HOMOLOGICAL MIRROR SYMMETRY
FOR THE GENUS TWO CURVE

PAUL SEIDEL

Abstract

We prove a form of Homological Mirror Symmetry for the genus two curve (on the symplectic or A-model side).

1. Introduction

Homological Mirror Symmetry (HMS) relates algebraic and symplectic geometry through their associated categorical structures. This relation is by no means straightforward, and its exploration has been the driving force behind several recent developments in algebraic geometry. Let's consider the simplest case, that of an elliptic curve. There, HMS provides a new viewpoint [58] on the well-known classification of vector bundles and their tensor category structure [6]. It also gives rise to nontrivial identities involving theta series for indefinite quadratic forms [50]. Finally, it sheds light on the action of $SL(2, \mathbb{Z})$ on the derived category of an elliptic curve, first introduced in [43]. Higher-dimensional instances of HMS also lead to consequences of the same kind, even though those tend to be somewhat harder. For instance, many symplectic manifolds have rich automorphism groups arising from monodromy, and by thinking of those in terms of HMS, new classes of autoequivalences of derived categories have been discovered [57, 29, 31]. A more recent example is the discovery of a new relation between coherent and constructible sheaves, in the context of toric varieties [26]. Other areas that have been influenced by HMS include: the study of spaces of stability conditions, tropical geometry, the theory of exceptional collections and mutation, and singularity theory.

Having considered its context, we'd now like to concentrate on the HMS conjecture itself (the following discussion is definitely not intended to be complete: it only lists a few papers which are relevant for the developments presented in the body of the paper). The conjecture comes in several related

but distinct versions, which apply in different geometric contexts. Kontsevich's original version [35] concerned Calabi-Yau varieties. There, we now have complete proofs of some instances [49, 54] (including the elliptic curve case mentioned above) and partial results for many more [36, 25]. Soon after, Kontsevich himself proposed an analogous conjecture for Fano varieties. This was gradually extended further, and it seems that varieties with effective anticanonical divisor provide a natural context [8]. The mirror in this case is not another variety but rather a Landau-Ginzburg theory, which means a variety together with a holomorphic function. Because of this asymmetry, the two directions of the mirror correspondence lead to substantially different mathematics. The one relevant for our purpose is where the Landau-Ginzburg theory is considered algebro-geometrically, through matrix factorizations or more generally Orlov's Landau-Ginzburg branes [48]. The corresponding symplectic geometry has been addressed in the toric case in [15, 16, 22, 23, 8]. To the best of this author's knowledge, none of those papers actually proves HMS in its full form, but in many cases it should follow from the results presented there together with additional steps which are fundamentally well understood.

More recently, Katzarkov [32, 37] has proposed a further extension of HMS encompassing some varieties of general type. As before, the mirror is a Landau-Ginzburg theory. Abouzaid, Auroux, Gross, Katzarkov, and Orlov have explored both directions of the correspondence, and accumulated large amounts of evidence (K-theory computations [1, 46] and more unpublished material). The aim of this paper is to prove one direction of Katzarkov's conjecture in the simplest possible case. This is inspired by the work we've just mentioned, and additionally by another instance of mirror symmetry in the literature, namely genus zero curves with three orbifold points (see [59, Section 7], and a little more recently [51]). Let M be a genus two curve, equipped with a symplectic structure. Its mirror is a three-dimensional Landau-Ginzburg theory $X \to \mathbb{C}$, whose zero fibre $H \subset X$ is the union of three rational surfaces. The singular set $\mathrm{Sing}(H)$ is the union of three rational curves, intersecting as shown in Figure 1 (the fact that this "looks like" a degeneration of the genus two curve is explained by an unpublished result of Gross-Katzarkov, which identifies the cohomology of M with that of the sheaf of vanishing cycles on H). Details of the construction of the mirror will be given later.

Let $\mathcal{F}(M)$ be the Fukaya category of M, and $D^\pi(\mathcal{F}(M))$ its split-closed (Karoubi completed) derived category. On the other side, take $D^b_{\mathrm{sing}}(H)$ to be the category of Landau-Ginzburg branes, and let $D^\pi_{\mathrm{sing}}(H)$ be the split-closure of that.

Theorem 1.1. *There is an equivalence of triangulated categories,*

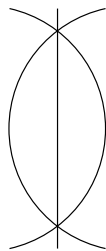$$(1.1) \qquad\qquad D^\pi(\mathcal{F}(M)) \cong D^\pi_{\mathrm{sing}}(H).$$

FIGURE 1

As one consequence, we have the maybe surprising result that the genus two mapping class group acts faithfully on $D^\pi_{\text{sing}}(H)$. An outline of the proof of Theorem 1.1 is given in the next section. For now, it is maybe enough to say that the argument relies on the fact that both categories can be described by A_∞-algebras of a very special form (A_∞-deformations of the exterior algebra, with an added group action). The determination of the exact A_∞-structure is then reduced to the computation of finitely many terms of a superpotential. In a wider context, Theorem 1.1 may raise more questions than it answers. First of all, Katzarkov's original construction, which embeds M as a holomorphic curve into $\mathbb{C}P^1 \times \mathbb{C}P^1$ and applies [28] to that situation, leads to a mirror which is similar but not quite the same as the one considered here. Presumably, the two resulting categories D^π_{sing} are equivalent, but that remains to be shown. Next, the approach followed here has a natural generalization to higher genus curves, and to some higher-dimensional manifolds. In a different direction, the use of split-closures is unsatisfactory, since that process is known to lose information [47].

Addenda: Several relevant preprints have appeared in the time since this one was originally written and distributed. A general approach to matrix factorizations similar to that in Sections 11–12 is given in [19]. Particularly relevant for us is [19, Theorem 4.3], which could replace the ad hoc computation [53] in the proof of Proposition 11.3. The results of [42] and [61] are generalized and put on a more systematic footing in [10]. In particular, [10, Theorem 2.10] covers our needs in that respect, since it directly implies Theorem 13.1. The generalization of our results to curves of any genus ≥ 2 is carried out in [20]. Finally, [33] contains, among other things, a detailed description of the mirror geometry in Katzarkov's original construction.

2. Overview

We now give a guided tour of the proof, simultaneously fixing the notation. Take $V = \mathbb{C}^3$. We write ξ_k for the standard basis vectors of V, thought of

as constant vector fields, and $v_k \in V^\vee$ for the dual basis of functions. The superpotential which is key to our considerations is the polynomial

$$(2.1) \qquad\qquad W = -v_1 v_2 v_3 + v_1^5 + v_2^5 + v_3^5 \in \mathbb{C}[V].$$

Take $Z \cong \mathbb{Z}/5$ to be the subgroup of $SL(V)$ generated by the diagonal matrix $\mathrm{diag}(\zeta, \zeta, \zeta^3)$, with $\zeta = \exp(2\pi i/5)$. Let $X \to \bar{X} = V/Z$ be the crepant resolution given by the Z-Hilbert scheme [44]. Our mirror Landau-Ginzburg model is the composition

$$(2.2) \qquad\qquad X \longrightarrow \bar{X} \xrightarrow{W} \mathbb{C}.$$

In particular, $H \subset X$ is the preimage of $\bar{H} = W^{-1}(0)/Z \subset \bar{X}$. It is an elementary exercise to determine the geometry of H, which is as described in the Introduction. This is done in Section 13.

At the same time, this construction yields a way to approach $D_{\mathrm{sing}}^\pi(H)$. Namely, a version of the derived McKay correpondence [42, 61] shows that this is equivalent to the equivariant category $D_{\mathrm{sing},Z}^\pi(W^{-1}(0))$. For simplicity, let's forget about the group action and just talk about $D_{\mathrm{sing}}^\pi(W^{-1}(0))$. A theorem of Orlov [47] shows that this category is split-generated by a single object, which is the skyscraper sheaf at the origin, denoted by $\mathcal{S}_{W^{-1}(0),0}$. Hence, the category can be completely reconstructed from the A_∞-structure on

$$(2.3) \qquad \mathrm{Hom}_{D_{\mathrm{sing}}^\pi(W^{-1}(0))}(\mathcal{S}_{W^{-1}(0),0}, \mathcal{S}_{W^{-1}(0),0}) \cong \Lambda(V).$$

Moreover, matrix factorizations give rise to a natural dg structure underlying this algebra. On general grounds, the A_∞-structure can be extracted from this by applying the Homological Perturbation Lemma (even though in practice, the computational complexity of computing the operations μ^d rises very rapidly with d). This is explained in Sections 11–12.

Switching to the other side, we represent M as a covering of a genus zero orbifold $\bar{M}$, where the covering group is $\Sigma = \mathrm{Hom}(Z, \mathbb{C}^*) \cong \mathbb{Z}/5$. We choose a collection of five curves $\{L_1, \ldots, L_5\}$ which split-generate $D^\pi(\mathcal{F}(M))$, and which all project to the same immersed curve $\bar{L} \subset \bar{M}$. If we again forget about the covering group action, all the desired information is contained in the A_∞-structure on the Floer cohomology

$$(2.4) \qquad\qquad HF^*(\bar{L}, \bar{L}) \cong \Lambda(V).$$

The first few A_∞-operations can be determined combinatorially by counting polygons (an idea that goes back at least to [35]). The relevant material is covered in Sections 6–10.

At this point, we've reduced both sides to the computation of a specific $\mathbb{Z}/2$-graded A_∞-deformation of the exterior algebra $\Lambda(V)$. The relevant deformation theory is governed by the differential graded Lie algebra of Hochschild

cochains. We apply a version of Kontsevich's Formality Theorem [34], and standard tools from Maurer-Cartan theory, to reduce this to a problem about polyvector fields, which means elements of $\mathbb{C}[[V]] \otimes \Lambda(V)$. In fact, in our case the A_∞-deformation is determined by a single function $W \in \mathbb{C}[[V]]$, which turns out to be precisely the polynomial defined above. The crucial ingredient is a technical result (Proposition 5.2) which shows that a specific isomorphism class of A_∞-deformations, denoted by $\mathcal{A}$, is characterized by the first few nontrivial A_∞-products. The underlying geometric idea is finite determinacy of function germs, which applies to any formal power series with an isolated critical point at the origin. This is the content of Sections 3–5.

3. Kontsevich formality

We begin by recalling some well-known generalities. Let $\mathfrak{g}$ be a dg Lie algebra over $\mathbb{C}$. A Maurer-Cartan element is an $\alpha \in \mathfrak{g}^1$ which satisfies

$$(3.1) \qquad \partial\alpha + \tfrac{1}{2}[\alpha, \alpha] = 0.$$

There is a natural Lie algebra homomorphism from $\mathfrak{g}^0$ to the space of affine vector fields on $\mathfrak{g}^1$, which associates to $\gamma \in \mathfrak{g}^0$ the infinitesimal gauge transformation $\alpha \mapsto -\partial\gamma + [\gamma, \alpha]$. These endomorphisms are tangent to (3.1). Hence, in situations where they can be exponentiated, we get a group action on the space of solutions to the Maurer-Cartan equation. Two extreme special cases are worth considering. First, if $[\cdot, \cdot] = 0$, the Maurer-Cartan equation is just the cocycle equation, and infinitesimal gauge transformations act by adding coboundaries. On the other hand, if $\partial = 0$, what we have is just the adjoint infinitesimal action of $\mathfrak{g}^0$ on the set of elements of $\mathfrak{g}^1$ satisfying $[\alpha, \alpha] = 0$.

It is convenient to introduce generalized morphisms between dg Lie algebras, technically known as L_∞-homomorphisms. Such a morphism $\Phi : \mathfrak{g} \to \mathfrak{h}$ consists of a sequence of multilinear maps $\Phi^k : \mathfrak{g}^{\otimes k} \to \mathfrak{h}$ of degree $1-k$, $k \geq 1$, which are antisymmetric in a suitably graded sense, and satisfy the equations spelled out in [39]. In particular, Φ^1 is a chain map, and induces a homomorphism of graded Lie algebras on the cohomology level. One advantage is that quasi-isomorphisms can be inverted in this context. The precise statement we need is this:

Lemma 3.1. *Let $\mathfrak{g}$ be a graded Lie algebra, $\mathfrak{h}$ a dg Lie algebra, and $\Psi : \mathfrak{g} \to \mathfrak{h}$ an L_∞-homomorphism. Suppose that we are given a chain map $\Phi^1 : \mathfrak{h} \to \mathfrak{g}$ and an $l : \mathfrak{h} \to \mathfrak{h}$ of degree -1, such that*

$$(3.2) \qquad \begin{aligned} \Phi^1 \circ \Psi^1 &= \mathrm{id}, \\ \Psi^1 \circ \Phi^1 - \mathrm{id} &= \partial l + l\partial. \end{aligned}$$

Then Φ^1 can be extended to an L_∞-homomorphism $\Phi : \mathfrak{h} \to \mathfrak{g}$. Moreover, the higher order terms of Φ are given by universal formulae, which depend only on Ψ, Φ^1 and l.

Proof. The first part of the construction uses only Ψ^1, Φ^1 and l. Given these, the Homological Perturbation Lemma constructs another L_∞-algebra $\tilde{\mathfrak{g}}$ with vanishing differential, whose underlying vector space is $\mathfrak{g}$, together with an L_∞-homomorphism $\tilde{\Phi} : \mathfrak{h} \to \tilde{\mathfrak{g}}$ whose first order term is Φ^1. For explicit formulae see [40] (that reference concerns A_∞-algebras, but the L_∞-case is parallel). Now $\tilde{\Phi} \circ \Psi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ is an L_∞-homomorphism whose first order term is the identity. Hence it admits a unique exact inverse with the same property. Define $\Phi = (\tilde{\Phi} \circ \Psi)^{-1} \circ \tilde{\Phi}$.   □

To make the various pieces come together, we need to place ourselves in a framework where certain convergence properties are guaranteed. Assume that $\mathfrak{g}$ is filtered pronilpotent, which means that it comes with a complete decreasing filtration $L_\bullet \mathfrak{g}$ such that $L_1 \mathfrak{g} = \mathfrak{g}$ and

$$(3.3) \qquad \partial(L_r \mathfrak{g}) \subset L_r \mathfrak{g}, \quad [L_r \mathfrak{g}, L_s \mathfrak{g}] \subset L_{r+s} \mathfrak{g}.$$

Then $\mathfrak{g}^0$ is a pronilpotent Lie algebra, hence can be exponentiated to a prounipotent group by using the Baker-Campbell-Hausdorff formula. This group will act on the set of Maurer-Cartan elements. We call two elements equivalent if they lie in the same orbit (one can also define this relation through a suitable notion of homotopy between Maurer-Cartan elements). Deligne's basic idea (see [27] and references therein) is that this is a good model for many kinds of deformation theory.

