Open access • Journal Article • DOI:10.1007/S11587-011-0124-Y

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Published on: 01 Nov 2012 - Ricerche Di Matematica (Springer Milan)

Topics: Singular integral, Bounded function, Homogenization (chemistry) and Multiple integral

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HOMOGENIZATION OF UNBOUNDED SINGULAR INTEGRALS IN $W^{1,\infty}$

OMAR ANZA HAFSA AND JEAN-PHILIPPE MANDALLENA

ABSTRACT. We study homogenization by Γ-convergence, with respect to the L^1-strong convergence, of periodic multiple integrals in $W^{1,\infty}$ when the integrand can take infinite values outside of a convex bounded open set of matrices.

1. INTRODUCTION

In this paper we are concerned with homogenization by Γ-convergence of multiple integrals of type

$$\int_\Omega W\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx, \tag{1.1}$$

where $\varepsilon > 0$ is a (small) parameter, $\Omega \subset \mathbb{R}^d$ is a bounded open set with Lipschitz boundary, $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ and $W : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0, \infty]$ is a Borel measurable function which is p-coercive, 1-periodic with respect to its first variable, not necessarily convex with respect to its second variable and infinite outside a convex bounded open set $C \subset \mathbb{M}^{m\times d}$ such that $0 \in C$.

In the scalar case, i.e., $\min\{d, m\} = 1$, a wide literature exists on homogenization problems with constraints on the gradient, whose techniques cannot be generalized to the vector case, i.e., $\min\{d, m\} > 1$, (see the book [CDA02] and the reference therein). Thus, constraints on the gradient relating to problems of hyperelasticity cannot be treated with methods from the scalar framework. It is then of interest to develop techniques in the vector case for the homogenization of multiple integrals like (1.1) when the integrand can take infinite values: this is the general purpose of the present paper. For a recent work in the same spirit, we refer the reader to [AHLM] (see also [BB00, Syc05, AHM07, AHM08, AH10, Syc10] for the relaxation case).

In this paper, our main contribution (see Theorem 2.1 and Corollaries 2.2 and 2.4) is to prove that under certain assumptions, i.e., (2.3), (2.4) and (2.5), which are related to hyperelasticity but not consistent with the material frame indifference axiom (see §2.2 for more details), (1.1) Γ-converges, as the parameter ε tends to zero, to the homogeneous multiple integral

$$\int_\Omega W_{\text{hom}}(\nabla u(x))dx,$$

Key words and phrases. Homogenization, nonconvex singular integrands, constraints on the gradient, determinant type constraints, hyperelasticity.

where $u \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ and $W_{\rm hom} : \mathbb{M}^{m\times d} \to [0,\infty]$ is given by the formula (see also Remark 2.3)

$$W_{\rm hom}(\xi) = \begin{cases} \mathcal{ZH}W(\xi) := \displaystyle\inf_{\phi\in{\rm Aff}_0(Y;\mathbb{R}^m)} \int_Y \mathcal{H}W(\xi+\nabla\phi(y))dy & \text{if } \xi \in C \\ \displaystyle\varliminf_{t\to 1} \mathcal{ZH}W(t\xi) & \text{if } \xi \in \partial C \\ \infty & \text{otherwise} \end{cases}$$

with $\mathcal{H}W : \mathbb{M}^{m\times d} \to [0,\infty]$ defined by

$$\mathcal{H}W(\xi) := \inf_{k\geq 1} \inf_{\phi\in W_0^{1,\infty}(kY;\mathbb{R}^m)} \fint_{kY} W(x,\xi+\nabla\phi(x))dx$$

and ${\rm Aff}_0(Y;\mathbb{R}^m)$ denoting the space of continuous piecewise affine functions ϕ from $Y :=]0,1[^d$ to $\mathbb{R}^m$ such that $\phi = 0$ on the boundary ∂Y of Y.
The paper is organized as follows. In Section 2 we state the main results of the paper, i.e., Theorem 2.1 and Corollaries 2.2 and 2.4, and, although our assumptions are not compatible with the material frame indifference axiom, indicate how these results could be related to the framework of hyperelasticity (see Proposition 2.5). Section 3 is devoted to the statements and proofs of auxiliary results needed in the proof of Theorem 2.1. Finally, Theorem 2.1 is proved in Section 4.

2. Main results

2.1. **General results.** Let $d, m \geq 1$ be two integers, let $C \subset \mathbb{M}^{m\times d}$ be a convex bounded open set such that $0 \in C$ and let $W : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ be a Borel measurable function which is 1-periodic with respect to its first variable, i.e.,

$$W(x+z,\xi) = W(x,\xi) \text{ for all } x \in \mathbb{R}^d, \text{ all } z \in \mathbb{R}^d \text{ and all } \xi \in \mathbb{M}^{m\times d}, \tag{2.1}$$

and infinite outside of C, i.e.,

$$\text{dom}W(x,\cdot) = C \text{ for all } x \in \mathbb{R}^d \tag{2.2}$$

with $\text{dom}W(x,\cdot)$ denoting the effective domain of $W(x,\cdot)$. We define $M_W, \delta_W : [0,1] \to [0,\infty]$ and $\Delta_W : [0,1] \to]-\infty,\infty]$ by:

- ♦ $M_W(t) := \displaystyle\sup_{x\in\mathbb{R}^d} \sup_{\xi\in t\overline{C}} W(x,\xi)$;
- ♦ $\delta_W(t) := \displaystyle\inf_{x\in\mathbb{R}^d} \inf_{\xi\in\overline{C}\setminus t\overline{C}} W(x,\xi)$;
- ♦ $\Delta_W(t) := \displaystyle\sup_{x\in\mathbb{R}^d} \sup_{\xi\in C} W(x,t\xi) - W(x,\xi)$

and we consider the following three assertions:

- ♦ W is locally bounded in C, i.e.,

$$M_W(t) < \infty \text{ for all } t \in [0,1[; \tag{2.3}$$

- ♦ W is singular on the boundary ∂C of C, i.e.,

$$\lim_{t\to 1} \delta_W(t) = \infty; \tag{2.4}$$

- ♦ W is radially uniformly upper semicontinuous (ru-usc), i.e.,

$$\varlimsup_{t\to 1} \Delta_W(t) \leq 0. \tag{2.5}$$

Note that, under (2.3) and (2.4), it is easy to see that if $\mathrm{dom}W(x,\cdot)\subset\overline{C}$ for all $x\in\mathbb{R}^d$, then $\mathrm{dom}W(x,\cdot)=C$ for all $x\in\mathbb{R}^d$. Let $\Omega\subset\mathbb{R}^d$ be a bounded open set with Lipschitz boundary and let $I_\varepsilon,\widehat{\mathcal{H}I},\widehat{\mathcal{ZH}I}:W^{1,\infty}(\Omega;\mathbb{R}^m)\to[0,\infty]$ be defined by:

- $I_\varepsilon(u):=\displaystyle\int_\Omega W\left(\frac{x}{\varepsilon},\nabla u(x)\right)dx;$
- $\widehat{\mathcal{H}I}(u):=\displaystyle\int_\Omega \widehat{\mathcal{H}W}(\nabla u(x))dx;$
- $\widehat{\mathcal{ZH}I}(u):=\displaystyle\int_\Omega \widehat{\mathcal{ZH}W}(\nabla u(x))dx,$

where $\varepsilon>0$ is a (small) parameter and $\mathcal{H}W,\widehat{\mathcal{H}W},\mathcal{ZH}W,\widehat{\mathcal{ZH}W}:\mathbb{M}^{m\times d}\to[0,\infty]$ are given by:

- $\mathcal{H}W(\xi):=\inf\limits_{k\geq1}\inf\left\{\displaystyle⨍_{kY}W(x,\xi+\nabla\phi(x))dx:\phi\in W^{1,\infty}_0(kY;\mathbb{R}^m)\right\};$
- $\widehat{\mathcal{H}W}(\xi):=\varliminf\limits_{t\to1}\mathcal{H}W(t\xi);$
- $\mathcal{ZH}W(\xi):=\inf\left\{\displaystyle\int_Y\mathcal{H}W(\xi+\nabla\phi(y))dy:\phi\in\mathrm{Aff}_0(Y;\mathbb{R}^m)\right\};$
- $\widehat{\mathcal{ZH}W}(\xi):=\varliminf\limits_{t\to1}\mathcal{ZH}W(t\xi)$

with $Y:=]0,1[^d$ and $\mathrm{Aff}_0(Y;\mathbb{R}^m):=\big\{\phi\in\mathrm{Aff}(Y;\mathbb{R}^m):\phi=0\text{ on }\partial Y\big\}$ where $\mathrm{Aff}(Y;\mathbb{R}^m)$ denotes the space of continuous piecewise affine functions from Y to $\mathbb{R}^m$. The main result of the paper is the following.

Theorem 2.1. *Let $W:\mathbb{R}^d\times\mathbb{M}^{m\times d}\to[0,\infty]$ be a Borel measurable function satisfying* (2.1)*,* (2.2)*,* (2.3)*,* (2.4) *and* (2.5) *and let $u\in W^{1,\infty}(\Omega;\mathbb{R}^m)$.*

(i) *If $\{u_\varepsilon\}_\varepsilon\subset W^{1,\infty}(\Omega;\mathbb{R}^m)$ is such that $\|u_\varepsilon-u\|_{L^1(\Omega;\mathbb{R}^m)}\to0$, then*

$$\varliminf_{\varepsilon\to0}I_\varepsilon(u_\varepsilon)\geq\widehat{\mathcal{H}I}(u).$$

(ii) *There exists $\{u_\varepsilon\}_\varepsilon\subset W^{1,\infty}(\Omega;\mathbb{R}^m)$ such that $\|u_\varepsilon-u\|_{L^1(\Omega;\mathbb{R}^m)}\to0$ and*

$$\varlimsup_{\varepsilon\to0}I_\varepsilon(u_\varepsilon)\leq\widehat{\mathcal{ZH}I}(u).$$

Let $I_{\mathrm{hom}}:W^{1,\infty}(\Omega;\mathbb{R}^m)\to[0,\infty]$ be defined by

$$I_{\mathrm{hom}}(u):=\int_\Omega W_{\mathrm{hom}}(\nabla u(x))dx$$

with $W_{\mathrm{hom}}:\mathbb{M}^{m\times d}\to[0,\infty]$ given by

$$W_{\mathrm{hom}}(\xi):=\begin{cases}\mathcal{ZH}W(\xi) & \text{if }\xi\in C\\ \varliminf\limits_{t\to1}\mathcal{ZH}W(t\xi) & \text{if }\xi\in\partial C\\ \infty & \text{otherwise.}\end{cases}$$

The following homogenization result is a consequence of Theorem 2.1.

Corollary 2.2. *Let $W:\mathbb{R}^d\times\mathbb{M}^{m\times d}\to[0,\infty]$ be a Borel measurable function satisfying* (2.1)*,* (2.2)*,* (2.3)*,* (2.4) *and* (2.5)*. Then*

$$\Gamma(L^1)\text{-}\lim_{\varepsilon\to0}I_\varepsilon=I_{\mathrm{hom}}.$$

Proof. As $\widehat{\mathcal{ZH}I} \le \widehat{\mathcal{H}I}$, from Theorem 2.1 we deduce that

$$\left(\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} I_\varepsilon\right)(u) = \widehat{\mathcal{ZH}I}(u) = \int_\Omega \widehat{\mathcal{ZH}W}(\nabla u(x))dx$$

for all $u \in W^{1,\infty}(\Omega;\mathbb{R}^m)$, and the result follows from Corollary 3.8. ■

Remark 2.3. Under the assumption of Corollary 2.2 we have $W_{\rm hom} = \overline{\mathcal{H}W}$ with $\overline{\mathcal{H}W}$ denoting the lower semicontinuous envelope (lsc) of $\mathcal{H}W$. Indeed, as $\widehat{\mathcal{ZH}I} \le \widehat{\mathcal{H}I}$, from Theorem 2.1 we see that $\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} I_\varepsilon = \widehat{\mathcal{H}I}$, and consequently $\widehat{\mathcal{H}I} = I_{\rm hom}$ by Corollary 2.2. Thus $W_{\rm hom} = \widehat{\mathcal{H}W}$. On the other, by Remark 3.10, $t\mathcal{H}\mathbb{W} \subset {\rm int}(\mathcal{H}\mathbb{W})$ for all $t \in]0,1[$, where $\mathcal{H}\mathbb{W}$ denotes the effective domain of $\mathcal{H}W$. As W satisfies (2.5), from Proposition 3.7 we can assert that $\mathcal{H}W$ is ru-usc (see Definition 3.1) and so $\widehat{\mathcal{H}W} = \overline{\mathcal{H}W}$ by Theorem 3.5(iii).

To be complete, let us give the Dirichlet version of Corollary 2.2. For each $\varepsilon > 0$, let $J_\varepsilon : W_0^{1,\infty}(\Omega;\mathbb{R}^m) \to [0,\infty]$ be defined by

$$J_\varepsilon(u) := \begin{cases} I_\varepsilon(u) & \text{if } u \in W_0^{1,\infty}(\Omega;\mathbb{R}^m) \\ \infty & \text{otherwise.} \end{cases}$$

Using the Dirichlet version of Theorem 2.1 and arguing as in the proof of Corollary 2.2 we can establish the following result.

