functions  $(a_h)$  in  $M_\Omega$  which imply G-convergence. Furthermore, we give a representation formula for functions  $a \in M_\Omega$  showing that  $a(x, \xi)$  can be determined by the knowledge of the solutions  $v$  to the Dirichlet boundary value problems

$$\left. \begin{aligned} -\operatorname{div}(a(x, Dv + \xi)) &= 0 \quad \text{on } \Omega', \\ v &\in H_0^{1,p}(\Omega'), \end{aligned} \right\}$$

where  $\xi \in \mathbf{R}^n$  and  $\Omega'$  is an open subset of  $\Omega$ . More precisely, we prove that  $a(x, \xi)$  can be calculated by a differentiation process of the set function

$$\Omega' \rightarrow \int_{\Omega'} a(x, Dv(x) + \xi) \, dx$$

along a family of open subsets of  $\Omega$ . Similar results for minima of variational functionals were proved in [21] and [24] for the quadratic case, and in [19] for the general case.

The following theorem shows that the strong convergence  $(L^1(\Omega))^n$  of a sequence  $(a_h)$  in  $M_\Omega$  implies the G-convergence.

THEOREM 2.1. — *Let  $a_h$  and  $a \in M_\Omega$ . Assume that  $(a_h(\cdot, \xi))$  converges to  $a(\cdot, \xi)$  strongly in  $(L^1(\Omega))^n$  for every  $\xi \in \mathbf{R}^n$ . Then, the sequence  $(a_h)$  G-converges to  $a$ .*



*Proof.* — Let  $(f_h)$  be a sequence in  $H^{-1,q}(\Omega)$  converging to  $f$  strongly in  $H^{-1,q}(\Omega)$ . Let  $u_h$  be the solution to the equation

$$\left. \begin{aligned} -\operatorname{div}(a_h(x, Du_h)) &= f_h \quad \text{on } \Omega, \\ u_h &\in H_0^{1,p}(\Omega). \end{aligned} \right\}$$

By the definition of G-convergence we have to prove that (1.8), (1.9) and (1.10) are satisfied. Since  $(f_h)$  is bounded in  $H^{-1,q}(\Omega)$ , condition (1.4) implies that  $(u_h)$  is bounded in  $H_0^{1,p}(\Omega)$ , hence  $(a_h(\cdot, Du_h(\cdot)))$  is bounded in  $(L^q(\Omega))^n$  by (1.3). Therefore, up to a subsequence,

$$\begin{aligned} u_h &\rightarrow u \quad \text{weakly in } H_0^{1,p}(\Omega), \\ a_h(x, Du_h) &\rightarrow g \quad \text{weakly in } (L^q(\Omega))^n, \end{aligned}$$

with  $-\operatorname{div} g = f$ . We shall show that  $g(x) = a(x, Du(x))$  for a.e.  $x \in \Omega$ , hence  $u$  is the unique solution to (1.10). Therefore, the whole sequences  $(u_h)$  and  $(a_h(\cdot, Du_h(\cdot)))$  converge, and the proof of our assertion is complete. By the strong convergence in  $(L^1(\Omega))^n$  of the sequence  $(a_h(\cdot, \xi))$  to  $a(\cdot, \xi)$  and by the equicontinuity [see Definition 1.1 (iii)], there exists a subsequence, still denoted by  $(a_h)$ , such that  $(a_h(x, \xi))$  converges to  $a(x, \xi)$  for a.e.  $x \in \Omega$ , for every  $\xi \in \mathbf{R}^n$ . Furthermore, by taking the equiboundedness condition [Definition 1.1 (iii)] for  $a_h$  into account, the dominated convergence theorem implies that

$$a_h(\cdot, \xi) \rightarrow a(\cdot, \xi) \quad \text{strongly in } (L^q(\Omega))^n, \quad \text{for every } \xi \in \mathbf{R}^n.$$

Since  $a_h(x, \cdot)$  is monotone for a.e.  $x \in \Omega$ , we have

$$\int_{\Omega} (a_h(x, Du_h) - a_h(x, \xi), Du_h(x) - \xi) \varphi(x) dx \geq 0$$

for every  $\varphi \in \mathcal{C}_0^\infty(\Omega)$ ,  $\varphi \geq 0$ . Passing to the limit as  $h$  tends to  $+\infty$  we obtain by means of Lemma 1.7 that

$$\int_{\Omega} (g(x) - a(x, \xi), Du(x) - \xi) \varphi(x) dx \geq 0 \tag{2.1}$$

holds for every  $\varphi \in \mathcal{C}_0^\infty(\Omega)$ ,  $\varphi \geq 0$ . By a standard density argument, (2.1) implies that

$$(g(x) - a(x, \xi), Du(x) - \xi) \geq 0$$

for a.e.  $x \in \Omega$ , for every  $\xi \in \mathbf{R}^n$  [remind that  $a(x, \xi)$  is continuous with respect to  $\xi$  by (iii) in Definition 1.1]. By Minty's lemma (see, for example, [27], Chapter III, Lemma 1.5) it follows that

$$(g(x) - a(x, Du(x)), Du(x) - \xi) \geq 0$$

for a.e.  $x \in \Omega$ , for every  $\xi \in \mathbf{R}^n$ . Hence,  $g(x) = a(x, Du(x))$  for a.e.  $x \in \Omega$ , which completes the proof.  $\square$

In order to state the representation formula for functions in the class  $M_\Omega$ , given  $a \in M_\Omega$ , let us define the function

$$\Psi(\xi, \Omega') = \int_{\Omega'} a(x, Dv(x) + \xi) dx \tag{2.2}$$

for every  $\xi \in \mathbf{R}^n$ , for every open subset  $\Omega'$  of  $\Omega$ , where the function  $v$ , depending on  $\xi$  and  $\Omega'$ , is the unique solution to

$$\left. \begin{aligned} -\operatorname{div}(a(x, Dv + \xi)) &= 0 \quad \text{on } \Omega', \\ v &\in H_0^{1,p}(\Omega'). \end{aligned} \right\} \tag{2.3}$$

**THEOREM 2.2.** — *Let  $a \in M_\Omega$ . Then there exists a measurable subset  $N$  of  $\Omega$  with  $|N| = 0$  such that*

$$a(x_0, \xi) = \lim_{\rho \rightarrow 0^+} \frac{\Psi(\xi, A_\rho(x_0))}{|A_\rho(x_0)|} \tag{2.4}$$

for every  $\xi \in \mathbf{R}^n$ ,  $x_0 \in \Omega \setminus N$ , where  $|\cdot|$  denotes the Lebesgue measure,  $A_\rho(x_0) = x_0 + \rho A$ , and  $A$  is any bounded open subset of  $\mathbf{R}^n$ .

*Proof.* — By a standard density argument and by Lebesgue's differentiation theorem (see, for instance, [38], Theorem 8.8) there exists a measurable subset  $N$  of  $\Omega$  with  $|N| = 0$  such that

$$\lim_{\rho \rightarrow 0^+} \frac{1}{|A_\rho(x_0)|} \int_{A_\rho(x_0)} |a(x, \xi) - a(x_0, \xi)| dx = 0 \tag{2.5}$$

for every  $x_0 \in \Omega \setminus N$ , for every  $\xi \in \mathbf{R}^n$ , where  $A_\rho(x_0) = x_0 + \rho A$ , and  $A$  is any bounded open subset of  $\mathbf{R}^n$ . Given  $x_0 \in \Omega \setminus N$ ,  $\rho > 0$ ,  $\xi \in \mathbf{R}^n$  we denote by  $v$  the function depending on  $\xi$  and  $A_\rho(x_0)$  which is the unique solution to the Dirichlet boundary value problem

$$\left. \begin{aligned} -\operatorname{div}(a(x, Dv + \xi)) &= 0 \quad \text{on } A_\rho(x_0), \\ v &\in H_0^{1,p}(A_\rho(x_0)). \end{aligned} \right\} \tag{2.6}$$

By performing the change of variables  $y = \frac{x - x_0}{\rho}$ , problem (2.6) becomes

$$\left. \begin{aligned} -\operatorname{div}_y(a(x_0 + \rho y, D_y u(y) + \xi)) &= 0 \quad \text{on } A, \\ u(y) = \frac{1}{\rho} v(x_0 + \rho y) &\in H_0^{1,p}(A). \end{aligned} \right\} \tag{2.7}$$

We may suppose that  $\rho$  runs through a sequence  $(\rho_h)$  which tends to  $0^+$  as  $h$  tends to  $+\infty$ . Let us set  $a_h(y, \xi) = a(x_0 + \rho_h y, \xi)$  for every  $y \in A$ ,  $\xi \in \mathbf{R}^n$ . By (2.5) we have

$$a_h(\cdot, \xi) \rightarrow a(x_0, \xi) \quad \text{strongly in } (L^1(A))^n,$$

which guarantees by Theorem 2.1 that

$$(a_h) \text{ G-converges to } a \text{ on } A. \tag{2.8}$$

Since  $w=0$  is the unique solution to the Dirichlet problem

$$\left. \begin{aligned} -\operatorname{div}_y(a(x_0, D_y w(y) + \xi)) &= 0 \quad \text{on } A, \\ w &\in H_0^{1,p}(A), \end{aligned} \right\}$$

if  $u_h$  denotes the solution to (2.7) corresponding to  $\rho = \rho_h$ , the G-convergence condition (2.8) and Remark 1.4 imply that  $(u_h)$  converges to 0 weakly in  $H_0^{1,p}(A)$  and

$$a(x_0 + \rho_h y, D_y u_h(y) + \xi) \rightarrow a(x_0, \xi) \quad \text{weakly in } (L^q(A))^n. \tag{2.9}$$

By (2.9) we have then

$$a(x_0, \xi) = \lim_{h \rightarrow \infty} \frac{1}{|A|} \int_A a(x_0 + \rho_h y, D_y u_h(y) + \xi) dy,$$

which by a change of variables proves (2.4).  $\square$

The aim of the next theorem is to obtain a necessary and sufficient condition for the G-convergence of a sequence  $a_h \in M_\Omega$  by means of the convergence of the momenta related to the Dirichlet problems

$$\left. \begin{aligned} -\operatorname{div}(a_h(x, Dv + \xi)) &= 0 \quad \text{on } A_\rho(x_0), \\ v &\in H_0^{1,p}(A_\rho(x_0)), \end{aligned} \right\}$$

where  $\xi \in \mathbf{R}^n$ .

