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HOMOLOGICAL DIMENSION OF PULLBACKS

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By a ring, we always mean a commutative ring with identity. A commutative square of rings and ring homomorphisms

$$\begin{array}{ccc} A & \xrightarrow{i_1} & A_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ A_2 & \xrightarrow{j_2} & A_0 \end{array} \tag{1}$$

is said to be a cartesian square (or a pullback, or a fiber product) if given $a_1 \in A_1$, $a_2 \in A_2$ with $j_1(a_1) = j_2(a_2)$ there exists a unique element $a \in A$ such that $i_1(a) = a_1$ and $i_2(a) = a_2$ (note that if j_2 is a surjection then so is i_1, but not conversely). The ring A is called the fiber product of A_1 and A_2 over A_0.

For a ring A, $\operatorname{gldim} A$ and $\operatorname{wd} A$ will denote the global dimension of A and the weak global dimension of A, respectively. For an A-module M, the projective dimension of M, and the flat dimension of M are denoted by $\operatorname{pd}_A(M)$ and $\operatorname{fd}_A(M)$, respectively.

This paper is motivated by the results in Kirkman and Kuzmanovich [KK] which give an upper bound on the global dimension of a fiber product. In [KK, Theorem 2] Kirkman and Kuzmanovich showed that if (1) is a cartesian square with j_2 surjective, then

$$\operatorname{gldim} A \leqq \max_{i=1,2} \{\operatorname{gldim} A_i + \operatorname{fd}_A(A_i)\} \tag{$*$}$$

We give sufficient conditions for the fiber product of rings with global dimension $\leqq n$ to be a ring with global dimension $\leqq n$, that generalize the preceding result. We also give examples which show that, in a certain sense, our results are best possible. Indeed, we can cover cases where $(*)$ is a strict inequality.

Our main result is:

THEOREM 1. *Suppose given a pullback diagram* (1), *with i_1 is surjective, and such*

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that for all ideals a of A we have that $\mathrm{pd}_{A_i}(\mathrm{Tor}_j^A(A_i, A/a)) \leqq n-j$ *for* $0 \leqq j \leqq n$ *and* $i = 1, 2$. *Then*

$$\mathrm{gldim}\, A \leqq n$$

We will also prove the analogue of theorem 1 for weak global dimension:

THEOREM 2. *Let* (1) *be a pullback diagram in which* i_1 *is surjective and such that for all finitely generated A-ideals a we have that* $\mathrm{fd}_{A_i}(\mathrm{Tor}_j^A(A/a, A_i)) \leqq n-j$ *for* $0 \leqq j \leqq n$ *and* $i = 1, 2$. *Then*

$$\mathrm{wd}\, A \leqq n$$

We begin by giving sufficient conditions for an A-module M to have projective (flat) dimension $\leqq n$.

We need the following proposition.

PROPOSITION 3. *Let the diagram* (1) *be a pullback in which* i_1 *is surjective, M an A-module and suppose that* $\mathrm{pd}_{A_i}(\mathrm{Tor}_j^A(A_i, M)) \leq n-j$ *for* $0 \leqq j \leqq n$ *and* $i = 1, 2$. *Then if* $n \geqq 1$, *we have that*

$$\mathrm{pd}_{A_i}(A_i \otimes_A K_t) \leqq n-(t+1) \quad \textit{for} \quad 0 \leqq t \leqq n-1 \quad \textit{and} \quad i = 1, 2$$

where K_t *is a* tth *syzygy of M.*

PROOF. The proof is by induction on t. Let

(2) $$P: \quad \ldots \to P_3 \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

be a projective resolution of M.

For $t = 0$, if we tensor the exact sequence of A-modules

$$0 \to K_0 \to P_0 \xrightarrow{f_0} M \to 0$$

with A_i $(i = 1, 2)$, we obtain an exact sequence of A_i-modules.

(3) $$0 \to \mathrm{Tor}_1^A(A_i, M) \to A_i \otimes_A K_0 \to A_i \otimes_A P_0 \to A_i \otimes_A M \to 0$$

and an isomorphism

$$\mathrm{Tor}_j^A(A_i, K_0) \simeq \mathrm{Tor}_{j+1}^A(A_i, M), \; j \geqq 1.$$

Put $I_{i,0} = \ker(1_{A_0} \otimes f_0)$, and break up (3) into two exact sequences

(4) $$0 \to I_{i,0} \to A_i \otimes_A P_0 \to A_i \otimes_A M \to 0$$

and

(5) $$0 \to \mathrm{Tor}_1^A(A_i, M) \to A_i \otimes_i K_0 \to I_{i,0} \to 0$$

for $i = 1, 2$.

