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MICHEL BELLIEUD
GUY BOUCHITTÉ

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Homogenization of Elliptic Problems in a Fiber Reinforced Structure. Non Local Effects.

MICHEL BELLIEUD - GUY BOUCHITTÉ

Abstract. Let Ω be a bounded open subset of $\mathbb{R}^3$ and (a_ε) a sequence of functions on Ω taking very high values on a set of ε-periodically distributed fibers of radius r_ε ($r_\varepsilon \ll \varepsilon$). We study asymptotically as $\varepsilon \to 0$ the elliptic equation

$$-\mathrm{div}(a_\varepsilon |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) = f + \text{boundary conditions}$$

and find a non local effective equation deduced from a homogenized system of several elliptic equations.

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1. – Introduction and statement of the main result

Let Ω be a bounded smooth open subset of $\mathbb{R}^N$ and $p \in (1, +\infty)$. We are concerned with the homogenization of quasilinear elliptic problems $(\mathcal{P}_\varepsilon)$ well posed in $W^{1,p}(\Omega)$ of the form:

$$(\mathcal{P}_\varepsilon): \quad \begin{cases} -\mathrm{div}(\sigma_\varepsilon) = f & \text{on } \Omega \\ \sigma_\varepsilon = a_\varepsilon |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon & \\ u_\varepsilon = u_0 & \text{on } \Gamma_0 \\ \sigma_\varepsilon . n = g & \text{on } \Gamma_1 \end{cases}$$

where:
– the conductivity coefficient a_ε is ε-periodic and satisfies a uniform lower bound (i.e. $a_\varepsilon \geq c_0$ a.e. for some suitable constant $c_0 > 0$),
– Γ_0 is an open subset of $\partial\Omega$ (such that $H^{N-1}(\partial\Gamma_0) > 0$) , $\Gamma_1 = \partial\Omega \setminus \Gamma_0$, n is the unit exterior normal to $\partial\Omega$,
– the boundary data u_0 is Lipschitz and f, g are assumed to verify (for simplicity) $f \in L^{p'}(\Omega)$, $g \in L^{p'}(\Gamma_1)$ (p' conjugate exponent of p).

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Equivalently, we like to study asymptotically as $\varepsilon \to 0$ the variational problem:

$$\inf\left\{F_\varepsilon(u) - \int_\Omega f u\, dx - \int_{\Gamma_1} g u\, dH^{N-1}\right\} \tag{1.1}$$

where

$$F_\varepsilon(u) = \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega a_\varepsilon |\nabla u|^p dx & \text{if } u \in W^{1,p}(\Omega) \text{ and } u = u_0 \text{ on } \Gamma_0 \ , \\ +\infty & \text{otherwise.} \end{cases} \tag{1.2}$$

The behaviour of $\mathcal{P}_\varepsilon$ as $\varepsilon \to 0$ is well understood when (a_ε) is bounded in $L^\infty(\Omega)$ (see for Example [9] and [16] in the case $p = 2$) and leads to a limit equation of the same type $-\text{div }a_{eff}(\nabla u) = f$ where $a_{eff} : \mathbb{R}^N \to \mathbb{R}^N$ represents the so-called effective law of the homogenized system. In [6], it is shown that the same conclusion holds if the sequence (a_ε) is simply assumed to be bounded and uniformly integrable in $L^1(\Omega)$.

In fact very few results identifying the homogenized equation are known if the latter condition is violated. This happens for example when the conductivity coefficient a_ε becomes very high on small subsets T_ε of vanishing measure, in such a way that the integral $\int_{T_\varepsilon} a_\varepsilon dx$ remains constant. Except in the stratified case which has been intensively investigated by several authors (see for Example [5], [12]), we can not expect the limit equation to be of the same type as $(\mathcal{P}_\varepsilon)$. Indeed a non local term may appear (see Mosco's results [14], [15] in the case $p = 2$, and also [13]). Find new mathematical tools in order to indentify this non local term in practical situations seems to us to be a big task. We present here some results in this direction which have been already announced in [3].

1.1. – Notations and setting of the problem

In this paper, we consider the case of a fibered stucture in $\mathbb{R}^3$: the body is a cylindrical domain $\Omega := \omega\times]0, L[$, where ω is a bounded connected open subset of $\mathbb{R}^2$ with smooth boundary. We denote the bases of this cylinder $\omega_0 := \omega \times \{0\}$ and $\omega_L := \omega \times \{L\}$.

For every $\varepsilon > 0$, we consider a partition of ω into a set of periodically distributed cells of size ε :

$$Y_\varepsilon^i = (\varepsilon\, i_1, \varepsilon\, i_2) +]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[^2 \quad , \quad i = (i_1, i_2) \in I_\varepsilon \subset Z^2 \ ,$$

where $I_\varepsilon := \{i \in Z^2 \,;\, Y_\varepsilon^i \subset \omega\}$.

Given a small parameter r_ε (later we will assume $r_\varepsilon << \varepsilon$), we define:
- $D_\varepsilon^i :=$ two dimensional disk centered at $(\varepsilon\, i_1, \varepsilon\, i_2)$ of radius r_ε,
- $T_\varepsilon^i := D_\varepsilon^i\times]0, L[\quad , \quad T_\varepsilon := \cup_{i\in I_\varepsilon} T_\varepsilon^i$.

The fibers are represented by the set of thin parallel cylinders T_ε (see fig.1) and are filled up with a medium we assume to be homogeneous with a very high conductivity λ_ε ($\lambda_\varepsilon \to +\infty$). Note that by the definition of I_ε, we have $D_\varepsilon^i \subset \omega$ so that the fibers do not intersect the lateral part of the boundary. The matrix $\Omega \setminus T_\varepsilon$ is assumed to have a constant conductivity coefficient K we normalize to $K = 1$. Hence the diffusion coefficient $a_\varepsilon(x)$ in the equation $(\mathcal{P}_\varepsilon)$ is given by:

$$a_\varepsilon(x) = 1 \quad \text{if } x \in \Omega \setminus T_\varepsilon \quad , \quad \lambda_\varepsilon \quad \text{if } x \in T_\varepsilon .$$

Clearly the asymptotic behavior of (a_ε) in $L^1(\Omega)$ is characterized by the parameter:

$$(1.3) \qquad k := \lim_\varepsilon k_\varepsilon \quad \text{where} \quad k_\varepsilon := \lambda_\varepsilon \frac{r_\varepsilon^2}{\varepsilon^2}$$

If $0 \leq k < +\infty$, (a_ε) is bounded in $L^1(\Omega)$ and converges weakly* in the sense of measures to the constant function $1 + k\pi$ (notice that this convergence does not hold in $L^1(\Omega)$ weak unless $k = 0$).

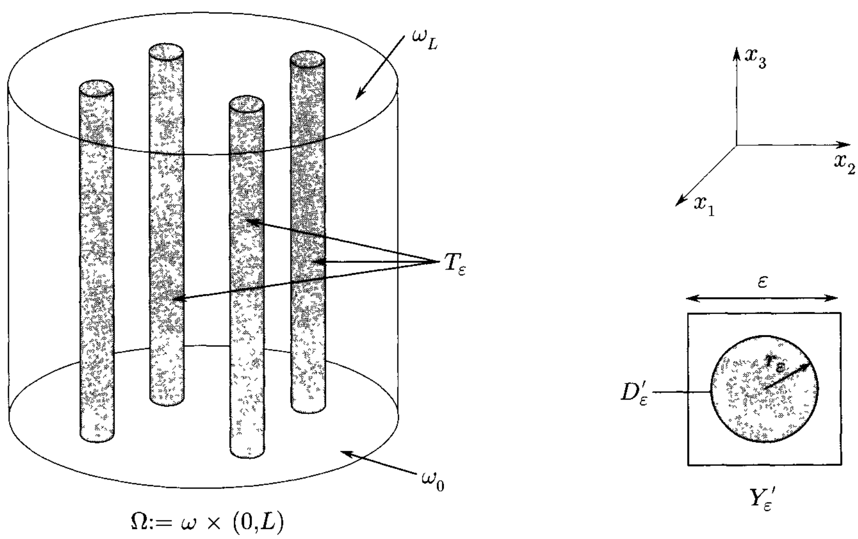

Fig. 1.

1.2. - Main Results

We show that the asymptotic behaviour of $(\mathcal{P}_\varepsilon)$ as $\varepsilon \to 0$ depends on k defined by (1.3) and on the parameter γ defined by:

$$(1.4) \qquad \gamma = \begin{cases} \lim_\varepsilon \varepsilon^{-2} |\mathrm{Log}(r_\varepsilon)|^{-1} & \text{if } p = 2 \\ \lim\limits_\varepsilon \left| \dfrac{2-p}{p-1} \right|^{p-1} r_\varepsilon^{2-p} \, \varepsilon^{-2} & \text{if } p \neq 2 \end{cases}$$

(note that, if $p > 2$, we have $\gamma = +\infty$ whatever r_ε).

As a limit problem, we find a system of two quasilinear elliptic equations in Ω deduced from the Euler equation of the following minimization problem:

$$(1.5) \qquad \inf\left\{\Phi(u,v) - \int_\Omega fu\,dx - \int_{\Gamma_1} gu\,dH^{N-1}\,;\ (u,v)\in (L^p(\Omega))^2\right\}$$

where

(1.6)

$$\Phi(u,v) = \begin{cases} \dfrac{1}{p}\left[\displaystyle\int_\Omega |\nabla u|^p dx + k\pi \int_\Omega \left|\frac{\partial v}{\partial x_3}\right|^p dx + 2\pi\gamma \int_\Omega (v-u)^p dx\right] \\ \qquad\qquad \text{if} \quad (u,v)\in W^{1,p}(\Omega)\times L^p(\omega, W^{1,p}(0,L)), \\ \qquad\qquad u = u_0 \quad \text{on} \quad \Gamma_0\ ,\quad v = u_0 \quad \text{on} \quad \Gamma_0\cap(\omega_0\cup\omega_L) \\ +\infty \qquad\qquad \text{otherwise}\ , \end{cases}$$

(and , in order to allow infinite values of k and γ, we adopt the convention: $0\times(+\infty) = 0$).

Here, the boundary data u_0 has been assumed to be Lipschitz in order to avoid the possibility that a concentration of fibers in the neighbourghood of Γ_0 forces the infimum value of problem $\mathcal{P}_\varepsilon$ to go to infinity as $\varepsilon\to 0$. Therefore also, having in mind the case $k = +\infty$, we add the following assumption

$$(1.7) \qquad \lim_\varepsilon k_\varepsilon r_\varepsilon \ = \ 0\ .$$

It is easy to check that the functional Φ is coercive in $W^{1,p}(\Omega)\times L^p(\omega, W^{1,p}(0,L))$ under the condition:

$$(1.8) \qquad k>0 \quad \text{and} \quad \{\gamma>0 \quad \text{or} \quad \omega_0\subset\Gamma_0 \quad \text{or} \quad \omega_L\subset\Gamma_0\}\ .$$

The variable u corresponds to the (strong) limit in L^p of the solutions u_ε. The new variable v corresponds to the weak-star limit in the space $\mathcal{M}_b(\Omega)$ of bounded Radon measures of the sequence (v_ε) defined by:

$$(1.9) \qquad v_\varepsilon(x) := \frac{|\Omega|}{|T_\varepsilon|}\,u_\varepsilon(x) \quad \text{if} \quad x\in T_\varepsilon \quad ,\quad v_\varepsilon(x) := 0 \qquad \text{otherwise}$$

(this means that $\lim_\varepsilon \int_\Omega \varphi v_\varepsilon dx = \int_\Omega \varphi v dx$ whenever φ belongs to the space $C_0(\Omega)$ of continuous functions vanishing on $\partial\Omega$).

A mathematical justification of the new scaling introduced in (1.9) is the following implication proved in Appendix (see the assertion i) of Lemma A1):

$$v\in C^0(\overline{\Omega})\ ,\ \frac{1}{T_\varepsilon}\int_{T_\varepsilon} |u_\varepsilon - v|\,dx\to 0 \quad \Rightarrow \quad v_\varepsilon\to v \quad \text{weakly* in } \mathcal{M}_b(\Omega)\ .$$

In that sense the variable v describes the average behavior of the restriction of u_ε to the fibers. Its contribution to the total energy $\Phi(u,v)$ comes through a diffusion term in the direction of the fibers (second term in the right hand side of (1.6)). The last integral in (1.6) represents the energy interaction between u and v in term of the capacitary parameter γ. Our main theorem extends the results of Caillerie and Dinari [8] obtained heuristically in the case $p = 2$, assuming a homogeneous Dirichlet boundary condition, by using a double-scaled matched asymptotic expansion.