**THEOREM 2.3.** — *Let  $(a_h)$  be a sequence in  $M_\Omega$ . Let  $\Psi_h$  be the function associated to  $a_h$  by (2.2). For every  $\rho > 0$  and  $x_0 \in \mathbf{R}^n$ , let  $A_\rho(x_0) = x_0 + \rho A$ , where  $A$  is any bounded open subset of  $\mathbf{R}^n$ . Let  $N$  be a measurable subset of  $\Omega$  with  $|N| = 0$ . Then, the following conditions are equivalent:*

(a) *the limit*

$$\lim_{h \rightarrow \infty} \Psi_h(\xi, A_\rho(x_0))$$

*exists for every  $\xi \in \mathbf{R}^n$  and for every  $x_0 \in \Omega \setminus N$ ;*

(b) *there exists a function  $a \in M'_\Omega$  such that  $(a_h)$  G-converges to  $a$ .*

*Moreover, if the previous conditions are satisfied, then*

$$a(x, \xi) = \lim_{\rho \rightarrow 0+} \lim_{h \rightarrow \infty} \frac{\Psi_h(\xi, A_\rho(x))}{|A_\rho(x)|} \tag{2.10}$$

*for a.e.  $x \in \Omega$ , for every  $\xi \in \mathbf{R}^n$ .*

*Proof.* — Assume (a). Let  $(a_{\tau(h)})$  be a subsequence of  $(a_h)$ . By the compactness theorem 1.6 there exist a further subsequence  $(a_{\tau(\sigma(h))})$  and  $a \in M'_\Omega$  such that

$$(a_{\tau(\sigma(h))}) \text{ G-converges to } a. \tag{2.11}$$

If we show that

$$a(x_0, \xi) = \lim_{\rho \rightarrow 0^+} \lim_{h \rightarrow \infty} \frac{\Psi_{\tau(\sigma(h))}(\xi, A_\rho(x_0))}{|A_\rho(x_0)|} \tag{2.12}$$

for a.e.  $x_0 \in \Omega$ , for every  $\xi \in \mathbf{R}^n$ , by the existence of the limit in (a) and the Urysohn property of the G-convergence (see [16], Remark 3.7) we may conclude that (b) and (2.10) hold.

Given  $\xi \in \mathbf{R}^n$ ,  $x_0 \in \Omega \setminus \mathbf{N}$ , and  $\rho > 0$ , we denote by  $v_h$  the solution to the Dirichlet problem

$$\left. \begin{aligned} -\operatorname{div}(a_{\tau(\sigma(h))}(x, Dv_h + \xi)) &= 0 \quad \text{on } A_\rho(x_0), \\ v_h &\in H_0^{1,p}(A_\rho(x_0)). \end{aligned} \right\}$$

Since  $v$  is the unique solution to the Dirichlet problem

$$\left. \begin{aligned} -\operatorname{div}(a(x, Dv + \xi)) &= 0 \quad \text{on } A_\rho(x_0), \\ v &\in H_0^{1,p}(A_\rho(x_0)), \end{aligned} \right\} \tag{2.13}$$

the G-convergence condition (2.11), Theorem 1.5 and Remark 1.4 imply

$$\begin{aligned} v_h &\rightarrow v \quad \text{weakly in } H_0^{1,p}(A_\rho(x_0)) \\ a_{\tau(\sigma(h))}(x, Dv_h + \xi) &\rightarrow a(x, Dv + \xi) \quad \text{weakly in } (L^q(A_\rho(x_0)))^n. \end{aligned}$$

Therefore

$$\Psi_{\tau(\sigma(h))}(\xi, A_\rho(x_0)) \rightarrow \int_{A_\rho(x_0)} a(x, Dv + \xi) dx = \Psi(\xi, A_\rho(x_0)).$$

Now, by the representation Theorem 2.2 we conclude that

$$a(x_0, \xi) = \lim_{\rho \rightarrow 0^+} \lim_{h \rightarrow \infty} \frac{\Psi_{\tau(\sigma(h))}(\xi, A_\rho(x_0))}{|A_\rho(x_0)|}$$

for a.e.  $x_0 \in \Omega$ , for every  $\xi \in \mathbf{R}^n$ , proving (2.12).

Assume (b). Then, Theorem 1.5 and Remark 1.4 guarantee that the solutions  $v_h$  to

$$\left. \begin{aligned} -\operatorname{div}(a_h(x, Dv_h + \xi)) &= 0 \quad \text{on } A_\rho(x_0), \\ v_h &\in H_0^{1,p}(A_\rho(x_0)) \end{aligned} \right\}$$

satisfy

$$\begin{aligned} v_h &\rightarrow v \quad \text{weakly in } H_0^{1,p}(A_\rho(x_0)) \\ a_h(x, Dv_h + \xi) &\rightarrow a(x, Dv + \xi) \quad \text{weakly in } (L^q(A_\rho(x_0)))^n, \end{aligned}$$

where  $v$  is the unique solution to (2.13). Hence, condition (a) follows immediately.  $\square$

*Remark 2.4.* – Theorem 2.3 provides a simple characterization of G-convergence in the special case of functions  $a_h$  in  $M_\Omega$  satisfying

$$-\operatorname{div}_x(a_h(x, \xi)) = 0 \quad \text{on } \Omega$$

for every  $h \in \mathbb{N}$  and  $\xi \in \mathbb{R}^n$ . In this case  $(a_h)$  G-converges to a function  $a \in M_\Omega$  if and only if  $(a_h(\cdot, \xi))$  tends to  $a(\cdot, \xi)$  weakly in  $(L^1(\Omega))^n$ , for every  $\xi \in \mathbb{R}^n$ . According to Theorem 2.3 this condition is clearly sufficient, since in this case

$$\Psi_h(\xi, A_p(x_0)) = \int_{A_p(x_0)} a_h(x, \xi) dx,$$

while its necessity follows easily from the local character of the G-convergence (see Theorem 1.5 and Remark 1.4). In the linear case, the previous characterization was proved in [13]. See also [12] for a similar result in the case  $a_h(x, \xi) = \partial_\xi f_h(x, \xi)$  with  $f_h$  convex in  $\xi$ .

### 3. HOMOGENIZATION OF QUASIPERIODIC OPERATORS

In this section we give a characterization of the G-limit of a sequence of functions  $a_h$  of the form

$$a_h(x, \xi) = a\left(\frac{x}{\varepsilon_h}, \xi\right),$$

with  $a \in M_{\mathbb{R}^n}$  verifying suitable hypotheses of quasiperiodicity in the first variable (see Definition 3.1). This result will be used in Section 4 to derive the homogenization theorem for general almost periodic operators.

DEFINITION 3.1 (see [30] 3.3). — A continuous function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is *quasiperiodic* if there exist  $m_1, \dots, m_n \in \mathbb{N}$  and a continuous function  $F: \mathbb{R}^N \rightarrow \mathbb{R}$ , where  $N = m_1 + \dots + m_n$ ,  $N \geq n$ , such that

$$f(x_1, \dots, x_n) = F(\underbrace{x_1, \dots, x_1}_{m_1}, \underbrace{x_2, \dots, x_2}_{m_2}, \dots, \underbrace{x_n, \dots, x_n}_{m_n}),$$

and  $F$  periodic of periods  $\frac{2\pi}{\lambda_1^1}, \frac{2\pi}{\lambda_1^2}, \dots, \frac{2\pi}{\lambda_1^{m_1}}, \frac{2\pi}{\lambda_2^1}, \dots, \frac{2\pi}{\lambda_2^{m_2}}, \dots, \frac{2\pi}{\lambda_n^1}, \dots, \frac{2\pi}{\lambda_n^{m_n}}$ , with

$\lambda_r^j \in ]0, +\infty[$ . It is not restrictive to assume that the frequencies  $\lambda_r^1, \dots, \lambda_r^{m_r}$  are linearly independent on  $\mathbb{Z}$  for every  $r = 1, \dots, n$ . This will be done constantly in the sequel. Under this assumption, Kronecker's lemma (see Appendix, Section A) guarantees that  $F$  is uniquely determined by  $f$ .

Given  $m_1, \dots, m_n$  as above and given  $\lambda \in \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_n}$ ,  $\lambda = (\lambda_1, \dots, \lambda_n)$  with  $\lambda_r = (\lambda_r^1, \dots, \lambda_r^{m_r})$  for  $r = 1, \dots, n$ , we denote by  $QP(\lambda)$  the set of all quasiperiodic functions  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  with frequencies  $\lambda$ . Furthermore, by  $Trig(\lambda)$  we indicate the set of all trigonometric polyno-

mials with frequencies  $\lambda$ ; *i. e.*, finite sums of terms of the form

$$P(x) = \operatorname{Re} \left( c \exp \left( i \sum_{l,r} k_r^l \lambda_r^l x_r \right) \right), \tag{3.1}$$

where  $k \in \mathbf{Z}^N$ ,  $c \in \mathbf{C}$ .

*Remark 3.2.* — Trigonometric polynomials are obviously quasiperiodic functions. Moreover, it can be proved that every function  $f$  of  $\operatorname{QP}(\lambda)$  is the uniform limit of a sequence of trigonometric polynomials belonging to  $\operatorname{Trig}(\lambda)$  (see [30] 3.3).

For every  $s > 0$ , for every  $z \in \mathbf{R}^n$ , let  $Q_s(z)$  be the cube of side length  $s$  and center  $z$ . For every  $f \in L^1_{\text{loc}}(\mathbf{R}^n)$  we define

$$\int f(x) dx = \limsup_{s \rightarrow \infty} \frac{1}{|Q_s(0)|} \int_{Q_s(0)} f(x) dx.$$

It can be seen easily that for every function  $f$  in  $\operatorname{QP}(\lambda)$  we have

$$\int f(x) dx = \lim_{s \rightarrow \infty} \frac{1}{|Q_s(z)|} \int_{Q_s(z)} f(x) dx$$

uniformly with respect to  $z$ . This limit is called the *mean value* of  $f$  (on  $\mathbf{R}^n$ ) (see [30], 2.3).