If P_0 is an A-projective module, it is well known that $A_i \otimes_A P_0$ is an A_i-projective module. Since $\mathrm{pd}_{A_i}(A_i \otimes_A M) \leqq n$, we obtain from (4) that $\mathrm{pd}_{A_i}(I_{i,0}) \leqq n-1$. In addition we have that $\mathrm{pd}_{A_i}(\mathrm{Tor}_1^A(A_i, M)) \leqq n-1$, and hence from (5) we obtain that $\mathrm{pd}_{A_i}(A_i \otimes_A K_0) \leqq n-1$ for $i = 1, 2$ as desired.

For $t \geqq 1$, consider the short exact sequences

$$0 \to K_t \to P_t \xrightarrow{f_t} K_{t-1} \to 0.$$

If we apply the functor $\mathrm{Tor}_*^A(A_i, -)$ ($i = 1, 2$) to short exact sequence above, we obtain exact sequences of A_i-modules

$$(6) \qquad 0 \to \mathrm{Tor}_1^A(A_i, K_{t-1}) \to A_i \otimes_A K_t \to A_i \otimes_A P_t \to A_i \otimes_A K_{t-1} \to 0$$

and isomorphisms

$$\mathrm{Tor}_j^A(A_i, K_t) \simeq \mathrm{Tor}_{j+1}^A(A_i, K_{t-1}),\ j \geqq 1.$$

Put $I_{i,t} = \ker(1_{A_i} \otimes f_t)$, and break up (6) into two exact sequences

$$(7) \qquad 0 \to I_{i,t} A_i \otimes_A P_t \xrightarrow{1_{A_i} \otimes f_t} A_i \otimes_A K_{t-1} \to 0$$

and

$$(8) \qquad 0 \to \mathrm{Tor}_1^A(A_i, K_{t-1}) \to A_i \otimes_A K_t \to I_{i,t} \to 0$$

for $i = 1, 2$.

By the induction hypothesis $\mathrm{pd}_{A_i}(A_i \otimes_A K_{t-1}) \leqq n - t$, thus (7) implies that $\mathrm{pd}_{A_i}(I_{i,t}) \leqq n - (t+1)$. Recall now that we have that $\mathrm{Tor}_i^A(A_i, K_{t-1}) \simeq \mathrm{Tor}_{t+1}^A(A_i, M)$. Since $\mathrm{pd}_{A_i}(\mathrm{Tor}_{t+1}^A(A_i, M)) \leqq n - (t+1)$ for $0 \leqq t \leqq n-1$, we obtain from (8) that $\mathrm{pd}_{A_i}(A_i \otimes_A K_t) \leqq n - (t+1)$ for $i = 1, 2$ as desired.

Now we can deduce the following proposition which genralizes theorem 2.3 of [W]. Theorem 1 is an immediate consequence of proposition 4.

Proposition 4. *Suppose given a pullback diagram* (1) *with* i_1 *surjective and let* M *be an* A*-module with* $\mathrm{pd}_{A_i}(\mathrm{Tor}_j^A(A_i, M)) \leqq n - j$ *for* $0 \leqq j \leqq n$ *and* $i = 1, 2$. *Then*

$$\mathrm{pd}_A(M) \leqq n.$$

Proof. For $n = 0$ we have, from [W, theorem 2.3], that M is a projective A-module iff $A_i \otimes_A M$ are projective A_i-modules ($i = 1, 2$).

For $n \geqq 1$, consider (2). We want to show that $K_{n-1} = \mathrm{im}(f_t)$ is an A-projective module. By proposition 3, we have that $A_i \otimes_A K_{n-1}$ are A_i-projective modules for $i = 1, 2$ as desired.

Remark. We can obtain similar results about flat dimension if we replace projective by flat, in the argument above. Thus:

Proposition 5. *Suppose given a pullback diagram* (1) *with* i_1 *surjective, and let*

M be an A-module with $\mathrm{fd}_{A_i}(\mathrm{Tor}_j^A(A_i, M)) \leqq n - j$ *for* $0 \leqq j \leqq n$ *and* $i = 1, 2$. *Then*

$$\mathrm{fd}_A(M) \leqq n.$$

PROOF OF THEOREM 1. Let a be an ideal of A with $\mathrm{pd}_{A_i}(\mathrm{Tor}_j^A(A_i, A/a)) \leqq n - j$ for $0 \leqq j \leqq n$ and $i = 1, 2$. It follows from proposition 4 that $\mathrm{pd}_A(A/a) \leqq n$, and since $\mathrm{gldim}\, A = \sup\{\mathrm{pd}_A(A/a) \mid a \text{ an ideal of } A\}$, we conclude that $\mathrm{gldim}\, A \leqq n$.