THEOREM A. *Assume that* (1.3)(1.4) *hold with* $k, \gamma \in (0, +\infty)$ *and let* u_ε *be the unique solution of* $\mathcal{P}_\varepsilon$. *Then* (u_ε) *converges weakly to* u *in* $W^{1,p}$ *and* (v_ε) *given by* (1.9) *converges weakly* to* v *in* $\mathcal{M}_b(\Omega)$ *where* (u, v) *is the unique solution in* $W^{1,p}(\Omega) \times L^p(\omega, W^{1,p}(0, L))$ *of the system:*

$$
(\mathrm{P}^{hom}) \qquad \begin{cases}
-\mathrm{div}(\sigma) + 2\pi\gamma|u - v|^{p-2}(u - v) = f, & \\
\sigma = |\nabla u|^{p-2}\nabla u & on\ \Omega \\
\dfrac{\partial}{\partial x_3}\left(\left|\dfrac{\partial v}{\partial x_3}\right|^{p-2}\dfrac{\partial v}{\partial x_3}\right) & \\
+\dfrac{2\gamma}{k}|u - v|^{p-2}(u - v) = 0 & on\ \Omega \\
u = u_0 & on\ \Gamma_0 \\
\sigma.n = g & on\ \Gamma_1 \\
v = u_0 & on\ \Gamma_0 \cap (\omega_0 \cup \omega_L) \\
\dfrac{\partial v}{\partial x_3} = 0 & on\ \Gamma_1 \cap (\omega_0 \cup \omega_L)
\end{cases}
$$

Moreover, (u, v) *is the unique solution of the variational problem:*

$$
(1.10) \qquad \inf\left\{\Phi(u, v) - \int_\Omega f u\, dx - \int_{\Gamma_1} g u\, dH^{N-1}\, ;\, (u, v) \in (L^p(\Omega))^2\right\}
$$

and the convergence of associated energies holds, i.e. $\lim_\varepsilon F_\varepsilon(u_\varepsilon) = \Phi(u, v)$.

To prove this result, we use variational methods and show that the sequence (F_ε) defined by (1.2) converges in an appropriate sense to the functional $\Phi(u, v)$. More precisely

THEOREM B. *Under* (1.3), (1.4), (1.7), (1.8), *we have:*
i) (compactness) *Let* (u_ε) *such that* $\sup F_\varepsilon(u_\varepsilon) < +\infty$. *Then* (u_ε) *is strongly relatively compact in* $L^p(\Omega)$ *and* (v_ε) *given by* (1.9) *is bounded* $L^1(\Omega)$ *and, up to a subsequence, converges weakly* to a function* $v \in L^p(\omega, W^{1,p}(0, L))$.
ii) (lower bound inequality) *For all sequences* (u_ε) *such that* $u_\varepsilon \to u$ *in* $L^p(\Omega)$ *and* $v_\varepsilon \to v$ *weakly*, one has:* $\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq \Phi(u, v)$.
iii) (upper bound inequality) *For all* $(u, v) \in \left(L^p(\Omega)\right)^2$, *there exists* (u_ε) *such that* $u_\varepsilon \to u$ *in* $L^p(\Omega)$, $v_\varepsilon \to v$ *weakly* in* $L^1(\Omega)$ *and* $\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \leq \Phi(u, v)$.

REMARK. i) The conclusions of Theorem A can be easily extended to some degenerate cases:
– If $k = 0$, the solution u_ε converges strongly in $W^{1,p}(\Omega)$ to the solution of:

$$
(1.11) \qquad -\Delta_p u = f \quad \text{on } \Omega \quad , \quad u = u_0 \quad \text{on } \Gamma_0 \quad , \quad \frac{\partial^p u}{\partial n} = g \quad on\ \Gamma_1 \ .
$$

In this case the fibers completely disappear and $\lim_\varepsilon F_\varepsilon(u_\varepsilon) = \frac{1}{p}\int_\Omega |\nabla u|^p dx$.
– If $\gamma = 0$, we get the same conclusions for the behaviour of u_ε and the limit equation is still (1.11). However the limit of the energies $F_\varepsilon(u_\varepsilon)$ will contain an extra term taking into account the diffusion with respect to the variable v (that is the term $\frac{k\pi}{p}\int_\Omega |\frac{\partial v}{\partial x_3}|^p dx$). In this case there is no connection at all between the variables u and v.
ii) The convergence of (u_ε) to u cannot be strong in $W^{1,p}(\Omega)$ unless one of the parameters γ or k vanishes. In fact, if $k > 0$, the term $2\pi\gamma\int_\Omega |v-u|^p dx$ bears the energy associated with this defect of strong compactness. On the other hand, if $v \not\equiv 0$, we cannot expect the convergence of (v_ε) to hold weakly in $L^1(\Omega)$. Indeed, by Dunford Pettis Theorem, this would imply that (v_ε) is uniformly integrable and so $v_\varepsilon = v_\varepsilon 1_{T_\varepsilon}$ would converge to 0 in $L^1(\Omega)$!
iii) In the case $k < +\infty$, the geometrical assumption that the fibers do not intersect $\partial\omega \times (0, L)$ can be relaxed (i.e. we can take $I_\varepsilon = \{i \in Z^2; Y_\varepsilon^i \cap \omega \neq \emptyset\}$). The conclusions of Theorem B still hold.

1.3. – Non local effects

Let us formaly eliminate the variable v by defining:

$$F(u) = \inf\left\{\Phi(u, v) \quad ; \quad v \in L^p(\omega, W^{1,p}(0, L))\right\} \tag{1.12}$$

Clearly the limit u of the sequence of solutions (u_ε) (see Theorem B) is the solution of the variational problem:

$$\inf\left\{F(u) - \int_\Omega f u\, dx - \int_{\Gamma_1} g u\, dH^{N-1} ;\ u \in W^{1,p}(\Omega)\right\}$$

Moreover, one may reformulate the assertions ii) and iii) of Theorem B as the Γ-convergence of the sequence (F_ε) to F in $L^p(\Omega)$(see for instance [1] or [11] for all details relative to this notion of convergence).

Let us stress the fact that, in general, $F(u)$ is not local. This means that $F(u)$ cannot be written as the integration over Ω of a local density of energy of the form $f(x, u(x), \nabla u(x), \dots)$ like in the classical variational models of homogenization theory.

Let us give some particular cases where an explicit formula for $F(u)$ can be obtained:
1) If $\gamma = 0$, the variables u and v are independent. One gets:

$$F(u) = \frac{1}{p}\left(\int_\Omega |\nabla u|^p dx + \frac{k\pi}{L^{p-1}}\int_G |u_0(x_1, x_2, L) - u_0(x_1, x_2, 0)|^p\, dx_1 dx_2\right)$$

where $G = \{(x_1, x_2) \in \omega \ ;\ (x_1, x_2, 0) \text{ and } (x_1, x_2, L) \in \Gamma_0\}$.

2) Assume $p = 2$ (then $(\mathcal{P}_\varepsilon)$ is linear) and that $k, \gamma \in (0, \infty)$. Then $F(u)$ is a Dirichlet form (see [14], [15]) and can be written, using Deny-Beurling's representation formula, as:

$$F(u) = \frac{1}{2}\left[\int_\Omega |\nabla u|^2 + p_{\gamma,k}(x_3)|u|^2] dx \right.$$
$$\left. + \int_\omega \left(\int_{(0,L)^2} \left(u(x', s) - u(x', t)\right)^2 K_{\gamma,k}(s, t) ds dt\right) dx'\right].$$

where $x' = (x_1, x_2)$ and $K_{\gamma,k}(s, t)$ is a symetric kernel[(1)]. If for example, we consider a homogeneous Dirichlet condition on $\omega_L \cup \omega_0$, we get $K_{\gamma,k}$ and $p_{\gamma,k}$ in closed form as follows (see details in Appendix A5):

$$(1.13) \quad \begin{cases} K_{\gamma,k}(s,t) = \dfrac{\pi\gamma\sqrt{\frac{2\gamma}{k}}}{\sinh\left(\sqrt{\frac{2\gamma}{k}}L\right)} \sinh\left(\sqrt{\dfrac{2\gamma}{k}}(L - s \vee t)\right) \sinh\left(\sqrt{\dfrac{2\gamma}{k}}(s \wedge t)\right) \\ \\ p_{\gamma,k}(s) = 2\pi\gamma \dfrac{\cosh\left(\sqrt{\frac{2\gamma}{k}}\left(s - \frac{L}{2}\right)\right)}{\cosh\left(\sqrt{\frac{2\gamma}{k}}\frac{L}{2}\right)} \end{cases}$$

3) If $\gamma = +\infty$ (or $p > 2$), which corresponds to an infinite average capacity, one gets $v = u$ and $F(u)$ is a classical diffusion energy (stronger in the direction of the fibers):

$$F(u) = \frac{1}{p}\int_\Omega \left(|\nabla u|^p + k\pi \left|\frac{\partial u}{\partial x_3}\right|^p\right) dx .$$

4) If $k = +\infty$ and assuming a homogeneous Dirichlet boundary condition on one of the basis ω_0 or ω_L, we obtain the same homogenized energy as in the case of a Dirichlet problem on a perforated domain where the so called strange term is present (see Cioranescu and Murat [10]):

$$F(u) = \frac{1}{p}\left(\int_\Omega |\nabla u|^p dx + 2\pi\gamma \int_\Omega |u|^p dx\right) .$$

Here, like in the case 3), the non locality of $F(u)$ has disappeared.

[(1)]The general Deny-Beurling's representation formula reads, for $u \in C^1$, as

$$F(u) = \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_i} \nu_{i,j}(dx) + \int_\Omega |u|^2 \mu(dx) + \iint_{\Omega\times\Omega} (u(x) - u(y))^2 J(dxdy) ,$$

where $\nu_{i,j}, \mu$ are Radon measures on Ω ($\nu_{i,j} = \nu_{j,i}$) and J is a symetric Radon measure on $\Omega^2 \setminus D$ (D being the diagonal of Ω^2). The so called jumping measure J corresponds to the non local part of the energy. In our case, we find $\nu_{i,j} = \frac{1}{2}\delta_{i,j}\, dx$, $\mu(dx) = \frac{1}{2} p_{\gamma,k}(x_3)\, dx$, and $J(dxdy) = \frac{1}{2}\Delta(dx'dy') \otimes K_{\gamma,k}(x_3, y_3) dx_3 dy_3$, where $\Delta(dx'dy')$ denotes the measure on ω^2 defined by $\iint_{\omega^2} \varphi(x', y')\Delta(dx'dy') = \int_\omega \varphi(x', x')dx'$.

1.4. – Some variants

The method which consists in introducing new variables to describe the homogenized system can be adapted to many other geometries. We give here two examples. In the first one, we need to consider more than one extra variable and the resulting system of equations leads to a situation quite similar to the one met in [17]. In the second example, we recover some results announced in [13]. The proofs rely exactly on the same arguments as the one developed in Section 2 and can be found in details in [2].

a) Case of coaxial fibers. The fibers are still parallel to the x_3-axis but are filled up now with several media of conductivities $\lambda_\varepsilon^{(l)}$ ($l \in \{1, m\}$). We assume that, on each elementary cylinder, the conductivity coefficient $a_\varepsilon(x)$ is radial, piece-wise constant (with respect to the distance to the axis). Let $(r_\varepsilon^{(l)})$, $(\lambda_\varepsilon^{(l)})$ be sequences such that, for $l \in \{1, m\}$

$$\lambda_\varepsilon^{(l)} \to +\infty \quad , \quad r_\varepsilon^{(l)} << r_\varepsilon^{(l-1)} \quad \text{as } \varepsilon \to 0 \qquad (r_\varepsilon^{(0)} := \varepsilon) \ .$$

Denoting $D_\varepsilon^{i,(l)}$ the two dimensional disk centered on x_ε^i of radius $r_\varepsilon^{(l)}$, we define (see fig.2)

$$T_\varepsilon^{(l)} := \left(\bigcup_{i \in I_\varepsilon} D_\varepsilon^{i,(l)} \right) \times (0, L) \ ,$$

$$a_\varepsilon(x) := \begin{cases} 1 & \text{if } x \in \Omega \setminus \left(\bigcup_{i \in I_\varepsilon} T_\varepsilon^{(l)} \right) \\ \lambda_\varepsilon^{(l)} & \text{if } x \in T_\varepsilon^{(l)} \setminus T_\varepsilon^{(l-1)} \end{cases}$$

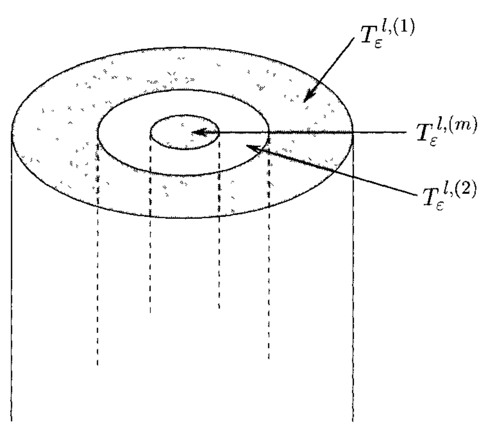

Fig. 2.