In the sequel it will be useful to consider the function  $j: \mathbf{R}^n \rightarrow \mathbf{R}^N$  defined by

$$j(x) = \underbrace{(x_1, \dots, x_1)}_{m_1}, \underbrace{(x_2, \dots, x_2)}_{m_2}, \dots, \underbrace{(x_n, \dots, x_n)}_{m_n}.$$

Then, introducing the variables

$$(y_1, \dots, y_N) = (y_1^1, \dots, y_1^{m_1}, \dots, y_n^1, \dots, y_n^{m_n})$$

on  $\mathbf{R}^N$ , for every  $v \in \mathcal{C}^1(\mathbf{R}^N)$  and  $u(x) = v(j(x))$  we get

$$D_r u(x) = \sum_{l=1}^{m_r} \frac{\partial v}{\partial y_r^l}(j(x)) \quad \text{for every } r = 1, \dots, n,$$

or briefly,  $Du(x) = (\partial v) \circ j(x)$ , where  $\partial = (\partial_1, \dots, \partial_n)$  with  $\partial_r = \sum_{l=1}^{m_r} \frac{\partial}{\partial y_r^l}$ .

Let us denote by  $\operatorname{Trig}_0(\lambda)$  the set of all trigonometric polynomials in  $\operatorname{Trig}(\lambda)$  with mean value 0. By Birkhoff's theorem (see Appendix, Theorem A) it is easy to prove that

$$\|u\|_{1,p} = \left( \sum_{r=1}^n \int |D_r u|^p dx \right)^{1/p}$$

is a norm on  $\text{Trig}_0(\lambda)$ . Since  $(\text{Trig}_0(\lambda), \|\cdot\|_{1,p})$  is not complete, we study its completion. To this aim let us introduce

$$T = \prod_{l,r} \left[ 0, \frac{2\pi}{\lambda_r^l} \right] \subset \mathbf{R}^N, \tag{3.2}$$

and let us denote by  $\text{Trig}(T)$  the set of all trigonometric polynomials in  $\mathbf{R}^N$  with period  $T$ . Let us also define the set  $\text{Trig}_0(T)$  of all functions  $v$  in  $\text{Trig}(T)$  with mean value zero. On  $\text{Trig}_0(T)$  we consider the norm

$$\|v\|_{\mathscr{W}} = \|\partial v\|_{(L^p(T))^n} = \left( \sum_{r=1}^n \frac{1}{|T|} \int_T |\partial_r v|^p dy \right)^{1/p};$$

in fact, if  $\partial v = 0$ , then  $u = v \circ j$  satisfies  $Du = 0$ . Hence,  $u$  is constant. Since  $v$  depends uniquely on  $u$  (see Definition 3.1),  $v$  is constant too; hence  $v = 0$ . By Birkhoff's theorem (see Appendix, Theorem A) the linear map

$$\text{Re}(c \exp(i \sum_{l,r} k_r^l \lambda_r^l x_r)) \mapsto \text{Re}(c \exp(i \sum_{l,r} k_r^l \lambda_r^l y_r^l))$$

is a bijective isometry between  $(\text{Trig}_0(\lambda), \|\cdot\|_{1,p})$  and  $(\text{Trig}_0(T), \|\cdot\|_{\mathscr{W}})$ , and will be denoted by  $J$ .

Let us denote by  $\mathscr{W}$  the completion of  $\text{Trig}_0(T)$  with respect to the norm  $\|\cdot\|_{\mathscr{W}}$ , which we can identify with the completion of  $\text{Trig}_0(\lambda)$  with respect to the norm  $\|\cdot\|_{1,p}$ . Finally, let us remark that the isometry

$$\begin{aligned} \partial: \text{Trig}_0(T) &\rightarrow (L^p(T))^n \\ u &\mapsto (\partial_1 u, \partial_2 u, \dots, \partial_n u) \end{aligned}$$

can be extended in a unique way to an isometry between  $\mathscr{W}$  and a closed subspace of  $(L^p(T))^n$ , which makes  $\mathscr{W}$  a reflexive Banach space.

Now, let us fix the frequencies  $\lambda$ , with  $\lambda_r^1, \dots, \lambda_r^{m_r}$  linearly independent on  $\mathbf{Z}$  for every  $r = 1, \dots, n$ , and let us fix  $a \in M_{\mathbf{R}^n}$  such that

$$a(\cdot, \xi) \in \text{QP}(\lambda) \quad \text{for every } \xi \in \mathbf{R}^n. \tag{3.3}$$

By definition it follows that there exists a unique function  $\tilde{a}(\cdot, \xi): \mathbf{R}^N \rightarrow \mathbf{R}^n$  such that  $a(x, \xi) = \tilde{a}(j(x), \xi)$  for every  $x \in \mathbf{R}^n$  and for every  $\xi \in \mathbf{R}^n$ ,  $\tilde{a}(\cdot, \xi)$  is  $T$ -periodic and continuous for every  $\xi \in \mathbf{R}^n$ . It follows that  $\tilde{a}(y, \cdot)$  satisfies (1.1) and (1.2) for every  $y \in \mathbf{R}^N$ . This is obvious for  $y = j(x)$ ,  $x \in \mathbf{R}^n$ , whereas the conclusion for a general  $y \in \mathbf{R}^N$  comes from Kronecker's lemma (see Appendix A) and the continuity of  $\tilde{a}(\cdot, \xi)$ .

The next proposition is an extension to the case  $p \neq 2$  of Lemma 1 in [36].

**PROPOSITION 3.3.** — *Let  $\xi \in \mathbf{R}^n$  be fixed, let  $a \in M_{\mathbf{R}^n}$  satisfying (3.3) and let  $\tilde{a}$  be the corresponding  $T$ -periodic function. Then, there exists a unique*

solution  $w^\xi \in \mathcal{W}$  to the problem

$$\left. \begin{aligned} \frac{1}{|\mathbb{T}|} \int_{\mathbb{T}} (\tilde{a}(y, \partial w^\xi + \xi), \partial h) dy = 0 \quad \text{for every } h \in \mathcal{W}, \\ w^\xi \in \mathcal{W}. \end{aligned} \right\} \quad (3.4)$$

Furthermore, for any  $\delta > 0$  there exist  $u_\delta^\xi \in \text{Trig}_0(\lambda)$  and a vector function  $g_\delta^\xi \in (\text{QP}(\lambda))^n$  such that

$$-\text{div}(a(x, Du_\delta^\xi + \xi)) = -\text{div } g_\delta^\xi \quad \text{in } \mathcal{D}'(\mathbb{R}^n), \quad (3.5)$$

$$\lim_{\delta \rightarrow 0^+} \|J(u_\delta^\xi) - w^\xi\|_{\mathcal{W}} = 0, \quad (3.6)$$

$$\lim_{\delta \rightarrow 0^+} \int |g_\delta^\xi|^q dx = 0 \quad (3.7)$$

hold.

*Proof.* — Given  $\xi \in \mathbb{R}^n$ , let  $\tilde{\mathcal{A}}^\xi: \mathcal{W} \rightarrow \mathcal{W}'$  be the operator defined by

$$\langle \tilde{\mathcal{A}}^\xi w, h \rangle = \frac{1}{|\mathbb{T}|} \int_{\mathbb{T}} (\tilde{a}(y, \partial w + \xi), \partial h) dy$$

for every  $w, h \in \mathcal{W}$ , where  $\langle \dots \rangle$  denotes here the duality pairing between  $\mathcal{W}$  and its dual space  $\mathcal{W}'$ . By the coerciveness and continuity properties of  $\tilde{\mathcal{A}}^\xi$  on  $\mathcal{W}$ , the theory of monotone operators on reflexive Banach spaces (see, for example [27]) implies immediately that there exists a unique  $w^\xi \in \mathcal{W}$  satisfying  $\tilde{\mathcal{A}}^\xi w^\xi = 0$ , which implies (3.4). By the density of  $J(\text{Trig}_0(\lambda))$  in  $\mathcal{W}$  there exists  $u_\delta^\xi \in \text{Trig}_0(\lambda)$  such that (3.6) is satisfied. By using (3.4) and the equicontinuity assumption on  $\tilde{a}$  we get

$$\begin{aligned} \|\tilde{\mathcal{A}}^\xi J(u_\delta^\xi)\|_{\mathcal{W}'} &= \|\tilde{\mathcal{A}}^\xi J(u_\delta^\xi) - \tilde{\mathcal{A}}^\xi w^\xi\|_{\mathcal{W}'} \\ &\leq c(1 + |\xi| + \|J(u_\delta^\xi)\|_{\mathcal{W}} + \|w^\xi\|_{\mathcal{W}})^{p-1-\alpha} \|J(u_\delta^\xi) - w^\xi\|_{\mathcal{W}}^\alpha. \end{aligned}$$

By (3.6) it follows that

$$\lim_{\delta \rightarrow 0^+} \|\tilde{\mathcal{A}}^\xi J(u_\delta^\xi)\|_{\mathcal{W}'} = 0. \quad (3.8)$$

Since  $x \mapsto a(x, Du_\delta^\xi + \xi)$  belongs to  $(\text{QP}(\lambda))^n$ , by Remark 3.2 there exist a function  $f_\delta^\xi \in (\text{Trig}(\lambda))^n$  and a quasiperiodic function  $h_\delta^\xi \in (\text{QP}(\lambda))^n$  such that

$$a(x, Du_\delta^\xi + \xi) = f_\delta^\xi + h_\delta^\xi, \quad (3.9)$$

and

$$\lim_{\delta \rightarrow 0^+} \int |h_\delta^\xi|^q dx = 0. \quad (3.10)$$

Since  $a(x, Du_\delta^\xi + \xi) = \tilde{a}(j(x), \partial J(u_\delta^\xi)(j(x)) + \xi)$ , by (3.8) and (3.10) we get

$$\lim_{\delta \rightarrow 0^+} \left\| \sum_{r=1}^n \partial_r (F_\delta^\xi)_r \right\|_{\mathcal{W}'} = 0, \quad (3.11)$$



where  $(F_\delta^\xi)_r$  is  $J((f_\delta^\xi)_r)$ , and

$$\left\| \sum_{r=1}^n \partial_r (F_\delta^\xi)_r \right\|_{\mathcal{W}'} = \sup_{\substack{v \in \mathcal{W} \\ \|v\|_{\mathcal{W}} \leq 1}} \frac{1}{|\Gamma|} \sum_{r=1}^n \int_{\Gamma} (F_\delta^\xi)_r \partial_r v dy.$$

