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Homogenization of Nonstationary Navier-Stokes Equations in a Domain with a Grained Boundary (*).

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Summary. - *We prove the convergence of the homogenization process for a nonstationary Navier-Stokes system in a porous medium. The result of homogenization is Darcy's law, as in the case of the Stokes equation, but the convergence of pressures is in a different function space.*

1. - Introduction.

In this paper we prove the convergence of the homogenization process for nonstationary Navier-Stokes equations in a porous medium under a nonhomogeneous boundary condition. Our proof is based on new a priori estimates for a pressure in $L_2(0, T; L_\beta(\Omega))$, $\beta \in (1, n/(n-1))$. The result of homogenization is Darcy's law, as in the case of homogenization for the Stokes equation (see TARTAR [7]). The convergence of velocities will be in the same space, but the convergence of pressures will be much weaker.

We use the standard notation required in homogenization theory. Let $Y =]0,1[^n$, $\mathcal{O} \subset Y$, be an open set strictly contained in Y and locally placed on one side of its boundary S (the boundary being a smooth $(n-1)$-dimensional manifold) and $Y^* = Y \setminus \mathcal{O}$. Let

$$Y_k = Y + k, \quad \mathcal{O}_k = \mathcal{O} + k, \quad Y_k^* = Y^* + k, \qquad k \in \boldsymbol{Z}^n .$$

Let $\Omega \subset \boldsymbol{R}^n$ be a bounded domain, locally placed on one side of its boundary Γ (being a piecewise smooth $(n-1)$-dimensional manifold). For sufficiently small $\varepsilon > 0$, we consider the sets

$$T_\varepsilon = \{k \in \boldsymbol{Z}^n : \varepsilon Y_k \subset \Omega\}, \quad K_\varepsilon = \{k \in \boldsymbol{Z}^n : \varepsilon Y_k \cap \Gamma \neq \emptyset\},$$

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and define

$$\mathcal{O}_\varepsilon = \bigcup_{k\in T_\varepsilon} \varepsilon\mathcal{O}_k\,, \quad S_\varepsilon = \partial\mathcal{O}_\varepsilon\,, \quad \Omega_\varepsilon = \Omega\setminus\overline{\mathcal{O}}_\varepsilon\,.$$

Obviously,

$$\overline{\Omega}_\varepsilon = \bigcup_{k\in T_\varepsilon}(\varepsilon\overline{Y}_k^*)\bigcup_{k\in T_\varepsilon}(\varepsilon\overline{Y}_k\cap\overline{\Omega}), \qquad \partial\Omega_\varepsilon = \Gamma\cup S_\varepsilon\,.$$

The domains $\mathcal{O}_\varepsilon$ and Ω_ε represent, respectively, the solid and fluid parts of a porous medium Ω. We shall consider the nonstationary flow of an incompressible viscous fluid in the domain Ω_ε, in two and three dimensions.

Let v^ε, p^ε and $\mu>0$ be the velocity, the pressure and the viscosity of the fluid, respectively, and let f^ε be the density of an external body force. Then v^ε and p^ε satisfy the conditions

$$\frac{\partial v^\varepsilon}{\partial t} + (v^\varepsilon\nabla)v^\varepsilon + \nabla p^\varepsilon - \mu\Delta v^\varepsilon = f^\varepsilon \quad \text{in } Q_{\varepsilon T} = \Omega_\varepsilon\times(0,T), \tag{1.1}$$

$$\operatorname{div} v^\varepsilon = 0 \quad \text{in } Q_{\varepsilon T}, \tag{1.2}$$

$$v^\varepsilon = h \quad \text{on } \Gamma_T = \Gamma\times(0,T), \tag{1.3}$$

$$v^\varepsilon = 0 \quad \text{on } S_{\varepsilon T} = S_\varepsilon\times(0,T), \tag{1.4}$$

$$v^\varepsilon(x,0) = v_0^\varepsilon(x) \quad \text{in } \Omega_\varepsilon, \tag{1.5}$$

where v_0^ε is a given initial velocity and h is a given function satisfying the condition

$$\int h(\cdot,t)\cdot\nu d\Gamma = 0.$$

Here ν denotes the unit outer normal on Γ.

We use the usual notation for Sobolev spaces from [3, 4, 8]. In particular, following [3] for a given bounded domain $D\subset R^n$, locally placed on one side of its boundary $\partial D\in C^{1,1}$ and for $1\leqslant\beta\leqslant+\infty$, we introduce the Banach spaces

$$W_\beta^1(D) = \{z\in L_\beta(D)\colon \nabla z\in L_\beta(D)^n\},$$

$$\overset{\circ}{W}{}_\beta^1(D) = \{z\in W_\beta^1(D)\colon z=0 \ \text{ on } \ \partial D\},$$

$$L_\beta^0(D) = \left\{z\in L_\beta(D)\colon \int_D z = 0\right\}, \tag{1.6}$$

$$W_{\beta,S}^1(D) = \{z\in W_\beta^1(D)\colon z=0 \ \text{ on } S, \ \text{being an open subset of } \partial D\}, \tag{1.7}$$

$$V_\beta(D) = \{z\in \overset{\circ}{W}{}_\beta^1(D)^n\colon \operatorname{div} z = 0\}. \tag{1.8}$$

The dual space of $\overset{\circ}{W}{}^1_\beta(D)$ is denoted by $_\circ W_\delta^{-1}(D)$, where $\beta^{-1}+\delta^{-1}=1$. For $\beta=2$, we use the notation $H^1(D)=W_2^1(D)$, $H_0^1(D)=W_2^1(D)$, $H_S^1(D)=W_{2,S}^1(D)$ and $H^{-1}(D)=W_2^{-1}(D)$. Furthermore, we define $H^{1/2}(\partial D)$ as an image of $H^1(D)$ under the trace operator.