In order to describe the limit associated with the sequence of problems $(\mathcal{P}_\varepsilon)$ (for $p > 1$), we need to introduce the new variables $v^{(1)}, v^{(2)}, \ldots, v^{(m)}$ defined by

$$v^{(l)} := \lim_\varepsilon \frac{|\Omega|}{|T_\varepsilon^{(l)}|} 1_{T_\varepsilon^{(l)}} u_\varepsilon \qquad \text{in } L^1(\Omega)\text{-weak} * .$$

Then, we can prove (see [2]) that the limit of $(\mathcal{P}_\varepsilon)$ is represented by a system of $m+1$ equations deduced from the Euler equation of the following minimization problem

$$\inf\left\{\Phi(u, v^{(1)}, v^{(2)}, \ldots, v^{(m)}) - \int_\Omega f u \, dx - \int_{\Gamma_1} g u \, dH^{N-1}\right\},$$

where:

$$\Phi(u, v^{(1)}, v^{(2)}, \ldots, v^{(m)}) = \begin{cases} \frac{1}{p}\left[\int_\Omega |\nabla u|^p dx + \sum_{l=1}^{l=m} k_l \pi \int_\Omega \left|\frac{\partial v^{(l)}}{\partial x_1}\right|^p dx \right. \\ \quad +2\pi\gamma_1 \int_\Omega |v^{(1)} - u|^p \, dx \\ \quad \left. + \sum_{l=2}^{l=m} 2\pi \gamma_l \int_\Omega |v^{(l)} - v^{(l-1)}|^p \, dx \right] \\ \quad \text{if } (u, (v^{(l)})) \in W^{1,p}(\Omega) \times \left[L^p(\omega, (W^{1,p}(0, L))\right]^m, \\ \quad u = u_0 \text{ on } \Gamma_0 \,, \ v^{(l)} = u_0 \text{ on } \Gamma_0 \cap (\omega_0 \cup \omega_L) \\ +\infty \qquad \text{otherwise} \,, \end{cases}$$

and

$$k_l := \lim_\varepsilon 1_\varepsilon^{(l)} \frac{(r_\varepsilon^{(l)})^2}{\varepsilon^2} \,,$$

$$\gamma_l := \begin{cases} \lim_\varepsilon (r_\varepsilon^{(l-1)})^{-2} |\mathrm{Log}(r_\varepsilon^{(l)})|^{-1} & \text{if } p = 2 \\ \lim_\varepsilon \left|\frac{2-p}{p-1}\right|^{p-1} (r_\varepsilon^{(l)})^{2-p} (r_\varepsilon^{(l-1)})^{-2} & \text{if } p \neq 2 \end{cases}$$

b) Case of fibers distributed in three orthogonal directions. Taking now for Ω any convex regular bounded open subset of $\mathbb{R}^3$, we denote by $T_\varepsilon^{(3)} (:= T_\varepsilon)$ the set of x_3-parralel fibers defined in sec. 1.1 and let $T_\varepsilon^{(1)}, T_\varepsilon^{(2)}$ the set of fibers deduced from $T_\varepsilon^{(3)}$ by rotation of the axis so that (see fig.3) $T_\varepsilon^{(l)}$ is x_l-parallel $(l = 1, 2, 3)$. Define:

$$T_\varepsilon := T_\varepsilon^{(1)} \cup T_\varepsilon^{(2)} \cup T_\varepsilon^{(3)} \,,$$

$$a_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \Omega \setminus T_\varepsilon \,, \\ \lambda_\varepsilon & \text{if } x \in T_\varepsilon . \end{cases}$$

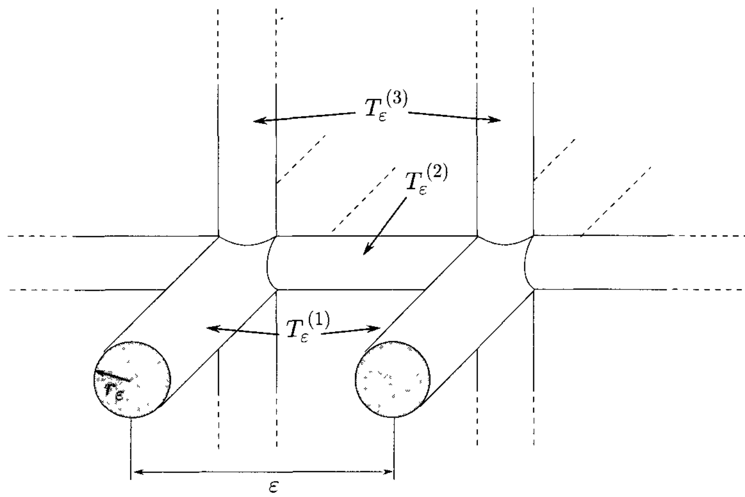

Fig. 3.

As before, we consider the problem $(\mathcal{P}_\varepsilon)$ (with $p > 1$) and the parameters k, γ defined by (1.3)(1.4). We assume that $0 < k < +\infty$ so that, (see part iii) of the remark after Theorem B), the fibers are allowed to intersect $\partial\Omega$ and the set of indices $I_\varepsilon^{(l)}$ corresponding to the fibers $T_\varepsilon^{(l)}$ can be defined as $I_\varepsilon^{(l)} := \{i \in Z^2;\, T_\varepsilon^{i,(l)} \cap \Omega \neq \emptyset\}$.

In view of the limit process as $\varepsilon \to 0$, it would be natural to introduce three new variables $v^{(1)}, v^{(2)}, v^{(3)}$ corresponding to the local behaviour of the solution u_ε in the neibourghood of each set of fibers $T_\varepsilon^{(1)}, T_\varepsilon^{(2)}, T_\varepsilon^{(3)}$, that is

$$v^{(l)} = \lim_\varepsilon v_\varepsilon^{(l)} \quad \text{in } L^1(\Omega)\text{-weak} * \quad \text{where} \quad v_\varepsilon^{(l)} := \frac{|\Omega|}{|T_\varepsilon^{(l)}|} 1_{T_\varepsilon^{(l)}} u_\varepsilon \ .$$

In fact, since the intersections $T_\varepsilon^{(l)} \cap T_\varepsilon^{(m)}$ are "large enough"[(2)], the following implication holds:

$$\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty \quad , \quad v_\varepsilon^{(l)} \overset{*}{\rightharpoonup} v^{(l)} \quad \Rightarrow \quad v^{(1)} = v^{(2)} = v^{(3)} \ .$$

Therefore, as in our main theorem, the limit problem associated with $(\mathcal{P}_\varepsilon)$ can be described through a functional Φ depending only on two variables u and v, where

$$v \ (= v^{(l)}) = \lim_\varepsilon v_\varepsilon \quad \text{in } L^1(\Omega)\text{-weak} * \quad , \quad v_\varepsilon := \frac{|\Omega|}{|T_\varepsilon|} 1_{T_\varepsilon} u_\varepsilon \ .$$

(2) It can be shown (see [2], in a more general setting), that a sufficient condition is that, for every $i, j \in \{1, 2, 3\}$, the ε-periodic set $T_\varepsilon^{(l)} \cap T_\varepsilon^{(m)}$ contains a ball of radius $b_\varepsilon >> r_\varepsilon\sqrt{\varepsilon}$.

We get that $(u_\varepsilon, v_\varepsilon)$ (u_ε solution of $(\mathcal{P}_\varepsilon)$) converges to the unique solution of (1.5) where

$$\Phi(u,v) = \begin{cases} \frac{1}{p}\left[\int_\Omega |\nabla u|^p dx + k\pi \int_\Omega \left(\left|\frac{\partial v}{\partial x_1}\right|^p + \left|\frac{\partial v}{\partial x_2}\right|^p + \left|\frac{\partial v}{\partial x_3}\right|^p\right) dx \right. \\ \left. +6\pi\gamma \int_\Omega |v-u|^p dx\right] \\ \quad \text{if } (u,v) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega), \\ \quad u = u_0 \text{ on } \Gamma_0 \,,\ v = u_0 \text{ on } \Gamma_0 \\ +\infty \text{ otherwise }, \end{cases}$$

Equivalently, we obtain the system of two elliptic equations:

$$(P^{hom}) \qquad \begin{cases} -\mathrm{div}(\sigma_u) + 6\pi\gamma |u-v|^{p-2}(u-v) = f & \text{on } \Omega \\ \mathrm{div}(\sigma_v) + \dfrac{6\gamma}{k}\ |u-v|^{p-2}(u-v) = 0 & \text{on } \Omega \\ \sigma_u = |\nabla u|^{p-2}\nabla u \ ,\ \sigma_v = \left(\left|\dfrac{\partial v}{\partial x_l}\right|^{p-2} \dfrac{\partial v}{\partial x_l}\right)_{l=1,2,3} & \text{on } \Omega \\ u = v = u_0 & \text{on } \Gamma_0 \\ \sigma_u . n = g \ ,\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \Gamma_1 \end{cases}$$

In the case $p = 2$ and assuming a Neumann homogeneous condition on $\partial\Omega$ (i.e. $\Gamma_0 = \emptyset$), we obtain (in a stationary setting) the homogenized system proposed in [13], where non local effects in space were pointed out in the context of the homogenization of parabolic boundary value problems.

2. – Proofs

In all this section, the letter C denotes a suitable positive constant which may vary from line to line.

Proof of Theorem A. Set $m_\varepsilon := \inf \mathcal{P}_\varepsilon$ and define, for every $u \in W^{1,p}(\Omega)$, $L(u) := \int_\Omega f u\, dx + \int_{\Gamma_1} g u\, dH^2$. Let $\tilde{u}_0$ denote a Lipschitz extension of u_0. Since $k < +\infty$, we have:

$$\text{(2.1)} \qquad \begin{aligned} \sup_\varepsilon\ m_\varepsilon &\le \sup_\varepsilon\ \{F_\varepsilon(\tilde{u}_0) - L(\tilde{u}_0)\} \\ &\le (Lip(\tilde{u}_0))^p \sup_\varepsilon \left\{\int_\Omega a_\varepsilon dx\right\} + |L(\tilde{u}_0)| < +\infty\ . \end{aligned}$$

We can assume that $a_\varepsilon \ge 1$ (recall $\lambda_\varepsilon \to \infty$) so that by using the boundary condition and Poincaré's inequality

$$\text{(2.2.)} \qquad F_\varepsilon(u) \ge \Psi(\|u\|_{W^{1,p}}(\Omega)) \quad \text{where } \Psi \text{ satisfies } \lim_{t\to\infty} \frac{\Psi(t)}{t} = +\infty$$

Then by the continuity of the linear form $L(\cdot)$ on $W^{1,p}$, (2.1) and (2.2), we deduce that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$. Apply the assertion i) of Theorem B to get (possibly after extraction of subsequences):

$$(2.3) \qquad u_\varepsilon \to \overline{u} \quad \text{weakly in } W^{1,p}(\Omega) \quad , \quad v_\varepsilon \overset{*}{\rightharpoonup} \overline{v} \quad \text{weakly* in } \mathcal{M}_b(\Omega) \ ,$$

where v_ε is defined by (1.9) and $(\overline{u}, \overline{v})$ belongs to $W^{1,p}(\Omega) \times L^p(\omega, W^{1,p}(0, L))$. By the assertion ii) of Theorem B, we obtain

$$(2.4) \qquad \liminf_\varepsilon m_\varepsilon = \liminf_\varepsilon F_\varepsilon(u_\varepsilon) - L(u_\varepsilon) \geq \Phi(\overline{u}, \overline{v}) - L(\overline{u}) \geq \inf(\mathcal{P}^{hom}) \ .$$

which proves by (2.1) that $m_0 := \inf(\mathcal{P}^{hom}) < +\infty$. Converserly let $r > \inf(\mathcal{P}^{hom})$ and (u, v) such that $\Phi(u, v) - L(u) < r$. By the assertion iii) of Theorem B, we can find a new sequence (u_ε) such that $u_\varepsilon \to u$ in $L^p(\Omega)$, $v_\varepsilon \to v$ weakly* and $\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \leq \Phi(u, v)$. As $\Phi(u, v)$ is finite, the convergence of u_ε holds also weakly in $W^{1,p}(\Omega)$ and by the continuity of $L(\cdot)$, one gets:

$$(2.5) \qquad \limsup_\varepsilon m_\varepsilon \leq \limsup_\varepsilon \left(F_\varepsilon(u_\varepsilon) - L(u_\varepsilon)\right) \leq \Phi(u, v) - L(u) \leq r \ .$$