Let $\mathfrak{g}, \mathfrak{h}$ be two filtered pronilpotent dg Lie algebras. A filtered L_∞-homomorphism $\Phi : \mathfrak{h} \to \mathfrak{g}$ consists of a family of maps as before, with the additional condition that

$$(3.4) \qquad \Phi^k(L_{r_k} \mathfrak{g} \otimes \cdots \otimes L_{r_1} \mathfrak{g}) \subset L_{r_k + \cdots + r_1} \mathfrak{g}.$$

There is an induced map on Maurer-Cartan elements, which preserves equivalence, namely:

$$(3.5) \qquad \alpha \longmapsto \sum_{k=1}^\infty \tfrac{1}{k!} \Phi^k(\alpha, \cdots, \alpha).$$

Lemma 3.2. *Suppose that $\Phi : \mathfrak{h} \to \mathfrak{g}$ is a filtered L_∞-homomorphism which is also a filtered quasi-isomorphism; the latter property means that Φ^1 induces quasi-isomorphisms of chain complexes $L_r \mathfrak{h} / L_{r+1} \mathfrak{h} \to L_r \mathfrak{g} / L_{r+1} \mathfrak{g}$ for any r. Then (3.5) induces a bijection between equivalence classes of Maurer-Cartan solutions.*

This is an adapted version of a result from [34, Section 4.4]. The proof given there does not immediately carry over to the filtered context. However, one

can prove the result by a more direct obstruction theory computation, where solutions of the Maurer-Cartan equation (or homotopies between them) are lifted from $\mathfrak{g}$ to $\mathfrak{h}$ up to errors which are of successively higher order in our filtration.

Kontsevich [34] used this framework to explore the relation between commutative and noncommutative geometry. We will summarize his result, with some minor modifications. On the commutative geometry side, let's temporarily generalize our notation to allow $V = \mathbb{C}^n$ for any n. By definition, the space of formal polyvector fields on V is

$$(3.6) \qquad \mathbb{C}[[V]] \otimes \Lambda(V) = \prod_{i,j} \mathrm{Sym}^i(V^\vee) \otimes \Lambda^j(V).$$

The (i, j) piece is given degree $j - 1$, and the whole space becomes a graded Lie algebra with the Schouten bracket
$$(3.7)$$
$$[f\,\xi_{i_1} \wedge \cdots \wedge \xi_{i_k}, g\,\xi_{j_1} \wedge \cdots \wedge \xi_{j_l}] =$$
$$\sum_q (-1)^{k-q-1} f\left(\partial_{i_q} g\right) \xi_{i_1} \wedge \cdots \wedge \widehat{\xi_{i_q}} \wedge \cdots \wedge \xi_{i_k} \wedge \xi_{j_1} \wedge \cdots \wedge \xi_{j_l} +$$
$$\sum_q (-1)^{l-q+(k-1)(l-1)} g\left(\partial_{j_q} f\right) \xi_{j_1} \wedge \cdots \wedge \widehat{\xi_{j_q}} \wedge \cdots \wedge \xi_{j_l} \wedge \xi_{i_1} \wedge \cdots \wedge \xi_{i_k}.$$

The Maurer-Cartan equation for $\alpha \in \mathbb{C}[[V]] \otimes \Lambda^2(V)$ says that the associated bracket $\{f, g\} = \alpha(df \wedge dg)$ satisfies the Jacobi identity, hence gives rise to a formal Poisson structure. Elements $\gamma \in \mathbb{C}[[V]] \otimes V$ are formal vector fields, acting by their Lie derivative. At least for vector fields vanishing at zero, the action can be exponentiated, and results in the obvious pushforward action of formal diffeomorphisms on Poisson brackets.

The analogue of polyvector fields in noncommutative geometry is given by Hochschild cohomology, which we now describe. Let A be a graded associative algebra over $\mathbb{C}$. Its Hochschild complex $CC(A, A)$ is the space of graded multilinear maps

$$(3.8) \qquad CC^d(A, A) = \prod_{i+j-1=d} \mathrm{Hom}^j(A^{\otimes i}, A).$$

The Hochschild differential is
$$(3.9)$$
$$(\partial\phi)^j(a_j, \ldots, a_1) = \sum_k (-1)^{|\phi|+|a_1|+\cdots+|a_k|+k} \phi^{j-1}(a_j, \ldots, a_{k+1}a_k, \ldots, a_1)$$
$$+ (-1)^{|\phi|+|a_1|+\cdots+|a_{j-1}|+j} a_j \phi^{j-1}(a_{j-1}, \ldots, a_1)$$
$$+ (-1)^{(|\phi|-1)(|a_1|-1)+1} \phi^{j-1}(a_j, \ldots, a_2)a_1,$$

and the Gerstenhaber bracket is

(3.10)
$$[\phi, \psi]^j(a_j, \ldots, a_1) = \sum_{k,l} (-1)^{|\psi|(|a_1| + \cdots + |a_k| - k)} \phi^{j-l+1}(a_j, \ldots,$$
$$\psi^l(a_{k+l}, \ldots, a_{k+1}), a_k, \ldots, a_1)$$
$$- \sum_{k,l} (-1)^{|\phi| \cdot |\psi| + |\phi|(|a_1| + \cdots + |a_k| - k)} \psi^{j-l+1}(a_j, \ldots,$$
$$\phi^l(a_{k+1}, \ldots, a_{k+1}), a_k, \ldots, a_1).$$

The cohomology of ∂ is just the Hochschild cohomology $HH(A, A)$, with the grading shifted down by 1 from the standard convention. Take $\alpha \in CC^1(A, A)$, which by definition is a sequence of maps $\alpha^j : A^{\otimes j} \to A$ of degree $2 - j$, $j \geq 0$. Set

(3.11)
$$\begin{cases} \mu^j = \alpha^j \text{ for } j \neq 2, \\ \mu^2(a_2, a_1) = \alpha^2(a_2, a_1) + (-1)^{|a_1|} a_2 a_1. \end{cases}$$

Then, the Maurer-Cartan equation for α says that μ satisfies the equations for a curved A_∞-structure; see for instance [24]. Suppose for technical simplicity that A is finite-dimensional in each degree, and take some $\gamma \in CC^0(A, A)$ whose constant term $\gamma^0 \in A^1$ vanishes. Define

(3.12)
$$\begin{cases} \phi^1 = \text{id} + \gamma^1 + \frac{1}{2}\gamma^1\gamma^1 + \cdots = \exp(\gamma^1), \\ \phi^2 = \gamma^2 + \frac{1}{2}\gamma^1\gamma^2 + \frac{1}{2}\gamma^2(\gamma^1 \otimes \text{id}) + \frac{1}{2}\gamma^2(\text{id} \otimes \gamma^1) + \frac{1}{3}\gamma^2(\gamma^1 \otimes \gamma^1) + \cdots, \\ \cdots \end{cases}$$

The general rule for ϕ^j is to sum up all possible ways of concatenating components of γ to get a j-linear map. If there are r components, and s ways of ordering the components compatibly with their appearance in the concatenation, then the constant in front of the associated term is $s(r!)^{-1}$ (this in particular ensures convergence of the sums). If α and $\tilde{\alpha}$ are two Maurer-Cartan elements which are related by the exponentiated action of γ, the associated curved A_∞-structures μ, $\tilde{\mu}$ are related by ϕ, which is an A_∞-isomorphism.

We now specialize to exterior algebras $A = \Lambda(V)$. A classical result [30] is that $HH(A, A) \cong \mathbb{C}[[V]] \otimes \Lambda(V)$. This isomorphism is induced by the Hochschild-Kostant-Rosenberg map, which is the projection $\Phi^1 : CC(A, A) \to \text{Hom}(T(V), \Lambda(V)) \to \mathbb{C}[[V]] \otimes \Lambda(V)$. Explicitly, thinking of $\Phi^1(\beta)$ as a $\Lambda(V)$-valued formal power series, we have

(3.13)
$$\Phi^1(\beta)(\xi) = \sum_{j=1}^{\infty} \beta^j(\xi, \ldots, \xi).$$

Kontsevich's formality theorem [34] says the following:

Theorem 3.3. *Φ^1 is the first term of an L_∞-homomorphism Φ. Moreover, Φ is equivariant with respect to the action of $GL(V)$ on both sides.*

Our formulation differs from the original one in two respects. First, it concerns exterior algebras instead of polynomial ones. However, the proof adapts in a straightforward way, by exchanging odd and even variables. Secondly, Kontsevich actually constructs an L_∞-homomorphism Ψ in the opposite direction. We want to use Lemma 3.1 to reverse direction, and that requires a choice of homotopy. By thinking of the classical grading of Hochschild cohomology, one sees that the homotopy can be taken to be a collection of maps $l^{i,j} : \operatorname{Hom}^j(A^{\otimes i}, A) \longrightarrow \operatorname{Hom}^j(A^{\otimes i-1}, A)$. Since each of these spaces is finite-dimensional and the group $GL(V)$ is reductive, one can average the homotopy to make it $GL(V)$-equivariant as well. Kontsevich's Ψ is $GL(V)$-equivariant, and because of the way in which the inverse is defined, the same will then hold for Φ.

Neither $\Lambda(V) \otimes \mathbb{C}[[V]]$ nor $CC(A, A)$ are pronilpotent, but one can remedy that by introducing an additional formal parameter $\hbar$, as in Kontsevich's original application to deformation quantization. This will also be the case here, but our parameter will have nonzero degree, which makes the situation somewhat less standard.

4. Finite determinacy

From now on, we again restrict to $V = \mathbb{C}^3$. Take the group $G \subset SL(V)$ which consists of diagonal matrices whose nonzero coefficients are fifth roots of unity. Because of the condition on the determinant, $G \cong (\mathbb{Z}/5)^2$. We will now tweak the previous framework by introducing the above-mentioned formal parameter with nonzero degree, while at the same time adding equivariance with respect to G. Namely, let $\mathfrak{g}$ be the graded vector space defined by

$$(4.1) \qquad \mathfrak{g}^d = \prod_{\substack{2i+j-4k=3d+3 \\ k \geq 0,\, i \geq d+2}} (\operatorname{Sym}^i(V^\vee) \otimes \Lambda^j(V))^G \, \hbar^k.$$

Even though the degrees are now different, their parities are the same as in our original discussion of $\mathbb{C}[[V]] \otimes \Lambda(V)$. The Nijenhuis bracket turns $\mathfrak{g}$ into a graded Lie algebra. It is filtered pronilpotent, with $L_r \mathfrak{g}^d$ being the part of (4.1) where $i \geq d+1+r$.

For elements of $\mathfrak{g}$ of a fixed degree d, the power of $\hbar$ occurring in each term $\operatorname{Sym}^i(V^\vee) \otimes \Lambda^j(V)$ is fixed to be $\frac{1}{4}(2i + j - 3d - 3)$. Hence, we can usually

omit it from the notation, as long as we still remember the inequality

$$2i + j \geq 3d + 3 \tag{4.2}$$

as well as the congruence

$$2i + j + d + 1 \equiv 0 \bmod 4. \tag{4.3}$$

Let $F_\bullet\mathbb{C}[[V]]$ be the complete decreasing filtration such that $F_r\mathbb{C}[[V]]$ consists of those power series with no terms of order strictly less than r. An element $\alpha \in \mathfrak{g}^1$ has the form $\alpha = (\alpha^0, \alpha^2)$, where $\alpha^0 \in F_3\mathbb{C}[[V]]$ is an odd formal function, and $\alpha^2 \in F_4\mathbb{C}[[V]] \otimes \Lambda^2(V)$ is an even formal bivector field. Here the terms even and odd refer to the action of $-1 \in GL(V)$ on polyvector fields. This property of α is an obvious consequence of (4.3). Similarly, an element $\gamma \in \mathfrak{g}^0$ can be written as $\gamma = (\gamma^1, \gamma^3)$, where $\gamma^1 \in F_3\mathbb{C}[[V]] \otimes V$ is even, and $\gamma^3 \in F_2\mathbb{C}[[V]] \otimes \Lambda^3(V)$ is odd. On top of that, we of course have the G-invariance condition.

Before starting actual computations, it is worth while to acquire some geometric intuition. The Maurer-Cartan equation decomposes into

$$\tfrac{1}{2}[\alpha^2, \alpha^2] = 0, \quad [\alpha^0, \alpha^2] = 0. \tag{4.4}$$

As before, the first part says that α^2 defines a Poisson bracket $\{\cdot, \cdot\}$. The second part says that $\{\alpha^0, \cdot\}$ is trivial, which means that the Poisson vector field associated to the function α^0 is identically zero. Equivalently, α^2 is a cocycle in the Koszul complex given by contraction with $d\alpha^0 \in \mathbb{C}[[V]] \otimes V^\vee$, which is

$$\tag{4.5}$$
$$0 \to \mathbb{C}[[V]] \otimes \Lambda^3(V) \xrightarrow{\iota_{d\alpha^0}} \mathbb{C}[[V]] \otimes \Lambda^2(V) \xrightarrow{\iota_{d\alpha^0}} \mathbb{C}[[V]] \otimes V \xrightarrow{\iota_{d\alpha^0}} \mathbb{C}[[V]] \to 0.$$

Considering degree zero elements, the exponentiated adjoint action of $\gamma = (\gamma^1, 0)$ is the usual action of formal diffeomorphisms on polyvector fields. The exponentiated adjoint action of $\gamma = (0, \gamma^3)$ is given by

$$(\alpha^0, \alpha^2) \longmapsto (\alpha^0, \alpha^2 + \iota_{d\alpha^0}\gamma^3). \tag{4.6}$$

Note that $(W, 0) \in \mathfrak{g}^1$ is a solution of the Maurer-Cartan equation. It turns out that any other solution which is sufficiently close to this one is actually equivalent to it. The precise statement is:

Lemma 4.1. *Any Maurer-Cartan element $\alpha = (\alpha^0, \alpha^2) \in \mathfrak{g}^1$ such that $\alpha^0 \equiv W \bmod F_7\mathbb{C}[[V]]$ is equivalent to $(W, 0)$.*

W has an isolated singularity at the origin, which in algebraic terms means that the ideal $I = (\partial_1 W, \ldots, \partial_3 W) \subset \mathbb{C}[[V]]$ is of finite codimension. As a consequence, any formal power series which agrees with W to sufficiently high order can be transformed into W by a formal change of coordinates. This phenomenon is known in singularity theory as finite determinacy

[60] (see [4, vol. I, p. 121] for an exposition). We will not appeal to these general results, but they've definitely guided our approach.

The explicit computation goes as follows. Elementary manipulation shows that

(4.7)
$$v_j v_k \in I + F_4 \mathbb{C}[[V]] \text{ for } j \neq k,$$
$$v_j^6 \in I \cdot F_2 \mathbb{C}[[V]] + F_8 \mathbb{C}[[V]].$$

Start with $W_7 = \alpha^0$. Because of its symmetry properties, this can contain no pure monomials v_k^7 or v_k^8. From the first part of (4.7) one sees that $W - W_7 \in I \cdot F_5 \mathbb{C}[[V]] + F_9 \mathbb{C}[[V]]$. By appropriately choosing $f_{5,1}, \ldots, f_{5,3} \in F_5 \mathbb{C}[[V]]$, one can achieve that

(4.8)
$$W_7(v_1 + f_{5,1}(v), \ldots, v_3 + f_{5,3}(v))$$
$$\equiv W_7 + f_{5,1} \partial_1 W + \cdots + f_{5,3} \partial_3 W \mod F_9 \mathbb{C}[[V]]$$
$$\equiv W \mod F_9 \mathbb{C}[[V]].$$

Here, the error term includes the differences $f_{5,k}(\partial_k W_7 - \partial_k W)$, as well as quadratic and higher terms in the Taylor expansion, all of which lie in $F_{11} \mathbb{C}[[V]]$. Moreover, one can a posteriori average the coordinate change to make it suitably equivariant. The result is a function $W_9 \equiv W \mod F_9 \mathbb{C}[[V]]$, with the same symmetry properties as W_7 itself.