Corollary 2.4. *Let* $W : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ *be a Borel measurable function satisfying* (2.1), (2.2), (2.3), (2.4) *and* (2.5). *Then*

$$\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} J_\varepsilon = J_{\rm hom}$$

with $J_{\rm hom} : W^{1,\infty}(\Omega;\mathbb{R}^m) \to [0,\infty]$ *given by*

$$J_{\rm hom}(u) := \begin{cases} I_{\rm hom}(u) & \textit{if } u \in W_0^{1,\infty}(\Omega;\mathbb{R}^m) \\ \infty & \textit{otherwise.} \end{cases}$$

To reduce technicalities and emphasize the essential difficulties, in the present paper we have restricted our attention on Theorem 2.1 and Corollary 2.2. The details of the proof of Corollary 2.4 are left to the reader.

2.2. **Towards applications in hyperelasticity.** Let $d \ge 1$ be an integer and let $\mathbb{B}$ be the unit open ball in $\mathbb{M}^{d\times d}$. Given a continuous function $g : \mathbb{M}^{d\times d} \to [0,\infty[$ and a convex function $h : [0,1[\to [0,\infty[$ such that

$$h(t) \ge \frac{ct^p}{1-t^p} \text{ for all } t \in [0,1[\text{ and some } c > 0 \text{ and } p > 0, \tag{2.6}$$

we consider $f : \mathbb{M}^{d\times d} \to [0,\infty]$ given by

$$f(\xi) := \begin{cases} g(I+\xi) + h(|\xi|) & \text{if } \xi \in \mathbb{B} \\ \infty & \text{otherwise,} \end{cases}$$

where I denotes the identity matrix in $\mathbb{M}^{d\times d}$. Given a 1-periodic function $a \in L^\infty(\mathbb{R}^d)$ such that $\alpha \le a(x) \le \beta$ for all $x \in \mathbb{R}^d$ and some $\beta > \alpha > 0$, we define $W : \mathbb{R}^d \times \mathbb{M}^{d\times d} \to [0,\infty]$ by

$$W(x,\xi) := a(x)f(\xi).$$

The following proposition makes clear the fact that such a W is consistent with the assumptions of Corollaries 2.2 and 2.4 as well as with some (but not all) conditions of hyperelasticity, i.e., the non-interpenetration of the matter, see Proposition 2.5(iv), and the necessity of an infinite amount of energy to compress a finite volume of matter into zero volume, see Proposition 2.5(v). However, since the effective domain of W is convex, it does not satisfy the material frame indifference axiom. Thus, we are still far from a result on homogenization that can be compared to Ball's lower semicontinuity theorem (see [Bal77]).

Proposition 2.5. *Let* $W : \mathbb{R}^d \times \mathbb{M}^{d\times d} \to [0,\infty]$ *be defined as above. Then*:

(i) W *is* 1*-periodic with respect to the first variable*;
(ii) W *satisfies* (2.2) *with* $C = \mathbb{B}$;
(iii) W *satisfies* (2.3), (2.4) *and* (2.5);
(iv) *for every* $(x,\xi) \in \mathbb{R}^d \times \mathbb{B}$, $W(x,\xi) < \infty$ *if and only if* $\det(I+\xi) > 0$;
(v) *for every* $x \in \mathbb{R}^d$, $W(x,\xi) \to \infty$ *as* $\det(I+\xi) \to 0$.

Proof. (i) and (ii) are obvious, and (iv) follows from the classical fact that $I+\mathbb{B} \subset \{\xi \in \mathbb{M}^{d\times d} : \det(I+\xi) > 0\}$.
(iii) Since $\overline{\mathbb{B}}$ is compact and g is continuous, g is bounded on $\overline{\mathbb{B}}$, i.e., there exists $C > 0$ such that $g(\xi) \le C$ for all $\xi \in \overline{\mathbb{B}}$. Given any $t \in [0,1[$, as $a(x) \le \beta$ for all $x \in \mathbb{R}^d$ it follows that

$$\sup_{x\in\mathbb{R}^d} \sup_{\xi\in t\overline{\mathbb{B}}} W(x,\xi) \le \beta C + \beta \sup_{|\xi|\le t} h(|\xi|).$$

But, h is convex and finite, hence $\sup_{|\xi|\le t} h(|\xi|) < \infty$, and (2.3) follows.
As $a(x) \ge \alpha$ for all $x \in \mathbb{R}^d$, using (2.6) we see that

$$\inf_{x\in\mathbb{R}^d} \inf_{\xi\in\overline{\mathbb{B}}\setminus t\overline{\mathbb{B}}} W(x,\xi) \ge \alpha \inf_{t<|\xi|<1} h(|\xi|) \ge \alpha c \frac{t^p}{1-t^p}$$

for all $t \in [0,1[$, which gives (2.4) because $\frac{t^p}{1-t^p} \to \infty$ as $t \to 1$.
For each $r > 0$, set $\omega(r) := \sup\{|g(\zeta_1) - g(\zeta_2)| : \zeta_1, \zeta_2 \in I + \overline{\mathbb{B}} \text{ and } |\zeta_1 - \zeta_2| \le r\}$. As g is uniformly continuous on $I + \overline{\mathbb{B}}$ we have $\omega(r) \to 0$ as $r \to 0$. Given any $t \in [0,1]$, as $a(x) \le \beta$ for each $x \in \mathbb{R}^d$ and h is convex, we see that

$$\begin{aligned} W(x,t\xi) - W(x,\xi) &= a(x)(g(I+t\xi) - g(I+\xi) + h(|t\xi|) - h(|\xi|) \\ &\le \beta\omega(1-t) + \beta(t-1)h(|\xi|) + \beta(1-t)h(0) \\ &\le \beta\omega(1-t) + \beta(1-t)h(0) \end{aligned}$$

for all $x \in \mathbb{R}^d$ and all $\xi \in \mathbb{B}$, hence

$$\sup_{x\in\mathbb{R}^d} \sup_{\xi\in\mathbb{B}} W(x,t\xi) - W(x,\xi) \le \beta\omega(1-t) + \beta(1-t)h(0)$$

for all $t \in [0,1]$, and (2.5) follows because $\omega(1-t) \to 0$ as $t \to 1$.
(v) Let $x \in \mathbb{R}^d$ and let $\{\xi_n\}_n \subset \mathbb{M}^{m\times d}$ be such that $\lim_{n\to\infty} \det(I+\xi_n) = 0$. Since $\mathrm{dom} W(x,\cdot) = \mathbb{B}$, without loss of generality we can assume that $\{\xi_n\}_n \subset \mathbb{B}$ and there exists $\xi \in \overline{\mathbb{B}}$ such that $\lim_{n\to\infty} |\xi_n - \xi| = 0$. By continuity of the determinant we obtain $\det(I+\xi) = 0$, and so $|\xi| = 1$. Thus

$$\lim_{n\to\infty} |\xi_n| = 1. \tag{2.7}$$

On the other hand, using (2.6) we see that $W(x,\xi_n) \ge \alpha c \frac{|\xi_n|^p}{1-|\xi_n|}$ for all $n \ge 1$, and consequently $\lim_{n\to\infty} W(x,\xi_n) = \infty$ because $\lim_{n\to\infty} \frac{|\xi_n|^p}{1-|\xi_n|} = \infty$ by (2.7). ■

3. Auxiliary results

3.1. **Ru-usc functions.** Let $U \subset \mathbb{R}^d$ be an open set and let $L : U\times\mathbb{M}^{m\times d} \to [0,\infty]$ be a Borel measurable function. For each $x \in U$, we denote the effective domain of $L(x,\cdot)$ by $\mathbb{L}_x$ and we define $\Delta_L : [0,1] \to]-\infty,\infty]$ by

$$\Delta_L(t) := \sup_{x\in U}\sup_{\xi\in\mathbb{L}_x} L(x,t\xi) - L(x,\xi).$$

Definition 3.1. We say that L is radially uniformly upper semicontinuous (ru-usc) if

$$\overline{\lim_{t\to 1}}\,\Delta_L(t) \le 0.$$

Remark 3.2. If L is ru-usc then

$$\overline{\lim_{t\to 1}}\, L(x,t\xi) \le L(x,\xi) \tag{3.1}$$

for all $x \in U$ and all $\xi \in \mathbb{L}_x$. Indeed, given $x \in U$ and $\xi \in \mathbb{L}_x$, we have

$$L(x,t\xi) \le \Delta_L(t) + L(x,\xi) \text{ for all } t \in [0,1],$$

which gives (3.1) since $\overline{\lim}_{t\to 1}\Delta_L(t) \le 0$.

Remark 3.3. If there exist $x \in U$ and $\xi \in \mathbb{L}_x$ such that $L(x,\cdot)$ is lsc at ξ then

$$\underline{\lim_{t\to 1}}\,\Delta_L(t) \ge 0. \tag{3.2}$$

Indeed, given such $x \in U$ and $\xi \in \mathbb{L}_x$, we have

$$\Delta_L(t) \ge L(x,t\xi) - L(x,\xi) \text{ for all } t \in [0,1],$$

which gives (3.2) since $\underline{\lim}_{t\to 1}(L(x,t\xi) - L(x,\xi)) \ge 0$.

The following lemma is essentially due to Wagner (see [Wag09]).

Lemma 3.4. *Assume that L is ru-usc and consider $x \in U$ such that*

$$t\overline{\mathbb{L}}_x \subset \mathbb{L}_x \text{ for all } t \in]0,1[, \tag{3.3}$$

where $\overline{\mathbb{L}}_x$ denotes the closure of $\mathbb{L}_x$. Then

$$\underline{\lim_{t\to 1}}\, L(x,t\xi) = \overline{\lim_{t\to 1}}\, L(x,t\xi)$$

for all $\xi \in \overline{\mathbb{L}}_x$.

Proof. Fix $\xi \in \overline{\mathbb{L}}_x$. It suffices to prove that

$$\overline{\lim_{t\to 1}}\, L(x,t\xi) \le \underline{\lim_{t\to 1}}\, L(x,t\xi). \tag{3.4}$$

Without loss of generality we can assume that $\underline{\lim}_{t\to 1} L(x,t\xi) < \infty$ and there exist $\{t_n\}_n, \{s_n\}_n \subset]0,1[$ such that:

- $t_n \to 1$, $s_n \to 1$ and $\frac{t_n}{s_n} \to 1$;
- $\overline{\lim_{t\to 1}}\, L(x,t\xi) = \lim_{n\to\infty} L(x,t_n\xi)$;
- $\underline{\lim_{t\to 1}}\, L(x,t\xi) = \lim_{n\to\infty} L(x,s_n\xi)$.

From (3.3) we see that for every $n \geq 1$, $s_n\xi \in \mathbb{L}_x$, and so we can assert that for every $n \geq 1$,

$$L(x, t_n\xi) \leq \Delta_L\left(\frac{t_n}{s_n}\right) + L(x, s_n\xi). \tag{3.5}$$

On the other hand, as L is ru-usc we have $\overline{\lim}_{n\to\infty} \Delta_L\left(\frac{t_n}{s_n}\right) \leq 1$, and (3.4) follows from (3.5) by letting $n \to \infty$. ■

Define $\widehat{L} : U \times \mathbb{M}^{m\times d} \to [0, \infty]$ by

$$\widehat{L}(x, \xi) := \varliminf_{t\to 1} L(x, t\xi).$$

The interest of Definition 3.1 comes from the following theorem.

Theorem 3.5. *If L is ru-usc and if for every $x \in U$,*

$$t\overline{\mathbb{L}}_x \subset \mathrm{int}(\mathbb{L}_x) \text{ for all } t \in]0, 1[\tag{3.6}$$

(in particular (3.3) holds) and $L(x, \cdot)$ is lsc on $\mathrm{int}(\mathbb{L}_x)$, where $\mathrm{int}(\mathbb{L}_x)$ denotes the interior of $\mathbb{L}_x$, then:

(i) $\widehat{L}(x, \xi) = \begin{cases} L(x, \xi) & \text{if } \xi \in \mathrm{int}(\mathbb{L}_x) \\ \lim_{t\to 1} L(x, t\xi) & \text{if } \xi \in \partial\mathbb{L}_x \\ \infty & \text{otherwise;} \end{cases}$

(ii) *$\widehat{L}$ is ru-usc;*

(iii) *for every $x \in U$, $\widehat{L}(x, \cdot)$ is the lsc envelope of $L(x, \cdot)$.*

Proof. (i) Lemma 3.4 shows that, for $x \in U$ and $\xi \in \overline{\mathbb{L}}_x$, $\widehat{L}(x, \xi) = \lim_{t\to 1} L(x, t\xi)$. From remark 3.2 we see that if $\xi \in \mathrm{int}(\mathbb{L}_x)$ then $\overline{\lim}_{t\to 1} L(x, t\xi) \leq L(x, \xi)$. On the other hand, from (3.6) it follows that if $\xi \in \mathrm{int}(\mathbb{L}_x)$ then $t\xi \in \mathrm{int}(\mathbb{L}_x)$ for all $t \in]0, 1[$. Thus, $\underline{\lim}_{t\to 1} L(x, t\xi) \geq L(x, \xi)$ whenever $\xi \in \mathrm{int}(\mathbb{L}_x)$ since $L(x, \cdot)$ is lsc on $\mathrm{int}(\mathbb{L}_x)$, and (i) follows.