From (2.4), (2.5) and by letting r tend to m_0, we deduce that:

$$\lim_\varepsilon m_\varepsilon = m_0 = \Phi(\overline{u}, \overline{v}) - L(\overline{u}) \ .$$

Hence we have proved that $(\overline{u}, \overline{v})$ is solution of the variational problem (1.10). The convergence of energies follows since:

$$\lim_\varepsilon F_\varepsilon(u_\varepsilon) = \lim_\varepsilon \big(m_\varepsilon + L(u_\varepsilon)\big) = m_0 + L(u) = \Phi(\overline{u}, \overline{v}) \ .$$

The final step consisting in deriving the Euler equation $(\mathcal{P}^{hom})$ associated with (1.10) is straightforward and left to the reader. □

PROOF OF THEOREM B. We will use successive claims whose proof sometimes refers to some lemmas stated in the Appendix. In the following, we choose a sequence R_ε such that $r_\varepsilon << R_\varepsilon << \varepsilon$ and denote by B_ε the subset of Ω consisting in the e_3-parallel tubes surrounding the fibers of outer radius R_ε. Then we write the total energy $F_\varepsilon(u)$ as the sum of three terms:

$$(2.6) \qquad F_\varepsilon(u) = \underbrace{\frac{1}{p}\int_{\Omega-(B_\varepsilon\cup T_\varepsilon)} |\nabla u|^p dx}_{F_\varepsilon^1(u)} + \underbrace{\frac{1}{p}\int_{B_\varepsilon} |\nabla u|^p dx}_{F_\varepsilon^2(u)} + \underbrace{\frac{k_\varepsilon \pi |\Omega_\varepsilon|}{p} \fint_{T_\varepsilon} |\nabla u|^p \, dx}_{F_\varepsilon^3(u_\varepsilon)} \ ,$$

where the symbol $\fint \ldots$ denotes the mean value with respect to the integration, k_ε is defined by (1.3) (recall $k_\varepsilon \to k$ as $\varepsilon \to 0$) and $\Omega_\varepsilon := \cup_{i\in I_\varepsilon} Y_\varepsilon^{\hat{\imath}} \times (0, L)$ satisfies $|\Omega_\varepsilon| \to |\Omega|$.

Preliminary estimates: Let (u_ε) be a sequence such that $\sup F_\varepsilon(u_\varepsilon) < +\infty$. Recalling that $k > 0$ by (1.8) (so $\lambda_\varepsilon \to +\infty$), we have

$$\sup_\varepsilon \fint_{T_\varepsilon} |\nabla u_\varepsilon|^p dx < +\infty \quad , \quad \sup_\varepsilon \int_\Omega |\nabla u_\varepsilon|^p dx < +\infty. \tag{2.7}$$

As $R_\varepsilon << \varepsilon$, the sequence $1_{B_\varepsilon \cup T_\varepsilon}$ converges strongly to 0 in $L^{p'}(\Omega)$. Then, by (2.7) and Hölder inequality,

$$(1-\chi_\varepsilon)\nabla u_\varepsilon \rightharpoonup 0 \quad \text{weakly in} L^p(\Omega) \ , \quad \text{where } \chi_\varepsilon := 1_{\Omega \setminus (B_\varepsilon \cup T_\varepsilon)} \ . \tag{2.8}$$

To estimate the second term $F_\varepsilon^2(u_\varepsilon)$, we need to construct on every x_3 section of Ω piece-wise constant approximants of the functions u_ε and v_ε (v_ε defined by (1.9)). To that aim, we write the section by $x_3 = 0$ of the boundary of B_ε as a union of circles of radius r_ε and R_ε (located in each cell Y_ε^i):

$$\partial B_\varepsilon \cap \{x_3 = 0\} = \bigcup_i C_{R_\varepsilon}^i \cup C_{r_\varepsilon}^i \ ,$$

and set:

$$\tilde{u}_\varepsilon(x_1, x_2, x_3) := \sum_i \left(\fint_{C_{R_\varepsilon}^i} u_\varepsilon^{x_3} ds \right) 1_{Y_\varepsilon^i}(x_1, x_2) \tag{2.9}$$

$$\tilde{v}_\varepsilon(x_1, x_2, x_3) := \sum_i \left(\fint_{C_{r_\varepsilon}^i} u_\varepsilon^{x_3} ds \right) 1_{Y_\varepsilon^i}(x_1, x_2) \ , \tag{2.10}$$

where $u_\varepsilon^{x_3}(\cdot,\cdot) := u_\varepsilon(\cdot,\cdot,x_3)$ is, by Fubini's Theorem, well defined in $W^{1,p}(\omega)$ for a.e. $x_3 \in (0, L)$. A lower bound can be then deduced for $F_\varepsilon^2(u_\varepsilon)$ by applying Lemma A3 to $u_\varepsilon^{x_3}$ on each bi-dimensional annulus $C_{R_\varepsilon, r_\varepsilon}^i := \{x'; r_\varepsilon < |x' - x_\varepsilon^i| < R_\varepsilon\}$ (where $i \in I_\varepsilon$ and by notation $x' := (x_1, x_2)$):

$$\int_{C_{R_\varepsilon, r_\varepsilon}^i} |\nabla u_\varepsilon|^p dx' \ \geq \int_{C_{R_\varepsilon, r_\varepsilon}^i} |\nabla_{x'} u^{x_3}|^p \, dx' \ \geq 2\pi\varepsilon^2 \gamma_\varepsilon |\tilde{u}_\varepsilon^{x_3} - \tilde{v}_\varepsilon^{x_3}|^p \ , \tag{2.11}$$

where (see Lemma A3) $\gamma_\varepsilon := \frac{\Gamma_p(r_\varepsilon, R_\varepsilon)}{\varepsilon^2}$. By (1.4), γ_ε tends to γ as $\varepsilon \to 0$. Summing (2.11) with respect to $i \in I_\varepsilon$ and integrating with respect to x_3, we are led to:

$$\begin{aligned} \int_{B_\varepsilon} |\nabla u_\varepsilon|^p dx &\geq \int_0^L \left(\sum_i 2\pi\varepsilon^2 \gamma_\varepsilon |\tilde{u}_\varepsilon^{x_3} - \tilde{v}_\varepsilon^{x_3}|^p \right) dx_3 \\ &\geq 2\pi\gamma_\varepsilon \int_0^L \left(\sum_i \int_{Y_\varepsilon^i} |\tilde{u}_\varepsilon^{x_3} - \tilde{v}_\varepsilon^{x_3}|^p \, dx' \right) dx_3 \ , \end{aligned}$$

so that:

$$F_\varepsilon^2(u_\varepsilon) \;\geq\; \frac{2\pi}{p}\gamma_\varepsilon \int_\Omega |\tilde u_\varepsilon - \tilde v_\varepsilon|^p\, dx \;. \tag{2.12}$$

We need now to estimate $|u_\varepsilon - \tilde u_\varepsilon|$ and $|u_\varepsilon - \tilde v_\varepsilon|$. To that aim, we use a Poincaré's type inequality proved in Appendix (Lemma A4). Apply Lemma A4 to $u_\varepsilon^{x_3}$ on the disk $D(x_i^\varepsilon, \frac{\varepsilon}{\sqrt2})$ with $R = \frac{\varepsilon}{\sqrt2}$, $\alpha = \frac{R_\varepsilon\sqrt2}{\varepsilon}$, sum with respect to $i \in I_\varepsilon$ and integrate with respect to x_3. We get:

$$\begin{aligned}
\int_\Omega |u_\varepsilon - \tilde u_\varepsilon|^p\, dx &= \sum_i \int_0^L \left(\int_{Y_\varepsilon^i} |u_\varepsilon^{x_3} - \tilde u_\varepsilon^{x_3}|^p dx' \right) dx_3 \\
&\leq \sum_i \int_0^L \left(\int_{D(x_\varepsilon^i, \frac{\varepsilon}{\sqrt2})} |u_\varepsilon^{x_3} - \tilde u_\varepsilon^{x_3}|^p dx' \right) dx_3 \\
&\leq \frac{C}{(\sqrt2)^p}\, \frac{\varepsilon^p}{h\left(\frac{R_\varepsilon\sqrt2}{\varepsilon}\right)} \sum_i \int_0^L \left(\int_{D\left(x_\varepsilon^i, \frac{\varepsilon}{\sqrt2}\right)} |\nabla_{x'} u_\varepsilon|^p dx' \right) dx_3 \\
&\leq \frac{2C}{(\sqrt2)^p}\, \frac{\varepsilon^p}{h\left(\frac{R_\varepsilon\sqrt2}{\varepsilon}\right)} \int_\Omega |\nabla u_\varepsilon|^p\, dx \;,
\end{aligned}$$

where the function h is defined by (3.9).
(here we have used the fact that the disks intersect each other no more than two times and so: $1 \leq \sum_i 1_{D(x_\varepsilon^i, \frac{\varepsilon}{\sqrt2})} \leq 2$ on Ω).

By (1.4),(2.7), (3.9) and recalling that $R_\varepsilon >> r_\varepsilon$, we deduce the implication

$$\gamma > 0 \quad\Longrightarrow\quad \tilde u_\varepsilon - u_\varepsilon \;\to\; 0 \quad \text{in} L^p(\Omega) \;. \tag{2.13}$$

On the same way, we may apply Lemma A4 to $u_\varepsilon^{x_3}$ in the disk $D_\varepsilon^i := D(x_i^\varepsilon, r_\varepsilon)$ with $R = r_\varepsilon$, $\alpha = 1$ to get:

$$\begin{aligned}
\int_{T_\varepsilon} |u_\varepsilon - \tilde v_\varepsilon|^p\, dx &= \sum_i \int_0^L \left(\int_{D_\varepsilon^i} |u_\varepsilon^{x_3} - \tilde v_\varepsilon^{x_3}|^p dx' \right) dx_3 \\
&\leq C r_\varepsilon^p \sum_i \int_0^L \left(\int_{D_\varepsilon^i} |\nabla_{x'} u_\varepsilon|^p dx' \right) dx_3 \\
&\leq C r_\varepsilon^p \int_{T_\varepsilon} |\nabla u_\varepsilon|^p\, dx \;.
\end{aligned}$$

Thus, by (2.7)

$$\lim_{\varepsilon\to0} \fint_{T_\varepsilon} |u_\varepsilon - \tilde v_\varepsilon|^p dx = 0 \;. \tag{2.14}$$

Noticing that (2.14) is still valid for $p = 1$ and since $\tilde{v}_\varepsilon$ is by construction piecewise constant with respect to x', we may apply Lemma A1.ii) to derive the following implication:

$$(2.15)\quad (\tilde{v}_\varepsilon) \text{ bounded in} L^1(\Omega) \implies \begin{cases} (v_\varepsilon) & \text{bounded in} L^1(\Omega) \\ v_\varepsilon - \tilde{v}_\varepsilon \to 0 & \text{weakly* in } \mathcal{M}_b(\Omega) \end{cases}$$

Proof of compactness The first part of assertion i) of Theorem B is quite obvious since, by the Dirichlet condition on Γ_0, (2.7) and Poincaré's inequality, (u_ε) is bounded in $W^{1,p}(\Omega)$, hence relatively compact in $L^p(\Omega)$ by Rellich theorem. As regards the relative compactness of the sequence (v_ε) defined by (1.9), according to assumption (1.8), we have to consider two cases:

First case: $\gamma > 0$ By (2.7) and (2.12) the sequence $(\tilde{u}_\varepsilon - \tilde{v}_\varepsilon)$ is bounded in $L^p(\Omega)$. By (2.13) and the boundedness of (u_ε) shown before, the sequence $(\tilde{u}_\varepsilon)$ is bounded in $L^p(\Omega)$. It follows that $(\tilde{v}_\varepsilon)$ is bounded in $L^p(\Omega)$ as well and, possibly passing to a subsequence, we can assume that $\tilde{v}_\varepsilon$ converges weakly to some $v \in L^p(\Omega)$. Then we can apply (2.15) to deduce that $v_\varepsilon \to v$ weakly* in $\mathcal{M}_b(\Omega)$.