From then on, one uses a slight variant of the same strategy. Suppose that for some odd $r \geq 9$ we have a function $W_r \equiv W \mod F_r \mathbb{C}[[V]]$, which is odd and G-invariant. By (4.7) one can write $W - W_r \in I \cdot F_{r-4} \mathbb{C}[[V]] + F_{r+2} \mathbb{C}[[V]]$. By appropriately choosing $f_{r-4,1}, \ldots, f_{r-4,3} \in F_{r-4} \mathbb{C}[[V]]$, one can achieve that $W_r(v_1 + f_{r-4,1}(v), \ldots, v_3 + f_{r-4,3}(v)) \equiv W \mod F_{r+2} \mathbb{C}[[V]]$. After averaging this coordinate transformation to make it equivariant, one gets a function W_{r+2} which can be used in the next step. This process yields an infinite sequence of coordinate changes, which are of increasingly high order, hence whose infinite composition converges. Alternatively, one can break off after a few steps and apply an equivariant version of the general finite determinacy theorem (see the references given above).

The conclusion is that, after acting by the exponential of some element $\gamma = (\gamma^1, 0) \in \mathfrak{g}^0$, we may assume that our Maurer-Cartan element is of the form (W, α^2), where $\alpha^2 \in F_4 \mathbb{C}[[V]] \otimes \Lambda^2(V)$. Finite-dimensionality of $\mathbb{C}[[V]]/I$ implies that the $\partial_k W$ form a regular sequence in $\mathbb{C}[[V]]$, which in turn implies that the complex (4.5) is a resolution of $\mathbb{C}[[V]]/I$ [63, Corollary 4.5.5]. Hence $\alpha^2 = -\iota_{dW} \gamma^3$, and again one can choose γ^3 to be odd and G-invariant. Moreover, by looking at the low degrees in the Taylor expansion, it follows that $\gamma^3 \in F_2 \mathbb{C}[[V]] \otimes \Lambda^3(V)$, hence lies in $\mathfrak{g}$. According to (4.6), the action

of the exponential of $(0, \gamma^3)$ transforms (W, α^2) into $(W, 0)$, which completes the proof of Lemma 4.1.

5. A classification theorem

We correspondingly modify the noncommutative geometry side. Take $V = \mathbb{C}^3$ and $A = \Lambda(V)$, with the same $G \subset SL(V) \subset \mathrm{Aut}(A)$ as before. Define a graded vector space $\mathfrak{h}$ by

$$(5.1) \qquad \mathfrak{h}^d \;=\; \prod_{\substack{3i+j-4k=3d+3 \\ k \geq 0,\, i \geq d+2}} \mathrm{Hom}^j(A^{\otimes i}, A)^G \, \hbar^k.$$

The parity of the grading agrees with the one previously used in our discussion for $CC(A, A)$. Hence, the Hochschild differential and Gerstenhaber bracket turn $\mathfrak{h}$ into a dg Lie algebra. It is filtered nilpotent, with $L_r \mathfrak{h}^d$ being the part of (5.1) where $i \geq d + 1 + r$.

Lemma 5.1. *There is a filtered L_∞-quasi-isomorphism $\Phi : \mathfrak{h} \to \mathfrak{g}$ whose first term Φ^1 is (the obvious $\hbar$-linear extension of) the Hochschild-Kostant-Rosenberg map* (3.13).

This is a direct consequence of Theorem 3.3. It is useful to think of $\mathfrak{h}$ as obtained from $CC(A, A)$ by the following process. One starts with $CC(A, A)[[\hbar]]$ with its traditional grading, and then modifies that to a $\mathbb{Q}$-grading by giving $\hbar$ degree $4/3$, and subtracting $2/3$ times the weight of the action of the central $\mathbb{C}^* \subset GL(V)$. Restrict to the subspace where this $\mathbb{Q}$-grading is integral, and where the weight of the $\mathbb{C}^*$-action is strictly less than four times the order of $\hbar$. In terms of (5.1) the latter condition says that $j < 4k$, which is equivalent to $i > d + 1$. Finally, take the G-invariant part. It is easy to check that the same process produces $\mathfrak{g}$ from $\mathbb{C}[[V]] \otimes \Lambda(V)$. Since the terms $\Phi^r : CC(A, A)[[\hbar]]^{\otimes r} \to \mathbb{C}[[V]] \otimes \Lambda(V)[[\hbar]]$ of the Kontsevich L_∞-homomorphism are $GL(V)$-equivariant and respect powers of $\hbar$, they restrict to maps $\mathfrak{h}^{\otimes r} \to \mathfrak{g}$ of the correct degree $1 - r$. The filtrations can be defined in similar terms, showing that Φ satisfies (3.4). Moreover, since parities of gradings are preserved, the restrictions satisfy the necessary symmetry and L_∞-homomorphism conditions.

We will use Lemma 5.1 to transfer classification problems for Maurer-Cartan solutions from $\mathfrak{h}$ to $\mathfrak{g}$. However, before bringing this theory to bear, let's look at the meaning of such solutions. A general $\alpha \in \mathfrak{h}^1$ consists of i-linear components α^i for $i \geq 3$, each of which is in turn of the form $\alpha^i = \alpha_0^i + \hbar \alpha_1^i + \cdots$, with

$$(5.2) \qquad \alpha_k^i \in \mathrm{Hom}^{6-3i+4k}(A^{\otimes i}, A)^G.$$

Note that for each fixed i, there are only finitely many k such that $\alpha_k^i \neq 0$, for degree reasons. Define multilinear maps $\mu^i : A^{\otimes i} \to A$ whose $\mathbb{Z}/2$-grading is i by setting

$$
(5.3) \qquad
\begin{cases}
\mu^1 = 0, \\
\mu^2(a_2, a_1) = (-1)^{|a_1|} a_2 \wedge a_1, \\
\mu^i = \alpha_0^i + \alpha_1^i + \cdots \text{ for } i \geq 3.
\end{cases}
$$

In parallel with our previous general discussion, α is a solution of the Maurer-Cartan equation iff μ is a $\mathbb{Z}/2$-graded A_∞-structure on A. Of course, this structure is automatically G-invariant as well. Next, suppose that we have two solutions of the Maurer-Cartan equation, related by the exponentiated action of some $\gamma \in \mathfrak{h}^0$. Then, the associated A_∞-structures $\mu, \tilde{\mu}$ are related by a G-equivariant $\mathbb{Z}/2$-graded A_∞-isomorphism ϕ, whose first term is $\phi^1 = \mathrm{id}$. One gets ϕ from γ by the formulae from (3.12), with the simplifications coming from $\gamma^1 = 0$.

Let's look explicitly at some of the simplest terms which α and γ can have. First of all,

$$
(5.4) \qquad
\begin{aligned}
\mathrm{Hom}^1(A^{\otimes 3}, A)^G &= 0, \\
\mathrm{Hom}^{-2}(A^{\otimes 3}, A)^G &= 0.
\end{aligned}
$$

The first part of this implies that $\alpha_1^3 = 0$. In view of that, the simplest nontrivial components of the Maurer-Cartan equation are

$$
(5.5) \qquad
\begin{aligned}
\partial \alpha_0^3 &= 0, \\
\partial \alpha_1^4 &= 0, \\
\partial \alpha_1^5 + [\alpha_0^3, \alpha_1^4] &= 0.
\end{aligned}
$$

α_0^3 is a cocycle, which under the Hochschild-Kostant-Rosenberg map goes to

$$
(5.6) \qquad \Phi^1(\alpha_0^3) \in \mathrm{Sym}^3(V^\vee)^G = \mathbb{C} \cdot v_1 v_2 v_3.
$$

The next term α_1^4 is again a cocycle, whose cohomology class is determined by

$$
(5.7) \qquad
\begin{aligned}
\Phi^1(\alpha_1^4) &\in \left(\mathrm{Sym}^4(V^\vee) \otimes \Lambda^2(V) \right)^G \\
&= \mathbb{C} \cdot (v_1^4 \otimes \xi_2 \wedge \xi_3) \oplus \mathbb{C} \cdot (v_2^4 \otimes \xi_3 \wedge \xi_1) \oplus \mathbb{C} \cdot (v_3^4 \otimes \xi_1 \wedge \xi_2).
\end{aligned}
$$

Since Φ^1 induces a Lie algebra homomorphism in cohomology, it follows from (5.5) that the Nijenhuis bracket $[\Phi^1(\alpha_0^3), \Phi^1(\alpha_1^4)] = 0$ vanishes, but that is only possible if one of the two classes involved is zero. The case of interest to us, which we will concentrate on from now on, is when $\Phi^1(\alpha_0^3) \neq 0$, which means that $\Phi^1(\alpha_1^4) = 0$. From the second part of (5.4), it follows a fortiori that there are no Hochschild coboundaries in $\mathrm{Hom}^{-2}(A^{\otimes 4}, A)^G$. Hence, vanishing of the

cohomology class of α_1^4 means that the cocycle itself is zero. That in turn means that α_1^5 is itself a cocycle, with cohomology class

$$(5.8) \qquad \Phi^1(\alpha_1^5) \in \mathrm{Sym}^5(V^\vee)^G = \mathbb{C} \cdot v_1^5 \oplus \mathbb{C} \cdot v_2^5 \oplus \mathbb{C} \cdot v_3^5.$$

Next, let's analyze the action of $\gamma \in \mathfrak{h}^0$. Both $\gamma_1^2 \in \mathrm{Hom}^1(A^{\otimes 2}, A)^G$ and $\gamma_1^3 \in \mathrm{Hom}^{-2}(A^{\otimes 3}, A)^G$ vanish by (5.4). Assuming as before that $\Phi^1(\alpha_0^3) \neq 0$, the infinitesimal gauge transformation $\beta = -\partial\gamma + [\gamma, \alpha]$ has components

$$
\begin{aligned}
\beta_0^3 &= -\partial\gamma_0^2, \\
(5.9) \qquad \beta_1^4 &= 0, \\
\beta_1^5 &= -\partial\gamma_1^4.
\end{aligned}
$$

Hence, the cohomology classes (5.6) and (5.8) are preserved, which means that they are invariants of the equivalence class of the Maurer-Cartan element.

Proposition 5.2. *Up to equivalence, there is a unique Maurer-Cartan element* $\alpha \in \mathfrak{h}^1$ *such that* $\Phi^1(\alpha_0^3) = -v_1 v_2 v_3$ *and* $\Phi^1(\alpha_1^5) = v_1^5 + v_2^5 + v_3^5$.

By construction, Φ is filtered and preserves powers of $\hbar$. Recall from the previous computation that $\alpha_1^3 = \alpha_1^4 = 0$. Moreover, $\Phi^r(\alpha_0^{i_r}, \ldots, \alpha_0^{i_1})$ vanishes for degree reasons (thinking back to the original grading as in Theorem 3.3), with the single exception of $\Phi^1(\alpha_0^3)$. Hence, the leading terms in the image of α under (3.5) are

$$
\begin{aligned}
\tilde{\alpha} &= \Phi^1(\alpha) + \tfrac{1}{2}\Phi^2(\alpha, \alpha) + \cdots \\
(5.10) \qquad &\equiv \Phi^1(\alpha^3 + \alpha^4 + \alpha^5) + \tfrac{1}{2}\Phi^2(\alpha^3, \alpha^3) + \Phi^2(\alpha^3, \alpha^4) \bmod L_4 \mathfrak{g}^1 \\
&\equiv \Phi^1(\alpha_0^3) + \hbar\Phi^1(\alpha_1^5) \bmod L_4 \mathfrak{g}^1 + (\hbar^2 \mathfrak{g})^1.
\end{aligned}
$$

Closer inspection shows that $L_4 \mathfrak{g}^1$ and $(\hbar^2 \mathfrak{g})^1$ coincide. Either of them consists of those elements $\tilde{\alpha} = (\tilde{\alpha}^0, \tilde{\alpha}^2) \in \mathfrak{g}^1$ such that $\tilde{\alpha}^0 \in F_7 \mathbb{C}[[V]]$ and $\tilde{\alpha}^2 \in F_6 \mathbb{C}[[V]] \otimes \Lambda^2(V)$. This means that $\tilde{\alpha}^0 \equiv W \bmod F_7 \mathbb{C}[[V]]$. Lemma 4.1 shows that such a Maurer-Cartan solution is unique up to equivalence. Hence, the same holds for the original solution α, by Lemma 3.2.

Even though that is basically a repetition, it may still make sense to reformulate the outcome in a way which is more directly relevant to applications. Let μ be a G-equivariant $\mathbb{Z}/2$-graded A_∞-structure on A, where $\mu^1 = 0$, μ^2 is the ordinary product up to sign changes as in (5.3), and where the higher order structures can be written as sums of components μ_k^i of degree $6 - 3i + 4k$. Assume moreover that for $\xi \in V \subset A$,

$$(5.11) \qquad \mu_0^3(\xi, \xi, \xi) = -\xi_1 \xi_2 \xi_3, \quad \mu_1^5(\xi, \xi, \xi, \xi, \xi) = \xi_1^5 + \xi_2^5 + \xi_3^5.$$

These requirements determine μ uniquely up to G-equivariant A_∞-isomorphisms. From now on, we will generally write $\mathcal{A}$ for any A_∞-algebra belonging to this isomorphism class.

6. General aspects of the Fukaya category

Temporarily, allow M to be a closed connected oriented surface of any genus ≥ 2. The Fukaya category $\mathcal{F}(M)$ is a $\mathbb{Z}/2$-graded A_∞-category over $\mathbb{C}$. We'll begin by giving a description of this category on the cohomological level, and then discuss some properties which can be stated independently of the more technical aspects of the chain level construction.

Let $S(TM) \to M$ be the tangent circle bundle; this can be defined without reference to a metric, as the bundle of oriented real lines in TM. Choose a symplectic form ω on M, and a one-form θ on $S(TM)$ whose exterior derivative is the pullback of ω; this exists because the tangent bundle has nonzero degree. Consider connected Lagrangian submanifolds, which are of course just simple closed curves $L \subset M$. Let $\sigma : L \to S(TM)|L$ be the section given by the tangent spaces of L, for some choice of orientation. We say that L is balanced if $\int_L \sigma^* \theta = 0$; this property is independent of the orientation, since the sections $\pm\sigma$ are fibrewise homotopic.

Remark 6.1. A nullhomologous curve is balanced iff it divides M into halves $M_\pm$ such that $\chi(M_+)/\text{area}(M_+) = \chi(M_-)/\text{area}(M_-)$. Thus, contractible curves can never be balanced. Every other isotopy class of curves contains a balanced representative, which is unique up to Hamiltonian isotopy.