(ii) Fix any $t \in]0, 1[$ any $x \in U$ and any $\xi \in \widehat{\mathbb{L}}_x$, where $\widehat{\mathbb{L}}_x$ denotes the effective domain of $\widehat{L}(x, \cdot)$. As $\widehat{\mathbb{L}}_x \subset \overline{\mathbb{L}}_x$ we have $\xi \in \overline{\mathbb{L}}_x$ and $t\xi \in \mathbb{L}_x$ since (3.3) holds. From Lemma 3.4 we can assert that:

- $\widehat{L}(x, \xi) = \lim_{s\to 1} L(x, s\xi)$;
- $\widehat{L}(x, t\xi) = \lim_{s\to 1} L(x, s(t\xi))$,

and consequently

$$\widehat{L}(x, t\xi) - \widehat{L}(x, \xi) = \lim_{s\to 1} L(x, t(s\xi)) - L(x, s\xi). \tag{3.7}$$

On the other hand, by (3.3) we have $s\xi \in \mathbb{L}_x$ for all $s \in]0, 1[$, and so

$$L(x, t(s\xi)) - L(x, s\xi) \leq \Delta_L(t) \text{ for all } s \in]0, 1[.$$

Letting $s \to 1$ and using (3.7) we deduce that $\Delta_{\widehat{L}}(t) \leq \Delta_L(t)$ for all $t \in]0, 1[$, which gives (ii) since L is ru-usc.

(iii) Given $x \in U$, we only need to prove that if $|\xi_n - \xi| \to 0$ then

$$\varliminf_{n\to\infty} L(x, \xi_n) \geq \widehat{L}(x, \xi). \tag{3.8}$$

Without loss of generality we can assume that

$$\varliminf_{n\to\infty} L(x, \xi_n) = \lim_{n\to\infty} L(x, \xi_n) < \infty, \text{ and so } \sup_{n\geq 1} L(x, \xi_n) < \infty.$$

Thus $\xi_n \in \mathbb{L}_x$ for all $n \geq 1$, hence $\xi \in \overline{\mathbb{L}}_x$, and so

$$\widehat{L}(x,\xi) = \lim_{t\to 1} L(x,t\xi)$$

by Lemma 3.4. Moreover, using (3.3) we see that, for any $t \in]0,1[$, $t\xi \in \mathbb{L}_x$ and $t\xi_n \in \mathbb{L}_x$ for all $n \geq 1$, and consequently

$$\varliminf_{n\to\infty} L(x,t\xi_n) \geq L(x,t\xi) \text{ for all } t \in]0,1[$$

because $L(x,\cdot)$ is lsc on $\mathbb{L}_x$ and $|t\xi_n - t\xi| \to 0$. It follows that

$$\varlimsup_{t\to 1} \varliminf_{n\to\infty} L(x,t\xi_n) \geq \widehat{L}(x,\xi). \tag{3.9}$$

On the other hand, for every $n \geq 1$ and every $t \in [0,1]$, we have

$$L(x,t\xi_n) \leq L(x,\xi_n) + \Delta_L(t).$$

As L is ru-usc, letting $n \to \infty$ and $t \to 1$ we obtain

$$\varlimsup_{t\to 1} \varliminf_{n\to\infty} L(x,t\xi_n) \leq \lim_{n\to\infty} L(x,\xi_n),$$

which gives (3.8) when combined with (3.9). ■

In what follows, given any bounded open set $A \subset \mathbb{R}^d$, we denote the space of continuous piecewise affine functions from A to $\mathbb{R}^m$ by $\mathrm{Aff}(A;\mathbb{R}^m)$, i.e., $u \in \mathrm{Aff}(A;\mathbb{R}^m)$ if and only if $u \in C(\overline{A};\mathbb{R}^m)$ and there exists a finite family $\{A_i\}_{i\in I}$ of open disjoint subsets of A such that $|A \setminus \cup_{i\in I} A_i| = 0$ and, for each $i \in I$, $|\partial A_i| = 0$ and $\nabla u(x) = \xi_i$ in A_i with $\xi_i \in \mathbb{M}^{m\times d}$. Define $\mathcal{Z}L : U \times \mathbb{M}^{m\times d} \to [0,\infty]$ by

$$\mathcal{Z}L(x,\xi) := \inf\left\{\int_Y L(x,\xi + \nabla\phi(y))dy : \phi \in \mathrm{Aff}_0(Y;\mathbb{R}^m)\right\}$$

with $Y :=]0,1[^d$ and $\mathrm{Aff}_0(Y;\mathbb{R}^m) := \big\{\phi \in \mathrm{Aff}(Y;\mathbb{R}^m) : \phi = 0 \text{ on } \partial Y\big\}$. Roughly, Proposition 3.6 shows that ru-usc functions have a nice behavior with respect to relaxation.

Proposition 3.6. *If L is ru-usc then $\mathcal{Z}L$ is ru-usc.*

Proof. Fix any $t \in [0,1]$, any $x \in U$ and any $\xi \in \mathcal{Z}\mathbb{L}_x$, where $\mathcal{Z}\mathbb{L}_x$ denotes the effective domain of $\mathcal{Z}L(x,\cdot)$. By definition, there exists $\{\phi_n\}_n \subset \mathrm{Aff}_0(Y;\mathbb{R}^m)$ such that:

- ♦ $\mathcal{Z}L(x,\xi) = \displaystyle\lim_{n\to\infty} \int_Y L\left(x,\xi + \nabla\phi_n(y)\right) dy$;
- ♦ $\xi + \nabla\phi_n(y) \in \mathbb{L}_x$ for all $n \geq 1$ and a.a. $y \in Y$.

Moreover, for every $n \geq 1$,

$$\mathcal{Z}L(x,t\xi) \leq \int_Y L\left(x,t(\xi + \nabla\phi_n(y))\right) dy$$

since $t\phi_n \in \mathrm{Aff}_0(Y;\mathbb{R}^m)$, and so

$$\mathcal{Z}L(x,t\xi) - \mathcal{Z}L(x,\xi) \leq \varliminf_{n\to\infty} \int_Y \big(L(x,t(\xi + \nabla\phi_n(y))) - L(x,\xi + \nabla\phi_n(y))\big)dy.$$

As L is ru-usc it follows that

$$\mathcal{Z}L(x,t\xi) - \mathcal{Z}L(x,\xi) \leq \Delta_L(t),$$

which implies that $\Delta_{\mathcal{Z}L}(t) \leq \Delta_L(t)$ for all $t \in [0,1]$, and the proof is complete. ■

Assume that $U = \mathbb{R}^d$ and define $\mathcal{H}L : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ by

$$\mathcal{H}L(\xi) := \inf_{k\geq 1} \inf \left\{ \fint_{kY} L(x, \xi + \nabla\phi(x))dx : \phi \in W_0^{1,\infty}(kY; \mathbb{R}^m) \right\}.$$

Roughly, Proposition 3.7 shows that ru-usc functions have a nice behavior with respect to homogenization.

Proposition 3.7. *If L is ru-usc then $\mathcal{H}L$ is ru-usc.*

Proof. Fix any $t \in [0,1]$ and any $\xi \in \mathcal{H}\mathbb{L}$, where $\mathcal{H}\mathbb{L}$ denotes the effective domain of $\mathcal{H}L$. By definition, there exists $\{k_n; \phi_n\}_n$ such that:

- $\phi_n \in W_0^{1,\infty}(k_nY; \mathbb{R}^m)$ for all $n \geq 1$;
- $\mathcal{H}L(\xi) = \lim_{n\to\infty} \fint_{k_nY} L(x, \xi + \nabla\phi_n(x))dx$;
- $\xi + \nabla\phi_n(x) \in \mathbb{L}_x$ for all $n \geq 1$ and a.a. $x \in k_nY$.

Moreover, for every $n \geq 1$,

$$\mathcal{H}L(t\xi) \leq \fint_{k_nY} L(x, t(\xi + \nabla\phi_n(x)))dx$$

since $t\phi_n \in W_0^{1,\infty}(k_nY; \mathbb{R}^m)$, and so

$$\mathcal{H}L(t\xi) - \mathcal{H}L(\xi) \leq \varliminf_{n\to\infty} \fint_{k_nY} \big(L(x, t(\xi + \nabla\phi_n(x))) - L(x, \xi + \nabla\phi_n(x))\big)dx.$$

As L is ru-usc it follows that

$$\mathcal{H}L(t\xi) - \mathcal{H}L(\xi) \leq \Delta_L(t),$$

which implies that $\Delta_{\mathcal{H}L}(t) \leq \Delta_L(t)$ for all $t \in [0,1]$, and the proof is complete. ■

As a consequence of Theorem 3.5 and Propositions 3.6 and 3.7 we have

Corollary 3.8. *Let $W : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ be a Borel measurable function satisfying* (2.2) *and* (2.4)*. If W is ru-usc then*

$$\widehat{\mathcal{Z}\mathcal{H}W}(\xi) = \begin{cases} \mathcal{Z}\mathcal{H}W(\xi) & \text{if } \xi \in C \\ \lim_{t\to 1} \mathcal{Z}\mathcal{H}W(t\xi) & \text{if } \xi \in \partial C \\ \infty & \text{otherwise.} \end{cases}$$

Proof. Denote the effective domain of $\mathcal{Z}\mathcal{H}W$ by $\mathcal{Z}\mathcal{H}\mathbb{W}$.

Step 1: we prove that $\mathcal{Z}\mathcal{H}W$ is ru-usc. First of all, we can assert that $\mathcal{Z}\mathcal{H}W$ is continuous on $\text{int}(\mathcal{Z}\mathcal{H}\mathbb{W})$ because of the following lemma due to Fonseca (see [Fon88]).

Lemma 3.9. *$\mathcal{Z}L$ is continuous on* $\text{int}(\mathcal{Z}\mathbb{L})$.

On the other hand, from Proposition 3.7 we see that $\mathcal{H}W$ is ru-usc, hence $\mathcal{Z}\mathcal{H}W$ is ru-usc by Proposition 3.6.

Step 2: we prove that $C \subset \mathcal{ZH}\mathbb{W} \subset \overline{C}$. As $\mathcal{ZH}W \le W$ and W satisfies (2.2) we have $C \subset \mathcal{ZH}\mathbb{W}$. Fix any $t \in [0, 1[$. Using (2.2) we see that

$$W(x, \xi) \ge \delta_W(t)\mathrm{dist}(\xi, \overline{C}) =: G_t(\xi) \text{ for all } (x, \xi) \in \mathbb{R}^d \times \mathbb{M}^{m\times d},$$

where $\mathrm{dist}(\xi, \overline{C}) := \inf\left\{|\xi - \zeta| : \zeta \in \overline{C}\right\}$. But $G_t : \mathbb{M}^{m\times d} \to [0, \infty]$ is convex, hence $\mathcal{ZH}G_t = G_t$, and consequently

$$\mathcal{ZH}W(\xi) \ge \delta_W(t)\mathrm{dist}(\xi, \overline{C}) \text{ for all } \xi \in \mathbb{M}^{m\times d} \text{ and all } t \in [0, 1[.$$

As W satisfies (2.4) we deduce that $\mathcal{ZH}W \ge \infty$ whenever $\xi \notin \overline{C}$, which shows that $\mathcal{ZH}\mathbb{W} \subset \overline{C}$.

Step 3: we prove that $t\overline{\mathcal{ZH}\mathbb{W}} \subset \mathrm{int}(\mathcal{ZH}\mathbb{W})$ for all $t \in]0, 1[$. Let $t \in]0, 1|$. As $\mathcal{ZH}\mathbb{W} \subset \overline{C}$ (by Step 2) we have $\overline{\mathcal{ZH}\mathbb{W}} \subset \overline{C}$, and so $t\overline{\mathcal{ZH}\mathbb{W}} \subset t\overline{C}$. But $t\overline{C} \subset C$ because C is open and convex and $0 \in C$, hence $t\overline{\mathcal{ZH}\mathbb{W}} \subset C$. On the other hand, as $C \subset \mathcal{ZH}\mathbb{W}$ (by Step 2) and C is open we have $C \subset \mathrm{int}(\mathcal{ZH}\mathbb{W})$, and consequently $t\overline{\mathcal{ZH}\mathbb{W}} \subset \mathrm{int}(\mathcal{ZH}\mathbb{W})$.

Step 4: application of Theorem 3.5. By Step 2 we deduce that $\mathrm{int}(\mathcal{ZH}\mathbb{W}) = C$ (because C is open) and $\partial(\mathcal{ZH}\mathbb{W}) = \partial C$, and, taking Step 3 into account, the result follows from Theorem 3.5. ■

Remark 3.10. Under (2.2) and (2.4) we have $t\overline{\mathcal{H}\mathbb{W}} \subset \mathrm{int}(\mathcal{H}\mathbb{W})$ for all $t \in]0, 1[$, where $\mathcal{H}\mathbb{W}$ denotes the effective domain of $\mathcal{H}W$. Indeed, as $\mathcal{H}W \le W$ and, for every $t \in [0, 1[$, G_t is both lsc and convex, arguing as in Step 2, it is easy to see that if (2.2) and (2.4) hold then $C \subset \mathcal{H}\mathbb{W} \subset \overline{C}$, and the result follows by the same method as in Step 3.