Second case: $\gamma = 0$ By (1.8), we can assume for example that $\omega_0 \subset \Gamma_0$. Hence $u_\varepsilon(x', 0) = u_0(x', 0)$ for a.e. $x' \in \omega$ and the following pointwise estimate holds:

$$|u_\varepsilon(x', x_3)|^p \le 2^{p-1}|u_0(x', 0)|^p + 2^{p-1}L^{p-1}\int_0^L \left|\frac{\partial u_\varepsilon}{dx_3}\right|^p dx_3 \ .$$

Then, since u_0 is bounded,

$$\fint_{T_\varepsilon} |u_\varepsilon|^p dx \le C + 2^{p-1}L^p \fint_{T_\varepsilon} \left|\frac{\partial u_\varepsilon}{dx_3}\right|^p dx \ ,$$

and (2.7) yields

$$\sup_\varepsilon \fint_{T_\varepsilon} |u_\varepsilon|^p dx < +\infty \ .$$

We may apply the assertion i) of Lemma A2 with $f_\varepsilon = u_\varepsilon$, $\mu_\varepsilon = \frac{|\Omega|}{|T_\varepsilon|} 1_{T_\varepsilon}\, dx$ and $\mu = 1_\Omega\, dx$. Noticing that $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ (in the sense of Radon measures) and that $f_\varepsilon \mu_\varepsilon = v_\varepsilon dx$, we deduce that, for a suitable subsequence still denoted (v_ε), there exists $v \in L^p(\Omega)$ such that $v_\varepsilon \overset{*}{\rightharpoonup} v$ in $\mathcal{M}_b(\Omega)$.

REMARK. By the arguments above, we have proved that any cluster point v of (v_ε) belongs to $L^p(\Omega)$. The fact that $v \in L^p(\omega, W^{1,p}(0, L))$ appears later in the proof of the lower bound inequality.

Proof of the lower bound inequality. Possibly passing to a subsequence, we can assume that:

$$\liminf_\varepsilon F_\varepsilon(u_\varepsilon) = \lim_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty \ .$$

and then using the compactness proved before that:

$$(2.16) \qquad u_\varepsilon \to u \quad \text{weakly in } W^{1,p}(\Omega) \quad , \quad v_\varepsilon \to v \quad \text{weakly* in } \mathcal{M}_b(\Omega) \ ,$$

for a suitable pair (u, v) in $W^{1,p}(\Omega) \times L^p(\Omega)$.

We look separetely at the three terms appearing in (2.6). A lower bound for the first one term can be easily deduced since, by (2.8), we have $\chi_\varepsilon \nabla u_\varepsilon \rightharpoonup \nabla u$ in $(L^p(\Omega))^3$. Then

$$(2.17) \qquad \liminf_\varepsilon F^1_\varepsilon(u_\varepsilon) = \liminf_\varepsilon \frac{1}{p} \int_\Omega |\chi_\varepsilon \nabla u_\varepsilon|^p dx \geq \frac{1}{p} \int_\Omega |\nabla u|^p dx \ .$$

Let us show that

$$(2.18) \qquad \liminf F^2_\varepsilon(u_\varepsilon) \geq \frac{1}{p} 2\pi\gamma \int_\Omega |u - v|^p \, dx.$$

Assume that $\gamma > 0$ (otherwise it is trivial). Then, by (2.13), $\tilde{u}_\varepsilon - u \to 0$ in $L^p(\Omega)$. Since the left hand side of (2.12) is bounded and $\gamma_\varepsilon \to \gamma$, the sequence $(\tilde{v}_\varepsilon)$ is bounded in $L^p(\Omega)$ and (2.15) yields that $\tilde{v}_\varepsilon - v \overset{*}{\rightharpoonup} 0$ in $\mathcal{M}_b(\Omega)$. Hence the sequence $(\tilde{v}_\varepsilon - \tilde{u}_\varepsilon)$ is bounded and converges weakly to $v - u$ in $L^p(\Omega)$. Then (2.18) follows from (2.12).

It remains now to show that:

$$(2.19) \qquad v \in L^p(\omega, W^{1,p}(0, L)) \ ,$$

$$(2.20) \qquad \liminf_\varepsilon F^3_\varepsilon(u_\varepsilon) \ \geq \ \frac{k\pi}{p} \int_\Omega \left| \frac{\partial v}{\partial x_3} \right|^p dx \ ,$$

$$(2.21) \qquad v = u_0 \qquad \text{on } \Gamma_0 \cap (\omega_0 \cup \omega_L) \ .$$

Indeed, collecting (2.17), (2.18), (2.20) and taking into account (2.19), (2.21), we will deduce:

$$\begin{aligned} \liminf_\varepsilon F_\varepsilon(u_\varepsilon) &\geq \liminf_\varepsilon F^1_\varepsilon(u_\varepsilon) + \liminf_\varepsilon F^2_\varepsilon(u_\varepsilon) \ + \liminf_\varepsilon F^3_\varepsilon(u_\varepsilon) \\ &\geq \frac{1}{p} \left[\int_\Omega |\nabla u|^p dx \ + \ 2\pi\gamma \int_\Omega |v - u|^p \, dx \ + \ k\pi \int_\Omega \left| \frac{\partial v}{x_3} \right|^p dx \right] \\ &\geq \Phi(u, v) \ , \end{aligned}$$

that is the assertion ii) of Theorem B.

Let us prove (2.19) and (2.20). Thanks to the estimate (2.7), we may apply Lemma A2 on $\mathbb{R}^3$ with $f_\varepsilon = \frac{\partial u_\varepsilon}{\partial x_3} 1_\Omega$, $\mu_\varepsilon = \frac{|\Omega|}{|T_\varepsilon|} 1_{T_\varepsilon} dx$ and $\mu = 1_\Omega dx$.

So, possibly after extracting a subsequence, we can assume that, for a suitable $\chi \in (L^p(\Omega))^3$, we have:

$$\left(\frac{\partial u_\varepsilon}{\partial x_3}\right)\frac{|\Omega|}{|T_\varepsilon|}1_{T_\varepsilon} \overset{*}{\rightharpoonup} \chi\, 1_\Omega \qquad \text{weakly*in } \mathcal{M}_b(\mathbb{R}^3)\ , \tag{2.22}$$

$$\liminf_\varepsilon\ |\Omega| \fint_{T_\varepsilon} \left|\frac{\partial u_\varepsilon}{\partial x_3}\right|^p dx = \liminf_\varepsilon \int_\Omega |f_\varepsilon|^p\, d\mu_\varepsilon \geq \int_\Omega |\chi|^p dx\ . \tag{2.23}$$

Let us test the convergence (2.22) for functions $\varphi \in C^1(\overline{\Omega})$. We have:

$$\lim_\varepsilon \fint_{T_\varepsilon} \varphi \frac{\partial u_\varepsilon}{\partial x_3} dx \ = \ \fint_\Omega \chi\varphi dx\ . \tag{2.24}$$

On the other hand , using Fubini's formula and integrating by parts with respect to x_3, we get:

$$\fint_{T_\varepsilon} \varphi \frac{\partial u_\varepsilon}{\partial x_3} dx = -\fint_{T_\varepsilon} u_\varepsilon \frac{\partial \varphi}{\partial x_3} dx + \frac{1}{L}\fint_{D_\varepsilon} \left[\varphi(x', L)u_\varepsilon(x', L) - \varphi(x', 0)u_\varepsilon(x', 0)\right] dx',$$

where $D_\varepsilon := \cup_i D_\varepsilon^i$. Then by (2.16) and (2.24), we have:

$$\begin{aligned} &\fint_\Omega \left(\chi\,\varphi + v\frac{\partial\varphi}{\partial x_3}\right) dx \\ &= \lim_\varepsilon \frac{1}{L} \fint_{D_\varepsilon} \left[\varphi(x', L)u_\varepsilon(x', L) - \varphi(x', 0)u_\varepsilon(x', 0)\right] dx'\ . \end{aligned} \tag{2.25}$$

Let us take first $\varphi \in \mathcal{D}(\Omega)$ so that the right hand side of (2.25) vanishes. We deduce that the distributional derivative $\frac{\partial v}{\partial x_3}$ coincides with χ, so that $\frac{\partial v}{\partial x_3} \in L^p(\Omega)$. As $v \in L^p(\Omega)$, we get (2.19). Then, recalling the definition of F_ε^3 in (2.6) and (1.3), (2.23) implies (2.20).

To prove (2.21), we choose now φ of the form $\varphi(x) = \theta(x')\Psi(x_3)$ with $\theta \in \mathcal{D}(G_0)$, where $G_0 := \{x' \in \omega;\ (x', 0) \in \Gamma_0\}$ and with $\Psi(0) = 1$, $\Psi(L) = 0$. Taking into account that $u_\varepsilon(x', 0) = u_0(x', 0)$ on G_0 and that θ is smooth, equality (2.25) becomes:

$$\fint_\Omega \varphi\frac{\partial v}{\partial x_3} + v\frac{\partial\varphi}{\partial x_3} dx = -\frac{1}{L}\fint_\omega \theta(x')u_0(x', 0)dx' \tag{2.26}$$

By (2.19), we may integrate by parts the left hand side of (2.26) to get

$$\fint_\omega \theta(x')v(x', 0)dx' = \fint_\omega \theta(x')u_0(x', 0)dx'\ .$$

This holds true for every $\theta \in \mathcal{D}(G_0)$, thus $v = u_0$ a.e.on $\Gamma_0 \cap \omega_0$. Similarly taking $\theta \in \mathcal{D}(G_L)$ where $G_L := \{x' \in \omega; (x', L) \in \Gamma_0\}$ and Ψ such that $\Psi(0) = 0, \Psi(L) = 1$, we obtain $v = u_0$ a.e. on $\Gamma_0 \cap \omega_L$. So (2.21) holds and the proof of the assertion ii) of Theorem B is achieved. □

Proof of the upper bound inequality. We are going to prove the assertion iii) of Theorem B using three steps. We may assume that $\Phi(u, v) < +\infty$ (otherwise the statement is trivial).

STEP 1. In this step, we assume first that u and v are uniformly Lipschitz on Ω and we construct explicitly an approximating sequence (u_ε) such that

$$\text{(2.27)} \quad u_\varepsilon = u \quad \text{on } \partial\omega \times (0, L), u_\varepsilon \to u \text{ in } L^p(\Omega), v_\varepsilon \stackrel{*}{\rightharpoonup} v \text{ in } \mathcal{M}_b(\Omega),$$

$$\text{(2.28)} \qquad \limsup_{\varepsilon \to 0} \frac{1}{p} \int_\Omega a_\varepsilon(x) |\nabla u_\varepsilon|^p \, dx \leq \Phi(u, v)$$

(here we forget the boundary constraint on $\Gamma_0 \cap (\omega_0 \cup \omega_L)$).

Let us denote by $\rho_\varepsilon(x')$ the Euclidean distance of $x' \in \omega$ to the set $\{x_\varepsilon^i, i \in I_\varepsilon\}$ (centers of the elementary cells Y_ε^i), i.e.: $\rho_\varepsilon(x') := |x' - x_\varepsilon^i|$ if $x' \in Y_\varepsilon^i$. We associate to $\rho_\varepsilon(x')$ a function $\theta_\varepsilon : \mathbb{R}^2 \to [0, 1]$ by setting

$$\theta_\varepsilon(x') := 1 \quad \text{if } \rho_\varepsilon(x') < r_\varepsilon \quad , \quad \theta_\varepsilon(x') := 0 \quad \text{if } \rho_\varepsilon(x') > R_\varepsilon \ ,$$

and for $r_\varepsilon < \rho_\varepsilon(x') < R_\varepsilon$ (which corresponds to the tubular set B_ε)

$$\text{(2.29)} \qquad \theta_\varepsilon(x') := \begin{cases} \dfrac{\operatorname{Log} R_\varepsilon - \operatorname{Log} \rho_\varepsilon}{\operatorname{Log} R_\varepsilon - \operatorname{Log} r_\varepsilon} & \text{if } p = 2 \ , \\[2ex] \dfrac{R_\varepsilon^s - \rho_\varepsilon^s}{R_\varepsilon^s - r_\varepsilon^s} & \text{if } p \neq 2 \quad \left(s = \dfrac{p-2}{p-1}\right) \ . \end{cases}$$

Then we consider the x'-piecewise constant approximations of v defined by

$$\text{(2.30)} \qquad w_\varepsilon(x', x_3) := \sum_{i \in I_\varepsilon} \left(\fint_{D_\varepsilon^i} v^{x_3}(y) \, dy \right) 1_{Y_\varepsilon^i}(x') \ ,$$

and define:

$$\text{(2.31)} \qquad u_\varepsilon(x) := (1 - \theta_\varepsilon(x')) \ u(x) + \theta_\varepsilon(x') \ w_\varepsilon(x) \ .$$

We claim that the sequence (u_ε) satisfies (2.27) and (2.28).