Objects of $\mathcal{F}(M)$ are balanced curves L equipped with orientations and *Spin* structures. On the cohomology level, the morphisms

$$(6.1) \qquad\qquad H(\hom_{\mathcal{F}(M)}(L_0, L_1)) = HF^*(L_0, L_1)$$

are the Lagrangian Floer cohomology groups. In particular, for any object L we have a canonical isomorphism $H(\hom_{\mathcal{F}(M)}(L, L)) = HF^*(L, L) \cong H^*(L; \mathbb{C})$. Still remaining on the cohomology level, composition of morphisms is given by Donaldson's holomorphic triangle product. In particular, the isomorphism $HF^*(L, L) \cong H^*(L; \mathbb{C})$ is compatible with the ring structure. Moreover, any two balanced curves which are isotopic (compatibly with the orientations and *Spin* structures) give rise to isomorphic objects of $H^0(\mathcal{F}(M))$.

Remark 6.2. In the case of the torus [49], the Fukaya category is defined over a Novikov field, which is a field of formal Laurent series with a parameter t. If one wants to define a Fukaya category of a higher genus surface containing *all* Hamiltonian isotopy classes of curves as objects, Novikov fields appear there as well. However, if one then restricts attention to balanced curves, all resulting series turn out to be finite (Laurent polynomials), hence one can set $t = 1$ and work over $\mathbb{C}$, which is what we are doing here. This phenomenon, usually called monotonicity, is familiar to symplectic geometers (it appears in the literature mainly in the context of Floer cohomology for Fano manifolds;

see [45], and for a formulation closer to the one adopted here, [62, Remark 3.1.4]).

The definition of the objects in the Fukaya category involves θ, or rather its equivalence class modulo exact one-forms (compare [1, Appendix A]). This has some consequences for functoriality. Namely, suppose that we have two choices of symplectic forms ω, $\tilde{\omega}$ and correspondingly one-forms θ, $\tilde{\theta}$. Given a symplectomorphism $\phi : (M, \omega) \to (M, \tilde{\omega})$, we can consider the induced map $S(D\phi) : S(TM) \to S(TM)$, which defines a class

(6.2)
$$[S(D\phi)^*\tilde{\theta} - \theta] \in H^1(S(TM); \mathbb{R}).$$

We say that ϕ is balanced if this class vanishes, in which case it induces a quasi-isomorphism between the associated Fukaya categories. Every connected component of the space of symplectomorphisms contains such representatives, and they are unique up to Hamiltonian isotopy. This, together with invariance under rescaling of ω and θ, implies that $\mathcal{F}(M)$ is independent of the additional choices up to quasi-isomorphisms. Hence, it is justifiable to talk of "the Fukaya category of M". For the same reason, the mapping class group of M acts on $\mathcal{F}(M)$.

Before continuing, we need to recall a few homological algebra notions from [38] or [56, Sections 3–4]. Given any $\mathbb{Z}/2$-graded A_∞-category $\mathcal{B}$, one can consider the associated dg category $mod(\mathcal{B})$ of right A_∞-modules. This comes with a canonical cohomologically full and faithful A_∞-functor $\mathcal{B} \to mod(\mathcal{B})$, the Yoneda embedding. The associated cohomology level category $Mod(\mathcal{B}) = H^0(mod(\mathcal{B}))$ is a triangulated category, with the property that the twofold shift functor is isomorphic to the identity, and moreover it is split-closed (also called Karoubi complete; this means that any idempotent endomorphism of an object leads to a splitting of that object as a direct sum). Take the smallest full subcategory of $Mod(\mathcal{B})$ which contains the image of the Yoneda embedding, is triangulated, and split-closed. We denote this by $D^\pi(\mathcal{B})$, and call it the split-closed derived category of $\mathcal{B}$.

Remark 6.3. The only point in the above discussion which might not be entirely familiar is the fact that $Mod(\mathcal{B})$ is split-closed. A short proof goes as follows. The dg category $mod(\mathcal{B})$ itself comes with a Yoneda embedding, which induces a functor $Mod(\mathcal{B}) \to Mod(mod(\mathcal{B}))$. In the other direction, we have a restriction functor $Mod(mod(\mathcal{B})) \to Mod(\mathcal{B})$; and the composition of the two is the identity on $Mod(\mathcal{B})$. From [56, Section 4b] we know that any idempotent endomorphism in $Mod(\mathcal{B})$ leads to a splitting of the associated Yoneda module in $Mod(mod(\mathcal{B}))$. The image of that under restriction is the desired splitting in $Mod(\mathcal{B})$.

Let $\mathcal{A} \subset \mathcal{B}$ be a full A_∞-subcategory. In that case, $D^\pi(\mathcal{A})$ is canonically equivalent to the smallest split-closed triangulated full subcategory of $D^\pi(\mathcal{B})$ containing all objects of $\mathcal{A}$. Objects of $\mathcal{B}$ which, up to isomorphism, lie in $D^\pi(\mathcal{A})$, are said to be split-generated by the objects of $\mathcal{A}$. If this holds for all of $\mathcal{B}$, which means that the embedding $D^\pi(\mathcal{A}) \to D^\pi(\mathcal{B})$ is an equivalence, we say that the objects of $\mathcal{A}$ split-generate $\mathcal{B}$.

To apply this to the Fukaya category, we need to recall some facts about the action of Dehn twists. Let L_0, L_1 be objects of $\mathcal{F}(M)$, where the *Spin* structure on L_1 is nontrivial. The Dehn twist τ_{L_1} is a balanced symplectic automorphism of M, hence $\tau_{L_1}(L_0)$ is again a balanced curve. We then have an exact triangle in $D^\pi(\mathcal{F}(M))$ of the form

$$(6.3) \qquad \begin{array}{ccc} HF^*(L_1, L_0) \otimes L_1 & \xrightarrow{\text{ev}} & L_0 \\ & \overset{[1]}{\nwarrow} & \downarrow \\ & & \tau_{L_1}(L_0). \end{array}$$

Here, $HF^*(L_1, L_0) \otimes L_1$ is a direct sum of copies of L_1 and its shifted version $L_1[1]$, with one summand for each generator of the Floer cohomology group, and ev is the canonical evaluation map. An equivalent statement is that $\tau_{L_1}(L_0)$ is isomorphic to the cone of ev. The construction of the exact triangle runs parallel to [56, Corollary 17.18], which means that it is based on a version of the long exact sequence from [55].

Lemma 6.4. *Let $\{L_1, \ldots, L_r\}$, $r \geq 1$, be objects of $\mathcal{F}(M)$ whose Spin structures are nontrivial. Let L_0 be another object, such that $\tau_{L_r} \cdots \tau_{L_1}(L_0)$ is isotopic to L_0 with the orientation reversed. Then L_0 is split-generated by $\{L_1, \ldots, L_r\}$.*

This is similar to [56, Proposition 19.7]. The composition of the vertical arrows in (6.3) yields a morphism

$$(6.4) \qquad L_0 \longrightarrow \tau_{L_r} \cdots \tau_{L_1}(L_0).$$

In the present situation, this morphism is an element of $HF^1(L_0, L_0) \cong H^1(L_0; \mathbb{C})$, hence its square is automatically zero. This is precisely what's needed to make the argument in [56, p. 70] go through.

Lemma 6.5. *Let $\{L_1, \ldots, L_r\}$ be objects of $\mathcal{F}(M)$ whose Spin structures are nontrivial, and such that $\tau_{L_r} \cdots \tau_{L_1}$ is isotopic to the identity. Then they split-generate $\mathcal{F}(M)$.*

The basic strategy is the same, but with an additional geometric step. Take an arbitrary L_0, and consider the analogue of (6.4), which this time is an element of $HF^0(L_0, L_0) \cong H^0(L_0; \mathbb{C})$. By construction of the exact triangle, this element admits the following description. Consider the Lefschetz fibration

with fibre M and vanishing cycles $\{L_1, \ldots, L_r\}$. Fix a generic almost complex structure which makes the fibration map pseudo-holomorphic, and consider the associated moduli space of pseudo-holomorphic sections. This may have components of different dimensions, but (due to the balancing condition, and the fact that the fibres contain no holomorphic spheres) the component of any fixed dimension is compact. By considering the evaluation map at a point, as in Gromov-Witten theory, one gets an even-dimensional cohomology class in M, which we call the section class. For any L_0, the morphism (6.4) is the image of the section class under the restriction map

$$(6.5) \qquad\qquad H^0(M; \mathbb{C}) \to H^0(L_0; \mathbb{C}).$$

Assume that the section class has a nontrivial component in $H^0(M; \mathbb{C})$. This means that through every point of M there is a pseudo-holomorphic section with zero self-intersection. Standard methods from four-dimensional symplectic topology [41] then ensure that these sections foliate the total space of our Lefschetz fibration, which is a contradiction. Hence the image of the section class under (6.5) vanishes, allowing one to proceed as before.

Remark 6.6. The section class itself is not zero in general. For instance, if the fibration is constructed by blowing up a Lefschetz pencil, every base point of the pencil gives rise to a section, which contributes 1 to the $H^2(M; \mathbb{C})$ component of the section class (however, taking the fibre connect sum of the fibration with itself corresponds to passing to the cup-square of the section class, which will kill it).

7. Technical aspects of the Fukaya category

In this section, we take a closer look at the definition of the Fukaya category. Since our target space is a surface, the naive idea is that after appealing to the uniformization theorem, the A_∞-structure maps should be computable purely combinatorially by counting polygons. This is true in many cases but fails to hold in general, due to transversality issues, which any proper definition must address. There are several approaches, all of which are ultimately equivalent (meaning that they give rise to different but quasi-equivalent A_∞-categories). We follow the "Morse-Bott" type approach, in a version which borrows some aspects of [52] and [17].

Fix a countable set $\mathcal{L}$ of balanced curves on M with the following properties. Each nontrivial isotopy class has at least one representative in $\mathcal{L}$. Moreover, any two distinct curves in $\mathcal{L}$ intersect transversally, and any three distinct curves have no common point. From now on, when defining $\mathcal{F}(M)$, we will only

allow curves taken from $\mathcal{L}$ (this is a technical contrivance, which is ultimately irrelevant: any two choices of $\mathcal{L}$ lead to quasi-equivalent A_∞-categories).

Suppose that L_0 and L_1 are objects, and that the underlying curves are distinct, hence transverse. In this case, the morphism space between them is the unperturbed Floer cochain complex

$$(7.1) \qquad \hom_{\mathcal{F}(M)}(L_0, L_1) = CF^*(L_0, L_1) = \bigoplus_x \mathbb{C}x,$$

where the sum is over all intersection points $x \in L_0 \cap L_1$. The generator associated to x is even if the local intersection number is -1, and odd otherwise. Next, suppose that $(L_0, \ldots, L_d)$ is a collection of objects, whose underlying curves are pairwise different. In this case, the coefficients of the A_∞-composition

$$\mu^d : CF^*(L_{d-1}, L_d) \otimes \cdots \otimes CF^*(L_0, L_1) \longrightarrow CF^*(L_0, L_d),$$

$$(7.2) \qquad \mu^d(x_d, \ldots, x_1) = \sum_{x_0} m(x_0, \ldots, x_d) x_0$$

are numbers $m(x_0, \ldots, x_d) \in \mathbb{Z}$ obtained by a signed count of immersed polygons. The construction is well known. We describe it briefly, and refer to [56, Section 13] for details; for other versions see [50, 14, 7].

Fix a Riemann surface structure on M, compatible with its symplectic orientation. Let $(\zeta_0, \ldots, \zeta_d)$, $d \geq 1$, be distinct boundary points on the closed disc $D \subset \mathbb{C}$, ordered in accordance with the boundary orientation. Consider holomorphic maps $u : D \setminus \{\zeta_0, \ldots, \zeta_d\} \to M$ which map the boundary sides to $(L_0, \ldots, L_d)$, and which extend continuously to D, taking each ζ_k to x_k. Each $(\zeta_0, \ldots, \zeta_d, u)$ has a virtual dimension, and we consider only ones of virtual dimension zero. The equivalence relation is that $(\zeta_0, \ldots, \zeta_d, u) \sim (\tilde{\zeta}_0, \ldots, \tilde{\zeta}_d, \tilde{u})$ if there is an automorphism $\phi : D \to D$ such that $\phi(\zeta_k) = \tilde{\zeta}_k$ and $u = \tilde{u} \circ \phi$. Denote the resulting space of equivalence classes by $\mathcal{M}(x_0, \ldots, x_d)$. Since constant maps u are excluded by our assumptions, automatic regularity [56, Lemma 13.2] ensures that each point of this space is regular, hence contributes ± 1 to $m(x_0, \ldots, x_d)$. To translate this into combinatorics, one observes first that points of $\mathcal{M}(x_0, \ldots, x_d)$ correspond bijectively to immersed polygons with sides on the L_k, and corners at x_k. To compute the sign with which each polygon contributes, we pick, for each L_k such that the *Spin* structure is nontrivial, a marked point $\circ_k \in L_k$ which is not an intersection point with any of the other curves in $\mathcal{L}$, as well as a trivialization of the *Spin* structure away from that point. If the L_k are oriented as in Figure 2(i), and if none of the points $\circ_k$ lie on the boundary of our immersed polygon, its contribution to $m(x_0, \ldots, x_d)$ is $+1$. The general rule is obtained from this by the following sign changes. The orientation of L_0 is irrelevant. Reversing the orientation of
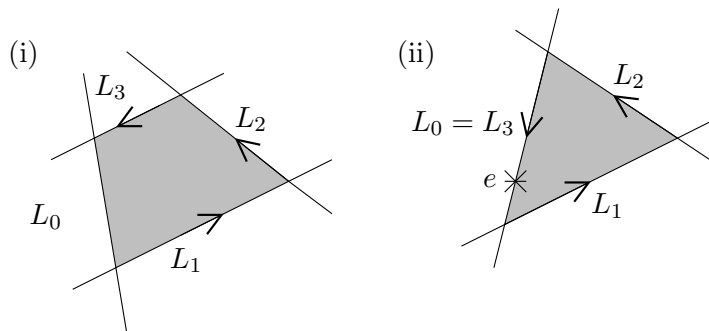
FIGURE 2

L_k, $0 < k < d$, changes the sign by $(-1)^{|x_k|}$. Reversing the orientation of L_d changes the sign by $(-1)^{|x_0|+|x_d|}$. Finally, for every time that the boundary of the polygon passes over one of the points $\circ_k$, for $0 \le k \le d$, we change the sign by (-1).