3.2. **Approximation of ru-usc functions with respect to homogenization.** Let $C \subset \mathbb{M}^{m\times d}$ be a convex bounded open set and let $L : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0, \infty]$ be a Borel measurable function such that $\mathrm{dom}L(x, \cdot) \subset \overline{C}$ for all $x \in \mathbb{R}^d$, where $\mathrm{dom}L(x, \cdot)$ denotes the effective domain of $L(x, \cdot)$. We also assume that L is locally bounded in C, i.e.,

$$M_L(t) < \infty \text{ for all } t \in [0, 1[\text{ with } M_L(t) := \sup_{x\in\mathbb{R}^d} \sup_{\xi\in t\overline{C}} L(x, \xi), \tag{3.10}$$

and singular on the boundary ∂C of C, i.e.,

$$\lim_{t\to 1} \delta_L(t) = \infty \text{ with } \delta_L(t) := \inf_{x\in\mathbb{R}^d} \inf_{\xi\in\overline{C}\setminus t\overline{C}} L(x, \xi) \text{ for all } t \in [0, 1[. \tag{3.11}$$

Under (3.10) and (3.11) we have $\mathrm{dom}L(x, \cdot) = C$ for all $x \in \mathbb{R}^d$. For each $t \in [0, 1[$, we define $L_t : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0, \infty]$ by

$$L_t(x, \xi) := \begin{cases} L(x, \xi) & \text{if } \xi \in t\overline{C} \\ \delta_L(t)\left(1 + \mathrm{dist}(\xi, \overline{C})\right) & \text{if } \xi \notin t\overline{C}. \end{cases} \tag{3.12}$$

As $0 \in C$ we have $\mathrm{dist}(\xi, \overline{C}) \le |\xi|$ for all $\xi \in \mathbb{M}^{m\times d}$, and so for every $t \in [0, 1[$, L_t is of 1-polynomial growth, i.e.,

$$L_t(x, \xi) \le \alpha_t(1 + |\xi|) \text{ for all } (x, \xi) \in \mathbb{R}^d \times \mathbb{M}^{m\times d}$$

with $\alpha_t := \max\{\delta_L(t), M_L(t)\}$. On the other hand, under (3.10) and (3.11), it is easy to see that $\{L_t\}_{t\in[0,1[}$ is increasing to L, i.e.,

$$L(x, \xi) = \lim_{t\uparrow 1} L_t(x, \xi) = \sup_{t\in[0,1[} L_t(x, \xi) \text{ for all } (x, \xi) \in \mathbb{R}^d \times \mathbb{M}^{m\times d}.$$

Moreover, we have

Proposition 3.11. *Under* (3.10), (3.11), *if L is ru-usc then*

$$\lim_{t\uparrow 1}\mathcal{H}L_t=\sup_{t\in[0,1[}\mathcal{H}L_t\geq\widehat{\mathcal{H}L},$$

where $\mathcal{H}L_t,\mathcal{H}L:\mathbb{M}^{m\times d},\widehat{\mathcal{H}L}\to[0,\infty]$ *are given by*

- $\mathcal{H}L_t(\xi):=\inf_{k\geq 1}\inf\left\{⨍_{kY}L_t(x,\xi+\nabla\phi(x))dx:\phi\in W_0^{1,\infty}(kY;\mathbb{R}^m)\right\};$
- $\mathcal{H}L(\xi):=\inf_{k\geq 1}\inf\left\{⨍_{kY}L(x,\xi+\nabla\phi(x))dx:\phi\in W_0^{1,\infty}(kY;\mathbb{R}^m)\right\};$
- $\widehat{\mathcal{H}L}(\xi):=\varliminf_{s\to 1}\mathcal{H}L(s\xi).$

Proof. Set $\mathcal{L}:=\sup_{t\in[0,1[}\mathcal{H}L_t$, fix $\xi\in\mathbb{M}^{m\times d}$ and, without loss of generality, assume that $\mathcal{L}(\xi)<\infty$. Noticing that (by a change of variable)

$$\mathcal{H}L_t(\xi)=\inf_{k\geq 1}\inf\left\{\int_Y L_t(ky,\xi+\nabla\phi(y))dy:\phi\in W_0^{1,\infty}(Y;\mathbb{R}^m)\right\},\tag{3.13}$$

given any $t\in[0,1[$, there exist $k_t\geq 1$ and $\phi_t\in W_0^{1,\infty}(Y;\mathbb{R}^m)$ such that

$$\int_Y L_t(k_ty,\xi+\nabla\phi_\delta(y))dy<\mathcal{H}L_t(\xi)+\frac{1}{\delta_L(t)}\leq\mathcal{L}(\xi)+\frac{1}{\delta_L(t)}.$$

Setting $A_t:=\big\{y\in Y:\xi+\nabla\phi_t(y)\in t\overline{C}\big\}$ and taking (3.12) into account, we see that:

$$\int_{A_t}L(k_ty,\xi+\nabla\phi_t(y))dy<\mathcal{L}(\xi)+\frac{1}{\delta_L(t)};\tag{3.14}$$

$$\int_{Y\setminus A_t}\mathrm{dist}(\xi+\nabla\phi_t(y),\overline{C})dy+|Y\setminus A_t|<\frac{1}{\delta_L(t)}\left(\mathcal{L}(\xi)+\frac{1}{\delta_L(t)}\right).\tag{3.15}$$

As $\mathcal{L}(\xi)<\infty$, from (3.15) we deduce that:

$$\lim_{t\to 1}\int_Y\mathrm{dist}(\xi+\nabla\phi_t(y),\overline{C})dy=0;\tag{3.16}$$

$$\lim_{t\to 1}|Y\setminus A_t|=0.\tag{3.17}$$

Recall, in our context, the following lemma due to Müller (see [Mül99, Theorem 4] for a proof, see also [Zha92]).

Lemma 3.12. *Given* $\xi\in\mathbb{M}^{m\times d}$ *and* $\{\phi_t\}_{t\in[0,1[}\in W_0^{1,\infty}(Y;\mathbb{R}^m)$, *if* (3.16) *holds then one can find another sequence* $\{\psi_t\}_{t\in[0,1[}\subset W_0^{1,\infty}(Y;\mathbb{R}^m)$ *such that*:

$$\lim_{t\to 1}\sup_{y\in Y}\mathrm{dist}(\xi+\nabla\psi_t(y),\overline{C})=0;\tag{3.18}$$

$$\lim_{t\to 1}|B_t|=0\text{ with }B_t:=\big\{y\in Y:\nabla\psi_t(y)\neq\nabla\phi_t(y)\big\}.\tag{3.19}$$

Let $\{\psi_t\}_{t\in[0,1[}\subset W_0^{1,\infty}(Y;\mathbb{R}^m)$ be given by Lemma 3.12. Fix any $s\in]0,1[$. Using (3.18) (and the fact that C is convex and $0\in C$) we can assert that there exists $t_s\in[0,1[$ such that for every $t\in[t_s,1[$ and every $y\in Y$,

$$\xi+\nabla\psi_t(y)\in\frac{1}{\sqrt{s}}\overline{C},\text{ i.e., }\sqrt{s}(\xi+\nabla\psi_t(y))\in\overline{C}.\tag{3.20}$$

Using (3.20) we see that for any $t \in [t_s, 1[$,

$$\begin{aligned}\int_Y L\left(k_t y, t(\xi + \nabla\psi_t(y))\right) dy &= \int_{(Y\setminus B_t)\cap A_t} L\left(k_t y, t(\xi + \nabla\phi_t(y))\right) dy \\ &\quad + \int_{(Y\setminus A_t)\cup B_t} L\left(k_t y, \sqrt{s}\sqrt{s}(\xi + \nabla\psi_t(y))\right) dy \\ &\leq \int_{A_t} L\left(k_t y, t(\xi + \nabla\phi_t(y))\right) dy \\ &\quad + M_L(\sqrt{s})\left(|Y \setminus A_t| + |B_t|\right).\end{aligned}$$

As $\xi + \nabla\phi_t(y) \in t\overline{C}$ for all $y \in A_t$ and $t\overline{C} \subset C$ (because C is open and convex and $0 \in C$) it follows that

$$\begin{aligned}\int_Y L\left(k_t y, s\xi + \nabla(s\psi_t)(y)\right) dy &\leq \int_{A_t} L\left(k_t y, \xi + \nabla\phi_t(y)\right) dy + \Delta_L(s) \\ &\quad + M_L(\sqrt{s})\left(|Y \setminus A_t| + |B_t|\right)\end{aligned}$$

with $\Delta_L(s) := \sup_{x\in\mathbb{R}^d}\sup_{\zeta\in C} L(x, s\zeta) - L(x,\zeta)$. Noticing that $s\psi_t \in W_0^{1,\infty}(Y;\mathbb{R}^m)$ and taking (3.13) and (3.14) into account, we deduce that

$$\mathcal{H}L(s\xi) \leq \mathcal{L}(\xi) + \frac{1}{\delta_L(t)} + \Delta_L(s) + M_L(\sqrt{s})\left(|Y \setminus A_t| + |B_t|\right)$$

for all $t \in [t_s, 1[$. As $M_L(\sqrt{s}) < \infty$ by (3.10), letting $t \to 1$ and using (3.11), (3.17) and (3.19), we obtain $\mathcal{H}L(s\xi) \leq \mathcal{L}(\xi) + \Delta_L(s)$ for all $s \in]0, 1[$, hence

$$\varliminf_{s\to 1} \mathcal{H}L(s\xi) \leq \mathcal{L}(\xi),$$

because L is ru-usc, i.e., $\varlimsup_{s\to 1} \Delta_L(s) \leq 0$, and the result follows from Theorem 3.5(iii). ■

3.3. **Weak star Γ-liminf of periodic integrals of 1-polynomial growth.** For the convenience of the reader, in what follows we recall classical techniques on subadditivity, localization and blow up (see for instance [LM02, AM04]). In particular, Proposition 3.16 below will be used in the proof of Theorem 2.1(i).

3.3.1. *A subadditive theorem.* Let $\mathcal{O}_b(\mathbb{R}^d)$ be the class of all bounded open subsets of $\mathbb{R}^d$. We begin with the following definition.

Definition 3.13. Let $\mathcal{S} : \mathcal{O}_b(\mathbb{R}^d) \to [0, \infty]$ be a set function.

(i) We say that $\mathcal{S}$ is subadditive if

$$\mathcal{S}(A) \leq \mathcal{S}(B) + \mathcal{S}(C)$$

for all $A, B, C \in \mathcal{O}_b(\mathbb{R}^d)$ with $B, C \subset A$, $B \cap C = \emptyset$ and $|A \setminus B \cup C| = 0$.

(ii) We say that $\mathcal{S}$ is $\mathbb{Z}^d$-invariant if

$$\mathcal{S}(A + z) = \mathcal{S}(A)$$

for all $A \in \mathcal{O}_b(\mathbb{R}^d)$ and all $z \in \mathbb{Z}^d$.

Let $\mathrm{Cub}(\mathbb{R}^d)$ be the class of all open cubes in $\mathbb{R}^d$ and let $Y :=]0, 1[^d$. The following theorem is due to Akcoglu and Krengel (see [AK81], see also [LM02] and [AM02, §B.1]).