PROOF OF THEOREM 2. From proposition 5 we have that $\mathrm{fd}_A(A/a) \leqq n$ for all finitely generated ideals a of A, and since $\mathrm{wd}\, A = \sup\{\mathrm{fd}_A(A/a) \mid a \text{ a finitely generated ideal of } A\}$, we conclude that $\mathrm{ws}\, A \leqq n$.

Now we can use theorem 2 to get an upper bound about weak global dimension, analogous to theorem 2 in [KK].

COROLLARY 6. *Let diagram* (1) *be a pullback in which* i_1 *is a surjection. Then*

$$\mathrm{wd}\, A \leqq \max_{i=1,2} \{\mathrm{wd}\, A_i + \mathrm{fd}_A(A_i)\}$$

PROOF. Let $n = \max_{i=1,2} \{\mathrm{wd}\, A_i + \mathrm{fd}_A(A_i)\}$.

Then for $j > \mathrm{fd}_A(A_i)$ we have that $\mathrm{Tor}_j^A(A_i, -) = 0$, and for $0 \leqq j \leqq \mathrm{fd}_A(A_i)$, we have that $\mathrm{pd}_{A_i}(\mathrm{Tor}_j^A(A_i, -) \leqq n - j\, (i = 1, 2)$, since $n - j \geqq n - \mathrm{fd}_A(A_i) \geqq \mathrm{wd}\, A_i$. Using theorem 2 we conclude that $\mathrm{wd}\, A \leqq n$.

The next corollaries show that we can obtain more precise results in some specific cases.

COROLLARY 7. *Let* (1) *be a pullback diagram in which* i_1 *is a surjection,* $\mathrm{gldim}\, A_i \leqq n$ *and* $\mathrm{fd}_A(A_i) \leqq 1$ *for* $i = 1, 2$. *Then*

$\mathrm{gldim}\, A \leqq n$ *iff* $\mathrm{pd}_{A_i}(\mathrm{Tor}_1^A(A_i, A/a)) \leqq n - 1$ *for all ideals* a *of* A *and* $i = 1, 2$.

PROOF. The only if assertion follows from theorem 1. We will prove the converse. Thus assume $\mathrm{gldim}\, A \leqq n$. Let M_i be an A_i-module $(i = 1, 2)$. *Then there is a change of rings spectral sequence*

$$E_2^{p,q} = \mathrm{Ext}_{A_i}^p(\mathrm{Tor}_q^A(A_i, A/a), M_i) \Rightarrow H^n = \mathrm{Ext}_A^n(A/a., M_i) \tag{9}$$

and from [CE, theorem 5.11] there is an exact sequence

$$\ldots \to H^{n+1} \to E_2^{n,1} \to E_2^{n+2,0} \to H^{n+2} \to \ldots \tag{10}$$

since $H^m = 0$ for $m > n$ and $E_2^{p,q} = 0$ for $p > n$ we conclude that $E_2^{n,1} = 0$ for all A_i-modules M_i and $i = 1, 2$ as desired.

COROLLARY 8. *Let* (1) *be a pullback diagram in which* i_1 *is surjective,* $\mathrm{wd}\, A_i \leqq n$ *and* $\mathrm{fd}_A(A_i) \leqq 1$ *for* $i = 1, 2$. *Then*

$$\operatorname{wd} A \leqq n \textit{ iff } \operatorname{fd}_{A_i}(\operatorname{Tor}_1^A(A/a, A_i)) \leqq n-1 \textit{ for all f.g. ideals } a \textit{ of } A \textit{ and } i = 1, 2.$$

PROOF. The only if assertion follows from theorem 2. We will prove the converse. Thus assume $\operatorname{wd} A \leqq n$. Let a_i be an finitely generated ideal of A_i $(i = 1, 2)$. Then there is a change of rings spectralsequence

$$(11) \qquad E_{p,q}^2 = \operatorname{Tor}_p^{A_i}(\operatorname{Tor}_q^A(A/a, A_i), A_i/a_i) \Rightarrow H_n = \operatorname{Tor}_n^A(A/a. A_i/a_i)$$

and from [R, Exercise 11.31] there is an exact sequence

$$(12) \qquad \ldots \to H_{n+2} \to E_{n+2,0}^2 \to E_{n,1}^2 \to H_{n+1} \to \ldots$$

Since $\operatorname{wd} A \leqq n$ and $H_m = 0$ for $m > n$ and since $\operatorname{wd} A_i \leqq n$, $E_{p,q}^2 = 0$ for $p > n$, then we have that $\operatorname{Tor}_n^{A_i}(\operatorname{Tor}_1^A(A/a., A_i), A_i/a_i) = 0$ for all finitely generated ideals a_i of A_i, which is equivalent to $\operatorname{fd}_{A_i}(\operatorname{Tor}_1^A(A_i, A/a)) \leqq n-1$ $(i = 1, 2)$.