Lemma 7.1. *Let S be a compact oriented surface with boundary, and $w : S \to M$ a map which takes each boundary component of S to a balanced curve. Then the normalized area $[w^*\omega]/\mathrm{area}(M)$ and the normalized relative Chern class $c_1^{\mathrm{rel}}(w^*TM)/\chi(M)$, both lying in $H^2(S, \partial S; \mathbb{R})$, agree.*

This is a straightforward fact, which is worth mentioning because it leads to the basic compactness result for the spaces $\mathcal{M}(x_0, \ldots, x_d)$. This is a "monotonicity" style consideration, which we summarize briefly. Consider two maps $u, \tilde{u}$ which contribute to $m(x_0, \ldots, x_d)$. By gluing together their domains topologically, we get a map $w : S \to M$, where S is a genus zero surface with $d + 1$ boundary circles, and where the images of the boundary components lie on the balanced curves L_k; this is unique up to homotopy within the class of such maps. An index theory argument shows that the relative Chern class of w is zero, hence by Lemma 7.1 that $\int w^*\omega$ vanishes. But by construction, that implies that the areas $\int u^*\omega$ and $\int \tilde{u}^*\omega$ are the same. From this, a compactness argument shows that $\mathcal{M}(x_0, \ldots, x_d)$ is a finite set. In fact, one can translate this argument into combinatorics, where it becomes elementary.

Now consider two objects L_0, L_1 such that the underlying curves agree. In that case, we fix a metric and a Morse function f_{01} on that curve, with a unique minimum and maximum which are both distinct from the intersection points with any other curve in $\mathcal{L}$. Denote the minimum by e and the maximum by q. We then define the morphism space to be the Morse cochain space

$$(7.3) \qquad \mathrm{hom}_{\mathcal{F}(M)}(L_0, L_1) = CM^*(f_{01}) = \mathbb{C}e \oplus \mathbb{C}q.$$

Suppose that L_0, L_1 actually have the same orientation and isomorphic *Spin* structures. Then, the $\mathbb{Z}/2$-grading on (7.3) coincides with the ordinary Morse index, and the differential μ^1 is the Morse differential of f_{01}, hence vanishes. This has to be suitably modified for the other cases. For instance, suppose that the orientations agree, but that the *Spin* structure on L_1 differs from that on L_0 by twisting with a nontrivial double cover $\xi \to L_0$. Then, μ^1 is the Morse differential with twisted coefficients in $\xi \otimes_{\mathbb{Z}/2} \mathbb{C}$, hence acyclic; see [56, Example 13.5] for further discussion.

Here is a simple class of higher order compositions involving (7.3). Take objects $(L_0, \ldots, L_d)$, $d \geq 2$, such that $L_{i-1} = L_i$ agree as curves for a single $0 < i \leq d$, and where the underlying curves are otherwise distinct (which means that there are d distinct curves among them). Denote by $f_{i-1,i}$ the function used to define $\hom_{\mathcal{F}(M)}(L_{i-1}, L_i)$. Choose intersection points $x_0 \in L_0 \cap L_d$, $x_k \in L_{k-1} \cap L_k$ (for $1 \leq k \leq d$ with $k \neq i$), and a critical point x_i of $f_{i-1,i}$. Consider again holomorphic maps $u : D \setminus \{\zeta_0, \ldots, \zeta_d\} \to M$, but where the extension at ζ_i is now smooth, and satisfies

$$(7.4) \qquad\qquad u(\zeta_i) \in W^u(x_i) \subset L_i,$$

$W^u(x_i)$ being the unstable manifold of x_i for the gradient flow. The moduli space of such maps of virtual dimension zero is again a finite set $\mathcal{M}(x_0, \ldots, x_d)$, and an appropriate signed count of points in it yields integers $m(x_0, \ldots, x_d) \in \mathbb{Z}$ which are the coefficients of the composition map
(7.5)
$$\mu^d : CF^*(L_{d-1}, L_d) \otimes \cdots \otimes CM^*(f_{i-1,i}) \otimes \cdots \otimes CF^*(L_0, L_1) \longrightarrow CF^*(L_0, L_d).$$

Again, this can be translated into combinatorics, as follows. Suppose first that $x_i = e$, where (7.4) reduces to the open condition $u(\zeta_i) \neq q$. Then, the only case where $(\zeta^0, \ldots, \zeta^d, u)$ has virtual dimension zero is when $d = 2$, and u is the constant map at a point of $L_0 \cap L_2$. Next, consider the case $x_i = q$, where (7.4) says that $u(\zeta_i) = q$. There, $m(x_0, \ldots, x_d)$ can be computed by a signed count of immersed d-gons with an additional marked point on the appropriate boundary side, whose image is q.

There is another case which can be treated in the same way. Take objects $(L_0, \ldots, L_d)$, $d \geq 2$, such that the first d have pairwise distinct underlying curves, but that $L_0 = L_d$ as curves. Let $f_{0,d}$ be the function used to define $\hom_{\mathcal{F}(M)}(L_0, L_d)$. Choose a critical point x_0 of that function, and intersection points $x_k \in L_{k-1} \cap L_k$, $k > 0$. In this case, the condition analogous to (7.4) involves the stable manifold $W^s(x_0)$:

$$(7.6) \qquad\qquad u(\zeta_0) \in W^s(x_0) \subset L_0.$$

Again, we have an appropriate moduli space $\mathcal{M}(x_0, \ldots, x_d)$ and a signed count $m(x_0, \ldots, x_d) \in \mathbb{Z}$, which defines the map

$$(7.7) \qquad \mu^d : CF^*(L_{d-1}, L_d) \otimes \cdots \otimes CF^*(L_0, L_1) \longrightarrow CM^*(f_{0,d}).$$

Translation into combinatorics now works as follows: if $x_0 = q$, only constant triangles contribute, while for $x_0 = e$, we are counting immersed d-gons with an additional marked boundary point going through e. Having reduced the computations to combinatorics in principle, it remains to describe the signs of each polygon. We will concentrate on the cases that actually occur in our application, and give the resulting formulae without proof (verification is tedious but not difficult, following the argument from [56]).

Constant maps: Take two curves $L_0 \neq L_1$. The constant triangle at any point $x \in L_0 \cap L_1$ contributes to the products

$$(7.8) \qquad \begin{aligned} &\mu^2(x, e), \ \mu^2(e, x) : CF^*(L_0, L_1) \longrightarrow CF^*(L_0, L_1), \\ &\mu^2(x, x) : CF^*(L_1, L_0) \otimes CF^*(L_0, L_1) \longrightarrow CM^*(f), \end{aligned}$$

where f is the function associated to the pair (L_0, L_0). In all three cases, there are no contributions from non-constant triangles (note that in the last case, we know a priori for degree reasons that the product must be a multiple of q). Taking signs into account, the consequence is that

$$(7.9) \qquad \begin{aligned} &\mu^2(x, e) = x, \ \mu^2(e, x) = (-1)^{|x|} x, \\ &\mu^2(x, x) = (-1)^{|x|} q, \end{aligned}$$

where $|x| \in \mathbb{Z}/2$ is the degree of the generator $x \in CF^*(L_0, L_1)$.

Non-constant maps: Take the case $L_0 = L_d$ (where the two are assumed to have the same orientation and *Spin* structure). Choose intersection points $x_1, \ldots, x_d$, and consider the contribution of a polygon to $m(e, x_1, \ldots, x_d)$. If the orientations are as in Figure 2(ii) and our polygon avoids all the points $\circ_i$ which describe the *Spin* structures, then its contribution is $+1$. The rules for sign changes are the same as for Figure 2(i), taking into account the fact that $x_0 = e$ is even.

Finally, we turn to the question of defining μ^d, $d \geq 2$, in general, where any number of the curves $(L_0, \ldots, L_d)$ may coincide. One chooses generators $(x_0, \ldots, x_d)$ of the associated Floer or Morse complexes as before, but the definition of the relevant moduli space $\mathcal{M}(x_0, \ldots, x_d)$ is somewhat more involved. A point of this moduli space consists of the following data (similar to, but not quite the same as, the "clusters" from [17]):

> First, we have a planar tree $T \subset \mathbb{R}^2$ with $d + 1$ semi-infinite edges, and with vertices which are at least trivalent. The components of $\mathbb{R}^2 \setminus T$ should be labeled by $(L_0, \ldots, L_d)$ in

accordance with their natural cyclic ordering. We require that
the two regions separated by any finite edge of T are marked
with objects which have the same underlying curve; see Figure
3(i).

Next, for every vertex v of T of valency $|v|$, we have a
collection of marked points $\{\zeta_0^v, \ldots, \zeta_{|v|-1}^v\}$ on the boundary
of the disc, and a holomorphic map $u_v : D \backslash \{\zeta_0^v, \ldots, \zeta_{|v|-1}^v\} \to$
M, extending continuously over D. The boundary conditions
for that map are given by the labels L_k in components of
$\mathbb{R}^2 \setminus T$ adjacent to v; see Figure 3(ii) for an illustration.

Now take a finite edge of T. Associated to this edge are its
two endpoints $v_{\pm}$, and also marked points $\zeta_{k_{\pm}}^{v_{\pm}}$. By assump-
tion, the two objects (L_i, L_j) labeling the components of $\mathbb{R}^2 \backslash T$
adjacent to our edge share the same underlying curve. Let $f_{i,j}$
be the Morse function used to construct $\hom_{\mathcal{F}(M)}(L_i, L_j)$. We
then ask that there should be a gradient flow line of $f_{i,j}$ of
some finite nonzero length, which goes from $u(\zeta_{k_-}^{v_-})$ to $u(\zeta_{k_+}^{v_+})$.
The implicit notational convention here is indicated by the di-
rection of the arrows in Figure 3(ii).

Finally, consider an infinite edge of T, with its unique asso-
ciated endpoint v and marked point ζ_k^v. If the curves labeling
the two components of $\mathbb{R}^2 \backslash T$ adjacent to our edge are distinct,
we ask that $u(\zeta_k^v)$ should lie at the relevant intersection point
x_i. Otherwise, we impose conditions as in (7.6) and (7.4).

It is easy to see that this generalizes the previous discussion: in all cases
we had considered before, the requirements only allow the star-shaped tree
T (with a single vertex, hence no finite edges). Unfortunately, in general
the spaces $\mathcal{M}(x_0, \ldots, x_d)$ are not regular (due to the failure of gradient flow
lines to intersect transversally, and to the appearance of constant holomorphic
maps which have excess dimension). Hence, one has to perturb this initial
definition either virtually, which leads to the construction of appropriate vir-
tual fundamental chains on the compactifications $\bar{\mathcal{M}}(x_0, \ldots, x_d)$ [24], or else
by perturbing the gradient flow equations and holomorphic map equations
themselves, in the manner of [56]. Fortunately, the only case of this more
complicated formalism which we need to determine explicitly is the product
structure on $\hom_{\mathcal{F}(M)}(L, L)$ for a single object L, which is given by

$$(7.10) \qquad \mu^2(e, e) = e, \ \mu^2(q, e) = q, \ \mu^2(e, q) = -q.$$

In this specific case, this also follows from the general fact that the product
reproduces the ordinary cup product on $H^*(L; \mathbb{C})$.
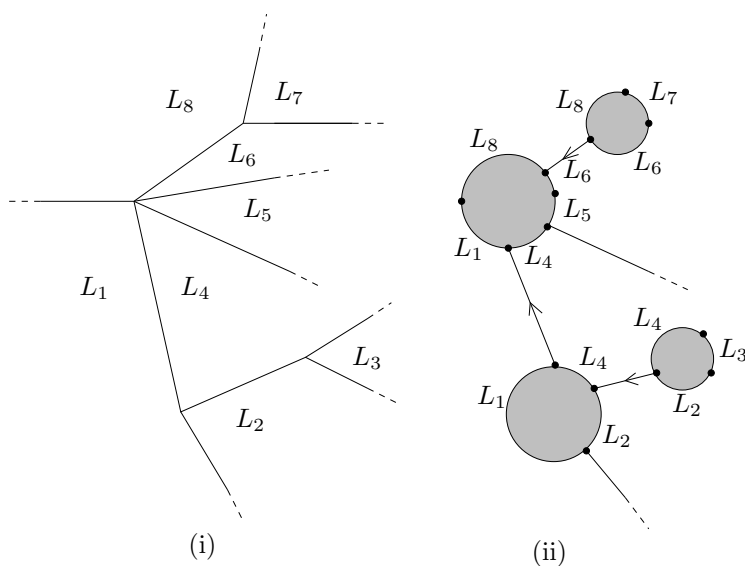
Coincidences of the underlying curves:
$L_1 = L_2 = L_4 = L_5, \; L_6 = L_8$

FIGURE 3

Remark 7.2. We want to quickly mention some other definitions of the Fukaya category. [24] also uses a Morse-Bott method, but where singular cohomology replaces Morse cohomology. In contrast, [56] uses Hamiltonian perturbations of the holomorphic map equation to treat $\hom_{\mathcal{F}(M)}(L_0, L_1)$ on the same footing for all pairs (L_0, L_1). The equivalence of any two approaches can be proved by constructing a "mixed" Fukaya category which contains two copies of each object, to which the two different methods are applied; compare the discussion in [56, Section 10a]. Strictly speaking, the only substantial property of the Fukaya category which we have borrowed from the literature is the existence of exact triangles (6.3), which quotes [56]. However, the argument leading to those triangles involves only the objects $(L_0, L_1, \tau_{L_1}(L_0))$, which moreover can be perturbed to be in general position. In that form, it carries over easily to any other framework, such as the one adopted here.

8. Gradings

The lack of an integer grading on $\mathcal{F}(M)$ is unavoidable, since it is directly related to the nonvanishing of $c_1(M)$. Nevertheless, one can partially improve the situation by thinking of $c_1(M)$ as being supported at finitely many points. Namely, let η^r be a nonzero meromorphic section of the r-th power of the canonical bundle $T^*M^{\otimes r}$, for $r \neq 0$. Let $D \subset M$ be the set of its zeros and poles, with the order of vanishing written as $\mathrm{ord}(\eta^r, z) \in \mathbb{Z}$ (a negative order signifies a pole). For every oriented $L \subset M \setminus D$ we get a map $L \to S^1$, defined by

$$(8.1) \qquad x \longmapsto \frac{\eta^r(X^{\otimes r})}{\|\eta^r(X^{\otimes r})\|}$$

where $X \in TL_x$ is nonzero and points in positive direction. An $1/r$-grading of L is a lift $a : L \to \mathbb{R}$ of this map. Let $\mathcal{F}(M, D)$ be a version of the Fukaya category, whose objects are Lagrangian submanifolds as before, with the added condition that they lie in $M \setminus D$, and moreover should come equipped with $1/r$-gradings. Nothing else changes, which in particular means that there is a full and faithful A_∞-functor $\mathcal{F}(M, D) \to \mathcal{F}(M)$.

In the presence of $1/r$-gradings, the generators x of $\hom_{\mathcal{F}(M)}(L_0, L_1)$ acquire additional integer indices $i^r(x) \in \mathbb{Z}$. In the case of (7.3) where the two underlying curves agree, the difference of their $1/r$-gradings is constant, $a_1(x) - a_0(x) = \pi d$ for some $d \in \mathbb{Z}$. One then sets $i^r(e) = d$, $i^r(q) = d + r$. In the other situation (7.1), let $\alpha \in (0, \pi)$ be the angle with which our Lagrangian submanifolds meet at x, counted clockwise from $TL_{0,x}$ to $TL_{1,x}$, and let $a_0(x), a_1(x)$ be the $1/r$-gradings at that point. Define

$$(8.2) \qquad i^r(x) = \frac{r\alpha(x) + a_1(x) - a_0(x)}{\pi}.$$

If r is odd, which is when orientations actually matter in (8.1), the parity of i^r agrees with the previously used $\mathbb{Z}/2$ grading.