Theorem 3.14. *Let $\mathcal{S} : \mathcal{O}_b(\mathbb{R}^d) \to [0,\infty]$ be a subadditive and $\mathbb{Z}^d$-invariant set function for which there exists $c > 0$ such that*

$$\mathcal{S}(A) \leq c|A| \tag{3.21}$$

for all $A \in \mathcal{O}_b(\mathbb{R}^d)$. Then, for every $Q \in \mathrm{Cub}(\mathbb{R}^d)$,

$$\lim_{\varepsilon\to 0} \frac{\mathcal{S}\left(\frac{1}{\varepsilon}Q\right)}{\left|\frac{1}{\varepsilon}Q\right|} = \inf_{k\geq 1} \frac{\mathcal{S}(kY)}{k^d}.$$

Proof. Fix $Q \in \mathrm{Cub}(\mathbb{R}^d)$. First of all, it is easy to see that, for each $k \geq 1$ and each $\varepsilon > 0$, there exist $k_\varepsilon \geq 1$ and $z_\varepsilon \in \mathbb{Z}^d$ such that $\lim_{\varepsilon\to 0} k_\varepsilon = \infty$ and

$$(k_\varepsilon - 2)kY + k(z_\varepsilon + \hat{e}) \subset \frac{1}{\varepsilon}Q \subset k_\varepsilon kY + kz_\varepsilon \tag{3.22}$$

with $\hat{e} := (1, 1, \cdots, 1)$. Fix any $k \geq 1$ and any $\varepsilon > 0$. As the set function $\mathcal{S}$ is subadditive and $\mathbb{Z}^d$-invariant, using the left inclusion in (3.22) we obtain

$$\mathcal{S}\left(\frac{1}{\varepsilon}Q\right) \leq (k_\varepsilon - 2)^d \mathcal{S}(kY) + \mathcal{S}\left(\frac{1}{\varepsilon}Q \setminus \left((k_\varepsilon - 2)k\overline{Y} + k(z_\varepsilon + \hat{e})\right)\right).$$

Moreover, it is clear that

$$\left|\left[\frac{1}{\varepsilon}Q \setminus \left((k_\varepsilon - 2)k\overline{Y} + k(z_\varepsilon + \hat{e})\right)\right] \setminus \underset{i\in I}{\cup}(A_i + q_i)\right| = 0$$

where $q_i \in \mathbb{Z}^d$ and $\{A_i\}_{i\in I}$ is a finite family of disjoint open subsets of kY with $\mathrm{card}(I) = k_\varepsilon^d - (k_\varepsilon - 2)^d$, and so

$$\mathcal{S}\left(\frac{1}{\varepsilon}Q\right) \leq (k_\varepsilon - 2)^d \mathcal{S}(kY) + c(k_\varepsilon^d - (k_\varepsilon - 2)^d)k^d$$

by (3.21). It follows that

$$\frac{\mathcal{S}\left(\frac{1}{\varepsilon}Q\right)}{\left|\frac{1}{\varepsilon}Q\right|} \leq \frac{\mathcal{S}(kY)}{k^d} + c\frac{k_\varepsilon^d - (k_\varepsilon - 2)^d}{(k_\varepsilon - 2)^d}$$

because $|\frac{1}{\varepsilon}Q| \geq (k_\varepsilon - 2)^d k^d$ by the left inequality in (3.22). Letting $\varepsilon \to 0$ and passing to the infimum on k, we obtain

$$\overline{\lim_{\varepsilon\to 0}} \frac{\mathcal{S}\left(\frac{1}{\varepsilon}Q\right)}{\left|\frac{1}{\varepsilon}Q\right|} \leq \inf_{k\geq 1} \frac{\mathcal{S}(kY)}{k^d}.$$

On the other hand, using the right inequality in (3.22) with $k = 1$, by subadditivity and $\mathbb{Z}^d$-invariance we have

$$\mathcal{S}(k_\varepsilon Y) \leq \mathcal{S}\left(\frac{1}{\varepsilon}Q\right) + \mathcal{S}\left((k_\varepsilon Y + z_\varepsilon) \setminus \frac{1}{\varepsilon}\overline{Q}\right).$$

As previously, since, up to a set of zero Lebesgue measure, the set $(k_\varepsilon Y + z_\varepsilon) \setminus \frac{1}{\varepsilon}\overline{Q}$ can be written as the disjoint union of $k_\varepsilon^d - (k_\varepsilon - 2)^d$ integer translations of open subsets of Y, by using (3.21), we deduce that

$$\mathcal{S}(k_\varepsilon Y) \leq \mathcal{S}\left(\frac{1}{\varepsilon}Q\right) + c(k_\varepsilon^d - (k_\varepsilon - 2)^d),$$

and consequently

$$\inf_{k\geq 1} \frac{\mathcal{S}(kY)}{k^d} \leq \frac{\mathcal{S}(k_\varepsilon Y)}{k_\varepsilon^d} \leq \frac{\mathcal{S}\left(\frac{1}{\varepsilon}Q\right)}{\left|\frac{1}{\varepsilon}Q\right|} + c\frac{k_\varepsilon^d - (k_\varepsilon - 2)^d}{k_\varepsilon^d}$$

because $|\frac{1}{\varepsilon}Q| \leq k_\varepsilon^d$ by the right inequality in (3.22) with $k = 1$. Letting $\varepsilon \to 0$ we obtain

$$\inf_{k\geq 1} \frac{\mathcal{S}(kY)}{k^d} \leq \varliminf_{\varepsilon\to 0} \frac{\mathcal{S}\left(\frac{1}{\varepsilon}Q\right)}{\left|\frac{1}{\varepsilon}Q\right|},$$

and the proof is complete. ■

Given a Borel measurable function $L : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0, \infty]$, for each $\xi \in \mathbb{M}^{m\times d}$, we define $\mathcal{S}_\xi : \mathcal{O}_b(\mathbb{R}^d) \to [0, \infty]$ by

$$\mathcal{S}_\xi(A) := \inf \left\{ \int_A L(x, \xi + \nabla\phi(x))dx : \phi \in W_0^{1,\infty}(A; \mathbb{R}^m) \right\}. \tag{3.23}$$

It is easy that the set function $\mathcal{S}_\xi$ is subbadditive. If we assume that L is 1-periodic with respect to the first variable,i.e.,

$$L(x + z, \xi) = L(x, \xi) \text{ for all } x \in \mathbb{R}^d, \text{ all } z \in \mathbb{R}^d \text{ and all } \xi \in \mathbb{M}^{m\times d}, \tag{3.24}$$

then $\mathcal{S}_\xi$ is $\mathbb{Z}^d$-invariant. Moreover, if L is of 1-polynomial growth, i.e.,

$$L(x, \xi) \leq \alpha(1 + |\xi|) \text{ for all } \xi \in \mathbb{M}^{m\times d} \text{ and some } \alpha > 0, \tag{3.25}$$

then $\mathcal{S}_\xi(A) \leq \alpha(1 + |\xi|)|A|$ for all $A \in \mathcal{O}_b(\mathbb{R}^d)$. From the above, we see that the following result is a direct consequence of Theorem 3.14.

Corollary 3.15. *If L satisfies* (3.24) *and* (3.25), *then for every $\xi \in \mathbb{M}^{m\times d}$ and every $Q \in \mathrm{Cub}(\mathbb{R}^d)$,*

$$\lim_{\varepsilon\to 0} \frac{\mathcal{S}_\xi\left(\frac{1}{\varepsilon}Q\right)}{\left|\frac{1}{\varepsilon}Q\right|} = \inf_{k\geq 1} \frac{\mathcal{S}_\xi(kY)}{k^d}.$$

3.3.2. *Localization and blow up techniques.* In what follows, "$\overset{*}{\rightharpoonup}$" denotes the weak star convergence in $W^{1,\infty}(\Omega; \mathbb{R}^m)$.

Proposition 3.16. *Let $L : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0, \infty]$ be a Borel measurable function, let $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ and let $\{u_\varepsilon\}_\varepsilon \subset W^{1,\infty}(\Omega; \mathbb{R}^m)$ be such that $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega; \mathbb{R}^m)$. If L satisfies* (3.24) *and* (3.25), *then*

$$\varliminf_{\varepsilon\to 0} \int_\Omega L\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon(x)\right) dx \geq \int_\Omega \mathcal{H}L(\nabla u(x))dx \tag{3.26}$$

with $\mathcal{H}L : \mathbb{M}^{m\times d} \to [0, \infty]$ defined by

$$\mathcal{H}L(\xi) := \inf_{k\geq 1} \frac{\mathcal{S}_\xi(kY)}{k^d},$$

where $\mathcal{S}_\xi : \mathcal{O}_b(\mathbb{R}^d) \to [0, \infty]$ is given by (3.23).

Proof. Without loss of generality we can assume that:

$$\varliminf_{\varepsilon\to 0} \int_\Omega L\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon(x)\right) dx = \lim_{\varepsilon\to 0} \int_\Omega L\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon(x)\right) dx < \infty; \tag{3.27}$$

$$\sup_\varepsilon \|\nabla u_\varepsilon\|_{L^\infty(\Omega;\mathbb{M}^{m\times d})} < \infty; \tag{3.28}$$

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - u\|_{L^\infty(\Omega;\mathbb{R}^m)} = 0. \tag{3.29}$$

Step 1: localization. For each $\varepsilon > 0$, we define $f_\varepsilon \in L^1(\Omega; [0, \infty])$ by

$$f_\varepsilon := L\left(\frac{\cdot}{\varepsilon}, \nabla u_\varepsilon(\cdot)\right).$$

By (3.27) we have $\sup_\varepsilon \|f_\varepsilon\|_{L^1(\Omega;[0,\infty])} < \infty$, and so there exist $f \in L^1(\Omega; [0, \infty])$ and a finite positive Radon measure μ_s with $|\mathrm{supp}(\mu_s)| = 0$ such that

$$f_\varepsilon dx \rightharpoonup f dx + \mu_s \text{ in the sense of measures,}$$

where for a.e. $x_0 \in \Omega$,

$$f(x_0) = \lim_{\rho\to 0}\lim_{\varepsilon\to 0} \fint_{Q_\rho(x_0)} f_\varepsilon(x)dx = \lim_{\rho\to 0}\lim_{\varepsilon\to 0} \fint_{Q_\rho(x_0)} L\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon(x)\right) dx$$

with $Q_\rho(x_0) := x_0 + \rho Y$. By Alexandrov's theorem we have

$$\varliminf_{\varepsilon\to 0} \int_\Omega L\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) dx = \varliminf_{\varepsilon\to 0} \int_\Omega f_\varepsilon dx \geq \int_\Omega f dx + \mu_s(\Omega) \geq \int_\Omega f dx$$

and so, to prove (4.1) it suffices to show that for a.e. $x_0 \in \Omega$,

$$f(x_0) = \lim_{\rho\to 0}\lim_{\varepsilon\to 0} \fint_{Q_\rho(x_0)} L\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon(x)\right) dx \geq \mathcal{H}L(\nabla u(x_0)). \tag{3.30}$$

Step 2: cut-off method. Fix any $\delta \in]0, 1[$. Let $\phi \in C_c^\infty(Q_\rho(x_0); [0, 1])$ be a cut-off function between $Q_{\rho\delta}(x_0)$ and $Q_\rho(x_0)$ such that $\|\nabla\phi\|_{L^\infty(Q_\rho(x_0);\mathbb{R}^d)} \leq \frac{2}{\rho(1-\delta)}$. Setting

$$v_\varepsilon := \phi u_\varepsilon + (1 - \phi) l_{\nabla u(x_0)}$$

with $l_{\nabla u(x_0)}(x) := u(x_0) + \nabla u(x_0)(x - x_0)$, it follows that

$$\nabla v_\varepsilon := \begin{cases} \nabla u_\varepsilon & \text{on } Q_{\rho\delta}(x_0) \\ \phi\nabla u_\varepsilon + (1 - \phi)\nabla u(x_0) + \Psi_{\varepsilon,\rho} & \text{on } S_\rho \\ l_{\nabla u(x_0)} & \text{on } \partial Q_\rho(x_0), \end{cases} \tag{3.31}$$

with $S_\rho := Q_\rho(x_0) \setminus Q_{\rho\delta}(x_0)$ and $\Psi_{\varepsilon,\rho} := \nabla\phi \otimes \left(u_\varepsilon - l_{\nabla u(x_0)}\right)$, which, in particular, means that

$$v_\varepsilon - l_{\nabla u(x_0)} \in W_0^{1,p}(Q_\rho(x_0); \mathbb{R}^m). \tag{3.32}$$

As L is of 1-polynomial growth, i.e., L satisfies (3.25), we have

$$\begin{aligned} \fint_{Q_\rho(x_0)} L\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right) dx &\leq \fint_{Q_\rho(x_0)} L\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) dx \\ &\quad + \frac{1}{\rho^d} \int_{S_\rho} L\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right) dx \\ &\leq \fint_{Q_\rho(x_0)} L\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) dx + \alpha(1 - \delta^d) \\ &\quad + \frac{\alpha}{\rho^d} \int_{S_\rho} |\nabla v_\varepsilon| dx. \end{aligned}$$

On the other hand, for every $x \in S_\rho$, we have

$$\begin{aligned}|\nabla v_\varepsilon(x)| &\le |\nabla u_\varepsilon(x)| + |\nabla u(x_0)| + |\Psi_{\varepsilon,\rho}(x)|\\ &\le \sup_\varepsilon \|\nabla u_\varepsilon\|_{L^\infty(\Omega;\mathbb{M}^{m\times d})} + |\nabla u(x_0)| + \|\Psi_{\varepsilon,\rho}\|^p_{L^\infty(Q_\rho(x_0);\mathbb{M}^{m\times d})}\\ &\le c + |\nabla u(x_0)| + \frac{2}{\rho(1-\delta)}\|u_\varepsilon - u\|_{L^\infty(\Omega;\mathbb{R}^m)}\\ &\quad + \frac{2}{1-\delta}\frac{1}{\rho}\|u - l_{\nabla u(x_0)}\|_{L^\infty(Q_\rho(x_0);\mathbb{R}^m)}\end{aligned}$$

with $c := \sup_\varepsilon \|\nabla u_\varepsilon\|_{L^\infty(\Omega;\mathbb{M}^{m\times d})} < \infty$ by (3.28), and so

$$\begin{aligned}\frac{\alpha}{\rho^d}\int_{S_\rho} |\nabla v_\varepsilon| dx &\le \alpha(1-\delta^d)(c + |\nabla u(x_0)|) + \frac{2\alpha(1-\delta^d)}{\rho(1-\delta)}\|u_\varepsilon - u\|_{L^\infty(\Omega;\mathbb{R}^m)}\\ &\quad + \frac{2\alpha(1-\delta^d)}{1-\delta}\frac{1}{\rho}\|u - l_{\nabla u(x_0)}\|_{L^\infty(Q_\rho(x_0);\mathbb{R}^m)}.\end{aligned}$$