COROLLARY 9. *Let diagram* (1) *be a pullback in which* i_1 *is a surjection,* $wd\, A_i \leqq n$, $\operatorname{fd}_A(A_i) \leqq 1$, *and where for all finitely generated ideals* a_i *of* A_i *we have that* $\operatorname{fd}_A(A_i/a_i) \leqq n$ $(i = 1, 2)$, *Then*

$$\operatorname{wd} A \leqq n$$

PROOF. If we consider the sequences (11) and (12), then corollary 9 is an immediate consequence of corollary 8.

COROLLARY 10. *Let diagram* (1) *be a pullback in which* i_1 *is a surjection, and suppose that* $\operatorname{gldim} A_i \leqq 1$. *Then*

$$\operatorname{gldim} A \leqq n \textit{ iff } \operatorname{Tor}_n^A(A_i, A/a) \textit{ is } A_i\textit{-projective for all ideals } a \textit{ of } A \; (i = 1, 2)$$

PROOF. The only if assertion follows from theorem 1. We will prove the converse. Thus assume $\operatorname{gldim} A \leqq n$. Let $M = A/a$ in (2) where a is an ideal of A. We know that $\operatorname{Tor}_n^A(A_i, A/a) \simeq \operatorname{Tor}_1^A(A_i, K_{n-2})$ $(i = 1, 2)$. If we consider the sequences (7) and (8) for $t = n-1$, and recall that $\operatorname{pd}_{A_i}(A_i \otimes_A K_{n-1})$ is A_i-projective $(i = 1, 2)$, we obtain that $\operatorname{Tor}_n^A(A_i, A/a)$ is A_i-projective for $i = 1, 2$.

As an example where corollaries 6, 7 and 8 can be applied, we present the following.

EXAMPLE 1. Let V be a valuation domain with a non principal maximal ideal m. Explicit examples of such rings will be given below.

Consider the cartesian square, where V_1 and V_2 are two copies of V

$$\begin{array}{ccc} A & \xrightarrow{i_1} & V_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow \\ V_2 & \longrightarrow & V/m_0 \end{array}$$

and where the maps onto V/m are the natural ones. Then $A = \{(a,b) \in V_1 \times V_2 / a - b \in m\}$. The ring A is local with zero divisors and maximal ideal $J = m \times m$.

(i) We claim that $\mathrm{fd}_A(V_k) \leqq 1$ for $k = 1,2$.

PROOF. Let I be a finitely generated ideal of A. By considering $\min\{v(a_1) \mid (a_1,a_2) \in I\}$ where v is the valuation associated to V we can conclude that either $I = (a_1,a_2)A$ with $a_1 \neq 0$, $a_2 \neq 0$, (and since V is a domain, I is projective), or $I = (a,0)A \oplus (0,b)A$.

Thus for the proof of (i), we may assume that $I = (a,0)A$.

Consider the exact sequence

$$0 \to (0,m) \to A \to I \to 0. \tag{13}$$

Tensoring (13) with V_k, we obtain

$$0 \to \mathrm{Tor}_1^A(V_k,I) \to V_k \otimes_A (0,m) \xrightarrow{f_k} V_k \to V_k \otimes_A I \to 0 \text{ for } k = 1,2$$

For $k = 1$, we have that $V_1 \otimes_A (0,m) = 0$.

Indeed, since m is not a principal ideal, the set $\{v(m) \mid m \in m\}$ has no minimal elements, hence for every $m \in m$, there exists an element n in m, such that $v(m) > v(n)$. Recalling that the lattice of ideals of V are linearly ordered. we obtain that $m \in nV$. Thus we may write $m = n \cdot w$, where n and w are elements of m.

It follows that if $v \in V$, $m \in m$, then $v \otimes (0,n)(0,w) = 0$.

For $k = 2$, we have that f_2 is injective.