Let X be a given Banach space. The usual spaces of β-integrable, essentially bounded and continuous functions $z:[0,T]\to X$ we denote by $L_\beta(0,T;X)$, $L_\infty(0,T;X)$ and $C([0,T];X)$, respectively.

We make the following assumptions on data:

$$h\in C([0,T];W_\infty^1(\Omega)^n),\quad \frac{\partial h}{\partial t}\in C([0,T];W_6^1(\Omega)^n),\quad \operatorname{div} h=0, \tag{1.9}$$

$$\|\varepsilon v_0^\varepsilon\|_{L_2(\Omega_\varepsilon)^n}\leq C,\quad \operatorname{div} v_0^\varepsilon=0 \text{ in } \Omega_\varepsilon, \tag{1.10}$$

$$f^\varepsilon\in L_2(Q_{\varepsilon T})^n,\quad \varepsilon^2 f^\varepsilon\to f\quad \text{in } L_2(Q_T)^n,\quad \text{when } \varepsilon\to 0. \tag{1.11}$$

The crucial quantity in our estimates is $A(v_0^\varepsilon, f^\varepsilon, h)$ defined by

$$A^2(v_0^\varepsilon,f^\varepsilon,h)=\int_\Omega |v_0^\varepsilon|^2+\int_0^t\int_\Omega\left[|\varepsilon f^\varepsilon|^2+|\nabla h|^2+\left|\frac{1}{\varepsilon}h\right|^2+\left|\varepsilon\frac{\partial h}{\partial t}\right|^2+\left|\varepsilon^2\frac{\partial}{\partial t}\nabla h\right|^2\right]+ \\ +\sup_{0\leq t\leq T}\left[\|\varepsilon\nabla h(t)\|^2_{L_6(\Omega)^{n^2}}+\|h(t)\|^2_{L_6(\Omega)^n}\right]\left(1+\frac{1}{\varepsilon^2}\right). \tag{1.12}$$

The assumptions from (1.9) to (1.11) imply

$$A(v_0^\varepsilon,f^\varepsilon,h)\leq C/\varepsilon. \tag{1.13}$$

Under the assumptions (1.9)-(1.11), classical theory (see [4] or [8]) gives the existence of at least one weak solution $v^\varepsilon\in L_2(0,T;H^1(\Omega_\varepsilon)^n)$, $v^\varepsilon-h\in L_2(0,T;V_2(\Omega_\varepsilon))$, $v^\varepsilon\in L_\infty(0,T;L_2(\Omega_\varepsilon)^n)$, $(\partial/\partial t)(v^\varepsilon-h)\in L_{4/3}(0,T;(V_2(\Omega_\varepsilon))')$ for (1.1)-(1.5).

For brevity, we denote by C a generic constant not depending on ε and possibly having different values at different places.

2. – The operator R_q^ε.

In order to extend the pressure p^ε to the whole medium Ω, TARTAR ([7]) introduce an operator R^ε. It was an operator between $H^1(\Omega)^n$ and $H^1_{S_\varepsilon}(\Omega_\varepsilon)^n$. We need an operator between $W_q^1(\Omega)^n$ and $W^1_{q,S_\varepsilon}(\Omega_\varepsilon)^n$ with similar properties.

LEMMA 2.1. – There exists an operator

$$R_q\in\mathcal{L}(W_q^1(Y)^n,\ W^1_{q,S}(Y^*)^n),\qquad 1<q<+\infty, \tag{2.1}$$

with the properties

$$R_q w=w\quad \text{in a neighborhood of } \partial Y, \tag{2.2}$$

$$w=0\quad \text{on } S\Rightarrow R_q w=w, \tag{2.3}$$

$$\operatorname{div} w=0\Rightarrow \operatorname{div}(R_q w)=0. \tag{2.4}$$

REMARK 2.1. – The proof of Lemma 2.1 is very similar to that given in [7] for the case $q=2$. In addition to the technique used in [7], one needs L_q-regularity for the Stokes equation (see [2] or [8]). The same remark applies to other lemmas in this section.