Consider a moduli space $\mathcal{M}(x_0, \dots, x_d)$ which enters into the definition of the A_∞-structure of $\mathcal{F}(M, D)$. For simplicity, we assume that the Lagrangian submanifolds $(L_0, \dots, L_d)$ involved are pairwise distinct (similar arguments would apply in the more general case). Take a point $(\zeta^0, \dots, \zeta^d, u)$ in our moduli space. Since the boundary of u lies in $M \setminus D$, for any $z \in D$ there is a well-defined degree $\deg(u, z)$, namely the multiplicity with which u hits z. It is nonnegative, and vanishes iff $u^{-1}(z) = \emptyset$. Using the fact that our point has virtual dimension zero, and the index formula [56, Section 11], one sees that

$$(8.3) \quad i^r(x_0) - i^r(x_1) - \cdots - i^r(x_d) = r(2 - d) + 2 \sum_{z \in D} \mathrm{ord}(\eta^r, z) \deg(u, z).$$

Suppose for simplicity that η vanishes to the same order $m = \text{ord}(\eta^r, z) > 0$ at every point $z \in D$. One way to encode (8.3) is to equip $CF^*(L_0, L_1)$ with the grading given by the i^r, and then to write the composition maps in $\mathcal{F}(M, D)$ as

$$(8.4) \qquad\qquad \mu^i = \mu_0^i + \mu_1^i + \cdots$$

where the subscript denotes the total degree of u over D, and the term μ_k^i has degree $r(2-i) + 2km$ with respect to the $i^r(x_k)$. Equivalently, one could add a formal variable $\hbar$ of degree $2m$ to form $CF^*(L_0, L_1)[[\hbar]]$, and then get a homogeneous map $\mu_0^i + \hbar\mu_1^i + \cdots$ of degree $r(2-i)$. The sum in (8.4) is of course finite when applied to any given $(L_0, \dots, L_d)$. Correspondingly, each graded piece of $CF^*(L_0, L_1)[[\hbar]]$ only contains finitely many powers of $\hbar$.

9. Orbifolds

Part of our argument will involve working equivariantly with respect to the action of a finite group on the target surface. Even though it makes no fundamental difference, it can be more intuitive to think of this as working on the orbifold quotient, so we'll give a short discussion which takes this point of view into account. Take M as before. Let Γ be a finite group acting effectively on M by orientation-preserving diffeomorphisms, and let D be the finite set of points where the action is not free. Write $\bar{M} = M/\Gamma$ for the orbifold quotient, and $\bar{D}$ for its finite set of orbifold points. We can choose ω and θ to be Γ-invariant.

Take an embedded balanced curve $L \subset M \setminus D$, such that the collection $\{\gamma(L) : \gamma \in \Gamma\}$ is in general position, hence can be assumed to be part of our family $\mathcal{L}$. Then, the image curve $\bar{L} \subset \bar{M} \setminus \bar{D}$ is immersed and has only transversal double points. Equip L with an orientation, a *Spin* structure, and a Morse function f as in Section 7. We then define

$$(9.1) \qquad\qquad CF^*(\bar{L}, \bar{L}) = CM^*(f) \oplus \bigoplus_{\gamma \neq 1} CF^*(L, \gamma(L)).$$

Denote the summands by $CF^*(\bar{L}, \bar{L})^\gamma$, where the first term corresponds to $\gamma = 1$. We have formulated the definition in terms of L for convenience. In terms of $\bar{L}$ itself, we have two generators corresponding to the classical Morse chain complex, as well as a pair of generators for each self-intersection point (this is a general feature of Floer cohomology for immersed Lagrangian submanifolds, see [2, 3]). Suppose that $(\bar{x}_0, \dots, \bar{x}_d)$ are generators, belonging to the $(\gamma_0, \dots, \gamma_d)$ summands of (9.1), and $(x_0, \dots, x_d)$ their obvious lifts to M. The associated moduli space $\mathcal{M}(\bar{x}_0, \dots, \bar{x}_d)$ is empty if $\gamma_0 \neq \gamma_1\gamma_2 \cdots \gamma_d$, and

can be identified with $\mathcal{M}(x_0, x_1, \gamma_1(x_2), \gamma_1\gamma_2(x_3), \ldots, \gamma_1 \cdots \gamma_{d-1}(x_d))$ in the remaining case. Here, the assumption is that the Riemann surface structure, and other auxiliary choices, are made equivariantly. In simple situations where points of that moduli space are holomorphic discs, the image of any such disc under the quotient map $M \to \bar{M}$ is an "orbifold holomorphic disc" (meaning that it has appropriate ramification at all points of $\bar{D}$), and conversely all orbifold holomorphic discs lift to appropriate objects in M.

Remark 9.1. Transversality may seem to be an issue in the equivariant context, but a quick reflection shows that this is not the case in the present context, where the group action is free on the set of objects $\{\gamma(L)\}$. If one uses virtual perturbation methods as in [24], multivalued perturbations are built into the framework, and that naturally allows equivariance with respect to any given finite group action. Alternatively, let's consider using explicit perturbations as in [56]. Such perturbations are given by inhomogeneous terms which vary on the domain of our holomorphic maps, which means that we are looking at equations of the class

$$(9.2) \qquad\qquad \bar{\partial} u = \nu(z, u(z)).$$

The group Γ acts on the target space, but leaves z invariant, and that allows enough freedom even if one takes ν to be Γ-invariant and to vanish near D. This is also easy to see if one thinks in terms of $\bar{M}$.

It should be clear from the definition that there is a simple relationship between the A_∞-structures obtained by looking at $\bar{L} \subset \bar{M}$ and at all preimages $\gamma(L) \subset M$. To state this in a simple algebraic way, suppose that Γ is abelian, and consider its character group $G = \mathrm{Hom}(\Gamma, \mathbb{C}^*)$. We then have an action of G on $CF^*(\bar{L}, \bar{L})$, defined by $g \cdot x = g(\gamma)x$ for $x \in CF^*(\bar{L}, \bar{L})^\gamma$. By construction, the A_∞-structure is equivariant with respect to this action, and we have

$$(9.3) \qquad\qquad \bigoplus_{\gamma_0, \gamma_1} CF^*(\gamma_0(L), \gamma_1(L)) = CF^*(\bar{L}, \bar{L}) \rtimes G,$$

compatibly with the A_∞-structures, where $\rtimes$ is the semidirect product. This is familiar from other instances of mirror symmetry; see for instance [54]. Conversely, the left hand side of (9.3) carries an action of Γ, whose invariant part can be identified with $CF^*(\bar{L}, \bar{L})$.

Remark 9.2. Suppose that $\bar{M}$ has genus zero. Then $\Gamma = H_1^{\mathrm{orb}}(\bar{M})$ is a finite abelian group, and one can write $\bar{M} = M/\Gamma$. The associated group action can be interpreted in more familiar, if somewhat more abstract, terms

as follows. Suppose that we introduced a larger version of the Fukaya category of the orbifold, in which our curves carry flat $\mathbb{C}^*$-bundles. Then the Picard groupoid, which consists of flat $\mathbb{C}^*$-bundles defined on the entire orbifold, acts on that category by tensor product. In particular, if we have a curve $\bar{L}$ which is fixed under this action in a suitable sense, which means that its class in $H_1^{\mathrm{orb}}(\bar{M})$ is zero, then $G = H_{\mathrm{orb}}^1(\bar{M}; \mathbb{C}^*)$ acts on its endomorphism space $CF^*(\bar{L}, \bar{L})$.

10. The genus two case

From now on, we return to the specific case where M has genus 2. Initially, it will be convenient to represent this as a double cover of $S^2 = \mathbb{C} \cup \{\infty\}$ branched over six points, which are the fifth roots of unity and 0. A nonseparating simple closed curve in M which is invariant under the hyperelliptic involution projects to an embedded path in S^2 which connects two of the branch points. Similarly, the Dehn twist along the curve projects to the half-twist along the associated path. We start with the configuration of five curves in M whose images in S^2 form the pentagram, Figure 4(i).
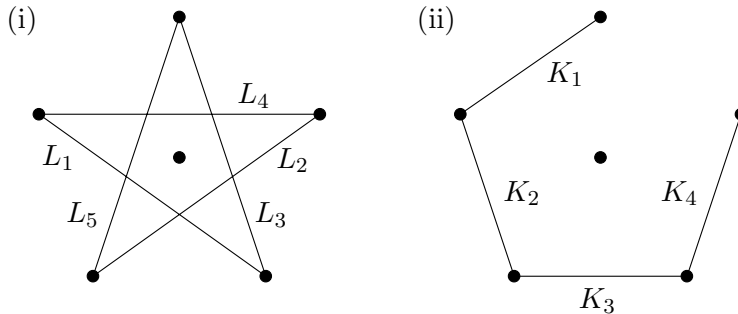

FIGURE 4

Lemma 10.1. *The curves* $\{L_1, \ldots, L_5\}$, *equipped with nontrivial Spin structures, split-generate* $D^\pi \mathcal{F}(M)$.

Proof. Consider first the collection $\{K_1, K_2, K_3, K_4\}$ from Figure 4(ii). The Dehn twists τ_{K_i} define a homomorphism from the braid group Br_5 to the mapping class group of M. Its kernel is infinite cyclic and generated by
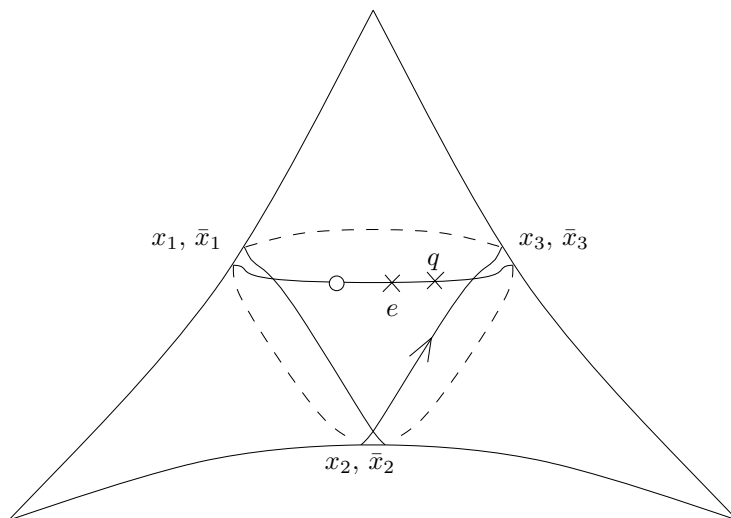
FIGURE 5

the central element $\Delta^4 \in Br_5$, which in particular means that its image $(\tau_{K_4} \cdots \tau_{K_1})^{10}$ is isotopic to the identity [12]. Lemma 6.5 shows that the K_i split-generate $D^{\pi}\mathcal{F}(M)$. On the other hand, one easily checks by hand that $\tau_{L_5} \cdots \tau_{L_1}(K_2)$ is isotopic to $K_2[1]$, and analogously for the other K_i. By Lemma 6.4, each K_i is split-generated by $\{L_1, \ldots, L_5\}$, which completes the argument. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

For the main computation we switch to a different picture. Take the action of $\Sigma = \mathbb{Z}/5$ on M which projects to the rotational action on S^2. The orbifold quotient $\bar{M} = M/\Sigma$ is a sphere with three orbifold points $\bar{D}$. Each $L = L_i$ projects to the same immersed curve $\bar{L} \subset \bar{M}$. We will use the previously introduced techniques to partially compute the A_∞-structure on $CF^*(\bar{L}, \bar{L})$. Note that all such computations can be equivalently thought of as being carried out on M, which means that the orbifold structure does not really introduce any new technical issues.

Generators: We have the Morse-theoretical generators e (minimum, even) and q (maximum, odd), together with a pair of generators coming from each self-intersection point, which we denote by $\bar{x}_k$ (even) and x_k (odd), see Figure 5 (we have perturbed that picture slightly to make the self-intersections more visible; the more natural picture would be the one with full symmetry between the front and back faces). Take $\Gamma = H_1^{\mathrm{orb}}(\bar{M})$, which one thinks of as the quotient of $(\mathbb{Z}/5)^3$ by its diagonal subgroup $\mathbb{Z}/5$. Since the class of our immersed curve in Γ is trivial, the generators of the Floer cochain complex come labeled

by weights which are elements of Γ. Moreover, there is a nontrivial holomorphic section η^3 of $(T^*\bar{M})^{\otimes 3}$, unique up to nonzero scalars, which has a double zero at each point of $\bar{D}$ (this is in fact the same as a meromorphic section of $(T^*S^2)^{\otimes 3}$ with double poles at our three points, with the different order due to considering it in orbifold charts: if $z = w^5$, then $z^{-2}dz^3 = 5^3 w^2 dw^3$). As a consequence, the generators acquire additional integer indices. All this data can be listed as follows:

(10.1)

generator	e	x_1	x_2	x_3
weight	$(0,0,0)$	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$
index	0	1	1	1

generator	$\bar{x}_1$	$\bar{x}_2$	$\bar{x}_3$	q
weight	$(0,1,1)$ $= (-1,0,0)$	$(1,0,1)$ $= (0,-1,0)$	$(1,1,0)$ $= (0,0,-1)$	$(1,1,1)$ $= (0,0,0)$
index	2	2	2	3

We know that the A_∞-structure is homogeneous with respect to weights. This immediately implies that $\mu^1 = 0$. Moreover, we have a decomposition as in (8.4), where μ_k^i has degree $6 - 3i + 4k$ with respect to the indices above. Concretely, μ_0^i counts the contributions from polygons in $\bar{M} \setminus \bar{D}$; the next term μ_1^i, that from polygons which meet $\bar{D}$ exactly once and have fivefold ramification at that point (which is the minimal order prescribed by the orbifold structure).

Triangles: For degree reasons, $\mu_k^2 = 0$ for all $k > 0$. According to (7.9) and (7.10), the contributions of the constant triangles are

$$
\begin{aligned}
&\mu^2(x_i, e) = x_i = -\mu^2(e, x_i), \\
&\mu^2(\bar{x}_i, e) = \bar{x}_i = \mu^2(e, \bar{x}_i), \\
&\mu^2(q, e) = q = -\mu^2(e, q), \\
&\mu^2(q, q) = 0, \\
&\mu^2(x_i, \bar{x}_i) = q = -\mu^2(\bar{x}_i, x_i).
\end{aligned}
$$

(10.2)

There are six (if one counts the ordering of their corners; otherwise, only two) non-constant triangles avoiding $\bar{D}$. To determine the sign of their contribution, we need to choose generic points $\circ$ on L, which represent the nontrivial *Spin* structure. Those being as in Figure 5, we get

$$
\begin{aligned}
&\mu_0^2(x_1, x_2) = \bar{x}_3 = -\mu_0^2(x_2, x_1), \\
&\mu_0^2(x_2, x_3) = \bar{x}_1 = -\mu_0^2(x_3, x_2), \\
&\mu_0^2(x_3, x_1) = \bar{x}_2 = -\mu_0^2(x_1, x_3).
\end{aligned}
$$

(10.3)

The triangle on the front part of Figure 5 goes through the e, hence can be thought of as a holomorphic map from the four-punctured disc which is smooth at one of the marked points. The resulting contribution is

$$(10.4) \qquad \mu_0^3(x_3, x_2, x_1) = -e.$$

All other expressions $\mu_0^3(x_{i_3}, x_{i_2}, x_{i_1})$ are zero: any such product can only be a multiple of e, for degree reasons, but the relevant spaces $\mathcal{M}(e, x_{i_3}, x_{i_2}, x_{i_1})$ are empty.