To see this, let x be an element of $V_2 \otimes_A (0,m)$, say $x = \sum_{i=0}^n (v_i \otimes (0,m_i))$, where $v_i \in V_2$, and $m_i \in m$. By considering $\{v_1, v_2, \ldots, v_n\}$, and recalling that the lattice of ideals of V are linearly ordered, we obtain an element $v \in V$, such that $v_i = \alpha_i \cdot v$, where $\alpha_i \in V$.

Thus $x = \sum_{i=0}^n (v_i \otimes (0, m_i)) = \sum_{i=0}^n (\alpha_i \cdot v \otimes (0, m_i)) = \sum_{i=0}^n (v \otimes (\alpha_i, \alpha_i)(0,m_i)) = v \otimes \sum_{i=1}^n (0, \alpha_i \cdot m_i)$, and $f_2(x) = v \cdot \sum_{i=1}^n (\alpha_i \cdot m_i)$. Since V_2 is a domain, it follows that f_2 is injective.

Thus $\mathrm{Tor}_1^A(V_k,I) = 0$ for every finitely generated ideal I of A and $k = 1,2$. Hence we can conclude that $\mathrm{fd}_A(V_k) \leqq 1$ $(k = 1,2)$.

In [V, theorem 3.4] W. Vasconcelos showed that

$$\mathrm{gldim}\, V \leqq \mathrm{gldim}\, A \leqq \mathrm{gldim}\, V + 1$$

(ii) Let k be a field, G be a totally ordered group, $G^+ = \{g \in G \mid g \geqq e\}$ (e is the neutral element of G) and let $V = k[[G^+]]$ be the ring of all formal power series, i.e. V consists of formal infinite sums $\alpha = \sum_{g \in G^+} \alpha_g g$, where $\alpha_g \in k$ and $\mathrm{supp}(\alpha) = \{g \in G^+ \mid \alpha_g \neq 0\}$ is well ordered. An element $\alpha \neq 0$ of V, may be written in the form $\alpha = \beta g(e + \varphi)$, with $\beta \in k, g \in G^+, \varphi \in V$, and $\varphi_e = 0$ ($(e + \varphi)$ is a unit of V, and, $(e + \varphi)^{-1} = e + \sum_{n=1}^{\infty}(-\varphi)^n$).

We can think of V as the ring of all power series in a symbol x with exponents the well ordered subsets in G^+, i.e., if $r \in V$, we can write $r = x^{\alpha}u$, where $\alpha \in G^+$ and u is a unit in V. The ring V is a valuation domain (more information about this ring can be found in [F, p134] and in [S]).

Suppose that $|G| = \mathcal{N}_n$ ($|G|$ denotes the cardinality of G) and that $G^+ - \{e\}$ has no coinitial subset B with $|B| \leqq \mathcal{N}_{n-1}$, i.e., for all subsets B of $G^+ - \{e\}$ with $|B| \leqq \mathcal{N}_{n-1}$, there exists an element g in $G^+ - \{e\}$ (g not in B) such that $g < b$, for every element b in B. Then every ideal I of V can be generated by a set D with $|D| \leqq \mathcal{N}_n$. The maximal ideal m can not be generated by $\leqq \mathcal{N}_{n-1}$ elements. To see this, suppose that m has a set of generators D with $|D| \leqq \mathcal{N}_{n-1}$. If we let $B = \{g_i \mid x^{g_i}u \in D\}$ then $|B| \leqq \mathcal{N}_{n-1}$, and there exists an element $g \in G^+ - \{e\}$, such that $g < g_i$ for all $g_i \in B$, i.e. x^g is not in m, which is a contradiction.

In [OB-2, p227] B. Osofsky showed that for a ring R with no zero divisors and linearly ordered ideals, an R-ideal I has $\mathrm{pd}_R(I) = n + 1$ if and only if the smallest cardinality of a generating set of I is $\mathcal{N}_n$. From this we obtain that $\mathrm{pd}_V(m) = n + 1$ if and only if the smallest cardinality of a generating set of I is $\mathcal{N}_n$. From this we obtain that $\mathrm{pd}_V(m) = n + 1$ and $\mathrm{pd}_V(I) \leqq n + 1$ for every ideal I of V, and from these assertions we conclude that gldim $V = n + 2$.

(iii) Let I be the well ordered set of all ordinals $< \mathcal{N}_n$. Let G be the coproduct of I copies of $\mathbb{Z}$, i,e,m $G = \coprod_I \mathbb{Z}$. Order G lexicographically. Then we have that G is a totally ordered group with $|G| = \mathcal{N}_n$, and $G^+ - \{e\}$ has no coinitial subset of cardinality $< \mathcal{N}_n$. So that by (ii) we obtain that gldim $V = n + 2$.