For $w \in W_q^1(\Omega)^n$, $1<q<+\infty$, we define

$$w_k^\varepsilon(y) = w(\varepsilon y), \quad y \in Y_k, \quad k \in T_\varepsilon .$$

LEMMA 2.2. – The operator R_q^ε, defined by the formula

$$(R_q^\varepsilon w)(x) = \begin{cases} (R_q w_k^\varepsilon)(x/\varepsilon), & x \in \varepsilon Y_k^*, \quad k \in T_\varepsilon , \\ w(x), & x \in \varepsilon Y_k , \quad k \in K_\varepsilon , \end{cases} \tag{2.5}$$

has the properties

$$R_q^\varepsilon \in \mathcal{L}(W_q^1(\Omega)^n, W_{q,S_\varepsilon}^1(\Omega_\varepsilon)^n), \tag{2.6}$$

$$w=0 \quad \text{on } S_\varepsilon \Rightarrow R_q^\varepsilon w = w, \tag{2.7}$$

$$\operatorname{div} w = 0 \Rightarrow \operatorname{div}(R_q^\varepsilon w) = 0, \tag{2.8}$$

$$\|R_q^\varepsilon w\|_{L_q(\Omega_\varepsilon)^n} \leq C\{\|w\|_{L_q(\Omega)^n} + \varepsilon\|\nabla w\|_{L_q(\Omega)^{n^2}}\}, \tag{2.9}$$

$$\|\nabla(R_q^\varepsilon w)\|_{L_q(\Omega_\varepsilon)^{n^2}} \leq C\left\{\frac{1}{\varepsilon}\|w\|_{L_q(\Omega)^n} + \|\nabla w\|_{L_q(\Omega)^{n^2}}\right\}. \tag{2.10}$$

Because of (2.1), (2.6), one can consider $R_q w (R_q^\varepsilon w)$ extended by zero to $Y(\Omega)$.

LEMMA 2.3. – For each $w \in \overset{\circ}{W}{}_q^1(\Omega_\varepsilon)^n$, the inequality

$$\|w\|_{L_q(\Omega_\varepsilon)^n} \leq C\varepsilon\|\nabla w\|_{L_q(\Omega_\varepsilon)^{n^2}}, \qquad 1<q<+\infty, \tag{2.11}$$

holds true.

LEMMA 2.4. – Let $w \in L_\infty(0,T;W_q^1(\Omega)^n)$, $\partial w/\partial t \in L_\infty(0,T;W_q^1(\Omega)^n)$ and let R_q^ε be given by (2.5). Then one has

$$R_q^\varepsilon \in \mathcal{L}(L_\infty(0,T;W_q^1(\Omega)^n), \ L_\infty(0,T;W_{q,S_\varepsilon}^1(\Omega_\varepsilon)^n)), \tag{2.12}$$

$$\|R_q^\varepsilon w\|_{L_\infty(0,T;L_q(\Omega_\varepsilon)^n)} + \varepsilon\|\nabla R_q^\varepsilon w\|_{L_\infty(0,T;L_q(\Omega_\varepsilon)^{n^2})} \leq \leq C\{\|w\|_{L_\infty(0,T;L_q(\Omega)^n)} + \varepsilon\|\nabla w\|_{L_\infty(0,T;L_q(\Omega)^{n^2})}\}, \tag{2.13}$$

$$\left\|\frac{\partial}{\partial t}R_q^\varepsilon w\right\|_{L_\infty(0,T;L_q(\Omega_\varepsilon)^n)} \leq C\left\{\left\|\frac{\partial w}{\partial t}\right\|_{L_\infty(0,T;L_q(\Omega)^n)} + \varepsilon\left\|\nabla\frac{\partial w}{\partial t}\right\|_{L_\infty(0,T;L_q(\Omega)^{n^2})}\right\}. \tag{2.14}$$

3. – Uniform estimates.

LEMMA 3.1. – Let v^ε be a weak solution for (1.1)-(1.5) and let ε be «small enough» [1]. Then we have

$$\|v^\varepsilon\|_{L_2(Q_T)^n} \leq C, \tag{3.1}$$

$$\|\nabla v^\varepsilon\|_{L_2(Q_T)^{n^2}} \leq C/\varepsilon, \tag{3.2}$$

$$\|v^\varepsilon\|_{L_\infty(0,T;L_2(\Omega)^n)} \leq C/\varepsilon. \tag{3.3}$$

PROOF. – We define the function b^ε by setting

$$b^\varepsilon = R_6^\varepsilon h. \tag{3.4}$$

A direct consequence of Lemmas 2.2 and 2.4 is as follows:

$$\begin{cases} \operatorname{div} b^\varepsilon = 0, \\ \|\nabla b^\varepsilon\|_{L_\infty(0,T;L_6(\Omega)^{n^2})} \leq C/\varepsilon, \\ \|b^\varepsilon\|_{L_\infty(0,T;L_6(\Omega)^n)} \leq C, \\ \left\|\dfrac{\partial b^\varepsilon}{\partial t}\right\|_{L_\infty(0,T;L_6(\Omega)^n)} \leq C. \end{cases} \tag{3.5}$$

We introduce a function u^ε by setting

$$u^\varepsilon = v^\varepsilon - b^\varepsilon. \tag{3.6}$$

Then $\{u^\varepsilon, p^\varepsilon\}$ is a solution for

$$\begin{cases} \nabla p^\varepsilon + \dfrac{\partial u^\varepsilon}{\partial t} - \mu\Delta u^\varepsilon + (u^\varepsilon\nabla)u^\varepsilon + (b^\varepsilon\nabla)u^\varepsilon + (u^\varepsilon\nabla)b^\varepsilon = \\ \qquad = f^\varepsilon - \dfrac{\partial b^\varepsilon}{\partial t} + \mu\Delta b^\varepsilon - (b^\varepsilon\nabla)b^\varepsilon & \text{in } Q_{\varepsilon T}, \\ \operatorname{div} u^\varepsilon = 0 & \text{in } Q_{\varepsilon T}, \\ u^\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \times (0,T), \\ u^\varepsilon(x,0) = v_0^\varepsilon(x) - b^\varepsilon(x,0) & \text{in } \Omega_\varepsilon. \end{cases} \tag{3.7}$$

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(1) See (3.8) and (3.9).

