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Homogenization of eigenvalue problems in perforated domains

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Abstract. In this paper, we treat some eigenvalue problems in periodically perforated domains and study the asymptotic behaviour of the eigenvalues and the eigenvectors when the number of holes in the domain increases to infinity. Using the method of asymptotic expansion, we give explicit formula for the homogenized coefficients and expansion for eigenvalues and eigenvectors. If we denote by ϵ the size of each hole in the domain, then we obtain the following aysmptotic expansion for the eigenvalues :

Dirichlet : $\lambda_\epsilon = \epsilon^{-2}\lambda + \lambda_0 + O(\epsilon)$,

Stekloff : $\lambda_\epsilon = \epsilon\lambda_1 + O(\epsilon^2)$,

Neumann : $\lambda_\epsilon = \lambda_0 + \epsilon\lambda_1 + O(\epsilon^2)$.

Using the method of energy, we prove a theorem of convergence in each case considered here. We briefly study correctors in the case of Neumann eigenvalue problem.

Keywords. Homogenization ; correctors ; eigenvalues ; eigenvectors.

1. Introduction

The theory of homogenization has been developed by many authors in recent years. For a historic introduction and for a complete bibliography of the subject, the reader is referred to the book of Bensoussan *et al* [3]. The method of asymptotic development introduced in this book can also be applied to problems in a periodically perforated domain. For the treatment of homogenization problems in such domains, the reader is referred to the works of Lions [13], Duvaut [9], Cioranescu [7], Cioranescu and Saint Jean Paulin [8].

The study of such problems is important from theoretical as well as numerical point of view. Because of the complicated structure of the perforated domains, any kind of calculation is difficult to perform. For example, if we treat the Dirichlet problem, we have to impose the boundary condition on the boundary of the holes which are many in number. So, we would like to "approximate" the given problem by a "homogenized" problem on the domain without holes, By the method of asymptotic development, a problem on a periodically perforated domain is reduced to solving problems in the "basic cell" and in the domain without holes.

This paper is divided into three parts :

Part A : Dirichlet eigenvalue problem

Part B : Stekloff eigenvalue problem

Part C : Neumann eigenvalue problem.

Our aim is to describe the asymptotic behaviour of the various eigenvalues when the number of holes in the domain increases to infinity. In each case we explicitly write down the "homogenized operator" with the help of the method of asymptotic development and prove a homogenization theorem using the energy method introduced by Tartar [14] and prolongation operators of Cioranescu and Saint Jean Paulin [8].

We treat here the case of Laplacian operator. But one can extend the results to the case of elliptic, self-adjoint operators with periodic coefficients of the form

$$A^\epsilon \equiv - \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x/\epsilon)\frac{\partial}{\partial x_j}\right).$$

The eigenvalue problem corresponding to A^ϵ in a fixed domain has been studied by Kesavan [12]. The results of this paper were announced in Vanninathan [15], [16].

2. Notations and hypothesis

First, we consider a reference cell :

$$(2.1) \qquad Y = \prod_{i=1}^{N} (0, y_i^0) \subset \mathbf{R}^N, y_i^0 > 0.$$

Let τ_i $(i = 1, 2, \ldots, M)$ be connected bounded open subsets of $\mathbf{R}^N$ with sufficiently smooth boundaries and which lie locally on one side of the boundary. Then the holes in Y are $\bar{\tau}_i \cap Y$ $(i = 1, 2, \ldots M)$ and their union is denoted by T :

$$(2.2) \qquad T = \bigcup_{i=1}^{M} (\bar{\tau}_i \cap Y).$$

Let

$$(2.3) \qquad Y^* = Y - T.$$

Let S denote the boundary of T in Y^*. For details see figure 1.

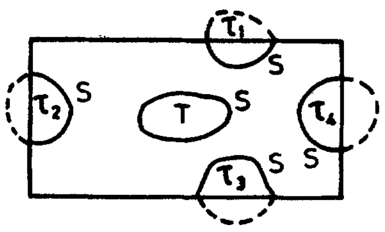

Figure 1

We assume that

(H1) Y^* is connected.

For a function f given on Y, we define the average of f on Y:

$$(2.4) \qquad m_Y(f) = \frac{1}{|Y|} \int_Y f(y)\, dy,$$

where $|Y|$ denotes the Lebesgue measure of Y. Let ϵ be a small positive parameter which goes to zero. We denote by $f^\epsilon(x)$ or $f(x/\epsilon)$, the function defined in $\mathbf{R}^N$ in a periodic fashion with period $\epsilon\, y_i^0$ in the direction x_i.

Let us define now, "the holes" in $\mathbf{R}^N$ corresponding to ϵ, starting from those in Y. For that we introduce the function χ as follows:

$$(2.5) \qquad \chi(y) = \begin{cases} 1 & \text{if } y \in Y^*, \\ 0 & \text{if } y \in T. \end{cases}$$

We consider also the characteristic function of T:

$$(2.6) \qquad \chi_T(y) = \begin{cases} 1 & \text{if } y \in T, \\ 0 & \text{if } y \in Y^*. \end{cases}$$

Put

$$(2.7) \qquad \theta = m_Y(\chi) = \frac{|Y^*|}{|Y|}.$$

The "holes" in $\mathbf{R}^N$ are the connected components of the set

$$\{x \in \mathbf{R}^N / \chi^\epsilon(x) = 0\}.$$

Finally, let us be given a bounded connected open subset Ω of $\mathbf{R}^N$ whose boundary is sufficiently smooth and Ω lies locally on one side of its boundary. The "holes" in Ω are then defined by

(2.8) T_ϵ = connected components in Ω of $\{x \in \Omega / \chi^\epsilon(x) = 0\}$. The perforated domain Ω_ϵ with which we work is

$$(2.9) \qquad \Omega_\epsilon = \Omega - T_\epsilon.$$

Let S_ϵ denote the boundary of T_ϵ in Ω_ϵ.

We make the following restrictions on the geometry of Ω_ϵ.

(H2) Ω_ϵ is connected.

(H3) Each hole in T_ϵ has regular boundary.

In the problems we consider here, there is one more restriction on the geometry of Ω and the holes (*cf.* Cioranescu and Saint Jean Paulin [8]).

(H4) The holes T_ϵ do not meet $\partial\Omega$, the boundary of Ω.

We need, in fact, in Part A a stronger hypothesis. Given any hole T' in Y, we can as before construct the holes T'_ϵ in Ω periodically. Set

$$(2.10) \qquad \Omega_{T'_\epsilon} = \Omega - T'_\epsilon.$$

With this notation, it is evident that

$$(2.11) \qquad \Omega_\epsilon = \Omega_{T_\epsilon}.$$

The stronger hypothesis is the following:

(H5) $\begin{cases} \text{there exists a hole } T' \text{ in } Y \text{ such that } T \subset\subset T' \text{ and the holes } T'_\epsilon \text{ do} \\ \text{not intersect } \partial\Omega. \end{cases}$

Remark (2.1). The hypothesis (H4) is severe on the geometry of the domain Ω and the hole T. One such example of Ω is a finite union of cells homothetic to Y and with the hole T placed in the middle of Y.

Summation Convention. We adopt the usual summation convention with respect to the repeated indices.

Part A: Dirichlet eigenvalue problem

3. Problem to be treated

With the above notations, we consider the following eigenvalue problem:

$$(3.1) \qquad \begin{cases} \text{Find } (u_\epsilon, \lambda_\epsilon) \in H_0^1(\Omega_\epsilon) \times \mathbf{R} \text{ such that} \\ -\triangle u_\epsilon = \lambda_\epsilon u_\epsilon \text{ in } \Omega_\epsilon, \\ (u_\epsilon, u_\epsilon)_\epsilon = 1, \end{cases}$$

where $(.,.)_\epsilon$ denotes the inner product in $L^2(\Omega_\epsilon)$ and $H_0^1(\Omega_\epsilon) = \{v \in L^2(\Omega_\epsilon);\ \partial v/\partial x_i \in L^2(\Omega_\epsilon)$ for $i = 1, 2, \ldots N$ and $v = 0$ on $\partial\Omega\}$.

The variational formulation of this problem is the following :

$$(3.2) \qquad \begin{cases} \text{Find } (u_\epsilon, \lambda_\epsilon) \in H_0^1(\Omega_\epsilon) \times \mathbf{R} \text{ such that} \\ a^\epsilon(u_\epsilon, v) = \lambda_\epsilon (u_\epsilon, v)_\epsilon \text{ for } v \in H_0^1(\Omega_\epsilon), \\ (u_\epsilon, u_\epsilon)_\epsilon = 1, \end{cases}$$

with the bilinear form $a^\epsilon(.,.)$ defined by

$$(3.3) \qquad a^\epsilon(u, v) = \int_{\Omega_\epsilon} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx.$$

According to spectral theory, there exist a sequence of eigenvalues $\{\lambda_\epsilon^l\}_{l=1}^\infty$ and a sequence of corresponding eigenvectors $\{u_\epsilon^l\}_{l=1}^\infty$ such that

$$(3.4) \qquad \begin{cases} 0 < \lambda_\epsilon^1 < \lambda_\epsilon^2 \leqslant \lambda_\epsilon^3 \ldots \to \infty, \\ \lambda_\epsilon^l \text{ is of finite multiplicity for each } l, \\ \text{and } \{u_\epsilon^l\}_{l=1}^\infty \text{ form an orthonormal basis in } L^2(\Omega_\epsilon). \end{cases}$$

We can characterise the eigenvalues λ_ϵ^l with the help of Rayleigh quotient (*cf.* Weinstein and Steinger [17]).

$$(3.5) \qquad R_\epsilon(v) = \frac{a^\epsilon(v, v)}{(v, v)_\epsilon} \text{ for } v \in H_0^1(\Omega_\epsilon),\ v \neq 0.$$

The mimimax principle for the eigenvalues states that

$$(3.6) \qquad \begin{cases} \lambda_\epsilon^l = \min\{\max_{v \in S_l} R_\epsilon(v);\ S_l \subset H_0^1(\Omega_\epsilon),\ \dim S_l = l\}, \\ \quad = \max_{v \in E_\epsilon(l)} R_\epsilon(v) \\ \quad = \max\{R_\epsilon(v); (v, u_\epsilon^i)_\epsilon = 0,\ i = 1, 2, \ldots, l-1\}, \end{cases}$$

where $E_\epsilon(l)$ is the subspace of $H_0^1(\Omega_\epsilon)$ spanned by $\{u_\epsilon^1, \ldots, u_\epsilon^l\}$.

This part is devoted to the study of the behaviour of λ_ϵ^l and u_ϵ^l when $\epsilon \to 0$. We prove, in particular, that λ_ϵ^l is of order ϵ^{-2} and that $\{\lambda_\epsilon^l - \epsilon^{-2}\lambda\} \to l$th eigenvalue of the "homogenized problem" where λ is the first eigenvalue in the cell

Y^*. As we will see, some weighted Sobolev spaces and their properties are used in this study.

