Homological mirror symmetry for Calabi-Yau hypersurfaces in projective space

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Outline

Holomorphic curves

- Let (M, ω) be a **Kähler manifold**: a complex manifold with a compatible symplectic form ω.
- Given a Riemann surface Σ, we consider the moduli space of **holomorphic curves**:

$$\{u : \Sigma \to M \text{ holomorphic map}\} / \text{reparametrization}.$$

- Gromov realized (1985) that holomorphic curves come in finite-dimensional families.

Counting curves

Counting the zero-dimensional part of such a moduli space (maybe with some point constraints) gives us numbers which are invariants of (M, ω) – the **Gromov-Witten invariants.** For example:

- Number of degree-1 curves (lines) $u : \mathbb{CP}^1 \to \mathbb{CP}^n$, passing through two generic points: 1.
- Number of degree-2 curves (conics) $u : \mathbb{CP}^1 \to \mathbb{CP}^2$, passing through five generic points: 1.
- Number of lines on a cubic surface: 27.

Curve-counting on the quintic three-fold

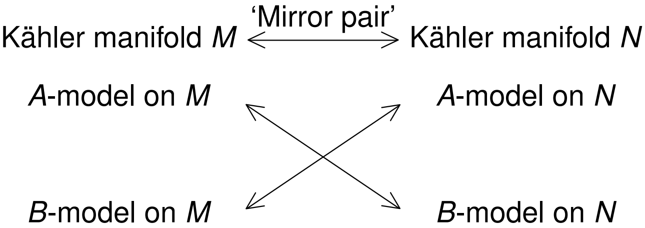
- Number of lines on a quintic three-fold: 2875.
- Number of conics on a quintic three-fold: 609250.
- Number of cubics on a quintic three-fold: 317206375.
- In 1991, the number of degree-d rational curves on the quintic three-fold was unknown, for $d \geq 4$.

A and B models

- Physics: study string theory on a **Calabi-Yau** Kähler manifold (M, ω, Ω).
- Calabi-Yau means there is a holomorphic volume form $\Omega \in \Omega^{n,0}(M)$.
- There are two models for closed-string theory on (M, ω, Ω):
  - The ‘A-model’ = Gromov-Witten invariants (depend on symplectic structure (M, ω));
  - The ‘B-model’ = periods of Ω (depend on complex structure (M, Ω)).

Mirror symmetry 1.0

Physicists noticed that there are many pairs of manifolds on which A- and B-models are exchanged:



Application to the quintic three-fold

In 1991, string theorists Candelas, de la Ossa, Green and Parkes used mirror symmetry to predict curve counts on the quintic three-fold M:

- They constructed a mirror N to M;
- The A-model (Gromov-Witten invariants) on M should correspond to the B-model on N;
- They explicitly computed the B-model on N (periods of the holomorphic volume form).

The results

- This gave a prediction for the number of degree-d curves on the quintic three-fold **for any** d.
- Their predictions agreed with the known results for $d = 1, 2, 3$.
- They furthermore predicted a rich structure (Frobenius manifold) underlying them.
- In 1996, Givental proved this version of mirror symmetry for all Calabi-Yau (and Fano) complete intersections in toric varieties, using equivariant localization.

Homological Mirror Symmetry

- In 1994, Kontsevich introduced a 'categorified' version of the mirror symmetry conjecture.
- The *A*-model should be the **Fukaya category** $\mathcal{F}(M)$ (a symplectic invariant).
- The *B*-model should be the **category of coherent sheaves** $Coh(M)$ (an algebraic invariant).

What HMS means

So, Calabi-Yau Kähler manifolds M and N should be mirror if there are equivalences of **derived** categories:

$$\begin{array}{ccc} D^{\pi}\mathcal{F}(M) & & D^{\pi}\mathcal{F}(N) \\ & \times & \\ D^{b}Coh(M) & & D^{b}Coh(N) \end{array}$$

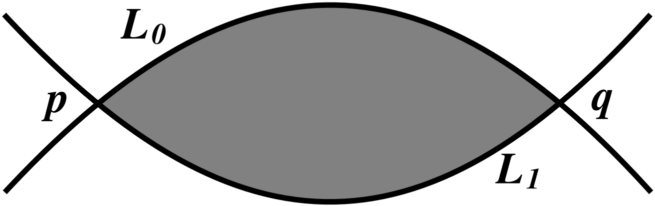
Taking the Hochschild cohomology of these categories recovers the old A- and B-models, so Mirror Symmetry 2.0 implies Mirror Symmetry 1.0 (but is much stronger!).