Using the energy inequality which corresponds to (3.7) (see [8], p. 291) and the inequalities

$$(3.8)\qquad \begin{cases} \left|\int_0^t\int_{\Omega_\varepsilon}(u^\varepsilon\nabla)b^\varepsilon u^\varepsilon\right| \leqslant n^2\varepsilon^{1/2}\|\nabla u^\varepsilon\|^2_{L^2(Q_t)^{n^2}} C(\Omega,S)[\varepsilon\|\nabla h\|_{L_2(Q_t)^{n^2}} + \|h\|_{L_2(Q_t)^n}], \\ \left|\int_0^t\int_{\Omega_\varepsilon}(b^\varepsilon\nabla)b^\varepsilon u^\varepsilon\right| \leqslant \frac{\mu}{8}\|\nabla u^\varepsilon\|^2_{L_2(Q_t)^{n^2}} + C\|\varepsilon\nabla b^\varepsilon\|^2_{L_\infty(0,T;L_6(\Omega)^{n^2})}, \end{cases}$$

one obtains

$$(3.9)\qquad \int_\Omega |u^\varepsilon(t)|^2 + (\mu - C(n,\Omega,S,\|h\|_{L_2(0,T;H^1(\Omega))})\varepsilon^{1/2})\int_0^t\int_\Omega |\nabla u^\varepsilon|^2 \leqslant$$

$$\leqslant C\cdot\left\{\int_\Omega |u^\varepsilon(0)|^2 + \int_0^t\int_\Omega\left[|\varepsilon f^\varepsilon|^2 + \left|\varepsilon\frac{\partial b^\varepsilon}{\partial t}\right|^2 + |\nabla b^\varepsilon|^2\right] + \|\varepsilon\nabla b^\varepsilon\|^2_{L_\infty(0,T;L_6(\Omega)^{n^2})}\right\}.$$

For ε smaller than some fixed ε_0, backward substitution, $v^\varepsilon = u^\varepsilon + b^\varepsilon$, implies

$$\int_\Omega |v^\varepsilon(t)|^2 + \int_0^t\int_\Omega |\nabla v^\varepsilon|^2 \leqslant$$

$$\leqslant C\cdot\left\{\int_\Omega |v_0^\varepsilon|^2 + \int_0^t\int_\Omega\left[|\varepsilon f^\varepsilon|^2 + |\nabla h|^2 + \left|\frac{1}{\varepsilon}h\right|^2 + \left|\varepsilon\frac{\partial h}{\partial t}\right|^2 + \left|\varepsilon^2\frac{\partial}{\partial t}\nabla h\right|^2\right] + \right.$$

$$\left. + \sup_{0\leqslant t\leqslant T}[\|\varepsilon\nabla h(t)\|^2_{L_6(\Omega)^{n^2}} + \|h(t)\|^2_{L_6(\Omega)^n}]\left(1 + \frac{1}{\varepsilon^2}\right)\right\} \leqslant CA^2(v_0^\varepsilon, f^\varepsilon, h) \leqslant C/\varepsilon^2. \qquad \text{Q.E.D.}$$

Our next step is to introduce a pressure. Following [8], p. 307, we introduce new functions U^ε, F^ε and Φ^ε and set

$$(3.10)\qquad U^\varepsilon(t) = \int_0^t v^\varepsilon(s)\,ds, \qquad F^\varepsilon(t) = \int_0^t f^\varepsilon(s)\,ds, \qquad \Phi^\varepsilon(t) = \int_0^t (v^\varepsilon(s)\nabla)v^\varepsilon(s)\,ds.$$

If v^ε is a weak solution for (1.1)-(1.5), then $U^\varepsilon \in C([0,T];H^1(\Omega_\varepsilon)^n)$ div $U^\varepsilon = 0$ and F^ε, $\Phi^\varepsilon \in C([0,T];V_2')$. The theory developed in TEMAM [8], p. 307, shows that there exist $P^\varepsilon \in C([0,T];L_2(\Omega_\varepsilon))$, $\nabla P^\varepsilon \in C([0,T];H^{-1}(\Omega_\varepsilon)^n)$, such that

$$(3.11)\qquad \nabla P^\varepsilon(t) + v^\varepsilon(t) - v_0^\varepsilon - \mu\Delta U^\varepsilon(t) + \Phi^\varepsilon(t) = F^\varepsilon(t) \quad \text{in } V_2', \quad \forall t\in[0,T].$$

LEMMA 3.2. – Let all assumptions from Lemma 3.1 be fulfilled. Then there exists an extension $\tilde{P}^\varepsilon \in C([0,T]; L^0_\beta(\Omega))$, $\forall \beta \in (1, n/(n-1))$ of the function P^ε such that the inequalities

$$\|\tilde{P}^\varepsilon\|_{C([0,T];L^0_\beta(\Omega))} \leq C/\varepsilon^2, \tag{3.12}$$

$$\|\nabla\tilde{P}^\varepsilon\|_{C([0,T];W^{-1}_\beta(\Omega)^n)} \leq C/\varepsilon^2, \tag{3.13}$$

hold true.