4. Eigenvalue problem in the cell Y^*

Let us define the space

(4.1) $W_0 = \{v \in H^1(Y^*)\,;\ v = 0$ on S and v is Y-periodic, i.e., v assumes same values on the opposite faces of $Y\}$

and the bilinear form

$$a(u, v) = \int_{Y^*} \frac{\partial u}{\partial y_i}\frac{\partial v}{\partial y_i}\, dy \text{ for } u, v \in H^1(Y^*). \tag{4.2}$$

The bilinear form being elliptic on W_0, the following problem is well posed :

$$\begin{cases} \text{Find } (\phi, \lambda) \in W_0 \times \mathbf{R} \text{ such that} \\ a(\phi, v) = \lambda(\phi, v)_{Y*} \text{ for } v \in W_0, \\ (\phi, \phi)_{Y*} = 1, \end{cases} \tag{4.3}$$

where $(.,.)_{Y*}$ denotes the scalar product in $L^2(Y^*)$.

In what follows, we consider only the first eigenvalue λ of the above problem. It is well known that λ is simple and the corresponding eigenvector ϕ has constant sign in Y^*. We choose the vector ϕ which is uniquely defined by (4.3) and

$$\phi > 0 \text{ in } Y^*. \tag{4.4}$$

Remark (4.1). We extend ϕ by zero in the interior of the holes T and we denote again by ϕ the extended function.

Remark (4.2). It follows from (4.3) that the function ϕ^ϵ defined periodically satisfies

$$\begin{cases} -\triangle\phi^\epsilon = \epsilon^{-2}\lambda\phi^\epsilon \text{ in } \Omega_\epsilon, \\ \phi^\epsilon = 0 \text{ on } S_\epsilon. \end{cases} \tag{4.5}$$

However, ϕ^ϵ is not zero on $\partial\Omega$.

5. Some weighted Sobolev spaces

We will see later that consideration of some weighted Sobolev spaces is very important in the study of the present problem. There is a vast literature on this subject: see for example Baouendi [1], Baouendi and Goulaouic [2], Geymont and Grisvard [10], Goudjo [11]. In this section, we define some weighted Sobolev spaces and state some of their properties which will be needed later.

We consider the following spaces with weights ϕ^ϵ and ϕ (ϕ being defined in § 4):

$$V_\epsilon = \left\{ v \in D'(\Omega_\epsilon)\,;\ \phi^\epsilon v \in L^2(\Omega_\epsilon),\ \phi^\epsilon \frac{\partial v}{\partial x_i} \in L^2(\Omega_\epsilon) \text{ for } i = 1, 2, \ldots, N \text{ and } v = 0 \text{ on } \partial\Omega \right\}, \tag{5.1}$$

(5.2) $$V=\left\{v\in D'(Y^*);\ \phi v\in L^2(Y^*),\ \phi\frac{\partial v}{\partial y_j}\in L^2(Y^*)\ \text{for}\right.$$

$$\left. j=1,2,\ldots,N \text{ and } v \text{ is } Y\text{-periodic}\right\}.$$

where, as usual,
$D'(\Omega)$ = space of distributions on Ω, and these spaces are provided with the following norms :

$$\|v\|_\epsilon=\left[\|\phi^\epsilon v\|^2_{L^2(\Omega_\epsilon)}+\sum_{i=1}^{N}\left\|\phi^\epsilon\frac{\partial v}{\partial x_i}\right\|^2_{L^2(\Omega_\epsilon)}\right]^{1/2},$$

$$\|v\|_o=\left[\|\phi\, v\|^2_{L^2(Y^*)}+\sum_{i=1}^{N}\left\|\phi\frac{\partial v}{\partial y_i}\right\|^2_{L^2(Y^*)}\right]^{1/2}.$$

Some of their propertieis are given in the following :

Proposition (5.1)

(1) The space $D(\Omega_\epsilon)$ of infinitely differentiable functions on Ω_ϵ with compact support in Ω_ϵ is dense in V_ϵ.

(2) We have a continuous inclusion

$$V_\epsilon\rightarrow L^2(\Omega_\epsilon)$$

and the continuity constant of this inclusion does not depend on ϵ: there exists a constant $c>0$ independent of ϵ such that

(5.5) $$\|v\|_{L^2(\Omega_\epsilon)}\leq c\|v\|_\epsilon\ \forall\, v\in V_\epsilon.$$

(3) The map $v\rightarrow\phi^\epsilon v$ defines an isomorphism of V_ϵ onto $H^1_0(\Omega_\epsilon)$.

(4) The inclusion $V_\epsilon\rightarrow L^2(\phi^\epsilon)$ is compact where

(5.6) $$L^2(\phi^\epsilon)=\{v\in D'(\Omega_\epsilon)\ ;\ \phi^\epsilon v\in L^2(\Omega_\epsilon)\}.$$

(5) $$v\rightarrow\left[\sum_{i=1}^{N}\left\|\phi^\epsilon\frac{\partial v}{\partial x_i}\right\|^2_{L^2(\Omega_\epsilon)}\right]^{1/2}$$

defines a norm on V_ϵ equivalent to the norm $\|\ \|_\epsilon$.

Proof. All these properties are proved in the works cited above. The only thing which is new is the inequality (5.5) with c independent of ϵ. But one can prove this, without much difficulty, from the continuity of the inclusion

(5.7) $$W(\phi)\rightarrow L^2(Y^*),$$

where the space $W(\phi)$ is defined by

(5.8) $$W(\phi)=\left\{v\in D'(Y^*);\ \phi\, v\in L^2(Y^*),\ \phi\frac{\partial v}{\partial y_i}\in L^2(Y^*)\ \text{for}\ i=1,2,\ \ldots,N.\right\}$$

Proposition (5.2)

(1) The space of functions $v \in c^\infty(\bar{Y}^*) \cap V$ which vanish in a neighbourhood of S are dense in V.

(2) One has the continuous inclusion

$$V \to L^2(Y^*).$$

(3) The map $v \to \phi v$ defines an isomorphism of V onto W_0.

(4) The inclusion

$$V \to L^2(\phi) \tag{5.9}$$

is compact where we define

$$L^2(\phi) = \{v \in D'(Y^*) \ ; \phi v \in L^2(Y^*)\}. \tag{5.10}$$

$$v \to \left[\sum_{i=1}^{N} \left\| \phi \frac{\partial v}{\partial y_i} \right\|^2_{L^2(Y^*)} \right]^{1/2} \tag{5}$$

defines a norm equivalent to the quotient norm on $V/\mathbf{R}$.

Now, we formulate the eigenvalue problems in the spaces V_ϵ. We define

$$\begin{cases} a(\phi^\epsilon \ ; u, v) = \displaystyle\int_{\Omega_\epsilon} \phi^{\epsilon^2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx \text{ for } u, v \in V_\epsilon, \\ (\phi^\epsilon \ ; u, v) = \displaystyle\int_{\Omega_\epsilon} \phi^{\epsilon^2} u v \, dx \, \forall \, u, v \in L^2(\phi^\epsilon). \end{cases} \tag{5.11}$$

By virtue of the properties (4) and (5) of the proposition (5.1), the following problem is well posed:

$$\begin{cases} \text{Find } (v_\epsilon, \mu_\epsilon) \in V_\epsilon \times \mathbf{R} \text{ such that} \\ a(\phi^\epsilon; v_\epsilon, v) = \mu_\epsilon (\phi^\epsilon \ ; v_\epsilon, v) \text{ for } v \in V_\epsilon, \\ (\phi^\epsilon; v_\epsilon, v_\epsilon) = 1. \end{cases} \tag{5.12}$$

Let $\{u_\epsilon^i\}_{i=1}^\infty$ be the sequence of eigenvalues and $\{v_\epsilon^i\}_{i=1}^\infty$ the sequence of corresponding vectors satisfying

$$\begin{cases} \text{(i) } 0 < \mu_\epsilon^1 \leq \mu_\epsilon^2 \leq \dots \to \infty, \\ \text{(ii) } \{v_\epsilon^i\}_{i=1}^\infty \text{ form an orthonormal basis in } L^2(\phi^\epsilon). \end{cases} \tag{5.13}$$

We have once again the characterization of μ_ϵ^i, analogous to (3.6), in terms of Rayleigh quotient

$$R(\phi^\epsilon \ ; v) = \frac{a(\phi^\epsilon \ ; v, v)}{(\phi^\epsilon \ ; v, v)} \text{ for } v \in V_\epsilon, \ v \neq 0.$$

6. Estimations on the eigenvalues

The following Lemma establishes one important relation between the eigenvalues λ_ϵ, λ and μ_ϵ.

Lemma (6.1). Let λ be the first eigenvalue of the problem (4.3) with the eigenvector ϕ. Then for all $v \in V_\epsilon$, we have

$$(6.1) \qquad \int_{\Omega_\epsilon} \frac{\partial}{\partial x_i}(\phi^\epsilon v) \frac{\partial}{\partial x_i}(\phi^\epsilon v)\, dx = \epsilon^{-2} \lambda \int_{\Omega_\epsilon} \phi^{\epsilon^2} v^2\, dx + \int_{\Omega_\epsilon} \phi^{\epsilon^2} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx.$$

Proof. Since $D(\Omega_\epsilon)$ is dense in V_ϵ, it suffices to verify (6.1) for all $v \in D(\Omega_\epsilon)$. On the one hand, we have

$$\int_{\Omega_\epsilon} \frac{\partial}{\partial x_i}(\phi^\epsilon v) \frac{\partial}{\partial x_i}(\phi^\epsilon v) = \int_{\Omega_\epsilon} \frac{\partial \phi^\epsilon}{\partial x_i} \frac{\partial}{\partial x_i}(\phi^\epsilon v^2)\, dx + \int_{\Omega_\epsilon} \phi\epsilon^2 \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx$$

and on the other hand, by multiplying (4.5) by $\phi^\epsilon v^2$ we obtain

$$\int_{\Omega_\epsilon} \frac{\partial \phi^\epsilon}{\partial x_i} \frac{\partial}{\partial x_i}(\phi^\epsilon v^2)\, dx = \epsilon^{-2} \lambda \int_{\Omega_\epsilon} \phi^{\epsilon^2} v^2\, dx,$$

and so we deduce (6.1) without difficulty.

Corollary (6.1)

Let $\{\lambda_\epsilon^l\}_{l=1}^\infty$, $\{\mu_\epsilon^l\}_{l=1}^\infty$ be the sequences of eigenvalues of the problems (3.2) and (5.12) respectively. Let λ be the first eigenvalue of the problem (4.3). Then we have

$$(6.2) \qquad \lambda_\epsilon^l = \epsilon^{-2} \lambda + \mu_\epsilon^l \text{ for } l \geq 1.$$

Proof. In fact, we obtain, from (6.1)

$$R_\epsilon(\phi^\epsilon v) = \epsilon^{-2} \lambda + R(\phi^\epsilon\,; v) \text{ for } v \in V_\epsilon.$$

Now we use the minimax principles for eigenvalues and the isomorphism of the proposition (5.1) (iii) to get the relation (6.2).

Since $\mu_\epsilon^l > 0$ for all $l \geq 1$ and $\epsilon > 0$, we see that the sequence $\{\lambda_\epsilon^l - \epsilon^{-2} \lambda\}_{\epsilon>0}$ is bounded below by zero. The following Proposition shows that it is bounded above.

Proposition (6.1). Let $\{\mu_\epsilon^l\}_{l=1}^\infty$ be the sequence of eigenvalues of the problem (5.12). Then for $l \geq 1$, $\{\mu_\epsilon^l\}_{\epsilon>0}$ is bounded independently of ϵ.