The Fukaya category $\mathcal{F}(M)$

- A submanifold $L \subset M$ is called **Lagrangian** if $\omega|_L = 0$, and $\dim(L) = \dim(M)/2$.
- Objects of $\mathcal{F}(M)$ are Lagrangian submanifolds of M.
- It is defined over the Novikov field Λ (elements of which are formal sums
$$\sum_{j=1}^{\infty} c_j r^{\lambda_j}$$
where $\{\lambda_j\} \subset \mathbb{R}$ is an increasing sequence, $\lambda_j \to \infty$).
- Morphism spaces are Λ-vector spaces generated by intersection points:
$$CF(L_0, L_1) := \Lambda \langle L_0 \cap L_1 \rangle.$$

The differential

- There is a differential on the morphism spaces, $\delta : CF(L_0, L_1) \to CF(L_0, L_1)$.
- Given $p, q \in L_0 \cap L_1$, the coefficient of q in δp is the number of holomorphic strips u like this:



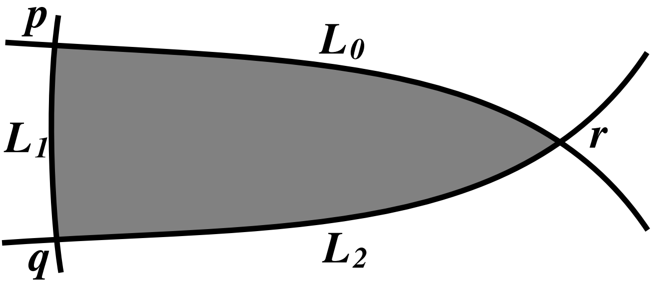
weighted by $r^{\omega(u)}$.

Compositions in $\mathcal{F}(M)$

Composition maps

$$CF(L_0, L_1) \otimes CF(L_1, L_2) \to CF(L_0, L_2)$$

are defined as follows: the coefficient of r in $p \bullet q$ is the number of holomorphic triangles u like this:



weighted by $r^{\omega(u)}$.

One way of proving Homological Mirror Symmetry

One way of proving that there is an equivalence

$$D^{\pi}\mathcal{F}(M) \cong D^{b}Coh(N),$$

is as follows:

- Find some finite collection of Lagrangians in M, and a corresponding collection of coherent sheaves in N;
- Show that their morphism spaces are equivalent;
- Show that the composition maps agree;
- Show that they 'generate' their respective categories.

The *A*-model

- Let $M^n \subset \mathbb{CP}^{n-1}$ be a smooth hypersurface of degree n. We will think of

$$M^n = \left\{ \sum_{j=1}^{n} z_j^n = 0 \right\} \subset \mathbb{CP}^{n-1}.$$

- The *A*-model is the Fukaya category, $\mathcal{F}(M^n)$, which is a $\mathbb{Z}$-graded Λ-linear A_∞ category.

The B-model

- Define

$$\widetilde{N}^n := \left\{ u_1 \ldots u_n + r \sum_j u_j^n = 0 \right\} \subset \mathbb{P}^{n-1}_{\Lambda}.$$

- $G_n \cong (\mathbb{Z}_n)^{n-2}$ acts on $\widetilde{N}^n$ (multiplying coordinates by nth roots of unity), and we define $N^n := \widetilde{N}^n / G_n$.
- Consider the category of coherent sheaves on N^n:

$$Coh(N^n) \cong Coh^{G_n}\left(\widetilde{N}^n\right).$$

Main result

Theorem (S.)

There is an equivalence of Λ-linear triangulated categories

$$D^{\pi}\mathcal{F}(M^n) \cong \Psi \cdot D^b Coh(N^n),$$

where Ψ is an automorphism (the 'mirror map')

$$\begin{aligned} \Psi : \Lambda &\to \Lambda, \text{ sending} \\ r &\mapsto \psi(r)r, \end{aligned}$$

where $\psi(r) \in \mathbb{C}[[r]]$ satisfies $\psi(0) = 1$. We are not yet able to determine the higher-order terms in $\psi(r)$.