Being  $\text{div} f_\delta^\xi \in \text{Trig}_0(\lambda)$ , we can write

$$\text{div} f_\delta^\xi(x) = \text{Re} \left( \sum_k c_k \exp \left( i \sum_{l,r} k_r^l \lambda_r^l x_r \right) \right),$$

where the sum runs over a finite set of non-zero vectors  $k \in \mathbb{Z}^N$ . It turns out that the function

$$w_\delta^\xi(x) = \text{Re} \left( \sum_k b_k \exp \left( i \sum_{l,r} k_r^l \lambda_r^l x_r \right) \right),$$

with  $b_k = -c_k / \left( \sum_r \left( \sum_l k_r^l \lambda_r^l \right)^2 \right)$ , is the unique solution in  $\text{Trig}_0(\lambda)$  to

$$\Delta w_\delta^\xi = \text{div} f_\delta^\xi \quad \text{on } \mathbb{R}^n.$$

If  $W_\delta^\xi = J(w_\delta^\xi) \in \text{Trig}(\Gamma)$ , then

$$\sum_{r=1}^n \partial_r^2 W_\delta^\xi = \sum_{r=1}^n \partial_r (F_\delta^\xi)_r \quad \text{in } \mathbb{R}^N$$

holds. By elliptic regularity (see Appendix, Section B)

$$\left( \frac{1}{|\Gamma|} \sum_{r=1}^n \int_{\Gamma} |\partial_r W_\delta^\xi|^q dx \right)^{1/q} \leq c \left\| \sum_{r=1}^n \partial_r (F_\delta^\xi)_r \right\|_{\mathcal{W}'},$$

with  $c > 0$  independent of  $\xi$  and  $\delta$ . We have then by Birkhoff's theorem

$$\left( \int |\mathbf{D}w_\delta^\xi|^q dx \right)^{1/q} \leq c \left\| \sum_{r=1}^n \partial_r (F_\delta^\xi)_r \right\|_{\mathcal{W}'}. \tag{3.12}$$

Hence, by setting

$$g_\delta^\xi = \mathbf{D}w_\delta^\xi + h_\delta^\xi$$

we have

$$\text{div} (a(x, \mathbf{D}u_\delta^\xi + \xi)) = \text{div} g_\delta^\xi$$

in the sense of distributions on  $\mathbb{R}^n$ , proving (3.5). Furthermore, by (3.10)-(3.12) we obtain (3.7).  $\square$

Let us consider the following Dirichlet boundary value problem

$$\left. \begin{aligned} -\text{div} \left( a \left( \frac{x}{\varepsilon_h}, \mathbf{D}u_h \right) \right) &= f \quad \text{on } \Omega, \\ u_h &\in \mathbf{H}_0^{1,p}(\Omega), \end{aligned} \right\} \tag{3.13}$$

where  $f \in \mathbf{H}^{-1,q}(\Omega)$  and  $(\varepsilon_h)$  is a sequence of positive real numbers converging to 0.

In this section we prove the convergence, as  $(\varepsilon_h)$  tends to  $0^+$ , of the solutions  $u_h$  of (3.13) to the solution  $u$  of the homogenized problem

$$\left. \begin{aligned} -\operatorname{div}(b(Du)) &= f \quad \text{on } \Omega, \\ u &\in H_0^{1,p}(\Omega). \end{aligned} \right\} \quad (3.14)$$

Furthermore, we give an asymptotic formula for the homogenized function  $b$  in terms of the solutions  $v_s^\xi$  to the following Dirichlet boundary value problems

$$\left. \begin{aligned} -\operatorname{div}(a(x, Dv_s^\xi + \xi)) &= 0 \quad \text{on } Q_s(z), \\ v_s^\xi &\in H_0^{1,p}(Q_s(z)). \end{aligned} \right\} \quad (3.15)$$

The convergence result above mentioned will follow from the next two theorems.

**THEOREM 3.4.** — *Let  $a \in M_{\mathbf{R}^n}$  satisfying (3.3) and let  $\tilde{a}$  be the corresponding  $T$ -periodic function. Let  $b: \mathbf{R}^n \rightarrow \mathbf{R}^n$  be the function defined by*

$$b(\xi) = \frac{1}{|T|} \int_T \tilde{a}(y, \partial w^\xi + \xi) dy,$$

where  $w^\xi$  is the solution to (3.4). Then, for any family  $(z_s)_{s>0}$  in  $\mathbf{R}^n$ , we have

$$b(\xi) = \lim_{s \rightarrow \infty} \frac{1}{s^n} \int_{Q_s(z_s)} a(x, Dv_s^\xi + \xi) dx,$$

where  $v_s^\xi$  is the unique solution to (3.15) with  $z = z_s$ .

**THEOREM 3.5.** — *Let  $a \in M_{\mathbf{R}^n}$  satisfying (3.3). Let  $(\varepsilon_h)$  be a sequence of positive real numbers converging to 0 and let  $a_h(x, \xi) = a\left(\frac{x}{\varepsilon_h}, \xi\right)$  for every  $x \in \Omega$  and for every  $\xi \in \mathbf{R}^n$ . Then  $(a_h)$   $G$ -converges to  $b$ , where  $b$  is the function defined in Theorem 3.4.*

The proofs of these theorems are quite technical and are therefore given at the end of the section as a consequence of the next proposition and some remarks stated in the sequel.

For every  $s > 0$ , and for every family  $(z_s)$  in  $\mathbf{R}^n$ , let us define

$$g_s(\xi) = \frac{1}{s^n} \int_{Q_s(z_s)} a(x, Dv_s^\xi + \xi) dx, \quad (3.16)$$

where  $v_s^\xi$  is the solution to (3.15) on  $Q_s(z_s)$ . By (iii) and (iv) in Definition 1.1, using Young's and Hölder's inequalities, it turns out that the function  $v_s^\xi$  satisfies the following estimates

$$\left. \begin{aligned} \|Dv_s^\xi + \xi\|_{(L^p(Q_s(z_s)))^n} &\leq c s^n (1 + |\xi|^p), \\ \|a(x, Dv_s^\xi + \xi)\|_{(L^q(Q_s(z_s)))^n} &\leq c s^n (1 + |\xi|^p) \end{aligned} \right\} \quad (3.17)$$

with  $c$  independent of  $z_s$ . Here and henceforth, we will denote by  $c$  any constant depending at most on  $c_1, c_2, \alpha, \beta, n, p$ , that can change from line to line. By (3.17), we have

$$|g_s(\xi)| \leq c(1 + |\xi|^{p/q}).$$

PROPOSITION 3.6. — *Let  $\xi \in \mathbf{R}^n$  be fixed. Let  $u_s^\xi$  be as in Proposition 3.3 and let  $v_s^\xi \in H_0^{1,p}(Q_s(z_s))$  be the unique solution to (3.15) with  $z$  replaced by  $z_s$ . Then*

$$\lim_{\delta \rightarrow 0+} \limsup_{s \rightarrow \infty} \frac{1}{s^n} \int_{Q_s(z_s)} |Dv_s^\xi - Du_\delta^\xi|^p dx = 0 \tag{3.18}$$

for every family  $(z_s)_{s>0}$  in  $\mathbf{R}^n$ .

*Proof.* — Given a family  $(z_s)_{s>0}$  we consider

$$\begin{aligned} I_{s,\delta} &\equiv \frac{1}{s^n} \int_{Q_s(z_s)} (a(x, Dv_s^\xi + \xi) - a(x, Du_\delta^\xi + \xi), Dv_s^\xi - Du_\delta^\xi) dx \\ &= \frac{1}{s^n} \int_{Q_s(z_s)} (a(x, Dv_s^\xi + \xi), Dv_s^\xi) dx \\ &\quad + \frac{1}{s^n} \int_{Q_s(z_s)} (a(x, Du_\delta^\xi + \xi), Du_\delta^\xi) dx \\ &\quad - \frac{1}{s^n} \int_{Q_s(z_s)} (a(x, Dv_s^\xi + \xi), Du_\delta^\xi) dx \\ &\quad - \frac{1}{s^n} \int_{Q_s(z_s)} (a(x, Du_\delta^\xi + \xi), Dv_s^\xi) dx \\ &\equiv I_s^1 + I_{s,\delta}^2 - I_{s,\delta}^3 - I_{s,\delta}^4. \end{aligned}$$

By (3.15),  $I_s^1 = 0$ . On the other hand,

$$\lim_{s \rightarrow \infty} I_{s,\delta}^2 = \int (a(x, Du_\delta^\xi + \xi), Du_\delta^\xi) dx.$$

By Birkhoff's theorem we get

$$\lim_{s \rightarrow \infty} I_{s,\delta}^2 = \frac{1}{|T|} \int_T (\tilde{a}(y, \partial J(u_\delta^\xi) + \xi), \partial J(u_\delta^\xi)) dy = \langle \tilde{\mathcal{A}}^\xi J(u_\delta^\xi), J(u_\delta^\xi) \rangle,$$

where  $\tilde{\mathcal{A}}^\xi$  is the operator defined in the proof of Proposition 3.3. Since by (3.6) we have

$$J(u_\delta^\xi) \rightarrow w^\xi \text{ strongly in } \mathcal{W},$$

the continuity of  $\tilde{\mathcal{A}}^\xi$  and the equality  $\tilde{\mathcal{A}}^\xi w^\xi = 0$  implies that

$$\lim_{\delta \rightarrow 0+} \lim_{s \rightarrow \infty} I_{s,\delta}^2 = 0.$$

Since

$$\frac{1}{|Q_s(z_s)|} \int_{Q_s(z_s)} |Du_\delta^\xi(x)|^p dx \rightarrow \int |Du_\delta^\xi(x)|^p dx,$$

and  $w_s(x) = \frac{1}{s} u_\delta^\xi(sx + z_s)$  converges to 0, as  $s \rightarrow +\infty$ , we obtain that the sequence  $(w_s)$  converges weakly to 0 in  $H^{1,p}(Q_1(0))$ . Applying Lemma 1.7, we obtain that the sequence of functions

$$\zeta_s(x) = (a(sx + z_s, Dv_s^\xi(sx + z_s) + \xi), Du_\delta^\xi(sx + z_s))$$

converges to 0 in  $\mathcal{D}'(Q_1(0))$  as  $s \rightarrow +\infty$ . On the other hand, by (1.6) and (3.17) we have

$$\|Dv_s^\xi(sx + z_s) + \xi\|_{(L^{p+\eta}(Q_1(0)))^\eta} \leq c(1 + |\xi|^p)^{(p+\eta)/p},$$

by (1.3) it follows that the sequence of functions

$$x \mapsto a(sx + z_s, Dv_s^\xi(sx + z_s) + \xi)$$

is uniformly bounded in some  $(L^\tau(Q_1(0)))^n$  for some  $\tau > q$ . This implies that there exists  $\sigma > 1$  such that

$$\|\zeta_s\|_{L^\sigma(Q_1(0))} \leq c,$$

where  $c$  is independent of  $s$ . Since  $(\zeta_s)$  converges to 0 in  $\mathcal{D}'(Q_1(0))$ , the above inequality implies that  $(\zeta_s)$  converges to 0 weakly in  $L^\sigma(Q_1(0))$ , as  $s \rightarrow +\infty$ . This fact gives then

$$\lim_{s \rightarrow \infty} I_{s,\delta}^3 = 0.$$

Let us show that

$$\lim_{\delta \rightarrow 0^+} \limsup_{s \rightarrow \infty} I_{s,\delta}^4 = 0.$$