CLAIM. gldim $A = n + 3$.

PROOF. We know that

$$n + 2 \leqq \operatorname{gldim} A \leqq n + 3$$

Consider the ideal $I = (\mathrm{a}, 0)A$ with a in m. We tensor the exact sequence of A-modules

$$0 \to I \to A \to A/I \to 0$$

with V_2, and we obtain an exact sequence

$$0 \to \mathrm{Tor}_1^A(V_2, A/I) \to V_2 \otimes_A I \xrightarrow{g} V_2 \to V_2 \otimes_A A/I \to 0$$

Since $g = 0$, we have that $\mathrm{Tor}_1^A(V_2, A/I) \simeq V_2 \otimes_A I$. From the exact sequence

$$0 \to V_2 \otimes_A (0, m) \xrightarrow{f} V_2 \to V_2 \otimes_A I \to 0$$

we see that $\mathrm{pd}_{V_2}(\mathrm{Tor}_1^A(V_2, A/I)) = n + 2$. This follows from the fact that

$V_2 \otimes_A (0, m) \simeq m$ and that, by (ii), $\mathrm{pd}_{V_2}(m) = n + 1$. Appealing to corollary 7, we can conclude that $\mathrm{gldim}\, A = n + 3$.

(iv) If we consider $G = \mathbb{Q}$, i.e., $V = k[[\mathbb{Q}^+]]$, we obtain that $\mathrm{gldim}\, A = 3$. This is the same ring that B. Osofsky studies in [OB-1, theorem 2.37].

EXAMPLE 2. (a) Let $T = k[[\mathbb{Q}^+]]$, m the maximal ideal of T, and $R = T \times_m T$ with maximal ideal J.

Consider the cartesian square

$$\begin{array}{ccc} S & \longrightarrow & T \\ \downarrow & & \downarrow \\ R & \longrightarrow & R/J \simeq T/m \simeq k \end{array}$$

where the maps onto k are the natural ones. Then $S = \{\mathrm{a}, \mathrm{b}, \mathrm{c}) \mid \mathrm{a}, \mathrm{b}, \mathrm{c} \in T$ and $\mathrm{a}_0 = \mathrm{b}_0 = \mathrm{c}_0\}$ where $\mathrm{a} = \sum_{i=0} \mathrm{a}_i x^{n_i}$, $\mathrm{b} = \sum_{i=0} \mathrm{b}_i x^{n_i}$, $\mathrm{c} = \sum_{i=0} \mathrm{c}_i x^{n_i}$ and $0 = n_0 < n_1 < n_2 < n_3 < \ldots$. The ring S is local with zero divisors and maximal ideal $m \times J$. We know that $\mathrm{gldim}\, T = 2$, $\mathrm{wd}\, T = 1$, $\mathrm{gldim}\, R = 3$ and $\mathrm{wd}\, R = 2$.

We will determine $\mathrm{gldim}\, S$.

First we shall show that $\mathrm{fd}_S T \leqq 1$ and $\mathrm{fd}_S R \leqq 1$.

To see this, let I be a finitely generated ideal of S, generated by $\{(\mathrm{r}_1\, \mathrm{v}_1, \mathrm{w}_1), (\mathrm{r}_2, \mathrm{v}_2, \mathrm{w}_2), (\mathrm{r}_3, \mathrm{v}_3, \mathrm{w}_3), \ldots, (\mathrm{r}_n, \mathrm{v}_n, \mathrm{w}_n)\}$. If v denotes the valuation associated to T, then by considering $\min\{v(\mathrm{r}_i)\}$ and $\min\{v(\mathrm{v}_i)\}$, we can conclude that either

$I = (\mathrm{a}, 0, 0)S \oplus (0, \mathrm{b}, \mathrm{c})S$, or

$I = (\mathrm{a}, \mathrm{b}, 0)S \oplus (0, 0, \mathrm{c})S$, or

$I = (\mathrm{a}, \mathrm{b}, \mathrm{c})S$ where $\mathrm{a} \neq 0$, $\mathrm{b} \neq 0$, $\mathrm{c} \neq 0$ (I is then S-projective since T is a domain).

Let $I_1 = (\mathrm{a}, 0, 0)S$, $I_2 = (0, \mathrm{b}, 0)S$, $I_3 = (0, \mathrm{b}, \mathrm{c})S$ and $I_4 = (\mathrm{a}, \mathrm{b}, 0)S$. It is sufficient to assume that $I = I_k$ for $k = 1, 2, 3, 4$.