Proof. We use the following characterizations of minimax principle :

$$(6.4) \qquad \mu_\epsilon^l = \min\{\max_{v \in S_l} R(\phi^\epsilon\,;\, v);\; S_l \subset V_\epsilon,\ \dim S_l = l\}.$$

We take S_l to be the vector space spanned by $w_1, w_2, \ldots, w_l$, the first l-eigenvectors of the following Dirichlet problem :

$$(6.5) \qquad \begin{cases} \text{Find } (w, \nu) \in H_0^1(\Omega) \times \mathbf{R} \text{ such that} \\ -\Delta w = \nu w \text{ in } \Omega. \end{cases}$$

It is not difficult to see that we have

$$(6.6) \qquad \dim(S_l/\Omega_\epsilon) = l \text{ for } \epsilon > 0.$$

[S_l/Ω_ϵ denotes the restriction to Ω_ϵ of functions belonging to S_l]. So, one can take S_l/Ω_ϵ in (6.4). (This is alright, since no boundary condition is required on S_ϵ for V_ϵ.) We obtain

(6.7) $$\mu_\epsilon^l \leq \max_{v \in S_l} R(\phi^\epsilon\ ; v).$$

We claim that the right hand side of the inequality (6.7) is bounded above by a constant independent of ϵ. In fact, on the contrary we would have, for a sequence $\epsilon_n \to 0$, a sequence $\{v_n\} \subset S_l$ such that

(6.8) $$\int_{\Omega_n} \phi_n^2 \nabla v_n \nabla v_n > n \int_{\Omega_n} \phi_n^2 v_n^2 \text{ for } n \geq 1,$$

(6.9) $$\int_{\Omega} v_n^2 = 1 \text{ for } n \geq 1.$$

Here we have set $\Omega_{\epsilon_n} = \Omega_n$ and $\phi^{\epsilon_n} = \phi_n$. Since S_l is of finite dimension, we have (for a subsequence)

$$v_n \to v \text{ in } H_0^1(\Omega) \text{ strong.}$$

Now, one can pass to the limit in (6.8) and obtain

$$m_y(\phi^2) \int_\Omega v^2 = 0.$$

But (6.9) implies that

$$\int_\Omega v^2 = 1.$$

This contradiction proves the Proposition.

7. Asymptotic development

The aim of this section is to find " the homogenized operator " for the problem (3.2) by the method of asymptotic development introduced in Bensoussan *et al* [3] and Lions [13].

We introduce one "fast" variable :

(7.1) $$y = x/\epsilon.$$

Then, the differential operator $\partial/\partial x_i$ applied to a function $\phi\ (x, y)$ becomes

(7.2) $$\frac{\partial}{\partial x_i} + \epsilon^{-1} \frac{\partial}{\partial y_i}\ .$$

So, the laplacian operator is transformed into

(7.3) $$\epsilon^{-2} \Delta_y + 2\epsilon^{-1} \Delta_{xy} + \Delta_x$$

where

(7.4) $$\Delta_y = \frac{\partial^2}{\partial y_i \partial y_i},\quad \Delta_{xy} = \frac{\partial^2}{\partial x_i \partial y_i},\quad \Delta_x = \frac{\partial^2}{\partial x_i \partial x_i}\ .$$

Taking into account Proposition (6.1) and Corollary (6.1), we propose the following Ansatz for the problem (3.1) :

(7.5) $u_\epsilon(x) = u_0(x, y) + \epsilon u_1(x, y) + \ldots \; y = x/\epsilon,$

(7.6) $\lambda_\epsilon = \epsilon^{-2}\lambda_{-2} + \epsilon^{-1}\lambda_{-1} + \lambda_0 + \ldots .$

We impose the following restrictions on the functions u_j which are defined for $x \in \Omega$ and $y \in Y^*$:

(7.7) $$\begin{cases} u_j(x, y) = 0 \text{ if } y \in S, \\ u_j(x, .) \text{ is } Y\text{-periodic in } y. \end{cases}$$

We substitute the expressions (7.5) and (7.6) in equation (3.1) and we identify the powers of ϵ. We obtain

(7.8) $$\begin{cases} u_0(x, y) = \phi(y)\,\tilde{u}_0(x), \\ u_1(x, y) = \phi^{(j)}(y)\dfrac{\partial \tilde{u}_0}{\partial x_j}(x) + \phi(y)\,\tilde{u}_1(x), \end{cases}$$

(7.9) $\lambda_{-2} = \lambda,$

(7.10) $\lambda_{-1} = 0,$

(7.11) $A\tilde{u}_0 = \lambda_0\tilde{u}_0,$

where

(1) λ is the first eigenvalue and ϕ the corresponding eigenvector of the problem (4.3).

(2) The functions $\phi^{(j)}$ $(j = 1, 2, .., N)$ are defined by

(7.12) $$a(\phi^{(j)}, v) - \lambda(\phi^{(j)}, v)_{Y^*} = 2\left(\frac{\partial \phi}{\partial y_j}, v\right)_{Y^*} \text{ for } v \in W_0,\ \phi^{(j)} \in W_0.$$

(3) The operator A (called homogenized operator) is defined by

(7.13) $$A \equiv -\, q_{ij}\frac{\partial^2}{\partial x_i \partial x_j}$$

the "homogenized coefficients" being defined by

(7.14) $$q_{ij} = \delta_{ij} + 2\int_{Y^*} \frac{\partial \phi^{(j)}}{\partial y_i}\phi\, dy \text{ for } i, j = 1, 2, \ldots, N.$$

Remark (7.1)

The relations (7.9) and (7.10) are in accordance with the results of Proposition (6.1) and Corollary (6.1).

Remark (7.2) Since

$$\left(\frac{\partial \phi}{\partial y_i}, \phi\right)_{Y^*} = 0 \text{ for } i = 1, 2, \ldots, N,$$

equations (7.12) can be solved for $\phi^{(j)}$ by Fredholm alternative.

The "homogenized problem" is an eigenvalue problem for the operator A :

$$(7.15)\qquad \begin{cases} \text{Find } (u, \mu) \in H_0^1(\Omega) \times \mathbf{R} \text{ such that} \\ Au = \mu u \text{ in } \Omega, \\ u \neq 0. \end{cases}$$

The preceding formal calculations show that the sequence $\{\lambda_\epsilon - \epsilon^{-2}\lambda\}_{\epsilon>0}$ converges to an eigenvalue of the problem (7.15). We prove this result later by the method of energy.

8. Ellipticity of the homogenized operator

The idea of proving the ellipticity of the operator consists of identifying the coefficients q_{ij} with the homogenized coefficients associated with the problem (5.12). So, we apply the asymptotic development method to the problem (5.12). First, we write the problem (5.12) in operator form : The solution $(v_\epsilon, \mu_\epsilon)$ is characterized by

$$(8.1)\qquad \begin{cases} (v_\epsilon, \mu_\epsilon) \in V_\epsilon \times \mathbf{R}, \\ -\dfrac{\partial}{\partial x_i}\left(\phi^{\epsilon^2}\dfrac{\partial v_\epsilon}{\partial x_i}\right) = \mu_\epsilon \phi^{\epsilon^2} v_\epsilon \text{ in } \Omega_\epsilon, \\ \int\limits_{\Omega_\epsilon} \phi^{\epsilon^2} v_\epsilon^2 = 1. \end{cases}$$

We develop v_ϵ and μ_ϵ in the following form :

$$(8.2)\qquad v_\epsilon(x) = v_0(x, y) + \epsilon v_1(x, y) + \dots\ y = x/\epsilon,$$

$$(8.3)\qquad \mu_\epsilon = \mu_0 + \epsilon\mu_1 + \dots,$$

where v_j is defined on $\Omega \times Y^*$ and it is Y-periodic in y.

We put these expressions in (8.1) and identify the powers of ϵ. We get the following results :

$$(8.4)\qquad v_0(x, y) = \bar{v}_0(x),$$

$$(8.5)\qquad v_1(x, y) = \psi^{(j)}(y)\frac{\partial \bar{v}_0}{\partial x_j}(x) + \bar{v}_1(x),$$

and the necessary and sufficient condition so that we solve for v_2 is

$$(8.6)\qquad B\bar{v}_0 = \mu_0 \bar{v}_0 \text{ in } \Omega,$$

where the operator B is defined by

$$(8.7)\qquad B \equiv -p_{ij}\frac{\partial^2}{\partial x_i \partial x_j},$$

with

$$(8.8)\qquad p_{ij} = \delta_{ij} + \int_{Y^*} \phi^2 \frac{\partial \psi^{(j)}}{\partial y_i}\, dy \text{ for } i, j = 1, 2, \dots, N.$$

The functions $\psi^{(j)}$ $(j = 1, 2, \ldots, N)$ are defined (upto an additive constant) to be solutions of the following variational problem :

$$(8.9)\quad \begin{cases} a(\phi;\, \psi^{(j)}, v) = -\, a(\phi;\, y_j, v)\ \forall\, v \in V, \\ \psi^{(j)} \in V, \end{cases}$$

the bilinear form $a(\phi;\, .,.)$ being defined by

$$(8.10)\quad a(\phi;\, u, v) = \int_{Y^*} \phi^2 \frac{\partial u}{\partial y_i} \frac{\partial v}{\partial y_i}\, dy\ \forall\, u, v \in V.$$

It is not difficult to see that the operator B is elliptic in the sense that there exists a constant $\alpha > 0$ such that

$$(8.11)\quad p_{ij}\xi_i\xi_j \geq \alpha\xi_i\xi_i \text{ for } \xi = (\xi_i) \in \mathbf{R}^N.$$

In fact, the coefficients p_{ij} can be expressed by the following formula :

$$(8.12)\quad p_{ij} = a(\phi;\, \psi^{(j)} + y_j, \psi^{(i)} + y_i) \text{ for } i, j = 1, 2, \ldots N,$$

taking into account the following relation

$$a(\phi;\, \psi^{(j)} + y_j, \psi^{(i)}) = 0 \text{ for } i, j = 1, 2, \ldots N.$$

Now, the inequality (8.11) is a simple consequence of (8.12).

Theorem (8.1)

Let (q_{ij}) and (p_{ij}) be defined by (7.14) and (8.8) respectively. Then we have

$$q_{ij} = p_{ij} \text{ for } i, j = 1, 2, \ldots N.$$

In particular, the operator A is symmetric and elliptic.

Proof. We prove that $\phi\psi^{(j)}$ is a solution to the problem (7.12). For that we use the isomorphism between the spaces V and W_0 given by the Proposition (5.2) (iii). Firstly, we have $\phi\psi^{(j)} \in W_0$.

By virtue of the Proposition (5.2) (i), it suffices to verify that

$$(8.15)\quad \int_{Y^*} \frac{\partial}{\partial y_i}(\phi\psi^{(j)}) \frac{\partial}{\partial y_i}(\phi v)\, dy = \lambda \int_{Y^*} \phi^2 \psi^{(j)} v\, dy + 2 \int_{Y^*} \frac{\partial \phi}{\partial y_j} \phi v\, dy,$$

for all $v \in c^\infty(\bar{Y}^*) \cap V$ which vanish in a neighbourhood of S. The relation (8.15) is easily proved by using the definitions of ϕ and $\psi^{(j)}$ and the following identity :

$$(8.16)\quad 2 \int_{Y^*} \phi \frac{\partial \phi}{\partial y_i} v\, dy = -\int_{Y^*} \phi^2 \frac{\partial v}{\partial y_i}\, dy\ \forall\, v \in V.$$

Now we take $\phi^{(j)} = \phi\psi^{(j)}$ in the formula (7.14) defining q_{ij}. We obtain

$$q_{ij} = \delta_{ij} + 2 \int_{Y^*} \frac{\partial}{\partial y_i}(\phi\psi^{(j)})\, \phi\, dy,$$

$$= \delta_{ij} + \int_{Y^*} \phi^2 \frac{\partial \psi^{(j)}}{\partial y_i} \, dy$$

$$= p_{ij},$$

and so the Theorem is proved.