Let  $(g_\delta^\xi)$  be as in Proposition 3.3. By applying Hölder's inequality and (3.17) we obtain

$$\begin{aligned} |I_{s,\delta}^4| &\leq c \left( \frac{1}{s^n} \int_{Q_s(z_s)} |g_\delta^\xi|^q dx \right)^{1/q} \left( \frac{1}{s^n} \int_{Q_s(z_s)} |Dv_s^\xi|^p dx \right)^{1/p} \\ &\leq c \left( \frac{1}{s^n} \int_{Q_s(z_s)} |g_\delta^\xi|^q dx \right)^{1/q} (1 + |\xi|). \end{aligned}$$

Now, by taking the limit first as  $s$  tends to  $+\infty$  and then as  $\delta$  tends to  $0^+$ , (3.7) implies that

$$\lim_{\delta \rightarrow 0} \limsup_{s \rightarrow \infty} |I_{s,\delta}^4| = 0.$$

Since the equicoerciveness assumption in Definition 1.1 (iii) guarantees

$$|I_{s, \delta}| \geq c_2 \frac{1}{s^n} \int_{Q_s(z_s)} (1 + |Dv_s^\xi + \xi| + |Du_\delta^\xi + \xi|)^{p-\beta} |Dv_s^\xi - Du_\delta^\xi|^\beta dx,$$

we obtain immediately

$$c |I_{s, \delta}|^{p/\beta} \left( \frac{1}{s^n} \int_{Q_s(z_s)} (1 + |Dv_s^\xi + \xi| + |Du_\delta^\xi + \xi|)^p dx \right)^{(\beta-p)/\beta} \geq \left( \frac{1}{s^n} \int_{Q_s(z_s)} |Dv_s^\xi - Du_\delta^\xi|^p dx \right).$$

Hence, by passing to the limit first as  $s$  tends to  $+\infty$ , and then as  $\delta \rightarrow 0^+$ , we get (3.18) and the proof of Proposition 3.6 is accomplished.  $\square$

*Proof of Theorem 3.5.* – Let  $x_0 \in \Omega$ , and let  $\xi \in \mathbb{R}^n$ . Consider

$$\Psi_h(\xi, Q_\rho(x_0)) = \int_{Q_\rho(x_0)} a\left(\frac{x}{\varepsilon_h}, Dz_{h,\rho}^\xi(x) + \xi\right) dx,$$

where  $z_{h,\rho}^\xi$  is the unique solution to

$$\left. \begin{aligned} -\operatorname{div}\left(a\left(\frac{x}{\varepsilon_h}, Dz_{h,\rho}^\xi + \xi\right)\right) &= 0 \quad \text{on } Q_\rho(x_0), \\ z_{h,\rho}^\xi &\in H_0^{1,p}(Q_\rho(x_0)). \end{aligned} \right\}$$

It follows immediately that

$$\Psi_h(\xi, Q_\rho(x_0)) = (\varepsilon_h)^n \int_{Q_{\rho/\varepsilon_h}(x_0/\varepsilon_h)} a(x, Dw_{h,\rho}^\xi + \xi) dx,$$

where  $w_{h,\rho}^\xi$  is the unique solution to

$$\left. \begin{aligned} -\operatorname{div}(a(x, Dw_{h,\rho}^\xi(x) + \xi)) &= 0 \quad \text{on } Q_{\rho/\varepsilon_h}\left(\frac{x_0}{\varepsilon_h}\right), \\ w_{h,\rho}^\xi &\in H_0^{1,p}\left(Q_{\rho/\varepsilon_h}\left(\frac{x_0}{\varepsilon_h}\right)\right). \end{aligned} \right\}$$

By Theorem 3.4 we conclude that

$$\lim_{h \rightarrow \infty} \Psi_h(\xi, Q_\rho(x_0)) = \rho^n b(\xi)$$

for every  $\rho > 0$ . Hence, the limit

$$\lim_{h \rightarrow \infty} \frac{\Psi_h(\xi, Q_\rho(x_0))}{\rho^n} = b(\xi)$$

is independent on  $\rho$ . This implies by Theorem 2.3 that  $(a_h)$  G-converges to  $b$  and concludes the proof.  $\square$

#### 4. HOMOGENIZATION OF ALMOST PERIODIC OPERATORS

In this section we prove the homogenization theorem for general almost periodic monotone operators defined by functions of the class  $M_{\mathbf{R}^n}$ .

DEFINITION 4.1. — A function  $f \in L^1_{\text{loc}}(\mathbf{R}^n)$  is *almost periodic* (in the sense of Besicovitch [3]) if there exists a sequence of trigonometric polynomials  $P_h: \mathbf{R}^n \rightarrow \mathbf{R}$  such that

$$\lim_{h \rightarrow \infty} \int |P_h(x) - f(x)| dx = 0. \quad (4.1)$$

It is easy to see that for every almost periodic function  $f$  and for every  $z \in \mathbf{R}^n$  we have

$$\int f(x) dx = \lim_{s \rightarrow \infty} \frac{1}{|Q_s(z)|} \int_{Q_s(z)} f(x) dx.$$

The limit is called the *mean value* of  $f$  (on  $\mathbf{R}^n$ ) and is, in general, not uniform with respect to  $z$ .

THEOREM 4.2. — Let  $a \in M_{\mathbf{R}^n}$  such that  $a(\cdot, \xi)$  is almost periodic for all  $\xi \in \mathbf{R}^n$  and let  $(\varepsilon_h)$  be a sequence of positive real numbers converging to 0.

Let us define  $a_h(x, \xi) = a\left(\frac{x}{\varepsilon_h}, \xi\right)$  for every  $x \in \Omega$  and  $\xi \in \mathbf{R}^n$ . Denote by  $v_s^\xi$  the solution to

$$\left. \begin{aligned} -\operatorname{div}(a(x, Dv_s^\xi + \xi)) &= 0 \quad \text{on } Q_s(0), \\ v_s^\xi &\in H_0^{1,p}(Q_s(0)). \end{aligned} \right\} \quad (4.2)$$

Then, for every  $\xi \in \mathbf{R}^n$  there exists the limit

$$b(\xi) = \lim_{s \rightarrow \infty} \frac{1}{s^n} \int_{Q_s(0)} a(x, Dv_s^\xi + \xi) dx. \quad (4.3)$$

Moreover, the map  $b$  belongs to  $M(\alpha, \beta, c_1, c_2)$  and  $(a_h)$  G-converges to  $b$ .

The proof of this theorem follows from the homogenization Theorem 3.5 for quasiperiodic functions, by means of an approximation result and a closure lemma, which are stated below.

With a slight change of notation we will write

$$b(x, \xi) = G\text{-}\lim_{\varepsilon \rightarrow 0^+} a\left(\frac{x}{\varepsilon}, \xi\right)$$

meaning that  $b(x, \xi)$  is the G-limit of  $a\left(\frac{x}{\varepsilon_h}, \xi\right)$  for every sequence  $(\varepsilon_h)$  which tends to  $0^+$ .

The following lemma states that homogenization is preserved under passage to the limit in the mean value.

LEMMA 4.3. — *Let  $(a_h)$  be a sequence of functions in  $M_{\mathbf{R}^n}$ , such that for every  $h \in \mathbf{N}$  the limit*

$$G-\lim_{\varepsilon \rightarrow 0^+} a_h\left(\frac{x}{\varepsilon}, \xi\right) = b_h(\xi) \tag{4.4}$$

*exists and is independent of  $x$ , and let  $a \in M_{\mathbf{R}^n}$  such that for every  $R > 0$*

$$\limsup_{h \rightarrow \infty} \int \sup_{|\xi| \leq R} |a_h(x, \xi) - a(x, \xi)| dx = 0. \tag{4.5}$$

*Then, the limit*

$$b(\xi) = \lim_{h \rightarrow \infty} b_h(\xi)$$

*exists and*

$$G-\lim_{\varepsilon \rightarrow 0^+} a\left(\frac{x}{\varepsilon}, \xi\right) = b(\xi).$$

The proof of Theorem 4.2 will be completed by the following approximation result.

LEMMA 4.4. — *Let  $a$  be a function of the class  $M_{\mathbf{R}^n}$  such that  $a(\cdot, \xi)$  is almost periodic for every  $\xi \in \mathbf{R}^n$ . Then, there exists a sequence  $(a_h)$  in  $M_{\mathbf{R}^n}$  of quasiperiodic functions satisfying (3.3) (with  $\lambda$  possibly depending on  $h$ ) such that for every  $R \geq 0$  we have*

$$\limsup_{h \rightarrow \infty} \int \sup_{|\xi| \leq R} |a_h(x, \xi) - a(x, \xi)| dx = 0. \tag{4.6}$$

Throughout this section the letter  $c$  will denote a positive constant depending at most on  $p, n, c_1, c_2, \alpha, \beta$ , and possibly on a fixed vector  $\xi \in \mathbf{R}^n$ . Its value can vary from line to line.

We begin by proving Lemma 4.4.

*Proof of Lemma 4.4.*

*Step 1 (discretization of the function  $a$  on bounded sets).* — Let  $(\gamma_h)$  be a sequence of natural numbers, and  $(\mu_h)$  a sequence of positive real numbers. For every  $h \in \mathbf{N}$ , let us set

$$I_h = \{ \mathbf{z} = (z_1, \dots, z_n) \in \mathbf{Z}^n : -h\gamma_h \leq z_j < h\gamma_h \text{ for all } j = 1, \dots, n \}.$$

For every  $\mathbf{z} \in I_h$ , let us define

$$\xi_{\mathbf{z}}^h = \frac{1}{\gamma_h} \mathbf{z}, \quad Q_{\mathbf{z}}^h = \xi_{\mathbf{z}}^h + \left[ 0, \frac{1}{\gamma_h} \right]^n, \quad a_{\mathbf{z}}^h(x) = a(x, \xi_{\mathbf{z}}^h),$$

and let us choose a trigonometric polynomial  $P_z^h: \mathbf{R}^n \rightarrow \mathbf{R}^n$  such that

$$\int |P_z^h(x) - a_z^h(x)| dx \leq \frac{1}{\mu_h}. \quad (4.7)$$

The choice of the sequences  $(\gamma_h) \subset \mathbf{N}$  and  $(\mu_h) \subset \mathbf{R}$  will be made in the sequel. Then, taking into account that

$$\bigcup_{z \in I_h} Q_z^h = [-h, h]^n,$$

we define the function  $\alpha_h: \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^n$  in the following way:

$$\alpha_h(x, \xi) = \begin{cases} P_z^h(x) & \text{if there exists } z \in I_h \text{ such that } \xi \in Q_z^h, \\ 0 & \text{if } \xi \notin [-h, h]^n. \end{cases}$$

Then, for every  $\xi \in \mathbf{R}^n$ , the function  $\alpha_h(\cdot, \xi)$  is quasiperiodic.