PROOF OF (2.27). By construction $\theta_\varepsilon = 0$ on $\partial\omega$ and so $u_\varepsilon = u$ on $\partial\omega \times (0, L)$. On the other hand, thanks to the smoothness of u, v and by the definitions $(2.29 - 31)$, we check easily that (C being a suitable constant):

$$\text{(2.32)} \quad u_\varepsilon = u \ \text{ on } \Omega \setminus (T_\varepsilon \cup B_\varepsilon), \ \ |u_\varepsilon| \leq C \ \text{ on } \Omega, \ \ |\nabla w_\varepsilon| \leq C \ \text{ on } B_\varepsilon,$$

$$\text{(2.33)} \quad u_\varepsilon = w_\varepsilon \ \text{ and } \ |w_\varepsilon - v| \leq C\, r_\varepsilon \ \text{ on } T_\varepsilon, \ \ |w_\varepsilon - v| \leq C\, R_\varepsilon \ \text{ on } B_\varepsilon.$$

It follows that $\int_\Omega |u_\varepsilon - u|^p dx \le C|B_\varepsilon \cup T_\varepsilon|$ and, since $R_\varepsilon << \varepsilon$, we get that $u_\varepsilon \to u$ in $L^p(\Omega)$. On the other hand, by (2.33), we have $⨍_{T_\varepsilon} |u_\varepsilon - v| dx \to 0$ and then, by the assertion i) of Lemma A1, v_ε $(:= \frac{|\Omega|}{|T_\varepsilon|} 1_{T_\varepsilon} u_\varepsilon) \overset{*}{\rightharpoonup} v$ in $\mathcal{M}_b(\Omega)$. □

PROOF OF (2.28). As in (2.6) we split the left hand side of (2.28) into three terms

$$\frac{1}{p}\int_\Omega a_\varepsilon(x)|\nabla u_\varepsilon|^p = F_\varepsilon^{(1)}(u_\varepsilon) + F_\varepsilon^{(2)}(u_\varepsilon) + F_\varepsilon^{(3)}(u_\varepsilon) \ .$$

An upper bound for the first term is quite trivial

$$\limsup_\varepsilon F_\varepsilon^{(1)}(u_\varepsilon) = \limsup_\varepsilon \int_{\Omega\setminus(B_\varepsilon \cup T_\varepsilon)} |\nabla u|^p\, dx = \int_\Omega |\nabla u|^p\, dx \ . \tag{2.34}$$

To majorize the second term, we notice that by (2.31)

$$\nabla u_\varepsilon = \nabla u\ ((1-\theta_\varepsilon(x')) + \nabla w_\varepsilon\ \theta_\varepsilon(x') + (w_\varepsilon - u)\ \nabla\theta_\varepsilon \ ,$$

so that, by (2.32)(2.33), we have a.e. on B_ε

$$\begin{aligned}|\nabla u_\varepsilon| &\le C + |\nabla_{x'}\theta_\varepsilon|(|w_\varepsilon - v| + |v-u|)\\ &\le C + CR_\varepsilon|\nabla_{x'}\theta_\varepsilon| + |v-u||\nabla_{x'}\theta_\varepsilon| \ .\end{aligned}$$

Hence, setting $f_\varepsilon(x) := |\nabla_{x'}\theta_\varepsilon|$, we get

$$\|\nabla u_\varepsilon\|_{L^p(B_\varepsilon)} \le C|B_\varepsilon|^{\frac{1}{p}} + CR_\varepsilon\|f_\varepsilon\|_{L^p(B_\varepsilon)} + \|f_\varepsilon(v-u)\|_{L^p(B_\varepsilon)} \ . \tag{2.35}$$

The sequence of non negative functions (f_ε) defined by $f_\varepsilon(x) := |\nabla_{x'}\theta_\varepsilon|$ is x_3- independent, vanishes on $\partial\omega$ and satisfies, for all $i \in I_\varepsilon$, $\int_{Y_\varepsilon^i} |f_\varepsilon|^p\, dx = 2\pi\, \Gamma_p(r_\varepsilon, R_\varepsilon)$ (see the definition (3.8)). One checks easily that, whatever the value of $p \in (1,\infty)$, we have

$$\lim_\varepsilon\ R_\varepsilon^p \int_\Omega f_\varepsilon^p\, dx = \lim_\varepsilon\ R_\varepsilon^p\, \varepsilon^{-2}\, \Gamma_p(r_\varepsilon, R_\varepsilon) = \ 0 \ ,$$

so that by (2.35)

$$\limsup_\varepsilon F_\varepsilon^{(2)}(u_\varepsilon) = \limsup_\varepsilon \int_\Omega\ f_\varepsilon^p\, |v-u|^p\, dx \ . \tag{2.36}$$

To evaluate the right hand side of (2.36), we need to consider two cases, according to the value of γ defined by (1.4):

Case 1: $\gamma < +\infty$ We find that (f_ε) is bounded in $L^p(\Omega)$ and that the sequence of Radon measures $f_\varepsilon^p\,dx$ converges tightly to $2\pi\,\gamma\,dx$ on Ω. Hence, since $|u-v| \in C^0(\overline{\Omega})$, (2.36) yields

$$\limsup_{\varepsilon} F_\varepsilon^{(2)}(u_\varepsilon) \leq 2\pi\,\gamma \int_\Omega |v-u|^p\,dx\ . \tag{2.37}$$

Case 2: $\gamma = +\infty$ As we have assumed that $\Phi(u,v) < +\infty$, we must have $u = v$, so that by (2.36), $\lim_\varepsilon F_\varepsilon^{(2)}(u_\varepsilon) = 0$ and (2.37) still holds (with the convention $0 \times (+\infty) = 0$).

Let us finally majorize the third term, that is $F_\varepsilon^3(u_\varepsilon)$. By (2.30)(2.31), we have

$$\nabla u_\varepsilon = \nabla w_\varepsilon = \left(0, 0, \frac{\partial w_\varepsilon}{\partial x_3}\right) \quad \text{a.e. on } T_\varepsilon\ . \tag{2.38}$$

Since v is smooth, we can use derivation under the integral to get

$$\frac{\partial w_\varepsilon}{\partial x_3} = \sum_i \left(\fint_{D_\varepsilon^i} \frac{\partial v}{\partial x_3}(\cdot,\cdot,x_3) \right) 1_{Y_\varepsilon^i}(x')\ .$$

From Jensen's inequality, we deduce

$$\left|\frac{\partial w_\varepsilon}{\partial x_3}\right|^p = \sum_i \left| \fint_{D_\varepsilon^i} \frac{\partial v}{\partial x_3}(\cdot,\cdot,x_3) \right|^p 1_{Y_\varepsilon^i}(x') \leq \sum_i \left| \fint_{D_\varepsilon^i} \left|\frac{\partial v}{\partial x_3}\right|^p(\cdot,\cdot,x_3) \right| 1_{Y_\varepsilon^i}(x')\ .$$

Integrating the last inequality on T_ε, we are led to

$$\fint_{T_\varepsilon} \left|\frac{\partial w_\varepsilon}{\partial x_3}\right|^p dx \leq \fint_{T_\varepsilon} \left|\frac{\partial v}{\partial x_3}\right|^p dx\ . \tag{2.39}$$

Noticing that the sequence of probability measures $\frac{1_{T_\varepsilon}}{|T_\varepsilon|}\,dx$ converges tightly to $\frac{1_\Omega}{|\Omega|}\,dx$, we may pass to the limit in the right hand side of (2.39), . Taking into account (2.38), we get

$$\limsup_{\varepsilon} \fint_{T_\varepsilon} |\nabla u_\varepsilon|^p\,dx \leq \fint_\Omega \left|\frac{\partial v}{\partial x_3}\right|^p dx\ . \tag{2.40}$$

Recalling (2.6) and the definition (1.4) of k, we distinguish two cases:

Case 1: $k < +\infty$ Then, by (2.40)

$$\text{(2.41)} \quad \limsup_{\varepsilon} F_{\varepsilon}^{3}(u_{\varepsilon}) = \frac{\pi|\Omega|}{p} \limsup_{\varepsilon} \ k_{\varepsilon} \fint_{T_{\varepsilon}} |\nabla u_{\varepsilon}|^{p}\, dx \leq \frac{k\pi}{p} \int_{\Omega} \left|\frac{\partial v}{\partial x_3}\right|^{p} dx \ .$$

Case 2: $k = +\infty$ Since $\Phi(u, v) < +\infty$, we must have $\frac{\partial v}{\partial x_3} = 0$ a.e. on Ω. In that case, by (2.30)(2.33), we find that, for every ε, $\nabla u_{\varepsilon} = 0$ on T_{ε}. Thus $F_{\varepsilon}^{(3)}(u_{\varepsilon}) = 0$ and (2.41) still holds.

Eventually, the proof of claim (2.28) is achieved since, collecting (2.34), (2.37) and (2.41), we get

$$\begin{aligned} \limsup_{\varepsilon} \frac{1}{p} \int_{\Omega} a_{\varepsilon}(x) |\nabla u_{\varepsilon}|^{p}\, dx &\leq \limsup_{\varepsilon} F_{\varepsilon}^{(1)}(u_{\varepsilon}) + \limsup_{\varepsilon} F_{\varepsilon}^{(2)}(u_{\varepsilon}) \\ &\quad + \limsup_{\varepsilon} F_{\varepsilon}^{(3)}(u_{\varepsilon}) \\ &\leq \Phi(u, v) \ . \end{aligned}$$

□

REMARK. Taking into account the lower bound inequality proved before, we have in fact $\lim_{\varepsilon} \int_{\Omega} a_{\varepsilon}(x) |\nabla u_{\varepsilon}|^{p} dx = \Phi(u, v)$. Moreover, by localizing this result on open subsets of Ω, we can prove, for every $\Psi \in C^{0}(\overline{\Omega})$, the following convergence:

$$\text{(2.42)} \quad \begin{aligned} &\lim_{\varepsilon} \int_{\Omega} a_{\varepsilon}(x) |\nabla u_{\varepsilon}|^{p}\, \Psi(x)\, dx \\ &= \int_{\Omega} \left(|\nabla u|^{p} + k\pi \left|\frac{\partial v}{\partial x_3}\right|^{p} + 2\pi\gamma |v - u|^{p} \right) \Psi(x)\, dx \ . \end{aligned}$$

Before going on with Step 2, let us introduce

$$\text{(2.43)} \qquad \Phi_s(u, v) := \inf \left\{ \liminf_{\varepsilon} F_{\varepsilon}(u_{\varepsilon}) \ ; \ (u_{\varepsilon}, v_{\varepsilon}) \xrightarrow{\tau} (u, v) \right\} \ ,$$

where τ denotes the product topology of $L^{p}(\Omega)$ strong by $\mathcal{M}_b(\Omega)$ weak*. Looking at the proof of the compactness, we see that , if $\Phi_s(u, v)$ is finite, then all approximating sequences defined by (2.43) are in a bounded subset of $L^{p}(\Omega) \times L^{1}(\Omega)$ which is metrizable for the topology τ. Therefore, by a classical diagonalization argument (see [1]) the infimum in (2.43) is achieved and the functional Φ_s is τ-lower semicontinuous. Then the upper bound inequality we have to prove reduces to

$$\text{(2.44)} \qquad \Phi_s(u, v) \ \leq \ \Phi(u, v) \ .$$

STEP 2. In this step, we prove (2.44) when u, v are smooth (and satisfy $\Phi(u,v) < +\infty$). Let $\Sigma_\varepsilon := \{x \in \Omega\ ;\ \operatorname{dist}(x, \Gamma_0) < r_\varepsilon\}$ and φ_ε a smooth function such that:

$$\varphi_\varepsilon = 1 \quad \text{on } \Gamma_0 \quad , \quad \varphi_\varepsilon = 0 \quad \text{on } \Omega_\varepsilon := \Omega \setminus \Sigma_\varepsilon \quad , \quad |\nabla \varphi_\varepsilon| \le \frac{C}{r_\varepsilon} \quad \text{on } \Omega \ .$$