9. Homogenization theorems

In this section using energy method we show how one passes to the limit as $\epsilon \to 0$ in the problem (5.12) which determines the correctors μ_ϵ. Before that, we need some more notations. Let us denote by T' any hole satisfying the hypothesis (H5). We denote by $P_\epsilon^{T'}$, the prolongation operator constructed from the hole T' satisfying the following condition :

$$(9.1) \quad \begin{cases} \text{For all } v \text{ in } H^1(\Omega_{T'_\epsilon}) \text{ with } v = 0 \text{ on } \partial\Omega, \text{ we have} \\ P_\epsilon^{T'} v \in H_0^1(\Omega) \text{ and there exists a constant } c > 0 \\ \text{independent of } \epsilon \text{ (but depending on the hole } T'\text{) such that} \\ | P_\epsilon^{T'} v |_{1,\Omega} \leq c \, | v |_{1,\Omega_{T'_\epsilon}}, \\ \text{where } \; | v |_{1,\Omega} = \int_\Omega \nabla v \cdot \nabla v \, dx. \end{cases}$$

The existence of the operator $P_\epsilon^{T'}$ is proved in Cioranescu and Saint Jean Paulin [8] and Cioranescu [7].

Lemma (9.1). Suppose for each $\epsilon > 0$, we are given w_ϵ in V_ϵ such that

(9.2) $\quad \| w_\epsilon \|_\epsilon \leqslant c$, independent of ϵ.

Then there exists a subsequence of ϵ (again denoted by ϵ) and a function w_0 in $H_0^1(\Omega)$ such that

(9.3) $\quad P_\epsilon^{T'} w_\epsilon \to w_0$ in $H_0^1(\Omega)$ weak,

for all holes T' verifying (H5).

Proof. It follows from the hypothesis (9.2) and from the fact that $\phi > 0$ in Y^* that there exists a constant $c_{T'}$ such that

(9.4) $\quad | w_\epsilon |_{1,\Omega_{T'_\epsilon}} \leq c_{T'}$.

Thanks to the inequality (9.1), we see that the sequence $\{P_\epsilon^{T'} w_\epsilon\}$ remains bounded in $H_0^1(\Omega)$. So, we can extract a subsequence of ϵ and a function $w_0^{T'}$ in $H_0^1(\Omega)$ both depending on T' such that

(9.5) $\quad P_\epsilon^{T'} w_\epsilon \to w_0^{T'}$ in $H_0^1(\Omega)$ weak,

for the extracted subsequence.

Now let us consider another hole satisfying (H5). By similar arguments as before, we obtain a subsequence of the subsequence already picked up and a function $w_0^{T''}$ in $H_0^1(\Omega)$ such that

(9.6) $\quad P_\epsilon^{T''} w_\epsilon \to w_0^{T''}$ in $H_0^1(\Omega)$ weak.

But we have the relation

(9.7) $\quad \chi_{Y-(T'\cup T'')}(x/\epsilon)\, P_\epsilon^{T'} w_\epsilon = \chi_{Y-(T'\cup T'')}(x/\epsilon)\, P_\epsilon^{T''} w_\epsilon$ in Ω

where $\chi_{Y-(T'\cup T'')}(y)$ is the characteristic function of $Y-(T'\cup T'')$. One can pass to the limit in (9.7) and we obtain

$$m_Y(\chi_{Y-(T'\cup T'')})\, w_0^{T'} = m_Y(\chi_{Y-(T'\cup T'')})\, w_0^{T''},$$

and so $w_0^{T'} = w_0^{T''}$ in Ω. By the uniqueness of the limit, we see that the proof is complete.

Lemma (9.2). Let $\{w_\epsilon\}_{\epsilon>0}$ be given as in the preceding Lemma. Then for the subsequence of ϵ and for the function w_0 in $H_0^1(\Omega)$ given by the Lemma (9.1) we have

(9.8) $\quad P_\epsilon^{T'} w_\epsilon \to w_0$ in $H_0^1(\Omega)$ weak for all holes T',

(9.9) $\quad \int_{\Omega_\epsilon} \phi(x/\epsilon) f(x/\epsilon)\, w_\epsilon\, dx \to \int_\Omega m_Y(\phi f)\, w_0\, dx,$

where f is given in $L^2(Y)$.

Proof. Let T' be a hole satisfying (H5). We write

(9.10) $\quad \int_{\Omega^\epsilon} \phi^\epsilon f^\epsilon w_\epsilon\, dx - \int_\Omega m_Y(\phi f)\, w_0\, dx$

$$= [\int_{\Omega_{T'_\epsilon}} \phi^\epsilon f^\epsilon w_\epsilon\, dx - \int_\Omega m_{Y-T'}(\phi f)\, w_0] + [\int_{\Omega_{T_\epsilon} - \Omega_{T'_\epsilon}} \phi^\epsilon f^\epsilon w_\epsilon\, dx$$

$$- \int_\Omega m_{T'-T}(\phi f)\, w_0\, dx],$$

where we have set

(9.11) $\quad m_U(g) = \dfrac{1}{|Y|}\int_U g\, dy$ for subsets U of Y.

We have the following estimate :

$$\left| \int_{\Omega_{T_\epsilon} - \Omega_{T'_\epsilon}} \phi^\epsilon f^\epsilon w_\epsilon\, dx \right| \le \|\phi\|_{L^\infty(T'-T)} \|f^\epsilon\|_{L^2(\Omega\epsilon)} \|w_\epsilon\|_{L^2(\Omega\epsilon)}.$$

Using now Proposition (5.1) (2), we deduce

(9.12) $\quad \left| \int_{\Omega_{T_\epsilon} - \Omega_{T'_\epsilon}} \phi^\epsilon f^\epsilon w_\epsilon\, dx \right| \le c\|\phi\|_{L^\infty(T'-T)},$

where c is a constant independent of T' and ϵ.
We also have

(9.13) $\quad \left| \int_\Omega m_{T'-T}(\phi f)\, w_0\, dx \right| \le c\|\phi\|_{L^\infty(T'-T)}.$

Since $\phi = 0$ on S, we can choose T' satisfying (H5) and $\| \phi \|_{L^\infty(T'-T)}$ sufficiently small. For this hole T', we have the convergence

$$(9.14) \quad \int_{\Omega_{T'_\epsilon}} \phi^\epsilon f^\epsilon P_\epsilon^{T'} w_\epsilon \, dx \to \int_\Omega m_{Y-T'}(\phi f) \, w_0 \, dx.$$

Combining all these results, we obtain (9.9).

Using similar arguments, we can prove the following :

Lemma (9.3). Let $\{w_\epsilon\}$ and $\{v_\epsilon\}$ be two sequences such that

$$(9.15) \quad \left.\begin{array}{l} \| w_\epsilon \|_\epsilon \leq c \\ \| v_\epsilon \|_\epsilon \leq c \end{array}\right\} \text{ independent of } \epsilon.$$

Then there exist a subsequence of ϵ (again denoted by ϵ) and functions w_0, v_0 in $H_0^1(\Omega)$ such that we have the following convergence for this subsequence of ϵ :

$$(9.16) \quad \begin{cases} P_\epsilon^{T'} w_\epsilon \to w_0 \text{ in } H_0^1(\Omega) \text{ weak}, \\ P_\epsilon^{T'} v_\epsilon \to v_0 \text{ in } H_0^1(\Omega) \text{ weak}, \end{cases}$$

for all holes T' satisfying (H5) and

$$(9.17) \quad \int_{\Omega_\epsilon} \phi^\epsilon f^\epsilon w_\epsilon v_\epsilon \, dx \to \int_\Omega m_Y(\phi f) \, w_0 v_0 \, dx,$$

for all f in $L^\infty(Y)$.

Now we have all the tools to prove the following homogenization theorem:

Theorem (9.1). We suppose that (H5) is satisfied. Let $(v_\epsilon, \mu_\epsilon)$ be a solution of the problem (5.12). We assume

$$(9.18) \quad \mu_\epsilon \to \mu \text{ as } \epsilon \to 0.$$

Then μ is an eigenvalue of the "homogenized problem" (7.15). Furthermore there is an eigenvector v_0 in $H_0^1(\Omega)$ corresponding to μ and a subsequence of ϵ (again denoted by ϵ) such that

$$(9.19) \quad P_\epsilon^{T'} v_\epsilon \to v_0 \text{ in } H_0^1(\Omega) \text{ weak for all holes } T'$$

satisfying (H5), $P_\epsilon^{T'}$ being the prolongation operator verifying (9.1).

Proof.

Step 1. Taking $v = v_\epsilon$ in (5.12), we obtain

$$(9.20) \quad \| v_\epsilon \|_\epsilon \leq c, \text{ independent of } \epsilon.$$

Set

$$(9.21) \quad \xi_i^\epsilon = \phi^\epsilon \frac{\partial v_\epsilon}{\partial x_i} \text{ in } \Omega_\epsilon, \; i = 1, 2, \ldots N.$$

It follows that

$$(9.22) \quad \| \xi_i^\epsilon \|_{L^2(\Omega_\epsilon)} \leq c, \; i = 1, 2, \ldots N,$$

and we have

$$(9.23) \quad -\operatorname{div}(\phi^\epsilon \xi^\epsilon) = \mu_\epsilon \phi^{\epsilon^2} v_\epsilon \text{ in } \Omega_\epsilon.$$

So, $\phi^\epsilon \xi^\epsilon \,.\, \nu_\epsilon$ admits a trace on $\partial\Omega_\epsilon$ where ν_ϵ is the exterior normal to $\partial\Omega_\epsilon$. It is not difficult to see that

(9.24) $\quad \phi^\epsilon \xi^\epsilon \,.\, \nu_\epsilon = 0$ on S_ϵ.

So, equation (9.23) is valid in Ω if we extend ϕ^ϵ by zero in the holes.

(9.25) $\quad -\operatorname{div}(\phi^\epsilon \xi^\epsilon) = \mu_\epsilon \phi^{\epsilon^2} v_\epsilon$ in Ω.

In (9.25), we take some arbitrary extensions for ξ^ϵ and v_ϵ. Using Lemma (9.1), we can extract a subsequence of ϵ (again denoted by ϵ) such that

(9.26) $\quad \phi^\epsilon \xi_i^\epsilon \to \xi_i$ in $L^2(\Omega)$ weak, $i = 1, 2, \ldots N$.

(9.27) $\quad P_\epsilon^{T'} v_\epsilon \to v_0$ in $H_0^1(\Omega)$ weak, for all holes

T' satisfying (H5).

Step 2. Using the technique of Lemma (9.2), we can pass to the limit in (9.25) to get

(9.28) $\quad -\operatorname{div}\xi = \mu\, m_Y(\phi^2)\, v_0$ in Ω.

Step 3. We introduce w as solution to the following variational problem :

(9.29) $\quad \begin{cases} a(\phi\,;\, w, v) = 0 \text{ for } v \in V, \\ w - \Pi \in V, \end{cases}$

where $\Pi(y)$ is a homogeneous polynomial of degree 1. Set

(9.30) $\quad \eta = \phi \nabla w$ in Y^*.