*Step 2 (projection of the function  $\alpha_h$  on  $M_{\mathbf{R}^n}$ ).* — For every  $h \in \mathbf{N}$ , let us define the function  $f_h: [0, +\infty[ \rightarrow [0, +\infty[$  by

$$f_h(\rho) = \min(1, e^{h!(h-\rho)}).$$

We also define the Hilbert space  $L_h^2$  of measurable functions  $u: \mathbf{R}^n \rightarrow \mathbf{R}^n$ , provided with the norm

$$\|u\|_h = \left( \int_{\mathbf{R}^n} |u(\xi)|^2 f_h(|\xi|) d\xi \right)^{1/2}.$$

The set  $M(\alpha, \beta, c_1, c_2)$  is a closed convex subset of  $L_h^2$ . We can define the projection  $\pi_h: L_h^2 \rightarrow M(\alpha, \beta, c_1, c_2)$ , and for every  $x \in \mathbf{R}^n$  the function

$$a_h(x, \cdot) = \pi_h(\alpha_h(x, \cdot)). \quad (4.8)$$

*Step 3 (quasiperiodicity of the function  $a_h$ ).* — Let us consider for every  $y \in \mathbf{R}^n$ ,  $N = N(h) \in \mathbf{N}$ , the function  $\tilde{a}_h(y, \cdot): \mathbf{R}^n \rightarrow \mathbf{R}^n$  defined by

$$\tilde{a}_h(y, \cdot) = \pi_h(\tilde{\alpha}_h(y, \cdot))$$

where, for every  $\xi \in \mathbf{R}^n$ ,  $\tilde{\alpha}_h(\cdot, \xi)$  is the periodic function given by the quasiperiodicity of  $\alpha_h(\cdot, \xi)$ . It follows clearly that

$$a_h(x, \xi) = \tilde{a}_h(j(x), \xi),$$

where  $j$  is the function related to the quasiperiodicity of  $a_h$  as in Section 3. Since the function  $\tilde{\alpha}_h(\cdot, \xi)$  is a periodic function, such is also  $\tilde{a}_h(\cdot, \xi)$ . Moreover, being  $\tilde{a}_h(y, \cdot)$  uniformly continuous with respect to  $\xi$ , it remains only to prove that  $\tilde{a}_h(\cdot, \xi)$  is continuous for every  $\xi \in \mathbf{R}^n$ .

Let  $\xi, \eta \in \mathbf{R}^n$ ; for all  $u, v \in M(\alpha, \beta, c_1, c_2)$  we have

$$|u(\xi) - v(\xi)| \leq 2c_1(1 + |\xi| + |\eta|)^{p-1-\alpha} |\xi - \eta|^\alpha + |u(\eta) - v(\eta)|.$$



Let  $0 < \delta \leq 1$ ; then an integration over  $B_\delta(\xi)$  gives

$$\begin{aligned} |u(\xi) - v(\xi)| |B_\delta(\xi)| &\leq 2c_1(1 + 2|\xi| + \delta)^{p-1-\alpha} \delta^\alpha |B_\delta(\xi)| \\ &\quad + \int_{B_\delta(\xi)} |u(\eta) - v(\eta)| d\eta \\ &\leq c(1 + |\xi|)^{p-1-\alpha} \delta^\alpha |B_\delta(\xi)| \\ &\quad + \left( \int_{B_\delta(\xi)} |u(\eta) - v(\eta)|^2 f_h^2(|\eta|) d\eta \right)^{1/2} \left( \int_{B_\delta(\xi)} (f_h(|\eta|))^{-2} d\eta \right)^{1/2}, \end{aligned}$$

so that we obtain

$$|u(\xi) - v(\xi)| \leq c \left[ (1 + |\xi|)^{p-1-\alpha} \delta^\alpha + \frac{\|u - v\|_h}{\delta^{n/2} f_h(|\xi| + 1)} \right].$$

In particular, since  $\pi_h$  is a Lipschitz function with constant 1 in  $L_h^2$ , for all  $u, v \in L_h^2$  and for all  $\xi \in \mathbf{R}^n$  we obtain

$$|\pi_h u(\xi) - \pi_h v(\xi)| \leq c \left[ (1 + |\xi|)^{p-1-\alpha} \delta^\alpha + \frac{\|u - v\|_h}{\delta^{n/2} f_h(|\xi| + 1)} \right]. \tag{4.9}$$

Let us remark that the function  $\tilde{\alpha}_h = \tilde{\alpha}_h(y, \xi)$  is uniformly continuous in  $y$ , uniformly with respect to  $\xi$ . Fixed  $\varepsilon > 0$ , let  $\rho > 0$  such that if  $|\tau| < \rho$ , then

$$|\tilde{\alpha}_h(y + \tau, \xi) - \tilde{\alpha}_h(y, \xi)| < \varepsilon$$

for all  $y \in \mathbf{R}^n$  and  $\xi \in \mathbf{R}^n$ . Then, for such a  $\tau$  we have

$$\begin{aligned} \|\tilde{\alpha}_h(y + \tau, \cdot) - \tilde{\alpha}_h(y, \cdot)\|_h &\leq \left( \int_{|1-h, h|^n} |\tilde{\alpha}_h(y + \tau, \eta) - \tilde{\alpha}_h(y, \eta)|^2 d\eta \right)^{1/2} \leq \varepsilon(2h)^{n/2}, \end{aligned}$$

so that, by (4.9) [with  $u(\xi) = \tilde{\alpha}_h(y + \tau, \xi)$  and  $v(\xi) = \tilde{\alpha}_h(y, \xi)$ ]

$$|\tilde{a}_h(y + \tau, \xi) - \tilde{a}_h(y, \xi)| \leq c \left[ (1 + |\xi|)^{p-1-\alpha} \delta^\alpha + \frac{\varepsilon(2h)^{n/2}}{\delta^{n/2} f_h(|\xi| + 1)} \right] \tag{4.10}$$

for all  $\xi \in \mathbf{R}^n$ . Since  $\varepsilon$  and  $\delta$  can be chosen independently arbitrarily small, (4.10) proves that  $\tilde{a}_h(\cdot, \xi)$  is uniformly continuous.

*Step 4 [proof of (4.6)].* – Fixed  $R \geq 0$ , let us consider for every  $x \in \mathbf{R}^n$  and  $h \in \mathbf{N}$  the function

$$g_h(x, R) = \sup_{|\xi| \leq R} |a_h(x, \xi) - a(x, \xi)|. \tag{4.11}$$

Since we are interested in the limit as  $h$  tends to  $+\infty$ , we will suppose  $h > R + 1$ , so that  $f_h(|\xi|) = 1$  for  $|\xi| \leq R$ .

Let us remark that  $a(x, \cdot) \in M(\alpha, \beta, c_1, c_2)$  for almost every  $x \in \mathbf{R}^n$ , since  $a \in M_{\mathbf{R}^n}$ , so that

$$a(x, \xi) = \pi_h(a(x, \xi))$$

for a. e.  $x \in \mathbf{R}^n$ , for every  $\xi \in \mathbf{R}^n$ , and for every  $h \in \mathbf{N}$ . By (4.9), we have for  $|\xi| \leq \mathbf{R}$  and  $0 < \delta \leq 1$

$$|a_h(x, \xi) - a(x, \xi)| = |\pi_h(\alpha_h(x, \xi)) - \pi_h(a(x, \xi))| \leq c[(1 + \mathbf{R})^{p-1-\alpha} \delta^\alpha + \delta^{-n/2} \|\alpha_h(x, \cdot) - a(x, \cdot)\|_h]. \quad (4.12)$$

Thus, we have to estimate

$$\|\alpha_h(x, \cdot) - a(x, \cdot)\|_h^2 \leq \int_{\{|-h, h\}^n} |\alpha_h(x, \eta) - a(x, \eta)|^2 d\eta + \int_{\{|\eta| > h\}} |\alpha_h(x, \eta) - a(x, \eta)|^2 f_h^2(|\eta|) d\eta \equiv \mathbf{J}_h^1 + \mathbf{J}_h^2. \quad (4.13)$$

By (1.3) we have

$$\mathbf{J}_h^2 \leq c \int_{\{|\eta| > h\}} |\eta|^{2p-2} f_h^2(|\eta|) d\eta \leq c \int_h^{+\infty} \rho^{2p+n-3} f_h^2(\rho) d\rho, \quad (4.14)$$

so that  $\lim_{h \rightarrow \infty} \mathbf{J}_h^2 = 0$ . By the definition of  $\alpha_h$ , (1.3), and the Hölder inequality we obtain

$$\begin{aligned} \mathbf{J}_h^1 &= \sum_{z \in I_h} \int_{Q_z^h} |\alpha_h(x, \eta) - a(x, \eta)|^2 d\eta = \sum_{z \in I_h} \int_{Q_z^h} |P_z^h(x) - a(x, \eta)|^2 d\eta \\ &\leq \sum_{z \in I_h} \int_{Q_z^h} 2(|P_z^h(x) - a(x, \xi_z^h)|^2 + |a(x, \xi_z^h) - a(x, \eta)|^2) d\eta \\ &\leq \sum_{z \in I_h} \int_{Q_z^h} 2(|P_z^h(x) - a_z^h(x)|^2 + [c_1(1 + |\xi_z^h| + |\eta|)^{p-1-\alpha} |\xi_z^h - \eta|^\alpha]^2) d\eta \\ &\leq \sum_{z \in I_h} 2 \left(\frac{1}{\gamma_h}\right)^n |P_z^h(x) - a_z^h(x)|^2 + ch^{n+2p-2-2\alpha} \left(\frac{1}{\gamma_h}\right)^{2\alpha}. \end{aligned} \quad (4.15)$$