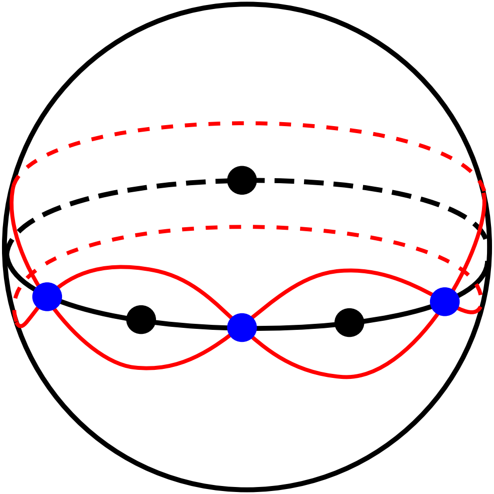
The Lagrangians

We consider the branched cover

$$M^n \cong \left\{ \sum_j z_j^n = 0 \right\} \;\to\; \left\{ \sum_j z_j = 0 \right\} \cong \mathbb{CP}^{n-2}$$
$$[z_1 : \ldots : z_n] \;\mapsto\; [z_1^n : \ldots : z_n^n],$$

branched along the divisors $D_j = \{z_j = 0\}$. We construct a single Lagrangian $L \subset \mathbb{CP}^{n-2} \setminus \cup D_j$ (the 'pair-of-pants'), and look at all of its lifts to M^n.

The one-dimensional case



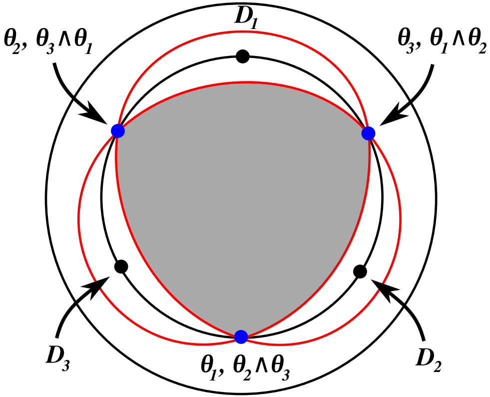
Computing $CF^*(L, L)$

- $CF^*(L, L) \cong \Lambda^* \mathbb{C}^n$ as an algebra.
- It has higher (A_∞) corrections, which correspond to terms

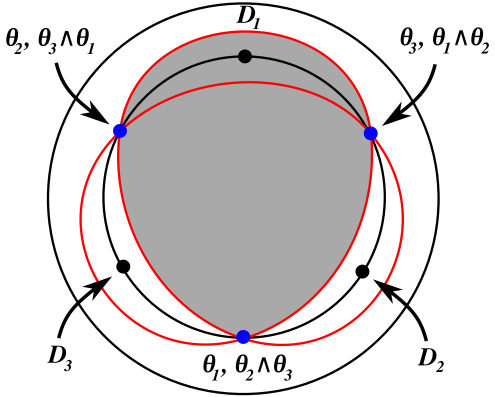
$$\begin{aligned} u_1 \dots u_n + r \sum_j u_j^n &\in \mathbb{C}[[u_1, \dots, u_n]] \otimes \Lambda^* \mathbb{C}^n \\ &\cong HH^*(\Lambda^* \mathbb{C}^n) \text{ (HKR isomorphism).} \end{aligned}$$

- They correspond to the defining equation of the mirror N^n.

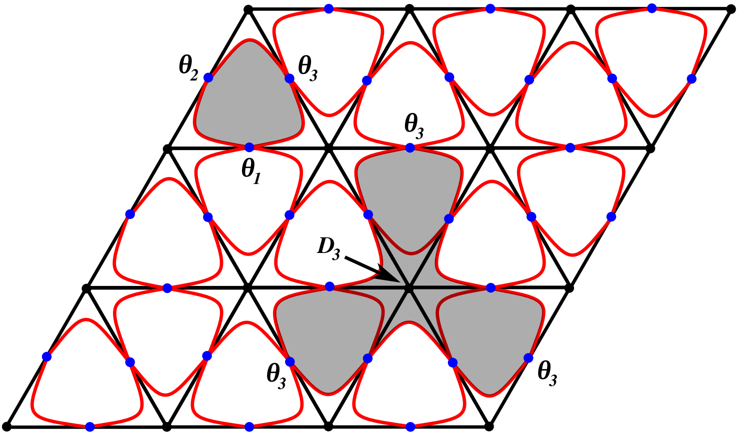
Holomorphic disks giving the exterior algebra



Holomorphic disks giving the higher-order terms



Lifts to N^1 = elliptic curve



The coherent sheaves

- We consider the restrictions of the Beilinson exceptional collection $\Omega^j(j)$ ($j = 0, \ldots, n-1$) to $\widetilde{N}^n$.
- There are $|G_n^*| = n^{n-2}$ ways of making each one into a G_n-equivariant coherent sheaf.
- These G_n-equivariant coherent sheaves on $\widetilde{N}^n$ are mirror to the lifts of the Lagrangian L to M^n.
- We can show that their morphisms and compositions agree, and they generate their respective categories.