Then η satisfies

(9.31) $\quad -\operatorname{div}(\phi\eta) = 0$ in Y^*.

Also we have $\phi\eta \,.\, \nu = 0$ on S where ν is the outer normal to S. So, we obtain

(9.32) $\quad -\operatorname{div}(\phi^\epsilon \eta^\epsilon) = 0$ in Ω.

We set

$$\psi(y) = w(y) - \Pi(y) \text{ for } y \in Y^*.$$

Then $\psi \in V$ and we take some extension of ψ in the hole and we extend ψ periodically throughout $\mathbf{R}^N$. We define

(9.33) $\quad w^\epsilon(x) = \epsilon\, w(x/\epsilon)$

$$= \Pi(x) + \epsilon\psi(x/\epsilon) \text{ in } \Omega.$$

Since $\psi \in L^2(Y)$, $\psi(x/\epsilon)$ remains bounded in $L^2(\Omega)$ and so

(9.34) $\quad w^\epsilon \to \Pi$ in $L^2(\Omega)$ strong.

Step 4. Let $g \in D(\Omega)$.

We take $v = gw^\epsilon$ in (5.12) and multiply (9.32) by gv_ϵ and subtract, we get

(9.35) $\quad \int\limits_{\Omega_\epsilon} \phi^\epsilon \xi_i^\epsilon w^\epsilon \dfrac{\partial g}{\partial x_i} - \int\limits_{\Omega_\epsilon} \phi^\epsilon \eta_i^\epsilon v_\epsilon \dfrac{\partial g}{\partial x_i} = \mu_\epsilon \int\limits_{\Omega_\epsilon} \phi^{\epsilon^2} v_\epsilon\, g\, w^\epsilon\, dx.$

Using the technique of Lemma (9.2), we pass to the limit in (9.35) to get

$$(9.36) \quad \int_{\Omega} \xi_i \, \Pi \, \frac{\partial g}{\partial x_i} - \int_{\Omega} m_Y (\phi \, \eta_i) \, v_0 \frac{\partial g}{\partial x_i} = \mu \, m_Y (\phi^2) \int_{\Omega} v_0 \, g \, \Pi \, dx.$$

It follows from (9.36) and (9.28) that

$$(9.37) \quad \xi_i \frac{\partial \Pi}{\partial x_i} = m_Y \left(\phi^2 \frac{\partial w}{\partial y_i} \right) \frac{\partial v_0}{\partial x_i} \text{ in } \Omega.$$

Step 5. Now we take $\Pi = y_j$ $(j = 1, 2, \ldots N)$. The corresponding test function $w = \psi^{(j)} + y_j$ where $\psi^{(j)}$ is given by (8.9). So, we get

$$(9.38) \quad \xi_j = m_Y (\phi^2) \, q_{ij} \frac{\partial v_0}{\partial x_i} \text{ for } j = 1, 2, \ldots N.$$

This when combined with (9.28) shows that

$$(9.39) \quad A v_0 = \mu \, v_0 \text{ in } \Omega.$$

Step 6. To complete the proof, we have to show that

$$(9.40) \quad v_0 \neq 0.$$

We have

$$(9.41) \quad \int_{\Omega_\epsilon} \phi^{\epsilon^2} v_\epsilon^2 = 1,$$

and at the limit we obtain

$$m_Y (\phi^2) \int_{\Omega} v_0^2 = 1,$$

and so (9.40) follows.

Theorem (9.2). We assume (H5). Let λ be the first eigenvalue of the problem (4.3) with eigenvector ϕ. Let $\{v_\epsilon^l, \mu_\epsilon^l\}_{l=1}^{\infty}$ be the spectrum of the problem (5.12) Let $\{\lambda_\epsilon^l\}_{l=1}^{\infty}$ be the sequence of eigenvalue of the problem (3.1). Then

(i) $\lambda_\epsilon^l - \epsilon^{-2} \lambda = \mu_\epsilon^l$ for $l \geq 1$.

(ii) $\mu_\epsilon^l \to l$th eigenvalue μ^l of the problem (7.15).

(iii) There exist a subsequence of ϵ (again denoted by ϵ) and eigenvectors $\{v^l\}$ of the problem (7.15) corresponding to $\{\mu^l\}$ such that

(9.42) $P_\epsilon^{T'} v_\epsilon \to v^l$ in $H_0^1(\Omega)$ weak for all holes T' satisfying (H5).

(9.43) $\{v^l\}$ form an orthogonal base in $L^2(\Omega)$.

(iv) If μ^l is a simple eigenvalue of (7.15), then given any eigenvector v_0^l corresponding to μ^l satisfying (9.42), we can choose an eigenvector $v_{\epsilon,0}^l$ of the problem (5.12) corresponding to μ_ϵ^l such that

(9.44) $P_\epsilon^{T'} v_{\epsilon,0}^l \to v_0^l$ in $H_0^1(\Omega)$ weak for the *whole sequence* of ϵ.

Proof. By Proposition (6.1), we can extract a subsequence of ϵ such that

(9.45) $\mu^l_\epsilon \to \mu^l$ as $\epsilon \to 0$.

According to Theorem (9.1), μ^l is an eigenvalue of the problem (7.15) with eigenvector v^l satisfying (9.42). To prove μ^l is the lth eigenvalue, it suffices to verify that

(1) $\{v^l\}$ is an orthogonal base in $L^2(\Omega)$.

(2) There is no other eigenvalue except $\{\mu^l\}$ for the problem (7.15) : $(u, \mu) \in H^1_0(\Omega) \times \mathbf{R}$ such that $Au = \mu u$.

We remark, first, that the eigenvectors $\{v^l\}$ are orthogonal in $L^2(\Omega)$. In fact, the passage to the limit in the relation

(9.46) $\int_{\Omega_\epsilon} \phi^{\epsilon^2} v^l_\epsilon v^m_\epsilon = \delta_{lm}$,

will give

(9.47) $m_Y(\phi^2) \int_\Omega v^l v^m = \delta_{lm}$.

We shall prove (2). Let μ be an eigenvalue of the problem (7.15) which is different from μ^l with eigenvector w satisfying

$$(9.48)\quad \begin{cases} Aw = \mu w \text{ in } \Omega, \\ \mu \neq \mu^l \text{ for any } l, \\ \int_\Omega w^2 = 1/(m_Y(\phi^2)), \\ \int_\Omega w v^l = 0 \text{ for each } l \geq 1. \end{cases}$$

We can choose l such that

(9.49) $\mu < \mu^{l+1}$.

Now, we define w_ϵ as follows :

$$(9.50)\quad \begin{cases} w_\epsilon \in V_\epsilon \\ a(\phi^\epsilon ; w_\epsilon, v) = \mu(\phi^\epsilon; w, v)\, \forall\, v \in V_\epsilon. \end{cases}$$

The proof of Theorem (9.1) shows that

(9.51) $P^{T'}_\epsilon w_\epsilon \to w$ in $H^1_0(\Omega)$ weak.

Set

$$(9.52)\quad \hat{w}_\epsilon = w_\epsilon - \sum_{i=1}^{l} (\phi^\epsilon; w_\epsilon, v^i_\epsilon)\, v^i_\epsilon.$$

We see easily that

(9.53) $(\phi^\epsilon ; \hat{w}_\epsilon, v^i_\epsilon) = 0$ for $i = 1, 2, \ldots l$,

and as a consequence

(9.54) $a(\phi^\epsilon ; \hat{w}_\epsilon, \hat{w}_\epsilon) \geq \mu^{l+1}_\epsilon (\phi^\epsilon ; \hat{w}_\epsilon, \hat{w}_\epsilon)$.

We can prove

$$(9.55)\quad \begin{cases} a(\phi^\epsilon; \hat{w}_\epsilon, \hat{w}_\epsilon) \to \mu, \\ (\phi^\epsilon ; \hat{w}_\epsilon, \hat{w}_\epsilon) \to m_Y(\phi^2) \int_\Omega w^2 = 1. \end{cases}$$

Hence (9.54) implies that

$$(9.56)\quad \mu \geq \mu^{l+1},$$

which is in contradiction with (9.49). This completes the proof of (2). In the same way, one can prove $v \in L^2(\Omega)$, $\int_\Omega v\, v^l = 0$ for all l implies $v = 0$. This proves that v^l is the lth eigenvalue of the problem (7.15).

To prove (iv). We note, first, that μ_ϵ^l is simple for ϵ sufficiently small. In fact, since we have $\mu_\epsilon^l \to \mu^l$ for all l, the multiplicity of μ_ϵ^l is less than or equal to that of μ^l. So, there are only two vectors $v_{\epsilon,0}^l$ and $-v_{\epsilon,0}^l$ which satisfy (5.12). We choose one which satisfies

$$(9.57)\quad (\phi^\epsilon ; v_{\epsilon,0}^l, v_0^l) > 0,$$

and this sequence must satisfy (9.44).

Part B: Stekloff eigenvalue problem

10. Problem to be treated

With the notations introduced in § 2, we consider the following eigenvalue problem:

$$(10.1)\quad \begin{cases} \text{Find } (u_\epsilon, \lambda_\epsilon) \in W_\epsilon \times \mathbf{R} \text{ such that} \\ -\triangle u_\epsilon = 0 \text{ in } \Omega_\epsilon, \\ \dfrac{\partial u_\epsilon}{\partial \nu_\epsilon} = \lambda_\epsilon u_\epsilon \text{ on } S_\epsilon, \\ u_\epsilon \neq 0, \end{cases}$$

where $\partial/\partial\nu_\epsilon$ is the exterior normal derivative to S_ϵ and

$$(10.2)\quad W_\epsilon = \{v \in H^1(\Omega_\epsilon);\ v = 0 \text{ on } \partial\Omega\}.$$

Problems of the type (10.1) have been studied by Bergmann and Schiffer [4] and Bramble and Osborn [6]. This problem can be put in the variational form as follows : Consider the following problem for $g \in L^2(S_\epsilon)$:

$$\begin{cases} a^\epsilon(w_\epsilon, v) = \int_{S_\epsilon} g\, v \text{ for } v \in W_\epsilon, \\ w_\epsilon \in W_\epsilon. \end{cases}$$

Set $G_\epsilon g = w_{\epsilon|S_\epsilon}$. Then G_ϵ is self-adjoint, compact operator in $L^2(S_\epsilon)$. We can write the eigenvalue problem

$$G_\epsilon u_\epsilon = \mu_\epsilon u_\epsilon,\ \mu_\epsilon = \frac{1}{\lambda_\epsilon}$$

as follows :

$$(10.2)\quad \begin{cases} a^\epsilon(u_\epsilon, v) = \lambda_\epsilon \int_{S_\epsilon} u_\epsilon v \text{ for } v \text{ in } W_\epsilon, \\ u_\epsilon \in W_\epsilon, \\ u_\epsilon \neq 0. \end{cases}$$

We denote by $\{\lambda_\epsilon^l\}_{l=1}^\infty$ the sequence of eigenvalues and by $\{u_\epsilon^l\}$ corresponding eigenvectors of this problem. We remark that $\{u_\epsilon^l\}$ form an orthogonal basis in $L^2(S_\epsilon)$. Our aim, in this part, is to study the asymptotic behaviour of λ_ϵ^l and $\{u_\epsilon^l\}$ as $\epsilon \to 0$. We prove that $\{\epsilon^{-1}\lambda_\epsilon^l\} \to l$th eigenvalue of the "homogenized problem". A very useful tool in this study will be a test function which will be defined in § 11.