Thus, for every $\varepsilon > 0$ and every $\rho > 0$,

$$\begin{aligned}\text{(3.33)}\quad \fint_{Q_\rho(x_0)} L\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right) dx &\le \fint_{Q_\rho(x_0)} L\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) dx\\ &\quad + \alpha(1-\delta^d)(1 + c + |\nabla u(x_0)|)\\ &\quad + \frac{2\alpha(1-\delta^d)}{\rho(1-\delta)}\|u_\varepsilon - u\|_{L^\infty(\Omega;\mathbb{R}^m)}\\ &\quad + \frac{2\alpha(1-\delta^d)}{1-\delta}\frac{1}{\rho}\|u - l_{\nabla u(x_0)}\|_{L^\infty(Q_\rho(x_0);\mathbb{R}^m)}.\end{aligned}$$

Step 3: passing to the limit. Taking (3.32) into account we see that for every $\varepsilon > 0$,

$$\fint_{Q_\rho(x_0)} L\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right) dx \ge \frac{1}{|Q_\rho(x_0)|}\mathcal{S}_{\nabla u(x_0)}\left(\frac{1}{\varepsilon}Q_\rho(x_0)\right),$$

where, for any $\xi \in \mathbb{M}^{m\times d}$ and any open set $A \subset \mathbb{R}^d$, $\mathcal{S}_\xi(A)$ is defined by (3.23). From Corollary 3.15 we deduce that

$$\text{(3.34)}\qquad \varlimsup_{\varepsilon\to 0}\fint_{Q_\rho(x_0)} L\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right) dx \ge \mathcal{H}L(\nabla u(x_0)) \text{ for all } \rho > 0.$$

On the other hand, as $u \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ we can assert that u is differentiable at x_0, hence

$$\text{(3.35)}\qquad \lim_{\rho\to 0}\frac{1}{\rho}\|u - l_{\nabla u(x_0)}\|_{L^\infty(Q_\rho(x_0);\mathbb{R}^m)} = 0.$$

Taking (3.33) into account, from (3.34), (3.29) and (3.35) we deduce that

$$\lim_{\rho\to 0}\lim_{\varepsilon\to 0}\fint_{Q_\rho(x_0)} L\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) dx \ge \mathcal{H}L(\nabla u(x_0)) + c'(\delta^d - 1)$$

with $c' := \alpha(1 + c + |\nabla u(x_0)|)$, and (3.30) follows by letting $\delta \to 1$. ■

3.4. **Approximation of integrals which are locally bounded.** Let $C \subset \mathbb{M}^{m\times d}$ be a convex bounded open set such that $0 \in C$ and let $L : \mathbb{M}^{m\times d} \to [0, \infty]$ be a Borel measurable function which is locally bounded in C, i.e.,

$$M_L(t) < \infty \text{ for all } t \in [0, 1[\text{ with } M_L(t) := \sup_{\xi \in t\overline{C}} L(\xi). \tag{3.36}$$

To prove Proposition 3.18 below, we need the following lemma whose proof can be found in [DM99, Theorem 10.16 and Corollary 10.21] (see also [AH10, Proposition 5.1]). (Recall that, given any bounded open set $A \subset \mathbb{R}^d$, $\mathrm{Aff}(A; \mathbb{R}^m)$ denotes the space of continuous piecewise affine functions from A to $\mathbb{R}^m$.)

Lemma 3.17. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary and let $v \in W^{1,\infty}(\Omega; \mathbb{R}^m)$. Given $t \in]0, 1[$, if $\nabla v(x) \in t\overline{C}$ for a.a. $x \in \Omega$ then there exists $\{\Omega_n; v_n\}_n$ such that:*

(3.37) Ω_n *is an open subset of* Ω *and* $|\partial\Omega_n| = 0$ *for all* $n \geq 1$;

(3.38) $\lim_{n\to\infty} |\Omega \setminus \Omega_n| = 0$;

(3.39) $v_n \in W^{1,\infty}(\Omega; \mathbb{R}^m), v_n|_{\Omega_n} \in \mathrm{Aff}(\Omega_n; \mathbb{R}^m)$ *and* $v_n = v$ *on* $\partial\Omega$ *for all* $n \geq 1$,

(3.40) $\nabla v_n(x) \in \left(t + \dfrac{1}{n}\right)\overline{C}$ *for a.a.* $x \in \Omega$ *and all* $n \geq 1$;

(3.41) $\lim_{n\to\infty} \|v_n - v\|_{W^{1,1}(\Omega;\mathbb{R}^m)} = 0$.

Proposition 3.18. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary and let $v \in W^{1,\infty}(\Omega; \mathbb{R}^m)$. Given $t \in]0, 1[$, if $\nabla v(x) \in t\overline{C}$ for a.a. $x \in \Omega$ and if L is continuous on C, then there exists $\{\Omega_n; v_n\}_n$ satisfying* (3.37)*,* (3.38)*,* (3.39)*,* (3.40)*,* (3.41) *and*

$$\lim_{n\to\infty} \int_\Omega L(\nabla v_n(x))dx = \int_\Omega L(\nabla v(x))dx. \tag{3.42}$$

Proof. Let $\{v_n\}_n \subset W^{1,\infty}(\Omega; \mathbb{R}^m)$ be given by Lemma 3.17. Taking (3.41) into account we can assert that, up to a subsequence,

$$\nabla v_n(x) \to \nabla v(x) \text{ for a.a. } x \in \Omega. \tag{3.43}$$

Given $\alpha_t \in]t, 1[$, there exists $n_t \geq 1$ such that $t + \frac{1}{n} < \alpha_t$ for all $n \geq n_t$, hence

$$\nabla v_n(x) \in \alpha_t\overline{C} \text{ for a.a. } x \in \Omega \text{ and all } n \geq n_t \tag{3.44}$$

by (3.40). Using (3.43) it follows that

$$\nabla v(x) \in \alpha_t\overline{C} \text{ for a.a. } x \in \Omega. \tag{3.45}$$

As $\alpha_t\overline{C} \subset C$ (because C is open and convex and $0 \in C$) and L is continuous on C, from (3.43), (3.44) and (3.45) we deduce that

$$L(\nabla v_n(x)) \to L(\nabla v(x)) \text{ for a.a. } x \in \Omega.$$

Moreover, from (3.44) we see that $L(\nabla v_n(x)) \leq M_L(\alpha_t)$ for a.a. $x \in \Omega$ and all $n \geq n_t$, where $M_L(\alpha_t) < \infty$ because L is locally bounded in C, i.e., L satisfies (3.36), and (3.42) follows from Lebesgue's dominated convergence theorem. ■

3.5. **Approximation of the relaxation formula.** Given a Borel measurable function $L : \mathbb{M}^{m\times d} \to [0,\infty]$ we consider $\mathcal{Z}L : \mathbb{M}^{m\times d} \to [0,\infty]$ defined by

$$\mathcal{Z}L(\xi) := \inf\left\{\int_Y L(\xi+\nabla\phi(y))dy : \phi \in \mathrm{Aff}_0(Y;\mathbb{R}^m)\right\}$$

with $Y :=]0,1[^d$ and $\mathrm{Aff}_0(Y;\mathbb{R}^m) := \big\{\phi \in \mathrm{Aff}(Y;\mathbb{R}^m) : \phi = 0 \text{ on } \partial Y\big\}$ where $\mathrm{Aff}(Y;\mathbb{R}^m)$ is the space of continuous piecewise affine functions from Y to $\mathbb{R}^m$. The following proposition is adapted from [AHM08, Lemma 3.1] (see also [AHM07]).

Proposition 3.19. *Given $\xi \in \mathbb{M}^{m\times d}$ and a bounded open set $A \subset \mathbb{R}^d$ there exists $\{\phi_k\}_k \subset \mathrm{Aff}_0(A;\mathbb{R}^m)$ such that:*

- $\lim\limits_{k\to\infty} \|\phi_k\|_{L^\infty(A;\mathbb{R}^m)} = 0$;
- $\lim\limits_{k\to\infty} \fint_A L(\xi+\nabla\phi_k(x))dx = \mathcal{Z}L(\xi)$.

Proof. Given $\xi \in \mathbb{M}^{m\times d}$ there exists $\{\phi_n\}_n \subset \mathrm{Aff}_0(Y;\mathbb{R}^m)$ such that

$$\lim_{n\to\infty} \int_Y L(\xi+\nabla\phi_n(y))dy = \mathcal{Z}L(\xi). \tag{3.46}$$

Fix any $n \geq 1$ and $k \geq 1$. By Vitali's covering theorem there exists a finite or countable family $\{a_i+\alpha_i Y\}_{i\in I}$ of disjoint subsets of A, where $a_i \in \mathbb{R}^d$ and $0 < \alpha_i < \frac{1}{k}$, such that $|A \setminus \cup_{i\in I}(a_i+\alpha_i Y)| = 0$ (and so $\sum_{i\in I}\alpha_i^d = |A|$). Define $\phi_{n,k} \in \mathrm{Aff}_0(A;\mathbb{R}^m)$ by

$$\phi_{n,k}(x) := \alpha_i\phi_n\left(\frac{x-a_i}{\alpha_i}\right) \text{ if } x \in a_i+\alpha_i Y.$$

Clearly $\|\phi_{n,k}\|_{L^\infty(A;\mathbb{R}^m)} \leq \frac{1}{k}\|\phi_n\|_{L^\infty(Y;\mathbb{R}^m)}$, hence $\lim_{k\to\infty}\|\phi_{n,k}\|_{L^\infty(A;\mathbb{R}^m)} = 0$ for all $k \geq 1$, and consequently

$$\lim_{n\to\infty}\lim_{k\to\infty}\|\phi_{n,k}\|_{L^\infty(A;\mathbb{R}^m)} = 0. \tag{3.47}$$

On the other hand, we have

$$\int_A L(\xi+\nabla\phi_{n,k}(x))dx = \sum_{i\in I}\alpha_i^d\int_Y L(\xi+\nabla\phi_n(y))dy = |A|\int_Y L(\xi+\nabla\phi_n(y))dy$$

for all $n \geq 1$ and all $k \geq 1$. Using (3.46) we deduce that

$$\lim_{n\to\infty}\lim_{k\to\infty}\fint_A L(\xi+\nabla\phi_{n,k}(x))dx = \mathcal{Z}L(\xi), \tag{3.48}$$

and the result follows from (3.47) and (3.48) by diagonalization. ■

3.6. **Approximation of the homogenization formula.** Given a convex bounded open set $C \subset \mathbb{M}^{m\times d}$ such that $0 \in C$ and a Borel measurable function $L : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ which is 1-periodic with respect to its first variable and locally bounded in C, i.e.,

$$M_L(t) < \infty \text{ for all } t \in [0,1[\text{ with } M_L(t) := \sup_{x\in\mathbb{R}^d}\sup_{\xi\in t\overline{C}} L(x,\xi), \tag{3.49}$$

we consider $\mathcal{H}L : \mathbb{M}^{m\times d} \to [0,\infty]$ defined by

$$\mathcal{H}L(\xi) := \inf_{k\geq 1}\inf\left\{\fint_{kY} L(x,\xi+\nabla\phi(x))dx : \phi \in W_0^{1,\infty}(kY;\mathbb{R}^m)\right\}.$$

The following proposition is adapted from [Mül87, Lemma 2.1(a)].

Proposition 3.20. *Let $t\in]0,1[$, let $\xi\in\mathbb{M}^{m\times d}$ and let $A\subset\mathbb{R}^d$ be a bounded open set. If $\xi\in t\overline{C}$ then there exists $\{\phi_\varepsilon\}_\varepsilon\subset W_0^{1,\infty}(A;\mathbb{R}^m)$ such that:*

- ♦ $\lim\limits_{\varepsilon\to0}\|\phi_\varepsilon\|_{L^1(A;\mathbb{R}^m)}=0$;
- ♦ $\lim\limits_{\varepsilon\to0}\fint_A L\left(\frac{x}{\varepsilon},\xi+\nabla\phi_\varepsilon(x)\right)dx=\mathcal{H}L(\xi)$.