We modify the approximating sequence (u_ε) defined by (2.31), by setting

$$u_\varepsilon^\# := u\,\varphi_\varepsilon \ + \ u_\varepsilon\,(1 - \varphi_\varepsilon) \ .$$

Clearly $u_\varepsilon^\# \to u$ in $L^p(\Omega)$ and $v_\varepsilon^\# := \frac{|\Omega|}{|T_\varepsilon|} 1_{T_\varepsilon} u\,\varphi_\varepsilon \ + v_\varepsilon\,(1 - \varphi_\varepsilon)$ converges weakly* to v in $\mathcal{M}_b(\Omega)$. As $\Phi(u,v) < +\infty$, we have $u_\varepsilon^\# = u = v = u_0$ on $\Gamma_0 \cap (\omega_0 \cap \omega_L)$. Since $u_\varepsilon = u$ on $\Omega \setminus (T_\varepsilon \cup B_\varepsilon)$ and we have assumed that the fibers do not intersect the lateral part of $\partial\Omega$, we have also $u_\varepsilon^\# = u = u_0$ on $\Gamma_0 \cap \partial\omega \times (0, L)$. Thus, according to (2.42),

$$\Phi_s(u, v) \le \limsup_\varepsilon F_\varepsilon(u_\varepsilon^\#) \ . \tag{2.45}$$

On the other hand, using (2.33) and the fact that $u = v$ on $\Gamma_0 \cap (\omega_0 \cap \omega_L)$, the following estimate holds on Σ_ε (recall that $u_\varepsilon = u$ on $\Omega \setminus (T_\varepsilon \cup B_\varepsilon)$)

$$|u_\varepsilon - u| \le \ |u - v| + |v - u_\varepsilon| \le \ C(r_\varepsilon 1_{T_\varepsilon} + R_\varepsilon 1_{B_\varepsilon}) \ .$$

Hence

$$\begin{aligned} |\nabla u_\varepsilon^\#|^p &\le C\left(|\nabla u|^p + |\nabla u_\varepsilon|^p + \frac{|u_\varepsilon - u|^p}{r_\varepsilon^p}\right) \\ &\le C\left(1 + |\nabla u_\varepsilon|^p + 1_{T_\varepsilon} + \left(\frac{R_\varepsilon}{r_\varepsilon}\right)^p 1_{B_\varepsilon}\right) \ . \end{aligned}$$

It follows that

$$\begin{aligned} &\int_{\Sigma_\varepsilon} a_\varepsilon(x) |\nabla u_\varepsilon^\#|^p \\ &\le C\left(|\Sigma_\varepsilon| + \int_{\Sigma_\varepsilon} a_\varepsilon(x)|\nabla u_\varepsilon|^p\,dx + k_\varepsilon |T_\varepsilon \cap \Sigma_\varepsilon| + \left(\frac{R_\varepsilon}{r_\varepsilon}\right)^p |\Sigma_\varepsilon|\right) \ . \end{aligned} \tag{2.46}$$

By (2.42) and denoting by f $(\in L^1(\Omega))$ the limit appearing in the right hand side of (2.42), we have $\limsup_\varepsilon \int_{\Sigma_\varepsilon} a_\varepsilon(x)\,|\nabla u_\varepsilon|^p\,dx \le \int_\Omega f\,\Psi\,dx$, for every $\Psi \in C^0(\overline{\Omega}, [0,1])$ such that $\Psi = 1$ on a small neibourghood of $\Gamma_0 \cap (\omega_0 \cup \omega_L)$. Hence, by taking a sequence (Ψ_k) such that $\Psi_k \searrow 0$, we get

$$\lim_\varepsilon \int_{\Sigma_\varepsilon} a_\varepsilon(x)|\nabla u_\varepsilon|^p\,dx = 0 \ .$$

On the other hand, by (1.3) and (1.7), we have $l_\varepsilon\, |T_\varepsilon \cap \Sigma_\varepsilon| \sim \pi\, k_\varepsilon\, r_\varepsilon \;\to 0$, and, choosing the sequence (R_ε) so that $r_\varepsilon << R_\varepsilon << (r_\varepsilon)^{1-\frac{1}{p}}$ (and $R_\varepsilon << \varepsilon$), we have $(\frac{R_\varepsilon}{r_\varepsilon})^p|\Sigma_\varepsilon| \to 0$. Thus the left hand side of (2.46) goes to 0 as $\varepsilon \to 0$. Then, by (2.45) and recalling (2.28), we conclude that

$$\begin{aligned}
\Phi_s(u,v) &\le \limsup_\varepsilon F_\varepsilon(u_\varepsilon^\#)\\
&\le \limsup_\varepsilon \frac{1}{p}\int_{\Sigma_\varepsilon} a_\varepsilon(x)|\nabla u_\varepsilon^\#|^p\,dx \;+ \limsup_\varepsilon \frac{1}{p}\int_{\Omega\setminus\Sigma_\varepsilon} a_\varepsilon(x)|\nabla u_\varepsilon|^p\,dx\\
&\le \limsup_\varepsilon \frac{1}{p}\int_\Omega a_\varepsilon(x)|\nabla u_\varepsilon|^p\,dx\\
&\le \Phi(u,v)\ .
\end{aligned}$$

□

STEP 3. Let us finally prove (2.43) in the general case. We may assume that $\Phi(u,v) < +\infty$ so that $(u,v) \in W^{1,p}(\Omega) \times L^p(\omega, W^{1,p}(0,L))$ and $u = u_0$ on Γ_0 and $v = u_0$ on $\Gamma_0 \cap (\omega_0 \cup \omega_L)$. By a standard approximation procedure, we can find sequences $(u_k), (v_k)$ in $C^1(\overline{\Omega})$ such that

$$\begin{aligned}
&u_k = u_0 \text{ on } \Gamma_0, \quad v_k = u_0 \text{ on } \Gamma_0 \cap (\omega_0 \cup \omega_L),\\
&(u_k, v_k) \to (u,v) \quad \text{strongly in } W^{1,p}(\Omega) \times L^p(\omega, W^{1,p}(0,L))\ .
\end{aligned}$$

Then one checks easily that $\Phi(u_k, v_k) \to \Phi(u,v)$. On the other hand, by Step 2, we have, for every k, $\Phi_s(u_k, v_k) \le \Phi(u_k, v_k)$. As the convergence of (u_k, v_k) to (u,v) holds also for the topology τ, by the lower semicontinuity of Φ_s, one gets

$$\Phi_s(u,v) \le \liminf_k \Phi_s(u_k,v_k) \le \limsup_k \Phi(u_k,v_k) \le \Phi(u,v)\ .$$

The Step 3 is completed and so the assertion iii) of Theorem B is proved. □

3. – APPENDIX

As in Section 2, in what follows, C denotes a suitable positive constant which may vary from line to line. The first lemma A1 connects the convergence of the new variable v_ε defined by (1.9) to the average behaviour of u_ε on the set of fibers T_ε.

LEMMA A1. i) *Let (u_ε) be a sequence in $L^1(\Omega)$ and $v \in C^0(\overline{\Omega})$ such that* $\lim_\varepsilon ⨍_{T_\varepsilon} |u_\varepsilon - v|dx = 0$. *Then (v_ε) defined by* (1.9) *is bounded in $L^1(\Omega)$ and $v_\varepsilon \to v$ weakly* in $\mathcal{M}_b(\Omega)$.*

ii) *Let* $(u_\varepsilon), (w_\varepsilon)$ *be sequences in* $L^1(\Omega)$ *such that:*

(3.1) $(w_\varepsilon)$is bounded in $L^1(\Omega)$,

(3.2) $\lim_\varepsilon \fint_{T_\varepsilon} |u_\varepsilon - w_\varepsilon| dx = 0$,

(3.3) $\begin{cases} \omega_\varepsilon(s) := \sup \left\{ |w_\varepsilon(x', s) - w_\varepsilon(y', s)| \,;\, i \in I_\varepsilon \,,\ (x', y') \in (Y_\varepsilon^i)^2 \right\} \\ \text{converges to } 0 \text{ in } L^1(0, L) \,. \end{cases}$

Then

$$\sup_\varepsilon \int_\Omega |v_\varepsilon| dx < +\infty \quad \text{and} \quad v_\varepsilon - w_\varepsilon \ \to \ 0 \quad \text{weakly* in } \mathcal{M}_b(\Omega) \ .$$

REMARK Condition (3.3) avoids strong oscillations of $(w_\varepsilon^{x_3})$ on the cells Y_ε^i. It is trivially satisfied in the case of the sequences $(\tilde{u}_\varepsilon), (\tilde{v}_\varepsilon)$ defined by (2.8), (2.9) and also if we take $w_\varepsilon = v$, v being a uniformly continuous function on Ω.

PROOF. By the previous remark, we have only to prove ii). By (3.2), the sequence $z_\varepsilon := \frac{|\Omega|}{|T_\varepsilon|} 1_{T_\varepsilon} w_\varepsilon$ satisfies $\lim_\varepsilon \|v_\varepsilon - z_\varepsilon\|_{L^1(\Omega)}) = 0$. Hence it is enough to show that (z_ε) is bounded in $L^1(\Omega)$ and that $z_\varepsilon - w_\varepsilon \ \to \ 0$ weakly* in $\mathcal{M}_b(\Omega)$. Thanks to (3.3), for a.e. $x_3 \in (0, L)$, the gap between the mean values on Y_ε^i of $w_\varepsilon^{x_3}$ and $z_\varepsilon^{x_3}$ (resp. $|w_\varepsilon|^{x_3}$ and $|z_\varepsilon|^{x_3}$) is majorized by $\omega_\varepsilon(x_3)$. Hence

$$\left| \int_{Y_\varepsilon^i} z_\varepsilon^{x_3} - \int_{Y_\varepsilon^i} w_\varepsilon^{x_3} \right| \le \varepsilon^2 \omega_\varepsilon(x_3) \tag{3.4}$$

$$\int_{Y_\varepsilon^i} |z_\varepsilon^{x_3}| \le \int_{Y_\varepsilon^i} |w_\varepsilon^{x_3}| + \varepsilon^2 \omega_\varepsilon(x_3) \tag{3.5}$$

Summing (3.5)with respect to $i \in I_\varepsilon$ and integrating with respect to x_3, we get:

$$\int_\Omega |z_\varepsilon(x)| dx \le \int_\Omega |w_\varepsilon(x)| dx + |\Omega| \fint_0^L \omega_\varepsilon(s) ds \ .$$

Thus, by (3.1), (z_ε) is bounded in $L^1(\Omega)$. On the other hand, let $\varphi \in C^0(\overline{\Omega})$ and its piecewise constant approximates defined by

$$\varphi_\varepsilon(x', x_3) = \sum_{i \in I_\varepsilon} \varphi_\varepsilon(x_\varepsilon^i, x_3) 1_{Y_\varepsilon^i}(x') \ ,$$

(recall that x_ε^i is the center of each bidimensional cell Y_ε^i). Then clearly $\varphi_\varepsilon \to \varphi$ uniformly on Ω and, by (3.4),

$$\left| \int_\Omega (z_\varepsilon - w_\varepsilon) \varphi_\varepsilon dx \right| \le \int_0^L \left(\sum_i \varepsilon^2 |\varphi(x_\varepsilon^i, s)| \right) \omega_\varepsilon(s) ds \le |\Omega| \, \|\varphi\|_\infty \fint_0^L \omega_\varepsilon(s) ds$$

so that

$$\begin{aligned}\left|\int_\Omega (z_\varepsilon - w_\varepsilon)\varphi dx\right| &\le \left|\int_\Omega (z_\varepsilon - w_\varepsilon)\varphi_\varepsilon dx\right| + \||\varphi_\varepsilon - \varphi|\|_\infty \int_\Omega |z_\varepsilon - w_\varepsilon| dx \\ &\le |\Omega|\ \|\varphi\|_\infty -\!\!\!\!\!\!\int_0^L \omega_\varepsilon(s)ds + \|\varphi_\varepsilon - \varphi\|_\infty \|z_\varepsilon - w_\varepsilon\|_{L^1(\Omega)} \ .\end{aligned}$$

By (3.3) and since $(z_\varepsilon - w_\varepsilon)$ is bounded in $L^1(\Omega)$, it follows that

$$\lim_\varepsilon \int_\Omega (z_\varepsilon - w_\varepsilon)\,\varphi\, dx = 0. \qquad \square$$

Our second lemma states, in a particular case, a lower bound inequality for convex functionals on measures (see [4], [7] for more general versions).