Pentagons: There are six "pentagons" hidden in the picture. Each of them hits exactly one of the points of $\bar{D}$ and has fivefold ramification there, and no ramification elsewhere, which means that it lifts to a genuine immersed pentagon in M. Figure 6 shows the image of one of the pentagons in the orbifold picture. Figure 7 shows two pentagons lifted to M and then projected to S^2 under the hyperelliptic quotient (the dots in the left hand picture are the corners, and the numbers local degrees; the right hand side shows a smoothed version of the boundary curve). We are not interested in their effect on μ_1^4 (this happens to cancel, but its vanishing is in fact a consequence of the previous computations and the A_∞ structure equations, hence yields no new information). Instead, we take the three pentagons whose boundary goes through e, and determine their contributions to μ_1^5, namely:

$$(10.5) \qquad \mu_1^5(x_1, x_1, x_1, x_1, x_1) = \mu_1^5(x_3, x_3, x_3, x_3, x_3) = -e,$$
$$\mu_1^5(x_2, x_2, x_2, x_2, x_2) = e.$$

The other $\mu_1^5(x_{i_5}, x_{i_4}, x_{i_3}, x_{i_2}, x_{i_1})$, where the indices are not all the same, vanish.

Identify $CF^*(\bar{L}, \bar{L}) \cong \Lambda(V)$ by mapping the generators as follows:

$CF^*(\bar{L}, \bar{L})$	e	x_1	x_2	x_3	$\bar{x}_1$	$\bar{x}_2$	$\bar{x}_3$	q
$\Lambda(V)$	1	$-\xi_1$	ξ_2	$-\xi_3$	$\xi_2 \wedge \xi_3$	$\xi_1 \wedge \xi_3$	$\xi_1 \wedge \xi_2$	$-\xi_1 \wedge \xi_2 \wedge \xi_3$

Then, the data above fit precisely into Proposition 5.2, proving that $CF^*(\bar{L}, \bar{L})$ is A_∞-isomorphic to $\mathcal{A}$. Moreover, this isomorphism is equivariant with respect to the action of $G = \mathrm{Hom}(\Gamma, \mathbb{C}^*) \subset SL(V)$. The orbifold covering $M \to \bar{M}$ is classified by a surjective homomorphism $\Gamma \to \Sigma$, whose dual is the subgroup $Z \subset G$. In view of (9.3) we then inherit an A_∞-isomorphism

$$(10.6) \qquad \bigoplus_{i,j} CF^*(L_i, L_j) \cong \mathcal{A} \rtimes Z.$$

We can consider the left hand as an A_∞-category with a single object, which is the formal direct sum of the L_i. Using Lemma 10.1 we arrive at the following description of the Fukaya category M on the derived level:

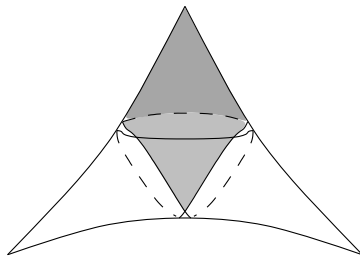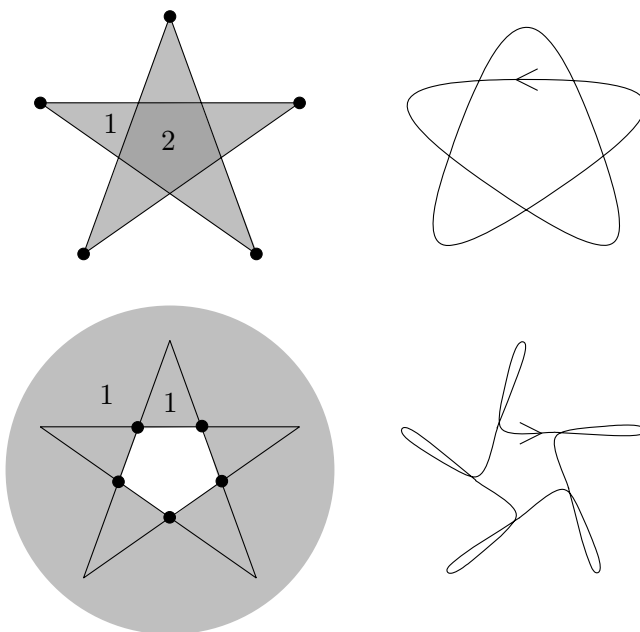**Corollary 10.2.** $D^\pi \mathcal{F}(M) \cong D^\pi(\mathcal{A} \rtimes Z)$.

FIGURE 6

FIGURE 7

11. Koszul duality

In this section, we again allow $V = \mathbb{C}^n$ for any n. Consider the space of differential forms $\Omega(V) = \mathbb{C}[V] \otimes \Lambda(V^\vee)$, with the grading reversed (negative), and equip it with the differential ι_η given by contraction with the Euler vector field $\eta = \sum_i v_i \xi_i$. Consider the dga

$$(11.1) \qquad\qquad B = \mathrm{Hom}_{\mathbb{C}[V]}(\Omega(V), \Omega(V))$$

with the induced differential ∂. Since $\Omega(V)$ is just the standard free Koszul resolution of the simple $\mathbb{C}[V]$-module $\mathbb{C}$, we have $H(B) = \mathrm{Ext}^*_{\mathbb{C}[V]}(\mathbb{C}, \mathbb{C}) \cong \Lambda(V)$. To write things down on the cochain level, let's identify

$$(11.2) \qquad B = \mathrm{Hom}_{\mathbb{C}}(\Lambda(V^\vee), \Omega(V)) = \Omega(V) \otimes \Lambda(V),$$

and correspondingly write the differential as

$$\partial : \Omega(V) \otimes \Lambda(V) \longrightarrow \Omega(V) \otimes \Lambda(V),$$

$$(11.3) \qquad \partial(f\beta \otimes \theta) = \sum_k v_k f\, \iota_{\xi_k}\beta \otimes \theta + (-1)^{|\beta|-1} v_k f\beta \otimes \xi_k \wedge \theta.$$

The projection $p : \Omega(V) \otimes \Lambda(V) \to \Lambda(V)$, where the target $\Lambda(V)$ carries the zero differential, is a chain homomorphism. In converse direction we have the inclusion

$$i : \Lambda(V) \longrightarrow \Omega(V) \otimes \Lambda(V),$$

$$(11.4) \qquad i(\theta) = \sum_{p \geq 0} \sum_{j_1 < \cdots < j_p} dv_{j_1} \wedge \cdots \wedge dv_{j_p} \otimes \xi_{j_p} \wedge \cdots \wedge \xi_{j_1} \wedge \theta.$$

Finally, there is a chain homotopy h between the identity and $p\,i$:

$$(11.5)$$
$$h : \Omega(V) \otimes \Lambda(V) \longrightarrow \Omega(V) \otimes \Lambda(V),$$

$$h(f\beta \otimes \theta) = 0 \ \text{ if } f\beta \text{ is a multiple of } 1, \text{ and otherwise}$$

$$= \sum_{p \geq 0} \frac{p!}{w(w+1)\cdots(w+p)} \sum_{j_1 < \cdots < j_p} df \wedge \beta \wedge dv_{j_1} \wedge \cdots \wedge dv_{j_p} \otimes \xi_{j_p} \wedge \cdots \wedge \xi_{j_1} \wedge \theta$$

where $w = r + s$ for $f \in \mathrm{Sym}^r(V^\vee)$ and $\beta \in \Lambda^s(V^\vee)$ (w is the weight of $f\beta$ with respect to the diagonal $\mathbb{C}^*$-action on $V^\vee$). Moreover, it is tacitly assumed that $h(f\beta \otimes \gamma)$ vanishes if $w = 0$. The entire structure constructed above also satisfies the so-called side conditions

$$(11.6) \qquad h^2 = 0, \ \ p\,h = 0, \ \ h\,i = 0.$$

Returning to the original definition (11.1), one could also write $i(\theta)$ as the action of θ by contraction, $\iota_\theta : \Omega(V) \to \Omega(V)$. From this, it follows that i is a map of differential graded algebras. Hence B is formal. This is the prototypical instance of a more general homogeneity phenomenon, known as Koszul duality (see for instance [11]). We will be interested in a deformed version of this statement. Namely, take a one-form $\gamma = \sum_k g_k dv_k \in \Omega^1(V)$, and change the differential on $\Omega(V)$ to $\iota_\eta - \gamma \wedge \cdot$, which of course reduces the grading to $\mathbb{Z}/2$. The square of the new differential is multiplication with the function

$$(11.7) \qquad W = -\gamma(\eta) \in \mathbb{C}[V],$$

which is a central element. Hence, the induced differential on B does indeed square to zero. We denote this differential by $\tilde{\partial}$, and the resulting dga structure by $\mathcal{B}$. Explicitly,

$$(11.8) \qquad (\tilde{\partial} - \partial)(f\beta \otimes \theta) = -f\,\gamma \wedge \beta \otimes \theta + (-1)^{|\beta|-1} \sum_k g_k f\beta \otimes \iota_{dv_k}\theta.$$

Starting from this, the Homological Perturbation Lemma [40] constructs an induced $\mathbb{Z}/2$-graded A_∞-structure $\mathcal{A}$ on $A = \Lambda(V)$, together with an A_∞-quasi-isomorphism from that structure to $\mathcal{B}$. An explicit formula for the differential is

$$(11.9) \qquad \mu^1(a) = (-1)^{|a|}p(\tilde{\partial} - \partial)i(a) + (-1)^{|a|}p(\tilde{\partial} - \partial)h(\tilde{\partial} - \partial)i(a) + \cdots$$

The general formula for μ^d is as a sum over ribbon trees with a root and d leaves, whose vertices may have valencies 2 or 3. Take such a tree and orient it in a way pointing from the leaves to the root, then attach an operation to each vertex and edge, as follows:
$$(11.10)$$

$$\begin{cases} \text{for a bivalent vertex,} & b \mapsto (-1)^{|b|}(\tilde{\partial} - \partial)(b) : B \to B, \\ \text{for a trivalent vertex,} & (b_2, b_1) \mapsto (-1)^{|b_1|}b_2b_1 : B \otimes B \to B, \\ \text{for a finite edge,} & b \mapsto (-1)^{|b|-1}h(b) : B \to B, \\ \text{for a semi-infinite incoming edge,} & a \mapsto i(a) : A \to B, \\ \text{for a semi-infinite outgoing edge,} & b \mapsto p(b) : B \to A. \end{cases}$$

Then compose these operations as prescribed by the tree itself, to get a multilinear map $A^{\otimes d} \to A$. For instance, the terms in (11.9) arise from the linear trees (chains of bivalent vertices, with one semi-infinite incoming and another semi-infinite outgoing end). Because both $\tilde{\partial} - \partial$ and h decrease the grading, trees containing sufficiently long chains contribute zero, and therefore all resulting sums are finite.

Lemma 11.1. *Fix some $r \geq 0$. Suppose that all g_k lie in $F_{r-1}\mathbb{C}[V]$, which means that they do not contain monomials of order $< r-1$. Then the resulting A_∞-structure on A agrees with the trivial (formal) one up to order $r - 1$.*

To see this, consider the grading of B by the order of its symmetric algebra part. This grading is preserved by the product structure, decreased by one under h, and increased by at least $r - 1$ under $\tilde{\partial} - \partial$. Hence, the multilinear map arising from a tree with d leaves and k bivalent vertices can be nonzero only if $k \leq (d-2)/(r-2)$. On the other hand, trees with $d \geq 3$ and $k = 0$ contribute zero, because $h(i(a_2)i(a_1)) = h(i(a_2 a_1)) = 0$.

Remark 11.2. The construction above gives explicit formulae for the entire A_∞-deformation of the exterior algebra induced by the superpotential W. These could be useful in other situations, such as ones considered in [5]. It should also be mentioned that there is another possible way of obtaining such formulae, namely by applying (3.5) and Kontsevich formality. I have not investigated the relation between the two approaches.

We now return to the usual special case, where $V = \mathbb{C}^3$ and W is as in (2.1). Take

$$(11.11) \qquad g_1 = -v_2 v_3/3 + v_1^4, \;\; g_2 = -v_1 v_3/3 + v_2^4, \;\; g_3 = -v_1 v_2/3 + v_3^4.$$

Proposition 11.3. *The resulting A_∞-algebra $\mathcal{A}$, with the obvious action of $G \subset GL(V)$, belongs to the quasi-isomorphism class singled out in Proposition 5.2.*

For this, it is convenient to make some temporary changes. Let's first modify the grading of B by giving the summand $\mathrm{Sym}^i(V^\vee) \otimes \Lambda^j(V^\vee) \otimes \Lambda^k(V)$ degree $2i - j + k$. The product is still compatible with this grading, but ∂ has degree 3 and h has degree -3. To make the remaining term $\tilde\partial - \partial$ have degree 3 as well, introduce a formal parameter $\hbar$ of degree -4, and write $g_1 = -v_2 v_3/3 + \hbar v_1^4$ and similarly for the other g_k. Then, the resulting A_∞-operations can be written as sums of terms μ_k^d, coming with $\hbar^k$, of degrees $6 - 3d + 4k$. Moreover, since $\tilde\partial - \partial$ is G-equivariant, and all the other data are equivariant for the entire group $GL(V)$, the A_∞-operations inherit G-symmetry.

Lemma 11.1 shows that μ^1 vanishes and μ^2 is the standard wedge product. It remains to compute the two higher order compositions in (5.11). This is elementary, using the explicit tree summation formulae discussed above, and the result is precisely as required. We omit the details, referring instead to [53].

12. Matrix factorizations

For the duration of this section, we consider the more general case where $V = \mathbb{C}^n$ for any n, and $W \in \mathbb{C}[V]$ is a polynomial such that the hypersurface $W^{-1}(0)$ has a single singular point, which lies at the origin. Orlov [48] associates to this hypersurface the category

$$(12.1) \qquad D_{\mathrm{sing}}^b(W^{-1}(0)) = D^b(W^{-1}(0))/\mathrm{Perf}(W^{-1}(0)).$$

The notation here is that $D^b(W^{-1}(0))$ is the bounded derived category of coherent sheaves; $\mathrm{Perf}(W^{-1}(0))$ the full triangulated subcategory of perfect

complexes; and the quotient is localization with respect to the class of morphisms whose cones lie in $\mathrm{Perf}(W^{-1}(0))$. By definition $D^b_{\mathrm{sing}}(W^{-1}(0))$ is triangulated. A deeper fact, based on cohomological properties of hypersurfaces, is that it is $\mathbb{Z}/2$-graded, which as before means that the twofold shift is isomorphic to the identity. Finally, the categories $D^b_{\mathrm{sing}}(W^{-1}(0))$ are not split-closed in general, but we can take the split-closure $D^\pi_{\mathrm{sing}}(W^{-1}(0))$, which is again naturally triangulated [9].