PROOF. – Let $\varphi \in \overset{\circ}{W}{}^1_\gamma(\Omega_\varepsilon)^n$, $\gamma > n$. Then the inequality

$$|\langle \Phi^\varepsilon(t), \varphi\rangle_{\Omega_\varepsilon}| \leq C\varepsilon^{1+\zeta}\|\nabla v^\varepsilon\|^2_{L_2(Q_t)^{n^2}}\|\nabla\varphi\|_{L_\gamma(\Omega_\varepsilon)^{n^2}},$$

where $0 < \zeta < 1 - n/\gamma$, implies

$$|\langle \nabla P^\varepsilon(t), \varphi\rangle_{\Omega_\varepsilon}| \leq C\{\varepsilon\|f^\varepsilon\|_{L_2(Q_T)^n} + \|v_0^\varepsilon\|_{L_2(\Omega)^n} + \varepsilon\|v^\varepsilon\|_{L_\infty(0,T;L_2(\Omega)^n)} + \|\nabla v^\varepsilon\|_{L_2(Q_T)^{n^2}} +$$
$$+\varepsilon^{1+\zeta}\|\nabla v^\varepsilon\|^2_{L_2(Q_T)^{n^2}}\}\|\nabla\varphi\|_{L_\gamma(\Omega_\varepsilon)^{n^2}}.$$

Therefore we may conclude

$$|\langle \nabla P^\varepsilon(t), \varphi\rangle_{\Omega_\varepsilon}| \leq \frac{C}{\varepsilon}\|\nabla\varphi\|_{L_\gamma(\Omega_\varepsilon)^{n^2}}, \qquad \forall\varphi \in \overset{\circ}{W}{}^1_\gamma(\Omega_\varepsilon)^n$$

and, consequently,

$$\left|\int_{\Omega_\varepsilon} P^\varepsilon(t)\,\mathrm{div}\,\varphi\right| = |\langle \nabla P^\varepsilon(t), \varphi\rangle_{\Omega_\varepsilon}| \leq \frac{C}{\varepsilon}\|\nabla\varphi\|_{L_\gamma(\Omega_\varepsilon)^{n^2}}, \qquad \forall\varphi \in \overset{\circ}{W}{}^1_\gamma(\Omega_\varepsilon)^n. \tag{3.14}$$

It is well known that $\forall g \in L^0_q(\Omega)$, $1 < q < +\infty$; there exists $\psi \in \overset{\circ}{W}{}^1_q(\Omega)^n$ such that $\mathrm{div}\,\psi = g$ and $\|\psi\|_{W^1_q(\Omega)^n} \leq C\|g\|_{L^0_q(\Omega)}$ (see [8], p. 35). We use this fact in order to extend the pressure. Let us define $\tilde{P}^\varepsilon \in L^0_\beta(\Omega)$, $1/\beta + 1/\gamma = 1$ by setting

$$\int_\Omega \tilde{P}^\varepsilon(t) g = \int_\Omega \tilde{P}^\varepsilon(t)\,\mathrm{div}\,\psi \equiv \int_{\Omega_\varepsilon} P^\varepsilon(t)\,\mathrm{div}\,(R^\varepsilon_\gamma\psi), \qquad \forall g \in L^0_\gamma(\Omega) \text{ and } \forall t \in [0,T]. \tag{3.15}$$

A direct consequence of (3.14) is

$$\left|\int_\Omega \tilde{P}^\varepsilon(t) g\right| \leq \frac{C}{\varepsilon^2}\|g\|_{L^0_\gamma(\Omega)}, \qquad \forall t \in [0,T] \text{ and } \forall g \in L^0_\gamma(\Omega).$$

Using the well-known facts that $L^0_\gamma(\Omega)$ is homeomorphic to $L_\gamma(\Omega)/\boldsymbol{R}$ and that $L^0_\beta(\Omega)$ is a dual space for $L_\gamma(\Omega)/\boldsymbol{R}$ (see [5]), we may conclude

$$\|\tilde{P}^\varepsilon(t)\|_{L^0_\beta(\Omega)} \leq C/\varepsilon^2.$$

Now it is easy to conclude that the assertions of Lemma 3.2 are true. Q.E.D.

4. – The convergence theorem.

Lemma 4.1. – There exist subsequences of $\{v^\varepsilon\}$, $\{U^\varepsilon\}$ and $\{\tilde{P}^\varepsilon\}$ (again denoted by $\{v^\varepsilon\}$, $\{U^\varepsilon\}$ and $\{\tilde{P}^\varepsilon\}$) and functions $v \in L_2(Q_T)^n$, $U \in C([0,T]; L_2(\Omega)^n)$, $P \in L_\infty(0,T; L_\beta^0(\Omega))$, $1 < \beta < n/(n-1)$, such that

$$v^\varepsilon \to v \quad \text{weakly in } L_2(Q_T)^n, \tag{4.1}$$

$$U^\varepsilon \to U \quad \text{weak* in } L_\infty(0,T; L_2(\Omega)^n), \tag{4.2}$$

$$\varepsilon^2 \tilde{P}^\varepsilon \to P \quad \text{weak* in } L_\infty(0,T; L_\beta^0(Q)), \tag{4.3}$$

as $\varepsilon \to 0$. The functions v and U satisfy the equations

$$\operatorname{div} v = 0 \quad \text{in } \Omega, \text{ (a.e.) on } (0,T), \tag{4.4}$$

$$\operatorname{div} U = 0 \quad \text{in } \Omega, \ \forall t \in [0,T], \tag{4.5}$$

$$U(t) = \int_0^t v(x,s)\, ds, \tag{4.6}$$

$$v \cdot \nu = h \cdot \nu \quad \text{on } \Gamma, \text{ (a.e.) on } (0,T), \tag{4.7}$$

$$U \cdot \nu = \int_0^t h \cdot \nu \quad \text{on } \Gamma, \ \forall t \in [0,T]. \tag{4.8}$$