Taking (4.12)-(4.15) into account, we obtain for  $|\xi| \leq \mathbf{R}$  and  $0 < \delta \leq 1$

$$|a_h(x, \xi) - a(x, \xi)| \leq c(1 + \mathbf{R})^{p-1-\alpha} \delta^\alpha + c \delta^{-n/2} (h^{n/2+p-1-\alpha} (\gamma_h)^{-\alpha} + (\gamma_h)^{-n/2} \sum_{z \in I_h} |P_z^h(x) - a_z^h(x)| + (\mathbf{J}_h^2)^{1/2});$$

an integration gives

$$\begin{aligned} \int g_h(x, \mathbf{R}) dx &\leq c(1 + \mathbf{R})^{p-1-\alpha} \delta^\alpha \\ &+ c \delta^{-n/2} \left( h^{n/2+p-1-\alpha} (\gamma_h)^{-\alpha} + (\gamma_h)^{-n/2} \sum_{z \in I_h} \int |P_z^h - a_z^h| dx + (\mathbf{J}_h^2)^{1/2} \right) \\ &\leq c(1 + \mathbf{R})^{p-1-\alpha} \delta^\alpha \\ &+ c \delta^{-n/2} \left( h^{n/2+p-1-\alpha} (\gamma_h)^{-\alpha} + h^n \frac{(\gamma_h)^{n/2}}{\mu_h} + (\mathbf{J}_h^2)^{1/2} \right). \end{aligned} \quad (4.16)$$

Now, we can choose the sequences  $(\gamma_h)$  and  $(\mu_h)$  in such a way that

$$\lim_{h \rightarrow \infty} h^{n/2+p-1-\alpha} (\gamma_h)^{-\alpha} = \lim_{h \rightarrow \infty} h^n \frac{(\gamma_h)^{n/2}}{\mu_h} = 0$$

(for example,  $\gamma_h = h!$  and  $\mu_h = h^{nh}$ ). Then, (4.16) gives

$$\limsup_{h \rightarrow \infty} \int g_h(x, R) dx \leq c(1+R)^{p-1-\alpha} \delta^\alpha.$$

By the arbitrariness of  $\delta$ , the proof is completed.  $\square$

*Proof of Lemma 4.3.* – By the representation Theorem 2.3 it is enough to prove that for every  $\xi \in \mathbf{R}^n$  and for every cube  $Q$  in  $\mathbf{R}^n$  the limit

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{|Q|} \int_Q a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) dx \tag{4.17}$$

exists and is independent of  $Q$ , where  $v_\varepsilon$  is the solution to the following boundary value problem

$$\left. \begin{aligned} -\operatorname{div}\left(a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right)\right) &= 0 \quad \text{on } Q, \\ v_\varepsilon &\in H_0^{1,p}(Q). \end{aligned} \right\} \tag{4.18}$$

Given a cube  $Q$  in  $\mathbf{R}^n$  and  $\xi \in \mathbf{R}^n$  let us prove that

$$\lim_{h \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0^+} \int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right| dx = 0, \tag{4.19}$$

where  $v_\varepsilon^h$  is the solution to the following boundary value problem

$$\left. \begin{aligned} -\operatorname{div}\left(a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right)\right) &= 0 \quad \text{on } Q, \\ v_\varepsilon^h &\in H_0^{1,p}(Q). \end{aligned} \right\} \tag{4.20}$$

For the sake of simplicity we drop in the notation any explicit dependence on  $\xi$  of  $v_\varepsilon$  and  $v_\varepsilon^h$  throughout this section. We can write

$$\begin{aligned} &\int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right| dx \\ &\leq \left| \int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) - a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right| dx \right| \\ &\quad + \left| \int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right| dx \right| \equiv J_{\varepsilon,h}^1 + J_{\varepsilon,h}^2. \end{aligned}$$

The estimate of  $J_{\varepsilon,h}^1$  and  $J_{\varepsilon,h}^2$  will be carried out in the following four steps.

*Step 1.* – Let us fix  $\xi \in \mathbf{R}^n$ . As in (3.17) one gets

$$\|Dv_\varepsilon^h + \xi\|_{(L^p(Q))^n} \leq c|Q|(1+|\xi|^p).$$

This estimate will be used frequently in the sequel. By the Meyers estimate (1.6) we have then

$$\|Dv_\varepsilon^h + \xi\|_{(L^{p+\eta}(Q))^n} \leq c.$$

Let us fix  $R \geq 0$ ,  $h \in \mathbb{N}$  and  $\varepsilon > 0$ , and let us define the set

$$U_R = \{x \in Q : |Dv_\varepsilon^h(x) + \xi| > R\}.$$

We have then

$$|U_R| R^p \leq \int_{U_R} |Dv_\varepsilon^h + \xi|^p dx,$$

so that, using the Hölder inequality, we obtain the estimate

$$\int_{U_R} (1 + |Dv_\varepsilon^h + \xi|^p) dx \leq |U_R| + |U_R|^{\eta/(p+\eta)} \|Dv_\varepsilon^h + \xi\|_{(L^{p+\eta}(Q))^n}^p \leq c(R^{-p} + R^{-p\eta/(p+\eta)}). \quad (4.21)$$

*Step 2 (estimate of  $J_{\varepsilon,h}^2$ ).* — Given  $R > 0$  by (1.3) and (4.12) we have

$$\begin{aligned} (J_{\varepsilon,h}^2)^q &\leq c \int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right|^q dx \\ &= c \int_{U_R} \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right|^q dx \\ &\quad + c \int_{Q \setminus U_R} \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right|^q dx \\ &\leq c \int_{U_R} (1 + |Dv_\varepsilon^h + \xi|^p) dx + c \int_{Q \setminus U_R} g_h\left(\frac{x}{\varepsilon}, R\right)^q dx \\ &\leq c \int_{U_R} (1 + |Dv_\varepsilon^h + \xi|^p) dx + c(1 + R) \int_Q g_h\left(\frac{x}{\varepsilon}, R\right) dx. \end{aligned}$$

By Step 1 we conclude then

$$J_{\varepsilon,h}^2 \leq c \left( (R^{-p} + R^{-p\eta/(p+\eta)}) + (1 + R) \int_Q g_h\left(\frac{x}{\varepsilon}, R\right) dx \right)^{1/q}. \quad (4.22)$$

*Step 3 (estimate of  $\|Dv_\varepsilon^h - Dv_\varepsilon\|_{(L^p(Q))^n}$ ).* — By (4.18) and (4.20) we have

$$\int_Q \left( a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right), Dv_\varepsilon - Dv_\varepsilon^h \right) dx = 0,$$

so that by (iii) and (iv) in Definition 1.1 and the Hölder inequality we have

$$0 = \int_Q \left( a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) - a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right), Dv_\varepsilon - Dv_\varepsilon^h \right) dx$$

$$\begin{aligned}
 & + \int_Q \left( a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right), Dv_\varepsilon - Dv_\varepsilon^h \right) dx \\
 & \geq c_2 \int_Q (1 + |Dv_\varepsilon + \xi| + |Dv_\varepsilon^h + \xi|)^{p-\beta} |Dv_\varepsilon - Dv_\varepsilon^h|^\beta dx \\
 & - \int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right| |Dv_\varepsilon - Dv_\varepsilon^h| dx \\
 & \geq c \left( |Q|^{1/p} + \|Dv_\varepsilon + \xi\|_{(L^p(Q))^n} \right. \\
 & \quad \left. + \|Dv_\varepsilon^h + \xi\|_{(L^p(Q))^n} \right)^{p-\beta} \|Dv_\varepsilon - Dv_\varepsilon^h\|_{(L^p(Q))^n}^\beta \\
 & - \left( \int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right|^q dx \right)^{1/q} \\
 & \quad \times \left( \int_Q |Dv_\varepsilon - Dv_\varepsilon^h|^p dx \right)^{1/p},
 \end{aligned}$$

so that by (4.22) we obtain

$$\begin{aligned}
 & \|Dv_\varepsilon^h - Dv_\varepsilon\|_{(L^p(Q))^n}^\beta \\
 & \leq c \left( (R^{-p} + R^{-p\eta/(p+\eta)}) + (1+R) \int_Q g_h\left(\frac{x}{\varepsilon}, R\right) dx \right); \tag{4.23}
 \end{aligned}$$

*Step 4 (estimate of  $J_{\varepsilon, h}^1$ ).* – By the equicontinuity condition of  $a$  and Hölder’s inequality we have

$$\begin{aligned}
 J_{\varepsilon, h}^1 & = \left| \int_Q \left( a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) - a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right) dx \right| \\
 & \leq c_1 \int_Q (1 + |Dv_\varepsilon + \xi| + |Dv_\varepsilon^h + \xi|)^{p-1-\alpha} |Dv_\varepsilon - Dv_\varepsilon^h|^\alpha dx \\
 & \leq c_1 |Q|^{1/p} \left( \int_Q (1 + |Dv_\varepsilon + \xi| + |Dv_\varepsilon^h + \xi|)^p dx \right)^{(p-1-\alpha)/p} \\
 & \quad \times \left( \int_Q |Dv_\varepsilon - Dv_\varepsilon^h|^p dx \right)^{\alpha/p}.
 \end{aligned}$$

Hence, by (4.23) we get

$$J_{\varepsilon, h}^1 \leq c \left( (R^{-p} + R^{-p\eta/(p+\eta)}) + (1+R) \int_Q g_h\left(\frac{x}{\varepsilon}, R\right) dx \right)^{\alpha/(\beta-1)}. \tag{4.24}$$

Taking into account that there exists  $s > 0$  such that  $Q \subset Q_s(0)$  and

$$\begin{aligned}
 \int_Q g_h\left(\frac{x}{\varepsilon}, R\right) dx & \leq \int_{Q_s(0)} g_h\left(\frac{x}{\varepsilon}, R\right) dx \\
 & \leq \varepsilon^n \int_{Q_{s/\varepsilon}(0)} g_h(x, R) dx \leq \frac{c}{|Q_{s/\varepsilon}(0)|} \int_{Q_{s/\varepsilon}(0)} g_h(x, R) dx,
 \end{aligned}$$

by passing in (4.22) and (4.24) first to the limit as  $\varepsilon$  tends to  $0^+$ , then as  $h$  tends to  $+\infty$ , and eventually as  $R$  tends to  $+\infty$  we obtain (4.19). By (4.4) and Theorem 2.3 this implies that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{|Q|} \int_Q a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) dx \\ = \lim_{h \rightarrow \infty} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{|Q|} \int_Q a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) dx \\ = \lim_{h \rightarrow \infty} b_h(\xi) = b(\xi). \end{aligned} \quad (4.25)$$

Since the first limit does not depend on  $Q$ , the proof of Lemma 4.3 can be concluded by applying again the representation Theorem 2.3.  $\square$

*Proof of Theorem 4.2.* — Let  $a \in M_{\mathbf{R}^n}$  such that  $a(\cdot, \xi)$  is almost periodic for every  $\xi \in \mathbf{R}^n$ . By Lemma 4.4 there exists a sequence  $(a_h)$  in  $M_{\mathbf{R}^n}$  satisfying (3.3) (with  $\lambda$  possibly depending on  $h$ ) such that condition (4.5) is satisfied. By the homogenization Theorem 3.5 we obtain that for every  $h \in \mathbf{N}$  the limit

$$G\text{-}\lim_{\varepsilon \rightarrow 0^+} a_h\left(\frac{x}{\varepsilon}, \xi\right) = b_h(\xi)$$

exists and is independent of  $x$ . Hence, by Lemma 4.3 the limit

$$G\text{-}\lim_{\varepsilon \rightarrow 0^+} a\left(\frac{x}{\varepsilon}, \xi\right) = b(\xi)$$

exists. Finally, the representation formula follows from (4.25).  $\square$

## APPENDIX

### A. Birkhoff's theorem for quasiperiodic functions

Before we give a direct proof of Birkhoff's Ergodic theorem in the quasiperiodic case, we state Kronecker's lemma (*see*, for instance, [30] 3.1).