LEMMA A2. *Let μ_ε and μ bounded Radon measures in $\mathbb{R}^N$ such that $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$. Let (f_ε) be a sequence of μ_ε-measurable functions such that $\sup_\varepsilon \int |f_\varepsilon|^p d\mu_\varepsilon < +\infty$. Then:*

i) *The sequence of measures $f_\varepsilon\, \mu_\varepsilon$ is $*$-weakly relatively compact and every cluster point ℓ is of the form $\ell = f\,\mu$ with $f \in L^p_\mu(\mathbb{R}^N)$.*

ii) *If $f_\varepsilon\, \mu_\varepsilon \overset{*}{\rightharpoonup} f\,\mu$, then* $\liminf_\varepsilon \int |f_\varepsilon|^p d\mu_\varepsilon \ge \int |f|^p d\mu$.

PROOF. By Hölder inequality, we have $\int |f_\varepsilon|\, d\mu_\varepsilon \le \left(\int |f_\varepsilon|^p\, d\mu_\varepsilon\right)^{\frac{1}{p}} (\mu_\varepsilon(\mathbb{R}^N))^{\frac{1}{p'}}$, so that the sequence $f_\varepsilon\, \mu_\varepsilon$ is uniformly bounded in variation, hence $*$-weakly relatively compact. Possibly passing to a subsequence, we can assume that $f_\varepsilon\, \mu_\varepsilon \to \ell$. Applying this convergence to a test function $\varphi \in C_0(\mathbb{R}^N)$ and using Fenchel's inequality, we get

$$\text{(3.6)} \qquad \begin{aligned}\liminf_\varepsilon \frac{1}{p}\int |f_\varepsilon|^p d\mu_\varepsilon &\ge \liminf_\varepsilon \left(\int f_\varepsilon\, \varphi\, d\mu_\varepsilon \ - \ \frac{1}{p'}\int |\varphi|^p d\mu_\varepsilon\right) \\ &\ge \ < \ell, \varphi > \ - \frac{1}{p'}\int |\varphi|^p d\mu\end{aligned}$$

As the left hand side of (3.6) is bounded, we deduce that

$$\sup\{< \ell, \varphi >; \varphi \in C_0, \|\varphi\|_{L^{p'}} \le 1\} < +\infty \ .$$

Thus ℓ, seen as an element of the dual space of $L^{p'}(\mathbb{R}^N)$, can be identified with an element $f \in L^p(\mathbb{R}^N)$ (i.e. $\ell = f\,\mu$). So (3.6) can be rewritten as:

$$\text{(3.7)} \qquad \liminf_\varepsilon \frac{1}{p}\int |f_\varepsilon|^p d\mu_\varepsilon \ge \int f\,\varphi\, d\mu \ - \ \frac{1}{p'}\int |\varphi|^{p'} d\mu \ .$$

Choosing φ in (3.7) converging to $|f|^{p-2}f$ in $L^{p'}$, we get the lower bound of the assertion ii). $\square$

In the next two lemmas, we are in $\mathbb{R}^2$ and denote for every $r > 0$ and $0 < r_1 < r_2$:

$$D_r \coloneqq \{x' \in \mathbb{R}^2 \ ; \ |x'| < r\} \ , \ C_r = \partial D_r \ , \ D(r_1, r_2) \coloneqq \{x' \in \mathbb{R}^2 \ ; r_1 < |x'| < r_2\} \ .$$

Lemma A3, crucial in the proof of the lower bound inequality, leads to a variant of Poincaré's inequality (Lemma A4).

LEMMA A3. *For every $u \in W^{1,p}(D(r_1, r_2))$, we have*

$$\int_{D(r_1,r_2)} |\nabla u|^p dx \ \geq \ 2\pi \, \Gamma_p(r_1, r_2) \left| \fint_{C_{r_2}} u \, ds - \fint_{C_{r_1}} u \, ds \right|^p ,$$

where

$$\Gamma_p(r_1, r_2) := \begin{cases} \dfrac{1}{\mathrm{Log}(\frac{r_2}{r_1})} & \text{if } p = 2 \\ \left(\dfrac{s}{r_2^s - r_1^s}\right)^{p-1} & \text{if } p \neq 2 \quad (s := \dfrac{p-2}{p-1}) \end{cases} \tag{3.8}$$

and $\fint_{C_r} u \, ds$ denotes the mean value of the trace of u on C_r with respect to the one dimensional Hausdorff measure.

PROOF. Using polar coordinates (r, θ), we can write

$$\int_{D(r_1,r_2)} |\nabla u|^p dx \ \geq \ \int_0^{2\pi} \int_{r_1}^{r_2} \left| \frac{\partial u}{\partial r} \right|^p r dr d\theta \ .$$

By a straightforward computation, we find that:

$$\Gamma_p(r_1, r_2) = \inf \left\{ \int_{r_1}^{r_2} |\varphi'|^p \, r dr \ ; \ \varphi(r_1) = 0 \ , \ \varphi(r_2) = 1 \right\}$$

(just solve the Euler equation of this one dimensional problem). Hence, for every $\theta \in [0, 2\pi)$, the following lower bound holds:

$$\int_{r_1}^{r_2} \left| \frac{\partial u}{\partial r} \right|^p r dr \ \geq \ \Gamma_p(r_1, r_2) \, |u(r_1, \theta) - u(r_2, \theta)|^p \ .$$

Integrating with respect to θ and using Jensen's inequality, we conclude that:

$$\begin{aligned} \int_{D(r_1,r_2)} |\nabla u|^p dx &\geq 2\pi \, \Gamma_p(r_1, r_2) \fint_0^{2\pi} |u(r_1, \theta) - u(r_2, \theta)|^p \, d\theta \\ &\geq 2\pi \, \Gamma_p(r_1, r_2) \left| \fint_{C_{r_2}} u \, ds - \fint_{C_{r_1}} u \, ds \right|^p . \end{aligned}$$

□

LEMMA A4. *There exists a constant $C > 0$ such that, for every $(R, \alpha) \in \mathbb{R}_+ \times (0, 1)$*

$$\forall u \in W^{1,p}(D_R) \quad , \quad \int_{D_R} \left| u - \fint_{C_{\alpha R}} u\, ds \right|^p dx \leq C \frac{R^p}{h(\alpha)} \int_{D_R} |\nabla u|^p dx \, .$$

where

$$h(\alpha) := \begin{cases} \alpha^{2-p} & \textit{if } 1<p<2 \\ \dfrac{1}{1 + |\log \alpha|} & \textit{if } p = 2 \\ 1 & \textit{if } p > 2 \end{cases} \tag{3.9}$$

PROOF. We need only to prove the inequality for $R = 1$, since the general case is deduced by making the change of variable $y = \frac{x}{R}$. Denote for every $r \in (0, 1)$, $\widehat{u}(r) := \fint_{C_r} u\, ds$. By Lemma A3, we have for every $r \in (0, 1)$

$$|\widehat{u}(r) - \widehat{u}(\alpha)|^p \leq \frac{C}{\Gamma_p(r, \alpha)} \int_{D_1} |\nabla u|^p\, dx \, . \tag{3.10}$$

On the other hand, integrating in polar coordinates on the disk D_1,

$$[u] := \fint_{D_1} u\, dx = \int_0^1 \widehat{u}(r) 2r\, dr \, .$$

Thus, by (3.10) and Hölder inequality

$$\begin{aligned} |[u] - \widehat{u}(\alpha)|^p &= \left| \int_0^1 [\widehat{u}(r) - \widehat{u}(\alpha)]\, 2r\, dr \right|^p \\ &\leq \int_0^1 |\widehat{u}(r) - \widehat{u}(\alpha)|^p\, 2r\, dr \\ &\leq C \left(\int_0^1 \frac{2r}{\Gamma_p(r, \alpha)} dr \right) \int_{D_1} |\nabla u|^p\, dx. \end{aligned}$$

Recalling (3.8) and using the substitution in $r = \alpha\rho$, we get

$$\int_0^1 \frac{r}{\Gamma_p(r, \alpha)} dr = \begin{cases} \dfrac{\alpha^p}{|s|^{p-1}} \displaystyle\int_0^{\frac{1}{\alpha}} \rho |\rho^s - 1|^{p-1}\, d\rho & \text{if } p \neq 2 \\ \alpha^2 \displaystyle\int_0^{\frac{1}{\alpha}} \rho |\mathrm{Log} \rho|\, d\rho & \text{if } p = 2 \, , \end{cases}$$

so that

$$|[u] - \widehat{u}(\alpha)|^p \leq \frac{C}{h(\alpha)} \int_{D_1} |\nabla u|^p\, dx \, .$$

From the classical Poincaré-Wirtinger inequality

$$\int_{D_1} |u - [u]|^p \, dx \le C \int_{D_1} |\nabla u|^p \, dx \ ,$$

we deduce

$$\int_{D_1} |u - \widehat{u}(\alpha)|^p \, dx \le 2^{p-1} \left[\int_{D_1} |u - [u]|^p \, dx \ + \int_{D_1} |[u] - \widehat{u}(\alpha)|^p \, dx \right]$$
$$\le C \left(1 + \frac{1}{h(\alpha)} \right) \int_{D_1} |\nabla u|^p \, dx \ .$$

Since $h(\alpha)$ is bounded on $(0, 1]$, the result follows for $R = 1$. □

A5 - Justification of (1.13). By (1.12), we have $F(u) = \Phi(u, v)$ where, for a.e. $x' \in \omega$, $v(x', \cdot)$ is the solution of the following one dimensional boundary value problem on $(0, L)$:

$$\left\{ \begin{array}{l} -w'' + \dfrac{2\gamma}{k} w = \dfrac{2\gamma}{k} u(x', \cdot) \\ w(0) = w(L) = 0 \end{array} \right. \tag{3.11}$$

The solution of (3.11) is given by

$$w(s) = \int_0^L G(s, t) u(x', t) \, dt \ , \tag{3.12}$$

where, for every $t \in (0, L)$, the Poisson kernel $G(\cdot, t)$ solves the equation

$$\left\{ \begin{array}{l} -\varphi'' + c_0^2 \varphi = c_0^2 \, \delta_t \qquad \left(c_0 = \sqrt{\dfrac{2\gamma}{k}} \right) \\ \varphi(0) = \varphi(L) = 0 \end{array} \right.$$

We find

$$G(s, t) = \frac{c_0 \sinh(c_0(L - s \vee t)) \sinh(c_0 \, (s \wedge t))}{\sinh(c_0 L)} \ . \tag{3.13}$$

Integrating by parts and thanks to (3.11), we obtain

$$\begin{aligned} k \pi \int_\Omega |\frac{\partial v}{\partial x_3}|^2 \, dx &= k \pi \int_\omega \left(\int_0^L |\frac{\partial v}{\partial x_3}|^2 \, dx_3 \right) dx' \\ &= k \pi \, c_0^2 \int_\omega \left(\int_0^L (-(v(x', s))^2 + u(x', s) v(x', s)) \, ds \right) dx' \\ &= 2 \pi \gamma \int_\Omega (uv - v^2) \, dx \end{aligned}$$

Thus, by (1.6) and substituting v by its expression (3.12), we get

$$
\begin{aligned}
2F(u) &= 2\Phi(u,v) \\
&= \int_\Omega |\nabla u|^2\,dx + 2\pi\gamma\int_\Omega u^2\,dx - 2\pi\gamma\int_\Omega uv\,dx \\
&= \int_\Omega |\nabla u|^2\,dx + 2\pi\gamma\int_\Omega u^2\,dx - 2\pi\gamma \\
&\quad\times \int_\omega \left(\int_{(0,L)^2} u(x',s)\,v(x',t)\,G(s,t)\,ds\,dt\right)dx' \\
&= \int_\Omega |\nabla u|^2\,dx + \pi\gamma\int_\omega\left(\int_{(0,L)^2}(u(x',s)-v(x',t))^2\,G(s,t)\,ds\,dt\right)dx' \\
&\quad + 2\pi\gamma\int_\omega\left[\int_0^L (u(x',s))^2\left(1-\int_0^L G(s,t)dt\right)ds\right]dx'\,.
\end{aligned}
$$

Replacing $G(s,t)$ by its explicit form (3.13), we deduce (1.13) after an easy computation. □

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Laboratoire ANLA UFR Science
Université de Toulon et du Var
BP 132 83957 La Garde Cedex, France
bouchitte@univ-tln.fr, Fax:94 14 26 33