Let w^i, π^i $(i = 1, \ldots, n)$ be a Y-periodic solution of the problem

$$-\nabla \pi^i + \Delta w^i + e^i = 0 \quad \text{in } Y^*, \tag{4.9}$$

$$\operatorname{div} w^i = 0 \quad \text{in } Y^*, \tag{4.10}$$

$$w^i = 0 \quad \text{on } S. \tag{4.11}$$

This problem has a unique solution $w^i \in H_s^1(Y^*)^n$, $\pi^i \in L_2(Y^*)$ (π^i defined up to a constant). Let

$$K = (K_{ij})_{i,j=1,\ldots,n}, \qquad K_{ij} = \int_Y (w^i)_j \, dy.$$

The matrix K (permeability tensor) is symmetric and positive definite (see [6]). Let

$$w^{i,\varepsilon}(x) = w^i\left(\frac{x}{\varepsilon}\right), \qquad \pi^{i,\varepsilon}(x) = \pi^i\left(\frac{x}{\varepsilon}\right). \tag{4.12}$$

Note that because of the basic lemma on the periodic extension

$$(w^{i,\varepsilon})_j \to K_{ij} \quad \text{weakly in } L_\gamma(\Omega),\ \frac{1}{\gamma} + \frac{1}{\beta} = 1, \text{ as } \varepsilon \to 0. \tag{4.13}$$

LEMMA 4.2. – ([7]). The functions (4.12) satisfy the conditions

$$-\varepsilon\nabla\pi^{i,\varepsilon}+\varepsilon^2\Delta w^{i,\varepsilon}+e^i=0 \quad \text{in } \Omega_\varepsilon, \tag{4.14}$$

$$\operatorname{div} w^{i,\varepsilon}=0 \quad \text{in } \Omega_\varepsilon, \tag{4.15}$$

$$w^{i,\varepsilon}=0 \quad \text{on } S_\varepsilon, \tag{4.16}$$

and the following inequalities holds:

$$\|\pi^{i,\varepsilon}\|_{L_2(\Omega_\varepsilon)}\leq C, \tag{4.17}$$

$$\|\nabla w^{i,\varepsilon}\|_{L_2(\Omega_\varepsilon)^{n^2}}\leq \frac{C}{\varepsilon}. \tag{4.18}$$

LEMMA 4.3. – (F. MURAT, private communication). Let $w^{i,\varepsilon}$ be defined by (4.12). Then

$$\|\nabla w^{i,\varepsilon}\|_{L_\delta(\Omega_\varepsilon)^{n^2}}\leq \frac{C}{\varepsilon}, \qquad \forall\delta\in(1,+\infty]. \tag{4.19}$$

Furthermore,

$$\varepsilon^2\nabla w^{i,\varepsilon}\nabla U^\varepsilon\to U_i \qquad \text{weak* in } L_\infty(0,T;L_{\delta'}(\Omega)^n), \tag{4.20}$$

as $\varepsilon\to 0$, $\delta'=2\delta/(\delta+2)$.

LEMMA 4.4. – Let $w^{i,\varepsilon}$ be given by (4.12) and $\tilde{P}^\varepsilon$ by (3.15). Then one has

$$\int_0^T\int_\Omega \varepsilon^2\tilde{P}^\varepsilon\nabla\varphi w^{i,\varepsilon}\to\int_0^T\int_\Omega P\nabla\varphi\cdot\int_{Y^*}w^i, \qquad \text{as } \varepsilon\to 0,\quad \forall\varphi\in C_0^\infty(Q_T). \tag{4.21}$$

PROOF. – Let us define $\omega^{i,\varepsilon}$ by setting

$$\omega^{i,\varepsilon}=w^{i,\varepsilon}-\int_\Omega w^{i,\varepsilon}\frac{1}{|\Omega|}. \tag{4.22}$$

Then we have

$$\omega^{i,\varepsilon}\to 0 \qquad \text{weakly in } L_\gamma^0(\Omega)^n,\quad \gamma^{-1}+\beta^{-1}=1, \tag{4.23}$$

$$\frac{1}{|\Omega|}\int_\Omega w^{i,\varepsilon}\to\int_{Y^*}w^i \qquad \text{in } \boldsymbol{R}. \tag{4.24}$$

Obviously, if we prove

$$\int_0^T\int_\Omega \varepsilon^2\tilde{P}^\varepsilon\nabla\varphi\omega^{i,\varepsilon}\to 0 \qquad \text{as } \varepsilon\to 0, \tag{4.25}$$

we easily obtain (4.21).