Proof. Consider $\{k_n;\hat{\phi}_n\}_n$ such that:

$$\hat{\phi}_n\in W_0^{1,\infty}(k_nY;\mathbb{R}^m)\text{ for all }n\geq1;$$

$$\lim_{n\to\infty}\fint_{k_nY}L(x,\xi+\nabla\hat{\phi}_n(x))dx=\mathcal{H}L(\xi). \tag{3.50}$$

For each $n\geq1$ and $\varepsilon>0$, denote the k_nY-periodic extension of $\hat{\phi}_n$ by ϕ_n, consider $A_{n,\varepsilon}\subset A$ given by

$$A_{n,\varepsilon}:=\bigcup_{z\in I_{n,\varepsilon}}\varepsilon(z+k_nY)$$

with $I_{n,\varepsilon}:=\left\{z\in\mathbb{Z}^d:\varepsilon(z+k_nY)\subset A\right\}$, where $\mathrm{card}(I_{n,\varepsilon})<\infty$ because A is bounded, and define $\phi_{n,\varepsilon}\in W_0^{1,\infty}(A;\mathbb{R}^m)$ by

$$\phi_{n,\varepsilon}(x):=\varepsilon\phi_n\left(\frac{x}{\varepsilon}\right)\text{ if }x\in A_{n,\varepsilon}.$$

Fix any $n\geq1$. It is easy to see that

$$\begin{aligned}\|\phi_{n,\varepsilon}\|_{L^1(A;\mathbb{R}^m)}&=\int_{A_{n,\varepsilon}}|\phi_{n,\varepsilon}(x)|dx\\&=\varepsilon\sum_{z\in I_{n,\varepsilon}}\int_{\varepsilon(z+k_nY)}\left|\phi_n\left(\frac{x}{\varepsilon}\right)\right|dx\\&\leq\varepsilon\frac{|A|}{k_n^d}\|\hat{\phi}_n\|_{L^1(k_nY;\mathbb{R}^m)}\end{aligned}$$

for all $\varepsilon>0$, and consequently $\lim_{\varepsilon\to0}\|\phi_{n,\varepsilon}\|_{L^1(A;\mathbb{R}^m)}=0$ for all $n\geq1$. It follows that

$$\lim_{n\to\infty}\lim_{\varepsilon\to0}\|\phi_{n,\varepsilon}\|_{L^1(A;\mathbb{R}^m)}=0. \tag{3.51}$$

On the other hand, for every $n\geq1$ and every $\varepsilon>0$, we have

$$\int_A L\left(\frac{x}{\varepsilon},\xi+\nabla\phi_{n,\varepsilon}(x)\right)dx=\int_{A_{n,\varepsilon}}L\left(\frac{x}{\varepsilon},\xi+\nabla\phi_{n,\varepsilon}(x)\right)dx+\int_{A\setminus A_{n,\varepsilon}}L\left(\frac{x}{\varepsilon},\xi\right)dx.$$

But

$$\begin{aligned}\int_{A_{n,\varepsilon}}L\left(\frac{x}{\varepsilon},\xi+\nabla\phi_{n,\varepsilon}(x)\right)dx&=\sum_{z\in I_{n,\varepsilon}}\int_{\varepsilon(z+k_nY)}L\left(\frac{x}{\varepsilon},\xi+\nabla\phi_n\left(\frac{x}{\varepsilon}\right)\right)dx\\&=|A_{n,\varepsilon}|\fint_{k_nY}L(x,\xi+\nabla\hat{\phi}_n(x))dx,\end{aligned}$$

and consequently

$$\begin{aligned}|A_{n,\varepsilon}|\mathcal{H}L(\xi)\leq\int_A L\left(\frac{x}{\varepsilon},\xi+\nabla\phi_{n,\varepsilon}(x)\right)dx\leq{}&|A|\fint_{k_nY}L(x,\xi+\nabla\hat{\phi}_n(x))dx\\&+|A\setminus A_{n,\varepsilon}|M_L(t)\end{aligned}$$

because $\xi\in t\overline{C}$. As $\lim_{\varepsilon\to0}|A\setminus A_{n,\varepsilon}|=0$ for any $n\geq1$, $M_L(t)<\infty$ by (3.49) and using (3.50) we see that:

- ♦ $\lim_{\varepsilon\to 0} |A \setminus A_{n,\varepsilon}|\mathcal{H}L(\xi) = 0$;
- ♦ $\lim_{n\to\infty}\lim_{\varepsilon\to 0}\left(\fint_{k_nY} L(x,\xi+\nabla\hat{\phi}_n)dx - \mathcal{H}L(\xi) + \frac{|A\setminus A_{n,\varepsilon}|}{|A|}M_L(t)\right)=0.$

Hence

$$\lim_{n\to\infty}\overline{\lim_{\varepsilon\to 0}}\left|\fint_A L\left(\frac{x}{\varepsilon},\xi+\nabla\phi_{n,\varepsilon}(x)\right)dx - \mathcal{H}L(\xi)\right| = 0, \tag{3.52}$$

and the result follows from (3.51) and (3.52) by diagonalization. ■

4. Proof of the homogenization theorem

In this section we prove Theorem 2.1.

4.1. **Proof of Theorem 2.1(i).** Fix $u \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ and $\{u_\varepsilon\}_\varepsilon \subset W^{1,\infty}(\Omega;\mathbb{R}^m)$ such that $\|u_\varepsilon - u\|_{L^1(\Omega;\mathbb{R}^m)} \to 0$. We have to prove that

$$\underline{\lim}_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) \geq \widehat{\mathcal{H}I}(u). \tag{4.1}$$

Without loss of generality we can assume that

$$\underline{\lim}_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) = \lim_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) < \infty, \text{ and so } \sup_\varepsilon I_\varepsilon(u_\varepsilon) < \infty.$$

Then, $\nabla u_\varepsilon(x) \in C$ for all $\varepsilon > 0$ and a.a. $x \in \Omega$ because $\mathrm{dom}W(x,\cdot) = C$ for all $x \in \mathbb{R}^d$, and so $\sup_\varepsilon \|\nabla u_\varepsilon\|_{L^\infty(\Omega;\mathbb{M}^{m\times d})} < \infty$. On the other hand, $\sup_\varepsilon \|u_\varepsilon\|_{L^1(\Omega;\mathbb{R}^m)} < \infty$ (since $\|u_\varepsilon - u\|_{L^1(\Omega;\mathbb{R}^m)} \to 0$) and by Poincaré-Wirtinger's inequality, there exists $c > 0$ such that $\sup_\varepsilon \|u_\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^m)} \leq c(\sup_\varepsilon \|u_\varepsilon\|_{L^1(\Omega;\mathbb{R}^m)} + \sup_\varepsilon \|\nabla u_\varepsilon\|_{L^\infty(\Omega;\mathbb{M}^{m\times d})})$. It follows that, up to a subsequence,

$$u_\varepsilon \overset{*}{\rightharpoonup} u \text{ in } W^{1,\infty}(\Omega;\mathbb{R}^m), \tag{4.2}$$

where "$\overset{*}{\rightharpoonup}$" denotes the weak star convergence in $W^{1,\infty}(\Omega;\mathbb{R}^m)$.

Step 1: approximation of W by periodic functions of 1-polynomial growth. For each $t \in [0,1[$, we define $W_t : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ by

$$W_t(x,\xi) := \begin{cases} W(x,\xi) & \text{if } \xi \in t\overline{C} \\ \delta_W(t)\left(1+\mathrm{dist}(\xi,\overline{C})\right) & \text{if } \xi \notin t\overline{C} \end{cases}$$

with $\delta_W(t) := \inf_{x\in\mathbb{R}^d}\inf_{\xi\in\overline{C}\setminus t\overline{C}} W(x,\xi)$. As W is 1-periodic with respect to the first variable, also is W_t for each $t \in [0,1[$, i.e.,

$$W_t(x+z,\xi) = W_t(x,\xi) \text{ for all } x \in \mathbb{R}^d, \text{ all } z \in \mathbb{R}^d \text{ and all } \xi \in \mathbb{M}^{m\times d}. \tag{4.3}$$

As $0 \in C$ we have $\mathrm{dist}(\xi,\overline{C}) \leq |\xi|$ for all $\xi \in \mathbb{M}^{m\times d}$, and so for every $t \in [0,1[$, W_t is of 1-polynomial growth, i.e.,

$$W_t(x,\xi) \leq \alpha_t(1+|\xi|) \text{ for all } (x,\xi) \in \mathbb{R}^d \times \mathbb{M}^{m\times d} \tag{4.4}$$

with $\alpha_t := \max\{\delta_W(t), M_W(t)\}$ where $M_W(t) := \sup_{x\in\mathbb{R}^d}\sup_{\xi\in t\overline{C}} W(x,\xi) < \infty$ by (2.3). On the other hand, under (2.4), it is easy to see that

$$W \geq W_s \geq W_t \text{ for all } s,t \in [0,1[\text{ with } s \geq t \tag{4.5}$$

and $W = \sup_{t\in[0,1[} W_t$, i.e., $\{W_t\}_{t\in[0,1[}$ is increasing to W.

Step 2: passing to the limit. First of all, using (4.5) we see that

$$\mathcal{H}W_s \geq \mathcal{H}W_t \text{ for all } s,t \in [0,1[\text{ with } s \geq t, \tag{4.6}$$

i.e., $\{\mathcal{H}W_t\}_{t\in[0,1[}$ is increasing, with $\mathcal{H}W_t : \mathbb{M}^{m\times d} \to [0,\infty]$ given by

$$\mathcal{H}W_t(\xi) := \inf_{k\geq 1} \inf \left\{ \fint_{kY} W_t(x, \xi + \nabla\phi(x))dx : \phi \in W_0^{1,\infty}(kY;\mathbb{R}^m) \right\}.$$

On the other hand, given any $t \in [0,1[$, by (4.5) we have

$$\varliminf_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) \geq \varliminf_{\varepsilon\to 0} \int_\Omega W_t\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon(x)\right) dx.$$

Taking (4.2), (4.3) and (4.4) into account, from Proposition 3.16 we deduce that

$$\varliminf_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) \geq \int_\Omega \mathcal{H}W_t(\nabla u(x))dx$$

for all $t \in [0,1[$, hence

$$\varliminf_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) \geq \int_\Omega \sup_{t\in[0,1[} \mathcal{H}W_t(\nabla u(x))dx$$

by using (4.6), and (4.1) follows from Proposition 3.11. ■

4.2. **Proof of Theorem 2.1(ii).** Let $u \in W^{1,\infty}(\Omega;\mathbb{R}^m)$. We have to prove that there exists $\{u_\varepsilon\}_\varepsilon \subset W^{1,\infty}(\Omega;\mathbb{R}^m)$ such that $\|u_\varepsilon - u\|_{L^1(\Omega;\mathbb{R}^m)} \to 0$ and

$$\varlimsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) \leq \widehat{\mathcal{Z}\mathcal{H}I}(u).$$

Without loss of generality we can assume that $\widehat{\mathcal{Z}\mathcal{H}I}(u) < \infty$, and so

$$\nabla u(x) \in \widehat{\mathcal{Z}\mathcal{H}\mathbb{W}} \text{ for a.a. } x \in \Omega, \tag{4.7}$$

where $\widehat{\mathcal{Z}\mathcal{H}\mathbb{W}}$ denotes the effective domain of $\widehat{\mathcal{Z}\mathcal{H}W}$.

Step 1: characterization of $\widehat{\mathcal{Z}\mathcal{H}W}$. As W is ru-usc, from Propositions 3.7 and 3.6 we can assert that $\mathcal{Z}\mathcal{H}W$ is ru-usc. Moreover, $\mathcal{Z}\mathcal{H}W$ is continuous on $\mathrm{int}(\mathcal{Z}\mathcal{H}\mathbb{W})$ by Lemma 3.9, and from Step 3 of Corollary 3.8 we see that $t\overline{\mathcal{Z}\mathcal{H}\mathbb{W}} \subset \mathrm{int}(\mathcal{Z}\mathcal{H}\mathbb{W})$ for all $t \in]0,1[$ (where $\mathcal{Z}\mathcal{H}\mathbb{W}$ denotes the effective domain of $\mathcal{Z}\mathcal{H}W$). Hence

$$\widehat{\mathcal{Z}\mathcal{H}W} \text{ is ru-usc, i.e., } \varlimsup_{t\to 1} \Delta_{\widehat{\mathcal{Z}\mathcal{H}W}}(t) \leq 0, \tag{4.8}$$

by Theorem 3.5(ii). On the other hand, using Corollary 3.8 we deduce that

$$\widehat{\mathcal{Z}\mathcal{H}W}(\xi) = \begin{cases} \mathcal{Z}\mathcal{H}W(\xi) & \text{if } \xi \in C \\ \lim\limits_{t\to 1} \mathcal{Z}\mathcal{H}W(t\xi) & \text{if } \xi \in \partial C \\ \infty & \text{otherwise.} \end{cases} \tag{4.9}$$

Step 2: approximation of $\widehat{\mathcal{ZHW}}$. First of all, it is clear that

$$\lim_{t\to 1} \|tu - u\|_{W^{1,1}(\Omega;\mathbb{R}^m)} = 0. \tag{4.10}$$

On the other hand, from (4.9) we see that $\widehat{\mathcal{ZH}\mathbb{W}} \subset \overline{C}$, and so $t\nabla u(x) \in C$ for a.a. $x \in \Omega$ because C is open and convex, $0 \in C$ and (4.7) holds. It follows that

$$\int_\Omega \mathcal{ZH}W(t\nabla u(x))dx \le \int_\Omega \widehat{\mathcal{ZH}W}(\nabla u(x))dx + |\Omega|\Delta_{\widehat{\mathcal{ZH}W}}(t)$$

for all $t \in]0,1[$, and consequently

$$\overline{\lim_{t\to 1}} \int_\Omega \mathcal{ZH}W(t\nabla u(x))dx \le \int_\Omega \widehat{\mathcal{ZH}W}(\nabla u(x))dx \tag{4.11}$$

because (4.8) holds.