We shall show that $\mathrm{Tor}_1^S(T, I_k) = 0$

For $k = 1$ we have an exact sequence

$$0 \to (0, J) \to S \to I_1 \to 0. \tag{14}$$

Tensoring (14) with T, we get

$$0 \to \mathrm{Tor}_1^S(T, I_1) \to T \otimes_S (0, J) \xrightarrow{f_1} T \to T \otimes_S I_1 \to 0.$$

But since J is not pricipal, $T \otimes_S (0, J) = 0$, and hence we have that $\mathrm{Tor}_1^S(T, I_1) = 0$.

For $k = 2$, there exists an exact sequence

$$0 \to (m, 0, m) \to S \to I_2 \to 0. \tag{15}$$

Tensoring (15) with T, we get

$$0 \to \operatorname{Tor}_1^S(T, I_2) \to T \otimes_S (m, 0, m) \xrightarrow{f_2} T \to T \otimes_S I_2 \to 0.$$

But since T is a valuation ring and $T \otimes_S (m, 0, m) \simeq T \otimes_S (m, 0, 0)$, f_2 is injective, hence we obtain that $\operatorname{Tor}_1^S(T, I_2) = 0$.

For $k = 3$, there exists an exact sequence

$$0 \to \operatorname{Tor}_1^S(T, I_3) \to T \otimes_S (m, 0, 0) \xrightarrow{f_3} T \to T \otimes_S I_3 \to 0.$$

Since f_3 is injective, we obtain that $\operatorname{Tor}_1^S(T, I_3) = 0$.

For $k = 4$, there exists an exact sequence

$$0 \to \operatorname{Tor}_1^S(T, I_4) \to T \otimes_S (0, 0, m) \xrightarrow{f_4} T \to T \otimes_S I_4 \to 0.$$

Since $T \otimes_S (0, 0, m) = 0$, we obtain that $\operatorname{Tor}_1^S(T, I_3) = 0$.

Now we show that $\operatorname{Tor}_1^S(R, I_k) = 0$.

For $k = 1$ tensoring (14) with R, we get

(16) $$0 \to \operatorname{Tor}_1^S(R, I_1) \to R \otimes_S (0, J) \xrightarrow{f_1} R \to R \otimes_S I_1 \to 0$$

Since $R \otimes_S (0, J) \simeq J$, f_1 is injective and it follows that $\operatorname{Tor}_1^S(R, I_1) = 0$.

For $k = 2$ tensoring (15) with R, we get

$$0 \to \operatorname{Tor}_1^S(R, I_2) \to R \otimes_S (m, 0, m) \xrightarrow{f_2} R \to R \otimes_S I_2 \to 0.$$

Since $R \otimes_S (m, 0, m) \simeq R \otimes_S (0, 0, m)$, f_2 is injective, and hence we obtain that $\operatorname{Tor}_1^S(R, I_2) = 0$.

With similar arguments, we can show that $\operatorname{Tor}_1^S(R, I_k) = 0$ for $k = 3, 4$.

We can shown that $\operatorname{Tor}_1^S(T, I) = 0$ and $\operatorname{Tor}_1^S(R, I) = 0$, for every finitely generated ideal I of S, so we can conclude that $\operatorname{fd}_S T \leqq 1$ and $\operatorname{fd}_S R \leqq 1$.

From [KK, theorem 2], [OB-1, proposition 2.36] and corollary 6 we have that

(17) $$3 \leqq \operatorname{gldim} S \leqq \max\{2 + 1, 3 + 1\} = 4 \text{ and}$$
$$2 \leqq \operatorname{wd} S \leqq \max\{1 + 1, 2 + 1\} = 3$$

CLAIM. $\operatorname{wd} S = 2$

PROOF. Consider the exact sequence

$$0 \to I_k \to S \to S/I_k \to 0.$$

Tensoring with R, we get

$$0 \to \operatorname{Tor}_1^S(R, S/I_k) \to R \otimes_S I_k \xrightarrow{g_k} R \to R \otimes_S S/I_k \to 0.$$

But $g_1 = 0$, so $\operatorname{Tor}_1^S(R, S/I_1) \simeq R \otimes_S I_1$. Since $R \otimes_S (0, J) \simeq J$, and J is R-flat by [OB-1, p53], we obtain from (16) and $\operatorname{fd}_S(R) \leqq 1$, that $\operatorname{fd}_S(R \otimes_S I_1) \leqq 1$.

For $k = 2, 3, 4$ we have that g_k is injective hence $\operatorname{Tor}_1^S(R, S/I_k) = 0$.

We have shown that $\operatorname{fd}_R(\operatorname{Tor}_1^S(R, S/I)) \leqq 1$ for all finitely generated ideals I of S.