There exists a sequence $\{\tilde{\psi}_k^{i,\varepsilon}\} \subset \overset{\circ}{W}{}_\gamma^1(\Omega)^n / V_\gamma(\Omega)$, such that $\operatorname{div} \tilde{\psi}_k^{i,\varepsilon} = \omega_k^{i,\varepsilon}$ and

$$\tilde{\psi}_k^{i,\varepsilon} \to 0 \quad \text{weakly in } \overset{\circ}{W}{}_\gamma^1(\Omega)^n / V_\gamma(\Omega). \tag{4.26}$$

Now it is easy to see that there exists a sequence $\{\psi_k^{i,\varepsilon}\}$, $\operatorname{div} \psi_k^{i,\varepsilon} = \omega_k^{i,\varepsilon}$ with the property

$$\|\psi_k^{i,\varepsilon}\|_{\overset{\circ}{W}{}_\gamma^1(\Omega)^n} = \|\tilde{\psi}_k^{i,\varepsilon}\|_{\overset{\circ}{W}{}_\gamma^1(\Omega)^n / V_\gamma(\Omega)}\,. \tag{4.27}$$

The sequence $\{\psi_k^{i,\varepsilon}\}$ is bounded in $\overset{\circ}{W}{}_\gamma^1(\Omega)^n$. Therefore, there exists a subsequence which convergences to zero weakly in $\overset{\circ}{W}{}_\gamma^1(\Omega)^n$ and strongly in $L_\gamma^0(\Omega)^n$.

Consequently, we have

$$\left| \int_0^T \int_\Omega \varepsilon^2 \tilde{P}^\varepsilon \frac{\partial \varphi}{\partial x_k} \omega_k^{i,\varepsilon} \right| \leq \left| \int_0^T \left\langle \varepsilon^2 \nabla \tilde{P}^\varepsilon, \frac{\partial \varphi}{\partial x_k} \psi_k^{i,\varepsilon} \right\rangle_\Omega \right| + \tag{4.28}$$

$$+ \left| \int_0^T \int_\Omega \varepsilon^2 \tilde{P}^\varepsilon \psi_k^{i,\varepsilon} \nabla \frac{\partial \varphi}{\partial x_k} \right| \leq C \|\varepsilon^2 \nabla \tilde{P}^\varepsilon\|_{L_\infty(0,T; W_\beta^{-1}(\Omega_\varepsilon)^n)} \times$$

$$\times \|R_\gamma^\varepsilon \psi_k^{i,\varepsilon}\|_{\overset{\circ}{W}{}_\gamma^1(\Omega_\varepsilon)^n} + C\|\varepsilon^2 \tilde{P}^\varepsilon\|_{L_\infty(0,T; L_\beta^0(\Omega))} \|\psi_k^{i,\varepsilon}\|_{L_\gamma(\Omega)} \leq C\{\|\psi_k^{i,\varepsilon}\|_{L_\gamma^0(\Omega)^n} + \varepsilon \|\nabla \psi_k^{i,\varepsilon}\|_{L_\gamma(\Omega)^{n^2}}\}\,.$$

(4.25) is now a direct consequence of (4.28). Q.E.D.

LEMMA 4.5. – The functions U and P, defined, respectively, by (4.2) and (4.3), satisfy the condition

$$U = \frac{K}{\mu}(F - \nabla P), \tag{4.29}$$

where

$$F(x,t) = \int_0^t f(x,s)\,ds\,. \tag{4.30}$$

PROOF. – We use eq. (3.11). After multiplying (3.11) by $\varepsilon^2 \varphi w^{i,\varepsilon}$, $\varphi \in C_0^\infty(Q_T)$, we find

$$\mu\varepsilon^2 \int_0^T \int_\Omega \nabla U^\varepsilon \nabla w^{i,\varepsilon} \varphi + \varepsilon^2 \mu \int_0^T \int_\Omega \nabla U^\varepsilon \nabla \varphi \otimes w^{i,\varepsilon} - \tag{4.31}$$

$$-\varepsilon^2 \int_0^T \int_\Omega \tilde{P}^\varepsilon \nabla \varphi w^{i,\varepsilon} + \varepsilon^2 \int_0^T \int_\Omega v^\varepsilon \varphi w^{i,\varepsilon} - \varepsilon^2 \int_0^T \int_\Omega v_0^\varepsilon \varphi w^{i,\varepsilon} + \varepsilon^2 \int_0^T \int_\Omega \Phi^\varepsilon(t) \varphi w^{i,\varepsilon} = \varepsilon^2 \int_0^T \int F^\varepsilon \varphi w^{i,\varepsilon}\,.$$

As a direct consequence of the inequalities (3.1), (3.2), (3.3), (3.12) and (4.19) we have

$$\varepsilon^2 \left| \int_0^T \int_\Omega \nabla U^\varepsilon \nabla \varphi \otimes w^{i,\varepsilon} \right| \leqslant C\varepsilon, \qquad \varepsilon^2 \left| \int_0^T \int_\Omega v^\varepsilon \varphi w^{i,\varepsilon} \right| \leqslant C\varepsilon,$$

$$\varepsilon^2 \left| \int_0^T \int_\Omega v_0^\varepsilon \varphi w^{i,\varepsilon} \right| \leqslant C\varepsilon, \qquad \varepsilon^2 \left| \int_0^T \int_\Omega \Phi^\varepsilon(t) \varphi w^{i,\varepsilon} \right| \leqslant C\varepsilon^\zeta .$$

Therefore, using Lemmas 4.3 and 4.4, we obtain

$$\int_0^T \int_\Omega [\mu U_i - (KF)_i]\varphi - \int_0^T \int_\Omega P(K\nabla\varphi)_i = 0 \tag{4.32}$$

and hence (4.29). Q.E.D.