Step 3: approximation of $\mathcal{ZHW}$. Fix any $t \in]0,1[$. From (4.9) we deduce that $\widehat{\mathcal{ZH}\mathbb{W}} \subset \overline{C}$, and so $\nabla(tu)(x) \in t\overline{C}$ for a.a. $x \in \Omega$ because (4.7) holds. Moreover, applying Lemma 3.9 with $L = \mathcal{H}W$, we can assert that $\mathcal{ZH}W$ is continuous on $\mathrm{int}(\mathcal{ZH}\mathbb{W})$. But, arguing as in Step 4 of Corollary 3.8 we see that $\mathrm{int}(\mathcal{ZH}\mathbb{W}) = C$, and consequently $\mathcal{ZH}W$ is continuous on C. From Proposition 3.18 it follows that there exists $\{\Omega_{n,t}; u_{n,t}\}_n$ where for each $n \ge 1$, $\Omega_{n,t}$ is an open subset of Ω and $u_{n,t} \in W^{1,\infty}(\Omega;\mathbb{R}^m)$, such that:

$$\lim_{n\to\infty} |\Omega \setminus \Omega_{n,t}| = 0; \tag{4.12}$$

$$u_{n,t}|_{\Omega_{n,t}} \in \mathrm{Aff}(\Omega_{n,t};\mathbb{R}^m) \text{ for all } n \ge 1; \tag{4.13}$$

$$\nabla u_{n,t}(x) \in \left(t + \frac{1}{n}\right)\overline{C} \text{ for a.a. } x \in \Omega \text{ and all } n \ge 1; \tag{4.14}$$

$$\lim_{n\to\infty} \|u_{n,t} - tu\|_{W^{1,1}(\Omega;\mathbb{R}^m)} = 0; \tag{4.15}$$

$$\lim_{n\to\infty} \int_\Omega \mathcal{ZH}W(\nabla u_{n,t}(x))dx = \int_\Omega \mathcal{ZH}W(t\nabla u(x))dx. \tag{4.16}$$

Consider $\alpha_t \in]t,1[$ and $n_t \ge 1$ such that $t + \frac{1}{n} < \alpha_t$ for all $n \ge n_t$. From (4.14) we see that

$$\nabla u_{n,t}(x) \in \alpha_t\overline{C} \text{ for a.a. } x \in \Omega \text{ and all } n \ge n_t. \tag{4.17}$$

Fix any $n \ge n_t$. As $u_{n,t}|_{\Omega_{n,t}} \in \mathrm{Aff}(\Omega_{n,t};\mathbb{R}^m)$ by (4.13), we can assert that there exists a finite family $\{U_i\}_{i\in I}$ of open disjoint subsets of $\Omega_{n,t}$ such that $|\Omega_{n,t} \setminus \cup_{i\in I} U_i| = 0$ and, for each $i \in I$, $|\partial U_i| = 0$ and $\nabla u_{n,t}(x) = \xi_i$ in U_i with $\xi_i \in \mathbb{M}^{m\times d}$. Thus

$$\int_{\Omega_{n,t}} \mathcal{ZH}W(\nabla u_{n,t}(x))dx = \sum_{i\in I} |U_i|\mathcal{ZH}W(\xi_i). \tag{4.18}$$

By Proposition 3.19, for each $i \in I$, there exists $\{\phi_{i,k}\}_k \subset \mathrm{Aff}_0(U_i;\mathbb{R}^m)$ such that:

$$\lim_{k\to\infty} \|\phi_{i,k}\|_{L^\infty(U_i;\mathbb{R}^m)} = 0; \tag{4.19}$$

$$\lim_{k\to\infty} \fint_{U_i} \mathcal{H}W(\xi_i + \nabla\phi_{i,k}(x))dx = \mathcal{ZH}W(\xi_i). \tag{4.20}$$

For each $k \ge 1$, define $u_{k,n,t} \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ by

$$u_{k,n,t}(x) := u_{n,t}(x) + \phi_{i,k}(x) \text{ if } x \in U_i.$$

Then:

$$u_{k,n,t}|_{\Omega_{n,t}} \in \mathrm{Aff}(\Omega_{n,t};\mathbb{R}^m); \tag{4.21}$$

$$\|u_{k,n,t}-u_{n,t}\|_{L^\infty(\Omega;\mathbb{R}^m)} = \max_{i\in I}\|\phi_{i,k}\|_{L^\infty(U_i;\mathbb{R}^m)}. \tag{4.22}$$

From (4.22) and (4.19) we deduce that

$$\lim_{k\to\infty}\|u_{k,n,t}-u_{n,t}\|_{L^\infty(\Omega;\mathbb{R}^m)} = 0. \tag{4.23}$$

On the other hand, taking (4.17) into account, for each $k\geq 1$, we have

$$\begin{aligned}\int_\Omega \mathcal{H}W(\nabla u_{k,n,t}(x))dx &= \int_{\Omega_{n,t}} \mathcal{H}W(\nabla u_{k,n,t}(x))dx + \int_{\Omega\setminus\Omega_{n,t}} \mathcal{H}W(\nabla u_{n,t}(x))dx\\ &\leq \sum_{i\in I}|U_i| -\!\!\!\!\!\!\int_{U_i}\mathcal{H}W(\xi_i+\nabla\phi_{i,k}(x))dx + |\Omega\setminus\Omega_{n,t}|M_W(\alpha_t)\end{aligned}$$

with $M_W(\alpha_t)<\infty$ by (2.3), and consequently

$$\overline{\lim_{k\to\infty}}\int_\Omega \mathcal{H}W(\nabla u_{k,n,t}(x))dx \leq \int_{\Omega_{n,t}}\mathcal{ZH}W(\nabla u_{n,t}(x))dx + |\Omega\setminus\Omega_{n,t}|M_W(\alpha_t) \tag{4.24}$$

by (4.20) and (4.18).

Step 4: approximation of $\boldsymbol{\mathcal{H}W}$. Fix any $k\geq 1$. As $\mathcal{ZH}I(u)<\infty$ and $M_W(\alpha_t)<\infty$, from (4.11), (4.16) and (4.24) we see that $\nabla u_{k,n,t}(x)\in\mathcal{H}\mathbb{W}$ for a.a. $x\in\Omega$, where $\mathcal{H}\mathbb{W}$ denotes the effective domain of $\mathcal{H}W$, hence:

$$t\nabla u_{k,n,t}(x)\in\mathcal{H}\mathbb{W} \text{ for a.a. } x\in\Omega; \tag{4.25}$$

$$t\nabla u_{k,n,t}(x)\in t\overline{C} \text{ for a.a. } x\in\Omega \tag{4.26}$$

because $C\subset\mathcal{H}\mathbb{W}\subset\overline{C}$ (see Remark 2.3) and C is open and convex and $0\in C$. Taking (4.25) into account it follows that

$$\int_\Omega \mathcal{H}W(t\nabla u_{k,n,t}(x))dx \leq \int_\Omega \mathcal{H}W(\nabla u_{k,n,t}(x))dx + |\Omega|\Delta_{\mathcal{H}W}(t) \tag{4.27}$$

with $\Delta_{\mathcal{H}W}(t) := \sup_{\xi\in\mathcal{H}\mathbb{W}}\mathcal{H}W(t\xi)-\mathcal{H}W(\xi)$. From (4.21) we see that $tu_{k,n,t}|_{\Omega_{n,t}}\in \mathrm{Aff}(\Omega_{n,t};\mathbb{R}^m)$, and so we can assert that there exists a finite family $\{V_j\}_{j\in J}$ of open disjoint subsets of $\Omega_{n,t}$ such that $|\Omega_{n,t}\setminus\cup_{j\in J}V_j|=0$ and, for each $j\in J$, $|\partial V_j|=0$ and

$$t\nabla u_{k,n,t}(x)=\zeta_j \text{ in } V_j \text{ with } \zeta_j\in\mathbb{M}^{m\times d}. \tag{4.28}$$

It follows that

$$\int_{\Omega_{n,t}}\mathcal{H}W(t\nabla u_{k,n,t}(x))dx = \sum_{j\in J}|V_j|\mathcal{H}W(\zeta_j). \tag{4.29}$$

From (4.28) and (4.26) we see that $\zeta_j\in t\overline{C}$ for all $j\in J$. Using Proposition 3.20 we deduce that for each $j\in J$, there exists $\{\psi_{j,\varepsilon}\}_\varepsilon\subset W_0^{1,\infty}(V_j;\mathbb{R}^m)$ such that:

$$\lim_{\varepsilon\to 0}\|\psi_{j,\varepsilon}\|_{L^1(V_j;\mathbb{R}^m)}=0; \tag{4.30}$$

$$\lim_{\varepsilon\to 0} -\!\!\!\!\!\!\int_{V_j} W\left(\frac{x}{\varepsilon},\zeta_j+\nabla\psi_{j,\varepsilon}(x)\right)dx = \mathcal{H}W(\zeta_j). \tag{4.31}$$

For each $\varepsilon>0$, define $u_{\varepsilon,k,n,t}\in W^{1,\infty}(\Omega;\mathbb{R}^m)$ by

$$u_{\varepsilon,k,n,t}(x) := tu_{k,n,t}(x)+\psi_{j,\varepsilon}(x) \text{ if } x\in V_j.$$

Then

$$\|u_{\varepsilon,k,n,t} - tu_{k,n,t}\|_{L^1(\Omega;\mathbb{R}^m)} = \sum_{j\in J}\|\psi_{j,\varepsilon}\|_{L^1(V_j;\mathbb{R}^m)},$$

and so

$$\lim_{\varepsilon\to 0}\|u_{\varepsilon,k,n,t} - tu_{k,n,t}\|_{L^1(\Omega;\mathbb{R}^m)} = 0 \tag{4.32}$$

by (4.30). On the other hand, Taking (4.26) into account, for each $\varepsilon > 0$, we have

$$\begin{aligned}\int_\Omega W\left(\frac{x}{\varepsilon},\nabla u_{\varepsilon,k,n,t}\right)dx &= \int_\Omega W\left(\frac{x}{\varepsilon},\nabla u_{\varepsilon,k,n,t}\right)dx + \int_{\Omega\setminus\Omega_{n,t}} W\left(\frac{x}{\varepsilon},t\nabla u_{k,n,t}\right)dx\\ &\le \sum_{j\in J}|V_j|\fint_{V_j} W\left(\frac{x}{\varepsilon},\zeta_j+\nabla\psi_{j,\varepsilon}\right)dx + |\Omega\setminus\Omega_{n,t}|M_W(t)\end{aligned}$$

with $M_W(t) < \infty$ by (2.3), and consequently

$$\overline{\lim_{\varepsilon\to 0}}\int_\Omega W\left(\frac{x}{\varepsilon},\nabla u_{\varepsilon,k,n,t}(x)\right)dx \le \int_{\Omega_{n,t}}\mathcal{H}W(t\nabla u_{k,n,t}(x))dx + |\Omega\setminus\Omega_{n,t}|M_W(t)$$

by (4.31) and (4.29). Using (4.27) we conclude that

$$\begin{aligned}\overline{\lim_{\varepsilon\to 0}}\int_\Omega W\left(\frac{x}{\varepsilon},\nabla u_{\varepsilon,k,n,t}(x)\right)dx &\le \int_\Omega \mathcal{H}W(\nabla u_{k,n,t}(x))dx\\ &\quad+|\Omega|\Delta_{\mathcal{H}W}(t) + |\Omega\setminus\Omega_{n,t}|M_W(t).\end{aligned}\tag{4.33}$$

Step 5: passing to the limit. On one hand, it is easy to see that

$$\begin{aligned}\|u_{\varepsilon,k,n,t}-u\|_{L^1(\Omega;\mathbb{R}^m)} &\le \|u_{\varepsilon,k,n,t} - tu_{k,n,t}\|_{L^1(\Omega;\mathbb{R}^m)} + t\|u_{k,n,t}-u_{n,t}\|_{L^1(\Omega;\mathbb{R}^m)}\\ &\quad+t\|u_{n,t}-tu\|_{L^1(\Omega;\mathbb{R}^m)} + (t+1)\|tu-u\|_{L^1(\Omega;\mathbb{R}^m)},\end{aligned}$$

and consequently

$$\lim_{t\to 1}\lim_{n\to\infty}\lim_{k\to\infty}\lim_{\varepsilon\to 0}\|u_{\varepsilon,k,n,t}-u\|_{L^1(\Omega;\mathbb{R}^m)} = 0 \tag{4.34}$$

by using (4.32), (4.23), (4.15) and (4.10). On the other hand, From (4.33), (4.24), (4.16) and (4.12) we deduce that

$$\overline{\lim_{n\to\infty}}\,\overline{\lim_{k\to\infty}}\,\overline{\lim_{\varepsilon\to 0}}\int_\Omega W\left(\frac{x}{\varepsilon},\nabla u_{\varepsilon,k,n,t}(x)\right)dx \le \int_\Omega \mathcal{ZH}W(t\nabla u(x))dx + |\Omega|\Delta_{\mathcal{H}W}(t).$$

But W is ru-usc, and so also is $\mathcal{H}W$, i.e., $\overline{\lim}_{t\to 1}\Delta_{\mathcal{H}W}(t) \le 0$, by Proposition 3.7, hence

$$\overline{\lim_{t\to 1}}\,\overline{\lim_{n\to\infty}}\,\overline{\lim_{k\to\infty}}\,\overline{\lim_{\varepsilon\to 0}}\int_\Omega W\left(\frac{x}{\varepsilon},\nabla u_{\varepsilon,k,n,t}(x)\right)dx \le \int_\Omega \widehat{\mathcal{ZH}W}(\nabla u(x))dx, \tag{4.35}$$

and the result follows from (4.34) and (4.35) by diagonalization. ■

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