Since wd $T = 1$ we have that $\mathrm{fd}_T(M) \leqq 1$ for all T-modules M. Then using corollary 8, we obtain that wd $S \leqq 2$. Using (17) we conclude that wd $S = 2$.

CLAIM. gldim $S = 3$

PROOF. By corollary 7 and (17), and given the fact that gldim $T = 2$, we only need to show that $\mathrm{pd}_R(\mathrm{Tor}_1^S(R, S/I)) \leqq 2$ for all ideals I of S.

If I is a finitely generated ideal of S, we have shown that $\mathrm{Tor}_1^S(R, S/I) = 0$ or $\mathrm{Tor}_1^S(R, S/I) \simeq R \otimes_S I_1$. Since $R \otimes_S (0, J) \simeq J$ and from [OB-1, p53] we know that $\mathrm{pd}_R(J) \leqq 1$, we conclude (from (16) and $\mathrm{fd}_S R \leqq 1$) that $\mathrm{pd}_R(\mathrm{Tor}_1^S(R, S/I)) \leqq 2$ for all finitely generated ideals I of S.

If I is not finitely generated, we have that either $\{v(\mathrm{r}), (\mathrm{r}, \mathrm{v}, \mathrm{w}) \in I\}$, or $\{v(\mathrm{v}), (\mathrm{r}, \mathrm{v}, \mathrm{w}) \in I\}$, or $\{v(\mathrm{w}), (\mathrm{r}, \mathrm{v}, \mathrm{w}) \in I\}$ has no minimal elements. Hence we can assume that

$$I = \sum_{i=0}^{\infty} (a_i, 0, 0)S \oplus \sum_{i=0}^{\infty} (0, b_i, 0)S \oplus \sum_{i=0}^{\infty} (0, 0, c_i)S$$

where the orders of a_i, b_i, c_i strictly decrease.

Thus $\mathrm{Tor}_1^S(R, S/I) \simeq R \otimes_S \sum_{i=0}^{\infty} (a_i, 0, 0)S$. With arguments similar to those used in [OB-1, p53] to show that $\mathrm{pd}_R(\sum_{i=0}^{\infty} a_i R) \leqq 1$, we can prove that

$\mathrm{pd}_S \sum_{i=0}^{\infty} (a_i, 0, 0)S) \leqq 1$. And since $\mathrm{fd}_S R \leqq 1$, we conclude that

$\mathrm{pd}_R(R \otimes_S \sum_{i=0}^{\infty} (a_i, 0, 0)S) \leqq 1$.

We have shown that $\mathrm{pd}_R(\mathrm{Tor}_1^S(R, S/I)) \leqq 2$ for all ideals I of S. It follows that gldim $S = 3$.

(b) Let R and T be rings as in (a).

Consider the cartesian square

$$\begin{array}{ccc} R^{(2)} & \longrightarrow & R \\ \downarrow & & \downarrow \\ R & \longrightarrow & R/J \end{array}$$

where the maps onto R/J are the natural ones.

Then $R^{(2)} = \{(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}) \in T \times T \times T \times T \mid \mathrm{a}_0 = \mathrm{b}_0 = \mathrm{c}_0 = \mathrm{d}_0\}$ where $\mathrm{a} = \sum_{i=0} \mathrm{a}_i x^{n_i}$, $\mathrm{b} = \sum_{i=0} b_i x^{n_i}$, $\mathrm{d} = \sum_{i=0} \mathrm{d}_i x^{n_i}$ and $0 = n_0 < n_1 < n_2 < \ldots$ is a local ring with zero divisors and maximal ideal $J \times J$.

With arguments similar to those in (a), we can show that

$$\mathrm{fd}_{R^{(2)}}(R) \leqq 1, \text{ wd } R^{(2)} = 2 \text{ and gldim } R^{(2)} = 3.$$

Summing up we have given examples of pullbacks

$$\begin{array}{ccc} A & \xrightarrow{i_1} & A_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ A_2 & \xrightarrow{j_2} & A_0 \end{array}$$

such that the matrix

$$\begin{pmatrix} \operatorname{gldim} A & \operatorname{gldim} A_1 \\ \operatorname{gldim} A_2 & \operatorname{gldim} A_0 \end{pmatrix}$$

takes the values

$$\begin{pmatrix} 3 & 3 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 3 & 0 \end{pmatrix}, \text{and} \begin{pmatrix} n+3 & n+2 \\ n+2 & 0 \end{pmatrix}$$

for $n \geqq 0$.

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