**KRONECKER'S LEMMA.** — *The set of vectors which are equivalent modulo  $T$  to vectors of the form  $j(x)$ , with  $x \in \mathbf{R}^n$ , is dense in  $\mathbf{R}^N$ .*

**THEOREM A.** — *Let  $f: \mathbf{R}^n \rightarrow \mathbf{R}$  be a quasiperiodic function in  $QP(\lambda)$  (see Section 3). Then*

$$\int f(x) dx = \frac{1}{|T|} \int_T F(y) dy,$$

where  $T$  is defined by (3.2).

*Proof.* — Let  $X = j(\mathbf{R}^n)$  and  $Z = X^\perp$ . Let us identify  $\mathbf{R}^N = X \oplus Z$ . Moreover, let us introduce

$$P_s^N(0) = j(Q_s(0)) \times B_s^Z(0),$$

where  $B_s^Z(0)$  is the ball in  $Z$  of radius  $s$  and center  $0$ , and define the set  $S$  by  $S = \{z \in B_s^Z(0) : z \text{ is equivalent modulo } T \text{ to vectors of the type } j(\tau) \text{ with } \tau \in \mathbf{R}^n\}$ . By Kronecker's lemma the set  $S$  is dense in  $B_s^Z(0)$ . Then, given  $z \in S$  there exists  $\tau = \tau(z) \in \mathbf{R}^n$  such that

$$F(j(x) + z) = f(x + \tau)$$

for every  $x \in \mathbf{R}^n$ . Since the limit

$$\lim_{s \rightarrow +\infty} \frac{1}{s^n} \int_{Q_s(\tau)} f(x) \, dx = \int f(x) \, dx$$

exists uniformly with respect to  $\tau$  (see [30] 2.3), given  $\varepsilon > 0$  we have

$$\left| \frac{1}{s^n} \int_{Q_s(0)} f(x + \tau) \, dx - \int f(x) \, dx \right| < \varepsilon$$

for every  $\tau \in \mathbf{R}^n$ , for  $s$  sufficiently large, so that for  $z \in S$

$$\left| \frac{1}{s^n} \int_{Q_s(0)} F(j(x) + z) \, dx - \int f(x) \, dx \right| < \varepsilon \tag{A.1}$$

for  $s$  sufficiently large. By the uniform continuity of  $F$  and the density of  $S$ , the estimate (A.1) holds for every  $z \in B_s^Z(0)$ . By Fubini's theorem we have

$$\begin{aligned} \frac{1}{|P_s^N(0)|} \int_{P_s^N(0)} F(y) \, dy &= \frac{1}{s^n} \int_{Q_s(0)} \left( \frac{1}{|B_s^Z(0)|} \int_{B_s^Z(0)} F(j(x) + z) \, dz \right) dx \\ &= \frac{1}{|B_s^Z(0)|} \int_{B_s^Z(0)} \left( \frac{1}{s^n} \int_{Q_s(0)} F(j(x) + z) \, dx \right) dz, \end{aligned}$$

which by (A.1) yields that

$$\left| \frac{1}{|P_s^N(0)|} \int_{P_s^N(0)} F(y) \, dy - \int f(x) \, dx \right| < \varepsilon.$$

This implies together with the periodicity of  $F$  that

$$\frac{1}{|T|} \int_T F(y) \, dy = \lim_{s \rightarrow +\infty} \frac{1}{|P_s^N(0)|} \int_{P_s^N(0)} F(y) \, dy = \int f(x) \, dx$$

and concludes the proof.  $\square$

**B. Regularity of the quasiperiodic solutions of  $\Delta u = \text{div } f$**

*Remark B. 1.* — Let  $\mathcal{W}$  be the completion of  $\text{Trig}_0(\mathbb{T})$  with respect to the norm  $\|\cdot\|_{\mathcal{W}}$  introduced in Section 3. Let us denote by  $\mathcal{W}'$  its dual. It turns out that for every  $S \in \mathcal{W}'$  there exists  $G \in (L^q(\mathbb{T}))^n$  such that

$$\langle S, v \rangle = \frac{1}{|\mathbb{T}|} \sum_{r=1}^n \int_{\mathbb{T}} G_r \partial_r v dy \quad \text{for every } v \in \mathcal{W}. \tag{B.1}$$

In fact, since by the map  $\partial$  the space  $\mathcal{W}$  is isometric to a subspace of  $(L^p(\mathbb{T}))^n$ , by the Hahn-Banach and Riesz theorems there exists a function  $G \in (L^q(\mathbb{T}))^n$  such that (B. 1) and

$$\|S\|_{\mathcal{W}'} = \left( \frac{1}{|\mathbb{T}|} \sum_{r=1}^n \int_{\mathbb{T}} |G_r|^q dy \right)^{1/q}$$

hold. We will write  $\|S\|_{\mathcal{W}'} = \left\| \sum_{r=1}^n \partial_r G_r \right\|_{\mathcal{W}'}$ .

**THEOREM B.** — Let  $F \in (\text{Trig}(\mathbb{T}))^n$  and let  $W \in \text{Trig}_0(\mathbb{T})$  be the unique solution to

$$\sum_{r=1}^n \partial_r^2 W = \sum_{r=1}^n \partial_r F_r \quad \text{in } \mathbf{R}^N. \tag{B.2}$$

Then,

$$\left( \frac{1}{|\mathbb{T}|} \sum_{r=1}^n \int_{\mathbb{T}} |\partial_r W|^q dy \right)^{1/q} \leq c \left\| \sum_{r=1}^n \partial_r F_r \right\|_{\mathcal{W}'}, \tag{B.3}$$

with  $c = c(\mathbb{T}) > 0$  and  $\partial_r = \sum_{l=1}^{m_r} \frac{\partial}{\partial y_r^l}$  as defined in Section 3.

*Proof.* — Let us define  $S \in \mathcal{W}'$  by

$$\langle S, v \rangle = \frac{1}{|\mathbb{T}|} \sum_{r=1}^n \int_{\mathbb{T}} F_r \partial_r v dy \quad \text{for every } v \in \mathcal{W}.$$

By Remark B. 1 there exists  $G \in (L^q(\mathbb{T}))^n$  such that

$$\langle S, v \rangle = \frac{1}{|\mathbb{T}|} \sum_{r=1}^n \int_{\mathbb{T}} G_r \partial_r v dy \quad \text{for every } v \in \mathcal{W}$$

and

$$\|G\|_{(L^q(\mathbb{T}))^n} = \|S\|_{\mathcal{W}'}. \tag{B.4}$$



By extending  $G$  on  $\mathbf{R}^N$  by periodicity we have that

$$\sum_{r=1}^n \partial_r^2 W = \sum_{r=1}^n \partial_r G_r \quad \text{in } \mathcal{D}'(\mathbf{R}^N). \tag{B.5}$$

Let us introduce on  $\mathbf{R}^N$  a new orthogonal base  $(E_l)_{1 \leq l \leq N}$  with

$$\begin{aligned} E_1 &= (\underbrace{1, \dots, 1}_{m_1}, 0, \dots, 0) \\ &\vdots \\ E_n &= (0, \dots, 0, \underbrace{1, \dots, 1}_{m_n}, 0). \end{aligned}$$

We shall denote the new coordinates by

$$(x, z) = (x_1, \dots, x_n, z_1, \dots, z_{N-n}) \in \mathbf{R}^N.$$

It turns out that

$$D_{x_l} u(x, z) = \partial_l u(y) \quad \text{for every } u \in \mathcal{W}, \quad \text{for every } l = 1, \dots, n,$$

where  $y = (x, z)$ . Hence by (B.5) the function  $W \in \mathcal{C}^\infty(\mathbf{R}^n \times \mathbf{R}^{N-n})$  satisfies for almost every  $z \in \mathbf{R}^{N-n}$

$$\Delta_x W(\cdot, z) = \operatorname{div}_x G(\cdot, z) \quad \text{in } \mathcal{D}'(\mathbf{R}^n).$$

By elliptic regularity (see, for instance [14]) for every  $A' \subset\subset A \subset\subset \mathbf{R}^n$  there exists a constant  $c = c(A', A)$ , such that

$$\int_{A'} |D_x W(x, z)|^q dx \leq c \int_A |G(x, z)|^q dx.$$

By Fubini's theorem we get

$$\int_B \int_{A'} |D_x W(x, z)|^q dx dz \leq c \int_B \int_A |G(x, z)|^q dx dz$$

for every  $B \subset\subset \mathbf{R}^{N-n}$ . Choosing  $A', A$  and  $B$  such that

$$T \subset B \times A' \subset B \times A \subset \prod_{l,r} \left] -\frac{2\pi}{\lambda_r^l}, \frac{4\pi}{\lambda_r^l} \right],$$

and using finally the periodicity of  $G$ , we get

$$\int_T |D_x W(x, z)|^q dx dz \leq c \int_T |G(x, z)|^q dx dz.$$

By (B.4) we obtain (B.3), and we can conclude the proof.  $\square$

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