11. Test function

We define ψ as the solution of the following variational problem :

$$(11.1)\quad \begin{cases} a(\psi, v) = -c_0 \int_{Y^*} v + \int_S v\, dy \text{ for } v \in W, \\ \psi \in W, \end{cases}$$

where

$$(11.2)\quad W = \{v \in H^1(Y^*);\ v \text{ is } Y \text{ periodic}\}, \text{ and}$$

$$(11.3)\quad c_0 = \frac{|S|}{|Y^*|}.$$

We define

$$(11.4)\quad \psi^\epsilon(x) = \psi(x/\epsilon) \text{ for } x \in \Omega_\epsilon,$$

and this will satisfy

$$(11.5)\quad \begin{cases} -\Delta\psi^\epsilon = c_0\, \epsilon^{-2} \text{ in } \Omega_\epsilon, \\ \dfrac{\partial\psi^\epsilon}{\partial\nu_\epsilon} = \epsilon^{-1} \text{ on } S_\epsilon, \end{cases}$$

and so, by Green's formula we obtain

$$(11.6)\quad \int_{S_\epsilon} g\, dy = \int_{\Omega_\epsilon} \frac{\partial\psi}{\partial y_i}(x/\epsilon)\frac{\partial g}{\partial x_i}(x) + c_0\, \epsilon^{-1}\int_{\Omega_\epsilon} g \text{ for } g \in W_\epsilon.$$

12. Estimates on the eigenvalues

Proposition (12.1). Let $\{\lambda_\epsilon^l\}$ be the sequence of eigenvalues of the problem (10.1). Then there exists a constant $c_l > 0$ independent of ϵ, such that

$$(12.1)\quad 0 < \lambda_\epsilon^l \leq c_l\, \epsilon \text{ for } \epsilon > 0.$$

Proof. We make use of the mini-max principle for the eigenvalues :

$$(12.2)\quad \lambda_\epsilon^l = \operatorname{Min}\left\{ \operatorname*{Max}_{v\in S_l} \frac{\int_{\Omega_\epsilon} \nabla v \cdot \nabla v}{\int_{S_\epsilon} v^2} \;;\quad \underset{\dim S_l = l}{S_l \subset W_\epsilon} \right\}.$$

As in Proposition (6.1), we take S_l to be the space spanned by the first l-eigenvectors of the problem $-\Delta w = \nu w$ in Ω, $w|_{\partial\Omega} = 0$. We get

$$(12.3)\quad \lambda_\epsilon^l \leq \nu^l \operatorname*{Max}_{v\in S_l} \frac{\int_\Omega v^2}{\int_{S_\epsilon} v^2},$$

where ν^l is the lth eigenvalue of the problem (6.5).

We prove now that there exists c_l such that

$$(12.4)\quad \operatorname*{Max}_{v\in S_l} \frac{\int_\Omega v^2}{\int_{S_\epsilon} v^2} \leq c_l \epsilon.$$

Suppose (12.4) is not true. Then we can extract a subsequence of ϵ (again denoted by ϵ) and $v_\epsilon \in S_l$ such that

$$(12.5)\quad \int_\Omega v_\epsilon^2 = 1,$$

$$(12.6)\quad \epsilon \int_{S_\epsilon} v_\epsilon^2 \to 0.$$

Since S_l is of finite dimension, the sequence $\{v_\epsilon\}$ remains bounded in $H_0^1(\Omega)$ also and it follows from (11.6) that

$$(12.7)\quad \int_{\Omega_\epsilon} v_\epsilon^2 \to 0.$$

If $v_\epsilon \to v_0$ in $L^2(\Omega)$ strong, then on the one hand (12.5) implies that

$$\int_\Omega v_0^2 = 1,$$

and on the other (12.7) implies

$$\int_\Omega v_0^2 = 0.$$

This contradiction proves (12.4) and hence the Proposition.

13. Asymptotic expansion

Taking into account the estimates we obtained in § 12, we normalize the eigenvector of the problem (10.1) as follows :

$$(13.1)\quad \int_{S_\epsilon} u_\epsilon^2 = \epsilon^{-1}.$$

Then we will have the following estimate :

$$(13.2)\quad \int_{\Omega_\epsilon} \nabla u_\epsilon \nabla u_\epsilon \leq c, \text{ independent of } \epsilon.$$

Considering all these estimates, we propose the following Ansatz to the problem (10.1).

(13.3) $u_\epsilon(x) = u_0(x, y) + \epsilon\, u_1(x, y) + \ldots\; y = x/\epsilon,$

(13.4) $\lambda_\epsilon = \epsilon\,\lambda_1 + \epsilon^2\,\lambda_2 + \ldots,$

Each u_j is defined on $\Omega \times Y^*$ and it is Y periodic in y. We put the expressions (13.3) (13.4) in equation (10.1). We get

(13.5) $-(\epsilon^{-2}\,\Delta_y + 2\epsilon^{-1}\,\Delta_{xy} + \Delta_x)(u_0 + \epsilon u_1 + \ldots) = 0,$

(13.6) $$\left(\epsilon^{-1}\frac{\partial}{\partial v_y} + v_i(y)\frac{\partial}{\partial x_j}\right)(u_0 + \epsilon u_1 + \ldots)$$
$$= (\epsilon\,\lambda_1 + \epsilon^2\,\lambda_2 + \ldots)(u_0 + u\epsilon_1 + \ldots)$$

where $v = (v_j)$ is the outer normal to S. For the notations used in this section see § 7. Now, we equate like powers of ϵ in the above relations and we solve the resulting equations. We obtain the following results :

(13.7) u_0 is independent of $y : u_0\,(x,\, y) = u_0\,(x)$.

(13.8) $$u_1(x, y) = -\chi^j(y)\frac{\partial u_0}{\partial x_j}(x) + \tilde{u}_1(x),$$

where χ^j are as defined below and $\tilde{u}_1$ is to be determined :

(13.9) $$\begin{cases} a(\chi^j, v) = \int_S v_j\, v\, dy \text{ for } v \in W, \\ \chi^j \in W. \end{cases}$$

We get the following equation satisfied by u_0 :

(13.10) $$-r_{ij}\frac{\partial^2 u_0}{\partial x_i \partial x_j} = \lambda_1 u_0 \text{ in } \Omega,$$

with

(13.11) $$r_{ij} = \frac{1}{|S|}\, a(\chi^i - y_i,\, \chi^j - y_j) \text{ for } i, j = 1, 2 \ldots, N.$$

It follows from (13.11) that (r_{ij}) is a symmetric, positive definite matrix and so the following eigenvalue problem (called homogenized problem in this case) is well posed :

(13.12) $$\begin{cases} \text{Find } (u, \lambda) \in H_0^1(\Omega) \times \mathbf{R} \text{ such that} \\ -r_{ij}\dfrac{\partial^2 u}{\partial x_i \partial x_j} = \lambda u \text{ in } \Omega, \\ u \neq 0. \end{cases}$$

The formal analysis so far shows that $\epsilon^{-1}\,\lambda_\epsilon \to$ to an eigenvalue of the problem (13.12). We prove this result in the later sections.

14. Results of homogenization

We denote by P_ϵ the prolongation operator constructed from the hole T_ϵ and P_ϵ satisfies (9.1). The following Theorem is analogous to Theorem (9.1).

Theorem (14.1). Let $l \geq 1$ be an integer. Let λ_ϵ^l be the lth eigenvalue of the problem (10.1) with u_ϵ^l be the corresponding eigenvector. Then $\{\epsilon^{-1}\lambda_\epsilon^l\}$ converges to an eigenvalue λ of the problem (13.12) and there exists a subsequence of ϵ (denoted by ϵ) such that

(14.1) $\quad P_\epsilon u_\epsilon^l \to u$ in $H_0^1(\Omega)$ weak,

where u is an eigenvector of the problem (13.12) corresponding to λ.

Proof.

Step 1. We set $\lambda_\epsilon = \lambda_\epsilon^l$, $u_\epsilon = u_\epsilon^l$ and

(14.2) $\quad \xi_i^\epsilon = \dfrac{\partial u_\epsilon}{\partial x_i}$ in Ω_ϵ for $i = 1, 2, \dots N$.

We put $\xi^\epsilon = 0$ in the holes. Because of the estimates (12.1) and (13.2), we can extract a subsequence of ϵ (denoted by ϵ again) such that

(14.3) $\quad \xi_i^\epsilon \to \xi_i$ in $L^2(\Omega)$ weak for $i = 1, 2, \dots N$.

(14.4) $\quad P_\epsilon u_\epsilon \to u$ in $H_0^1(\Omega)$ weak,

(14.5) $\quad \epsilon^{-1}\lambda_\epsilon \to \lambda$.

Step 2. It follows from (10.1) that

(14.6) $\quad \xi^\epsilon \,.\, \nu_\epsilon = \lambda_\epsilon u_\epsilon$ on S_ϵ,

(14.7) $\quad \operatorname{div} \xi^\epsilon = 0$ in Ω_ϵ.

We multiply equation (10.1) with $\eta \in D(\Omega)$:

$$\int_{\Omega_\epsilon} \nabla u_\epsilon \nabla \eta = \lambda_\epsilon \int_{S_\epsilon} u_\epsilon \eta$$

$$= \lambda_\epsilon \int_{\Omega_\epsilon} \frac{\partial \psi}{\partial y_i}(x/\epsilon)\frac{\partial}{\partial x_i}(u_\epsilon \eta) + c_0\, \epsilon^{-1}\lambda_\epsilon \int_{\Omega_\epsilon} u_\epsilon \eta,$$

and the passage to the limit will give us

$$\int_\Omega \xi \nabla \eta = c_0 \lambda \int_\Omega \theta\, u\, \eta,$$

which is equivalent to

(14.8) $\quad -\operatorname{div} \xi = c_0\, \lambda \theta\, u$ in Ω.

Step 3. We introduce w as solution to the following problem :

$$(14.9) \quad \begin{cases} a(w, v) = 0 \text{ for } v \in W \\ w - \Pi \in W, \end{cases}$$

where $\Pi(y)$ is a homogeneous polynomial of degree 1. We put

(14.10) $\eta = \nabla w$ in Y^*.

Then η satisfies

$$(14.11)\quad \begin{cases} \operatorname{div} \eta = 0 \text{ in } Y^*, \\ \eta \,.\, \nu = 0 \text{ on } S. \end{cases}$$

We extend η by zero in the hole and we put

(14.12) $\eta^\epsilon(x) = \eta(x/\epsilon)$ in Ω.

Then we will have

(14.13) $\operatorname{div} \eta^\epsilon = 0$ in Ω.

Next we define

(14.14) $w^\epsilon(x) = \epsilon\, w(x/\epsilon)$.

As before (see Step 3, Theorem (9.1)), we get

(14.15) $w^\epsilon \to \Pi$ in $L^2(\Omega)$ strong.