It is easy to conclude that, in fact, $P \in C([0,T]; H^1(\Omega))$ and $\nabla P \in C([0,T]; L_2(\Omega)^n)$. This enables us to differentiate (4.29) in the distribution sense in Q_T; setting

$$p = \frac{\partial}{\partial t} P, \tag{4.33}$$

we obtain

$$v = \frac{K}{\mu}(f - \nabla p), \qquad \text{(a.e.) on } Q_T, \tag{4.34}$$

and $p \in L_2(0,T; H^1(\Omega))$, $\nabla p \in L_2(0,T; L_2(\Omega)^n)$.

THEOREM 4.1. – There exists an extension $\tilde{P}^\varepsilon \in L_\infty(0,T; L_\beta^0(\Omega))$ of the function p^ε such that

$$v^\varepsilon \to v \qquad \text{weakly in } L_2(Q_T)^n,$$

$$\varepsilon^2 \frac{\partial}{\partial t} \tilde{P}^\varepsilon \to p \qquad \text{weakly in } W_2^{-1}(0,T; L_\beta^0(\Omega)),$$

(as $\varepsilon \to 0$), where v, p is the solution of the homogenized problem

$$\operatorname{div} v = 0 \qquad \text{in } L_2(Q_T), \tag{4.35}$$

$$v = \frac{K}{\mu}(f - \nabla p) \qquad \text{in } L_2(Q_T), \tag{4.36}$$

$$v \cdot \nu = h \cdot \nu \qquad \text{in } L_2(0,T; H^{1/2}(\Gamma)). \tag{4.37}$$

PROOF. – The conclusion follows immediately from (3.1), (3.2), (3.3) and (3.11). Because of the positive definiteness of the matrix K, the system (4.35)-(4.37) has a unique solution $v \in L_2(Q_T)^n$, $p \in L_2(0,T; H^1(\Omega) \cap L_2^0(\Omega))$. Therefore, the limits in (4.1), (4.2) and (4.3) do not depend on the subsequences $\{v^\varepsilon\}$, $\{\tilde{P}^\varepsilon\}$. Q.E.D.

5. – Some concluding remarks.

In our considerations, the viscosity coefficient is fixed, but it may also take a small value. Following [6], p. 142-144, we consider the problem

$$\frac{\partial v^\varepsilon}{\partial t} + (v^\varepsilon \nabla) v^\varepsilon - \mu \varepsilon^\gamma \Delta v^\varepsilon + \nabla p^\varepsilon = f \quad \text{in } Q_{\varepsilon T}, \tag{5.1}$$

$$\operatorname{div} v^\varepsilon = 0 \quad \text{in } Q_{\varepsilon T}, \tag{5.2}$$

$$v^\varepsilon(x, 0) = 0 \quad \text{in } \Omega_\varepsilon, \quad v^\varepsilon = 0 \text{ on } S_{\varepsilon T}. \tag{5.3}$$

Then, a straightforward consequence of the results in this article is as follows:

THEOREM 5.1. – Let $0 \leq \gamma < 3/2$ and let $f \in L_2(Q_T)^n$. Then there exists an extension $\tilde{P}^\varepsilon \in L_\infty(0, T; L^0_{\beta'}(\Omega))$, $1 < \beta' \leq 2$ for $\gamma < (6-n)/4$ and $1 < \beta' < n/(n-3+2\gamma)$ for $\gamma \geq (6-n)/4$ of the function P^ε, such that

$$\frac{v^\varepsilon}{\varepsilon^{2-\gamma}} \to v \quad \text{weakly in } L_2(Q_T)^n, \tag{5.4}$$

$$\tilde{P}^\varepsilon \to P = \int_0^t p \quad \text{weak* in } L_\infty(0, T; L^0_{\beta'}(\Omega)), \tag{5.5}$$

(as $\varepsilon \to 0$), where v, p is the solution of the homogenized problem

$$\begin{aligned} &\operatorname{div} v = 0 && \text{in } L_2(Q_T), \\ &v = \frac{K}{\mu}(f - \nabla p) && \text{in } L_2(Q_T), \\ &v \cdot \nu = 0 && \text{in } L_2(0, T; H^{1/2}(\Gamma)). \end{aligned}$$

The case $\gamma = 3/2$ is critical. Formal homogenization (see [6], p. 142-144) gives a nonlinear Darcy law (Forchheimer-type law). Unfortunately, it is no longer clear that Lemma 3.2 is valid. Hence the results in this paper do not cover this case. Also, let us remark that the term $\int_\Omega (v^\varepsilon \nabla) v^\varepsilon \varphi w^{i,\varepsilon}$ does not tend to zero any longer and it seems that it gives a second-order term in velocity.

Finally, it is easy to see that all these results can be extended to the case when one has $-\mu \operatorname{div}(A(x/\varepsilon)\nabla v^\varepsilon)$ instead of $-\mu \Delta v^\varepsilon$ (see [1] for more detail).

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