Step 4. Let $g \in D(\Omega)$. Multiply (14.7) by $g\,w^\epsilon$ and (14.13) by $g\,P_\epsilon u_\epsilon$ and subtract :

$$(14.16)\quad -\int_{\Omega_\epsilon} \xi_i^\epsilon \frac{\partial g}{\partial x_i} w^\epsilon + \int_{\Omega_\epsilon} \eta_i^\epsilon \frac{\partial g}{\partial x_i} P_\epsilon u_\epsilon + \int_{S_\epsilon} \lambda_\epsilon u_\epsilon g\, w^\epsilon = 0.$$

We can pass to the limit directly in the first two terms. For the third we use the formula (11.6). We get

$$(14.17)\quad -\int_\Omega \xi_i \frac{\partial g}{\partial x_i} \Pi + \int_\Omega m_Y\left(\chi \frac{\partial w}{\partial y_i}\right) \frac{\partial g}{\partial x_i} u + c_0 \lambda \theta \int_\Omega u\, g\, \Pi = 0$$

which gives

$$(14.18)\quad \xi_i \frac{\partial \Pi}{\partial x_i} = m_Y\left(\chi \frac{\partial w}{\partial y_i}\right) \frac{\partial u}{\partial x_i}\,.$$

Step 5. Now we take $\Pi = y_j$ $(j = 1, 2 \ldots N)$, The corresponding test function $w = -\chi^{(j)} + y_j$ where $\chi^{(j)}$ is defined by (13.9). So we get

$$(14.19)\quad \xi_j = \frac{1}{|Y|} \int_{Y^*} \left(\delta_{ij} - \frac{\partial \chi^{(j)}}{\partial y_i}\right) \frac{\partial u}{\partial x_i},$$

$$= \frac{|S|}{|Y|} r_{ij} \frac{\partial u}{\partial x_i}\,.$$

Combining this with (14.8) we obtain

$$(14.20)\quad - r_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = \lambda\, u.$$

Step 6. To complete the proof, we have to show that $u \neq 0$. To see this, we take $g = u_\epsilon^2$ in (11.6) :

$$\epsilon \int_{S_\epsilon} u_\epsilon^2 = c_0 \int_{\Omega_\epsilon} u_\epsilon^2 + \epsilon \int_{\Omega_\epsilon} \frac{\partial \psi}{\partial y_i}(x/\epsilon) \frac{\partial}{\partial x_i}(u_\epsilon^2).$$

Using (13.1), we pass to the limit in this relation and we obtain

(14.21) $1 = c_0 \theta \int_\Omega u^2,$

from which it follows that $u \neq 0$.

We now give our final result in the Stekloff case.

Theorem (14.2)

Let $\{\lambda_\epsilon^l\}$, $\{u_\epsilon^l\}$ be the sequences of eigenvalues and eigenvectors of the problem (10.1). We suppose that the eigenvectors are normalized according to (13.1): $\int_{S_\epsilon} u_\epsilon^2 = \epsilon^{-1}$. Then

(i) $\lambda_\epsilon^l \to \lambda^l$, the lth eigenvalue of the problem (13.12).

(ii) there exists a subsequence of ϵ (again denoted by ϵ) such that

(14.22) $P_\epsilon u_\epsilon^l \to u^l$ in $H_0^1(\Omega)$ weak,

where u^l satisfies (13.12) :

$$-r_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = \lambda^l u \text{ in } \Omega, \; u\,|_{\partial\Omega} = 0.$$

(iii) If λ^l is a simple eigenvalue of the problem (13.12) then given any eigenvector v^l associated to λ^l satisfying (14.2), we can choose an eigenvector v_ϵ^l of the problem (10.1) corresponding to λ_ϵ^l such that

(14.23) $\int_{S_\epsilon} v_\epsilon^{l^2} = \epsilon^{-1}$

(14.24) $P_\epsilon v_\epsilon^l \to v^l$ in $H_0^1(\Omega)$ weak *for the whole sequence.*

Proof. From the previous Theorem, we know that (ii) is true for a subsequence of ϵ. We obtain also the orthogonality condition for the eigenvectors by means of the method of Step 6 of Theorem (14.1) :

(14.25) $$\int_\Omega u^l u^m = \frac{1}{c_0 \theta} \delta_{lm} \text{ for } l, m \geq 1.$$

Finally, it remains to prove that the limiting point λ^l of $\{\lambda_\epsilon^l\}$ is the lth eigenvalue of (13.12). For this, it suffices to verify that

(14.26) $$\begin{cases} \text{(i) there is no eigenvalue other than } \{\lambda^l\}_{l=1}^{\infty} \text{ for the problem (13.12).} \\ \text{(ii) } \{u^l\}_{l=1}^{\infty} \text{ is an orthogonal basis in } L^2(\Omega). \end{cases}$$

In the following, we verify (14.26) (i) The proof of (14.26) (ii) is analogous.

Let us suppose that there is an eigenvalue μ different from λ^l with eigenvector u:

$$(14.27)\quad \begin{cases} -r_{ij}\dfrac{\partial^2 u}{\partial x_i \partial x_j} = \mu u \text{ in } \Omega, \\ u \in H_0^1(\Omega), \\ \mu \neq \lambda^l \text{ for all } l, \\ \int_\Omega u u^l = 0 \text{ for each } l, \\ \int_\Omega u^2 = 1. \end{cases}$$

We can choose an integer l such that

$$(14.28)\quad \mu < \lambda^{l+1}.$$

We define w_ϵ as the solution of

$$(14.29)\quad \begin{cases} -\triangle w_\epsilon = 0 \text{ in } \Omega_\epsilon, \\ w_\epsilon \in W_\epsilon, \\ \dfrac{\partial w_\epsilon}{\partial \nu_\epsilon} = \epsilon \mu u \text{ on } S_\epsilon. \end{cases}$$

From the proof of Theorem (14.1), it is seen that

$$(14.30)\quad P_\epsilon w_\epsilon \to u \text{ in } H_0^1(\Omega) \text{ weak.}$$

Now consider the element

$$(14.31)\quad v_\epsilon = w_\epsilon - \epsilon \sum_{k=1}^{l} [\int_{S_\epsilon} w_\epsilon u_\epsilon^k]\, u_\epsilon^k .$$

Since

$$\int_{S_\epsilon} v_\epsilon u_\epsilon^k = 0 \text{ for } k = 1, 2, \ldots, l,$$

we have

$$(14.32)\quad a^\epsilon(v_\epsilon, v_\epsilon) \geq \lambda_\epsilon^{l+1} \int_{S_\epsilon} v_\epsilon^2 .$$

We now pass to the limit in (14.32) using (11.6). We have, first

$$\epsilon \int_{S_\epsilon} w_\epsilon u_\epsilon^k = \epsilon \int_{\Omega_\epsilon} \frac{\partial \psi}{\partial y_i}(x/\epsilon)\frac{\partial}{\partial x_i}(w_\epsilon u_\epsilon^k) + c_0 \int_{\Omega_\epsilon} w_\epsilon u_\epsilon^k ,$$

and so

$$\epsilon \int_{S_\epsilon} w_\epsilon u_\epsilon^k \to c_0 \int_\Omega \theta u u^k = 0.$$

As a consequence,

$$(14.33)\quad P_\epsilon v_\epsilon \to u \text{ in } H_0^1(\Omega) \text{ weak.}$$

Using this and taking $g = \lambda_\epsilon^{l+1} v_\epsilon^2$ in (11.6), we get

(14.34) $\lambda_\epsilon^{l+1} \int_{S_\epsilon} v_\epsilon^2 \to c_0 \lambda^{l+1} \theta.$

By similar arguments, we prove that

(14.35) $a^\epsilon (v_\epsilon, v_\epsilon) \to c_0 \theta \mu.$

Now passing to the limit in (14.32) we get

(14.36) $\mu \geq \lambda^{l+1}$

which contradicts (14.28).

To prove (iii), we remark that λ_ϵ^l is simple for sufficiently small ϵ if λ^l is so. Afterwards, it suffices to pick up an eigenvector v_ϵ^l associated to λ_ϵ^l such that

(14.37) $\int_{S_\epsilon} v_\epsilon^l v^l > 0.$

Part C: Neumann eigenvalue problem

15. Problem to be studied

We consider the following eigenvalue problem

$$
(15.1) \quad \begin{cases} \text{Find } (u_\epsilon, \lambda_\epsilon) \in W_\epsilon \times \mathbf{R} \text{ such that} \\ -\triangle u_\epsilon = \lambda_\epsilon u_\epsilon \text{ in } \Omega_\epsilon, \\ \dfrac{\partial u_\epsilon}{\partial \nu_\epsilon} = 0 \text{ on } S_\epsilon, \\ \displaystyle\int_{\Omega_\epsilon} u_\epsilon^2 = 1, \end{cases}
$$

where we have used the notations of § 2 and § 10. Let us denote by $\{\lambda_\epsilon^l\}$ and $\{u_\epsilon^l\}$ the sequences of eigenvalues and the corresponding eigenvectors of the problem (15.1). We know that

$$
(15.2) \quad \begin{cases} \lambda_\epsilon^1 < \lambda_\epsilon^2 \leq \lambda_\epsilon^3 \ldots \to \infty, \\ \{u_\epsilon^l\} \text{form an orthonormal basis in } L^2(\Omega_\epsilon). \end{cases}
$$

Here, we study the asymptotic behaviour of λ_ϵ^l and u_ϵ^l as $\epsilon \to 0$. We prove that $\{\lambda_\epsilon^l\}$ converges to the lth eigenvalue of the "homogenized problem".

Since the method folllowed here is similar to Stekloff case, we do not give details of the proofs.

16. Estimates on the eigenvalues

Proposition (16.1). Let λ_ϵ^l be the lth eigenvalue of the problem (15.1). Then there exists a constant c_l independent of ϵ such that

$$\lambda_\epsilon^l \leq c_l.$$

Proof. As in the proof of Proposition (12.1), by the use of mini-max principle we get

$$(16.1)\quad \lambda_\epsilon^l \le \nu^l \operatorname*{Max}_{v \in S_l} \frac{\int_\Omega v^2}{\int_{\Omega_\epsilon} v^2}.$$

But we saw in the course of the proof of Proposition (12.1) that the right side of (16.1) is bounded above independently of ϵ and this completes the proof.

17. Asymptotic analysis

With the help of the estimates in § 16 and the normalization condition for the eigenvectors, we propose the following Ansatz for the problem (15.1) :

$$(17.1)\quad u_\epsilon(x) = u_0(x, y) + \epsilon\, u_1(x, y) + \ldots,\ y = x/\epsilon,$$

$$(17.2)\quad \lambda_\epsilon = \lambda_0 + \epsilon\, \lambda_1 + \ldots$$

where u_j is defined on $\Omega \times Y^*$ and it is Y periodic in y.

We substitute these expressions in equation* (15.1) :

$$(17.3)\quad -(\epsilon^{-2}\triangle_y + 2\epsilon^{-1}\triangle_{xy} + \triangle_x)(u_0 + \epsilon u_1 + \ldots) = (\lambda_0 + \epsilon\lambda_1 + \ldots)(u_0 + \epsilon u_1 + \ldots),$$

$$(17.4)\quad \left(\epsilon^{-1}\frac{\partial}{\partial v_y} + v_j\frac{\partial}{\partial x_j}\right)(u_0 + \epsilon\, u_1 + \ldots) = 0.$$

As before, we equate the powers of ϵ on either side and we obtain the following results :

(17.5) u_0 is independent of y : $u_0(x, y) = u_0(x)$.

$$(17.6)\quad u_1(x, y) = -\chi^j(y)\frac{\partial u_0}{\partial x_j}(x) + \tilde{u}_1(x),$$

where χ^j are defined in (13.9), We get the following equation satisfied by u_0 and λ_0 :

$$(17.7)\quad -s_{ij}\frac{\partial^2 u_0}{\partial x_i\,\partial x_j} = \lambda_0\, u_0 \text{ in } \Omega,$$

with

$$(17.8)\quad s_{ij} = \frac{1}{|Y^*|}\, a(\chi^i - y_i,\ \chi^j - y_j) \text{ for all } i, j.$$

So, the corresponding "homogenized problem" in this case can be formulated as follows :

$$(17.9)\quad \begin{cases} \text{Find } (u, \lambda) \in H_0^1(\Omega) \times \mathbf{R} \text{ such that} \\ -s_{ij}\dfrac{\partial^2 u}{\partial x_i\,\partial x_j} = \lambda u \text{ in } \Omega \\ \qquad u \ne 0. \end{cases}$$

In the following section, we justify the formal calculations done above.

18. Homogenization theorem

The proof of the following result proceeds along the lines of Theorems (14.1) and (14.2) and therefore omitted.

Theorem (18.1). Let $\{\lambda_\epsilon^l\}$, $\{u_\epsilon^l\}$ be the sequences of eigenvalues and the corresponding eigenvectors of the problem (15.1) satisfying (15.2). Then

(i) $\lambda_\epsilon^l \to \lambda^l$, the lth eigenvalue of the problem (17.9).

(ii) there exists a subsequence of ϵ (denoted again by ϵ) such that

$P_\epsilon u_\epsilon^l \to u^l$ in $H_0^1(\Omega)$ weak,

where u^l is an eigenvector corresponding to λ^l.

(iii) If λ^l is simple, then given any eigenvector v^l corresponding to λ^l such that

(18.1) $\int_\Omega v^{l^2} = 1/\theta,$

we can choose an eigenvector v_ϵ^l of the problem (15.1) corresponding to λ_ϵ^l such that $P_\epsilon v_\epsilon^l \to v^l$ in $H_0^1(\Omega)$ weak *for the whole sequence* of ϵ.

19. Correctors and error estimates

In this paragraph, we briefly mention a method of finding correctors of first order for simple eigenvalue of the Neumann problem. We will also see that the problem of error estimates is reduced to that of stationary problem for which we refer to Lions [13]. For the study of correctors in the homogenization theory, we refer to Bensoussan *et al* [3], Bourgat and Dervieux [5], Kesavan [12].

We consider a simple eigenvalue λ_0 of the problem (17.9) with eigenvector u_0 satisfying

(19.1) $\int_\Omega u_0^2 = 1/\theta.$

Let u_ϵ, λ_ϵ be eigenvector and eigenvalue of the problem (15.1) satisfying

(19.2) $P_\epsilon u_\epsilon \to u_0$ in $H_0^1(\Omega)$ weak for the whole sequence of ϵ,

(19.3) $\lambda_\epsilon \to \lambda_0.$

Let us define w_ϵ as the solution of

$$(19.4)\quad \begin{cases} -\triangle w_\epsilon = \lambda_0 u_0 \text{ in } \Omega_\epsilon, \\ w_\epsilon \in W_\epsilon, \\ \dfrac{\partial w_\epsilon}{\partial \nu_\epsilon} = 0 \text{ on } S_\epsilon. \end{cases}$$

The proof of Theorem (18.1) shows that

(19.5) $P_\epsilon w_\epsilon \to u_0$ in $H_0^1(\Omega)$ weak.

Multiplying (19.4) by u_ϵ and (15.1) by w_ϵ, we obtain

(19.6) $\lambda_\epsilon (u_\epsilon, w_\epsilon)_{\Omega_\epsilon} = \lambda_0 (u_0, u_\epsilon)_{\Omega_\epsilon}$,

where $(.,.)_{\Omega_\epsilon}$ denotes the scalar product in $L^2(\Omega_\epsilon)$.

First let us prove the following:

Theorem (19.1)

With λ_0, u_0, λ_ϵ, u_ϵ, w_ϵ defined as above, we have

(19.7) $| \lambda_\epsilon - \lambda_0 | \leq c \| w_\epsilon - u_0 \|_{L^2(\Omega_\epsilon)}$,

for sufficiently small ϵ and where $c > 0$ is a constant independent of ϵ.

Proof. We have

$$\lambda_\epsilon - \lambda_0 = \frac{\lambda_0}{(u_\epsilon, w_\epsilon)_{\Omega_\epsilon}} (u_\epsilon, u_0 - w_\epsilon)_{\Omega_\epsilon}.$$

But $(u_\epsilon, w_\epsilon)_{\Omega_\epsilon} \to 1$ as $\epsilon \to 0$ and so we obtain (19.7).

Now, we give a result which estimates the error between the eigenvectors by choosing an eigenvector corresponding to λ_ϵ. For that, we first define z_ϵ uniquely as follows :

$$(19.8) \quad \begin{cases} a^\epsilon (z_\epsilon, v) - \lambda_\epsilon (z_\epsilon, v)_{\Omega_\epsilon} = \lambda_\epsilon (u_\epsilon, v)_{\Omega_\epsilon} - \lambda_0 (u_0, v)_{\Omega_\epsilon} \text{ for } v \in W_\epsilon, \\ z_\epsilon \in W_\epsilon, \\ (z_\epsilon, u_\epsilon)_{\Omega_\epsilon} = 0. \end{cases}$$

Then

(19.9) $\tilde{u}_\epsilon = z_\epsilon + w_\epsilon$,

is an eigenvector of the problem (15.1) corresponding to λ_ϵ. We have the following estimate for z_ϵ:

Lemma (19.1). There is a constant $c > 0$ independent of ϵ such that

(19.10) $| z_\epsilon |_{1, \Omega_\epsilon} \leq c \| \lambda_\epsilon w_\epsilon - \lambda_0 u_0 \|_{L^2(\Omega_\epsilon)}$.

Proof. Suppose (19.10) is not true. Then there will be a subsequence of ϵ (again denoted by ϵ) such that

(19.11) $| z_\epsilon |_{1, \Omega_\epsilon} = 1$,

(19.12) $\| \lambda_\epsilon w_\epsilon - \lambda_0 u_0 \|_{L^2(\Omega_\epsilon)} \to 0$.

We can pass to the limit in (19.8) as we did in equation (15.1). We obtain

(19.13) $P_\epsilon z_\epsilon \to z_0$ in $H_0^1(\Omega)$ weak,

where z_0 satisfies

$$(19.14) \quad \begin{cases} - s_{ij} \dfrac{\partial^2 z_0}{\partial x_i \partial x_j} = \lambda_0 z_0 \text{ in } \Omega, \\ \int_\Omega z_0 u_0 = 0. \end{cases}$$

Since λ_0 is simple, this implies that

(19.15) $z_0 = 0.$

On the other hand, taking $v = z_\epsilon$ in (19.18), we get

$$1 = \lambda_\epsilon (z_\epsilon, z_\epsilon)_{\Omega_\epsilon} + (\lambda_\epsilon w_\epsilon - \lambda_0 u_0, z_\epsilon)_{\Omega_\epsilon},$$

which, at the limit, gives a contradiction.

Theorem (19.2)

Let λ_0, λ_ϵ, w_ϵ, u_0, $\tilde{u}_\epsilon$ be defined as above. Then there is a constant $c > 0$ independent of ϵ such that

(19.16) $\| \tilde{u}_\epsilon - u_0 \|_{L^2(\Omega_\epsilon)} \leq c \| w_\epsilon - u_0 \|_{L^2(\Omega_\epsilon)}.$

Proof. We write

$$\tilde{}_\epsilon - u_0 = z_\epsilon + w_\epsilon - u_0.$$

The proof is completed by using (19.11) and Theorem (19.1) and the fact that

$$\| z_\epsilon \|_{L^2(\Omega_\epsilon)} \leq c \mid z_\epsilon \mid_{1, (\Omega_\epsilon)}.$$

Now we give a first order corrector for the eigenvalue λ_ϵ. We use (19.6). We express

$$\lambda_\epsilon = \lambda_0 + \epsilon \lambda_1 + \ldots$$

$$u_\epsilon = u_0 + \epsilon u_1 + \ldots$$

$$w_\epsilon = u_0 + \epsilon w_1 + \ldots$$

We put these expressions in (19.6) and identify powers of ϵ : we get

(19.17) $$\lambda_1 = - \lambda_0 \frac{(u_0, w_1)_{\Omega_\epsilon}}{(u_0, u_0)_{\Omega_\epsilon}},$$

where w_1 is the first order corrector for w_ϵ. The estimate we give below is better than that given by Theorem (19.1).

Theorem (19.3)

Let λ_ϵ, λ_0, w_ϵ, u_0 be as in the previous Theorem. Let λ_1 be [defined by (19.17). Then we have

$$| \lambda_\epsilon - \lambda_0 - \epsilon \lambda_1 | \leq c \{ \epsilon \| w_\epsilon - u_0 \|_{L^2(\Omega_\epsilon)} + \| w_\epsilon - u_0 \|^2_{L^2(\Omega_\epsilon)}$$
$$+ \| w_\epsilon - u_0 - \epsilon w_1 \|_{L^2(\Omega_\epsilon)} \},$$

where c is a constant independent of ϵ.

Proof. We have

$$\lambda_\epsilon - \lambda_0 - \epsilon\lambda_1 = \frac{\lambda_0}{(u_0, u_0)_{\Omega_\epsilon}} (u_0, u_0 + \epsilon w_1 - w_\epsilon)_{\Omega_\epsilon}$$

$$- (\lambda_\epsilon - \lambda_0) \frac{(u_0, \epsilon w_1)_{\Omega_\epsilon}}{(u_0, u_0)_{\Omega_\epsilon}} + \left[\frac{\lambda_\epsilon}{(u_0, u_0)_{\Omega_\epsilon}} (u_0, w_\epsilon)_{\Omega_\epsilon} - \lambda_0\right].$$

We only have to estimate the last term. But using (19.6), one can express the last term as follows:

$$\frac{\lambda_\epsilon}{(u_0, u_0)_{\Omega_\epsilon}} (u_0, w_\epsilon)_{\Omega_\epsilon} - \lambda_0 = \frac{\lambda_\epsilon}{(u_0, u_0)_{\Omega_\epsilon}} (u_0 - \tilde{u}_\epsilon, w_\epsilon - u_0)_{\Omega_\epsilon}$$

$$+ \frac{\lambda_\epsilon - \lambda_0}{(u_0, u_0)_{\Omega_\epsilon}} (u_0, u_0 - \tilde{u}_\epsilon) \leq c \, \| w_\epsilon - u_0 \|^2_{L^2(\Omega_\epsilon)},$$

by Theorems (19.1) and (19.2). This completes the proof.

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The author thanks the referee for having brought Rauch's article " The Mathematical Theory of Crushed Ice ", Partial Differential Equations and Related topics, Springer Verlag, *Lecture Notes in Mathematics*, No. 446, 1975, to his notice. In this article, Rauch considers what he calls the crushed ice problem; obviously this is related to our Dirichlet problem. Rauch shows, in dimension 3, that the first eigenvalue $\lambda_1 \to \infty$ when $n \to \infty$, $r \to 0$, $nr^3 =$ constant and $nr^2 \to \infty$ where n is the number of holes and r is the size of each hole. In our case, $n = 0\,(\epsilon^{-3})$, $r = 0\,(\epsilon)$ and so $nr^2 = 0\,(\epsilon^{-1}) \to \infty$ as $\epsilon \to 0$. Therefore, it is not surprising that $\lambda_1 \to \infty$ for the Dirichlet problem. However, our method even gives an asymptotic expansion valid not only for the first eigenvalue but also for all eigenvalues.

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