Lemma 12.1. $D^\pi_{\mathrm{sing}}(W^{-1}(0))$ *is split-generated by the skyscraper sheaf at the origin,* $\mathcal{S}_{W^{-1}(0),0}$.

Proof. A result of Orlov [47] says that any object in that category is a direct summand of the image of an object of $D^b_{\mathrm{sing}}(W^{-1}(0))$ whose cohomology sheaves are supported on the singular locus of $W^{-1}(0)$, in this case the origin. On the other hand, any such complex can be built from shifted copies of $\mathcal{S}_{W^{-1}(0),0}$ through repeated mapping cones. Since the projection functor $D^b(W^{-1}(0)) \to D^b_{\mathrm{sing}}(W^{-1}(0))$ is exact by definition, this behaviour transfers to the quotient category. $\qquad\square$

Matrix factorizations [21], which historically predate $D^b_{\mathrm{sing}}(W^{-1}(0))$, can be used to construct a chain level model for that category. With W as before, a matrix factorization is a $\mathbb{Z}/2$-graded projective $\mathbb{C}[V]$-module E together with an odd $\mathbb{C}[V]$-linear differential δ_E such that $\delta_E^2 = W \cdot \mathrm{id}_E$. Matrix factorizations form a $\mathbb{Z}/2$-graded differential graded category $MF(W)$, and this admits mapping cones, hence the cohomological category $H^0(MF(W))$ is naturally triangulated.

Theorem 12.2 (Orlov [48, Theorem 3.9])**.** *There is an equivalence of triangulated categories,*

$$(12.2) \qquad\qquad H^0(MF(W)) \cong D^b_{\mathrm{sing}}(W^{-1}(0)).$$

On the level of objects, this takes a matrix factorization E to the coherent sheaf corresponding to the $\mathbb{C}[V]/W$-module $\mathrm{coker}(\delta_E^1 : E^1 \to E^0)$.

We will now make the connection with the material from the previous section. Suppose that W is written in the form (11.7), and take $E = \Omega(V)$ with its natural $\mathbb{Z}/2$-grading, and with the differential $\delta_E = \iota_\eta - \gamma \wedge \cdot$. This is a matrix factorization, and its endomorphism algebra in the category $MF(W)$ is the previously considered dga $\mathcal{B}$.

Lemma 12.3. *In* $D^b_{\mathrm{sing}}(W^{-1}(0))$, $\mathrm{coker}(\delta_E^1)$ *is isomorphic to* $\mathcal{S}_{W^{-1}(0),0}$.

Proof. Take the chain complex of vector bundles on $W^{-1}(0)$ given by

$$(12.3) \qquad\qquad C^i = \begin{cases} \bigoplus_{j=0}^{[n/2]} \Omega^{2j-i}(V)|W^{-1}(0), & i \leq 0, \\ 0, & i > 0, \end{cases}$$

with $\delta_C^i = \iota_\eta - \gamma \wedge \cdot$ for $i < 0$. This has the property that $\mathrm{coker}(\delta_E^1) = H^0(C)$. Consider the decreasing filtration $F_\bullet C$ whose pieces $F_m C$ consists of those summands in (12.3) with $j \geq m$. Passing to induced differential on the graded spaces of that filtration means that we keep only the ι_η term. In particular, $C/F_1 C$ is the standard Koszul resolution of $\mathcal{S}_{V,0}$ restricted to $W^{-1}(0)$. Hence, its cohomology sheaves are the derived restrictions

$$(12.4) \quad H^i(C/F_1 C) = \mathcal{S}_{V,0} \overset{\mathrm{L}^{-i}}{\otimes}_{\mathcal{O}(V)} \mathcal{O}(W^{-1}(0)) = \begin{cases} \mathcal{S}_{W^{-1}(0),0}, & i = 0, -1, \\ 0, & \text{otherwise.} \end{cases}$$

The other quotients $F_m C / F_{m+1} C$, $m > 0$, are truncations of the same resolution, which means that

$$(12.5)$$
$$H^i(F_m C / F_{m+1} C) = \begin{cases} \ker\left(\Omega^{2m}(V)|W^{-1}(0) \xrightarrow{\iota_\eta} \Omega^{2m-1}(V)|W^{-1}(0)\right), & i = 0, \\ 0, & i \neq 0. \end{cases}$$

Therefore, $H^i(F_1 C)$ is again zero in degrees $i \neq 0$, and $H^0(F_1 C)$ is a successive extension of torsion-free sheaves, hence itself torsion-free. Finally, consider the long exact sequence

$$(12.6) \qquad \cdots H^{-1}(C/F_1 C) \to H^0(F_1 C) \to H^0(C) \to H^0(C/F_1 C) \to 0.$$

Because $H^{-1}(C/F_1 C)$ is torsion, the leftmost arrow necessarily vanishes, which means that $H^0(C)$ is an extension of $\mathcal{S}_{W^{-1}(0),0}$ by $H^0(F_1 C)$. We know that in the derived category $H^0(F_1 C)$ is isomorphic to $F_1 C$, which is a perfect complex, hence maps to zero when passing to $D_{\mathrm{sing}}^b(W^{-1}(0))$. This yields an isomorphism $H^0(C) \cong \mathcal{S}_{W^{-1}(0),0}$ in that category. $\qquad \square$

We have now found a split-generator for the split-closure of $H^0(MF(W))$, and know that its endomorphism dga is $\mathcal{B}$. By the same general arguments as in Section 6, this implies:

Corollary 12.4. $D_{\mathrm{sing}}^\pi(W^{-1}(0)) \cong D^\pi(\mathcal{B})$.

We will also need an equivariant version of this discussion, which is fairly straightforward. Suppose that W and γ are invariant under the action of a finite group $Z \subset GL(V)$. Defining equivariant categories $D_{\mathrm{sing},Z}^b(W^{-1}(0))$ and $MF_Z(W)$ in the obvious way, the analogue of Theorem 12.2 holds [61, Proposition 6.2]. The skyscraper sheaf at the origin, with its natural equivariant structure, is no longer a split-generator. Instead, one should consider $\mathcal{S}_{W^{-1}(0),0} \otimes \mathbb{C}[Z]$, and the analogous equivariant matrix factorization $\Omega(V) \otimes \mathbb{C}[Z]$, whose endomorphism dga is the semidirect product $\mathcal{B} \rtimes Z$. The outcome is that

$$(12.7) \qquad\qquad D_{\mathrm{sing},Z}^\pi(W^{-1}(0)) \cong D^\pi(\mathcal{B} \rtimes Z).$$

13. The McKay correspondence

We now return to $V = \mathbb{C}^3$ with the action of $\mathbb{Z}/5 \cong Z \subset SL(V)$. The quotient has a canonical crepant resolution, namely the G-Hilbert scheme [44]

$$
(13.1) \qquad X = \mathrm{Hilb}_Z(V) \longrightarrow \bar{X} = V/Z.
$$

The paper [18] gives an elementary toric description of X. Namely, take $N_\mathbb{R} = \mathbb{R}^3$, and let $N \subset N_\mathbb{R}$ be the lattice generated by $\mathbb{Z}^3$ together with $\frac{1}{5}(1,1,3)$. Let $\bar{\Delta}$ be the fan consisting of the single cone $N \cap \mathbb{Z}^3_+$ and its faces. This describes the affine toric variety $\bar{X}$. Now take the elementary simplex in $N_\mathbb{R}$, which is the one spanned by $\{(1,0,0),(0,1,0),(0,0,1)\}$, and triangulate it as follows:

$$(13.2)$$
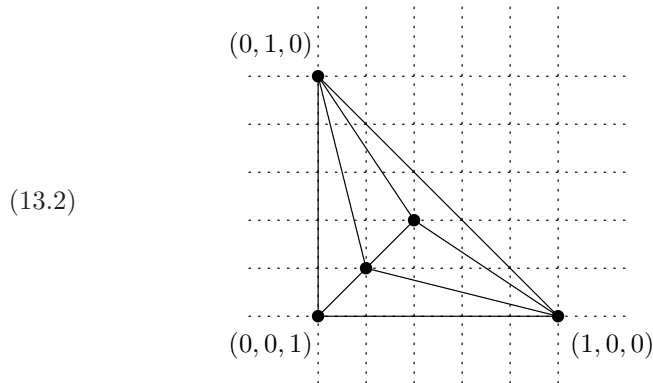
The cone over this triangulation yields a subdivision of $\bar{\Delta}$, whose fan Δ is the one describing X. Inspection of this picture shows that the preimage of the origin has two components, which are $\mathbb{C}P^2$ and a Hirzebruch surface F_3, intersecting each other in a rational curve.

We want to be pedestrian, and work through the standard construction of X from Δ. Let $M_\mathbb{R} = N_\mathbb{R}^\vee$ and $M = N^\vee$ be the dual space and dual lattice, respectively. Explicitly, $M_\mathbb{R} = \mathbb{R}^3$, and M consists of those points $m \in \mathbb{Z}^3$ where $\frac{1}{5}(m_1 + m_2 + 3m_3) \in \mathbb{Z}$. The cones $\sigma^\vee$ dual to the five maximal cones $\sigma \in \Delta$ are given by

$$
\begin{aligned}
&\{m_2 \geq 0,\ m_1 + m_2 + 3m_3 \geq 0,\ 2m_1 + 2m_2 + m_3 \geq 0\}, \\
&\{m_1 \geq 0,\ m_1 + m_2 + 3m_3 \geq 0,\ 2m_1 + 2m_2 + m_3 \geq 0\}, \\
(13.3) \quad &\{m_2 \geq 0,\ m_3 \geq 0,\ m_1 + m_2 + 3m_3 \geq 0\}, \\
&\{m_1 \geq 0,\ m_3 \geq 0,\ m_1 + m_2 + 3m_3 \geq 0\}, \\
&\{m_1 \geq 0,\ m_2 \geq 0,\ 2m_1 + 2m_2 + m_3 \geq 0\}.
\end{aligned}
$$

When intersected with M, these give five semigroups $\sigma^\vee \cap M \cong \mathbb{Z}_+^3$, whose generators are

$$\{(3,0,-1),\ (-1,0,2),\ (-1,1,0)\},$$
$$\{(0,3,-1),\ (0,-1,2),\ (1,-1,0)\},$$
(13.4) $$\{(5,0,0),\ (-3,0,1),\ (-1,1,0)\},$$
$$\{(0,5,0),\ (0,-3,1),\ (1,-1,0)\},$$
$$\{(0,1,-2),\ (1,0,-2),\ (0,0,5)\}.$$

These correspond to a covering of X by toric charts, each of which is a copy of $\mathbb{C}^3$. Specifically, if one takes the generators listed in (13.4) to correspond to a basis of monomials in $\mathbb{C}[\sigma^\vee \cap M] \cong \mathbb{C}[a_1, a_2, a_3]$, the transformations from each chart to the previous one are

$$(a_1, a_2, a_3) \longmapsto (a_1 a_3^3, a_2 a_3^{-1}, a_3^{-1}),$$
$$(a_1, a_2, a_3) \longmapsto (a_2^{-1}, a_1 a_2^2, a_3),$$
(13.5) $$(a_1, a_2, a_3) \longmapsto (a_1 a_3^5, a_2 a_3^{-3}, a_3^{-1}),$$
$$(a_1, a_2, a_3) \longmapsto (a_1^5 a_3^2, a_1^{-3} a_3^{-1}, a_1^{-1} a_2).$$

The Z-invariant hypersurface $W^{-1}(0)$ descends to $\bar{H} \subset \bar{X}$, which is a singular surface that turns out to be rational. We are interested in its preimage under the resolution (13.1), denoted by H. In the charts constructed above, the defining equation for H is, respectively,

$$a_1 a_2 (a_3 - a_1 - a_1 a_3^5 - a_2^2) = 0,$$
$$a_1 a_2 (a_3 - a_1 a_3^5 - a_1 - a_2^2) = 0,$$
(13.6) $$a_1 (a_2 a_3 - 1 - a_3^5 - a_1^2 a_2^5) = 0,$$
$$a_1 (a_2 a_3 - a_3^5 - 1 - a_1^2 a_2^5) = 0,$$
$$a_3 (a_1 a_2 - a_2^5 a_3 - a_1^5 a_3 - 1) = 0.$$

This is a surface with three components H_1, H_2, H_3 and only normal crossing singularities. The first component is $H_1 \cong \mathbb{C}P^2$, which is $\{a_1 = 0\}$ in the first and second charts, and $\{a_3 = 0\}$ in the fifth chart. The intersections $H_{12}, H_{13} \subset H_1$ are a line and smooth conic, respectively, which are in general position. The second component is $H_2 \cong F_3$, which is $\{a_2 = 0\}$ in the first and second charts, and $\{a_1 = 0\}$ in the third and fourth charts. Let's explicitly identify H_2 with the ruled surface $P(\mathcal{O} \oplus \mathcal{O}(-3)) \to \mathbb{C}P^1$, where we have sections $S_- = P(\mathcal{O} \oplus \{0\})$ and $S_+ = P(\{0\} \oplus \mathcal{O}(-3))$ with self-intersection ± 3. Then $H_{12} = S_-$, while H_{23} is a section intersecting S_- transversally over $0, \infty \in \mathbb{C}P^1$, and intersecting S_+ transversally over the five points $[1 : \eta] \in \mathbb{C}P^1$, $\eta \in \sqrt[5]{1}$. These properties determine $H_{23} \subset H_2$ uniquely up

to the fibrewise $\mathbb{C}^*$-action on the ruling. In particular, it has self-intersection number 7. The last component is the non-toric one, which is the proper transform of $\bar{H}$. One can analyze this through the map $H_3 \to \mathbb{C}P^2$, given by $(a_1, a_2, a_3) \mapsto [1 : ia_2 : -a_3]$ in the first of our five charts. Inside $\mathbb{C}P^2$ take again a smooth conic and a line, which concretely are given by $z_1^2 + z_0 z_2 = 0$ and $z_1 = 0$ in homogeneous coordinates $[z_0 : z_1 : z_2]$. Now blow up the ten points $[1 : \zeta : \zeta^2]$ for $\zeta \in \sqrt[10]{1}$. Furthermore, blow up the five points $[1 : 0 : \eta]$ for $\eta \in \sqrt[5]{1}$, and another five points infinitely close to them, corresponding to functions whose derivative vanishes along our line. The outcome of this blowup process is a compactification of H_3. The intersections H_{13} and H_{23} are the proper transforms of our conic and line, respectively. In particular, their selfintersection numbers are -6 and -9, respectively.

Consider the category $D_{\mathrm{sing}}^b(H)$ of Landau-Ginzburg branes, defined as in (12.1), and its split-closure $D_{\mathrm{sing}}^\pi(H)$.

Theorem 13.1. *There is an equivalence of triangulated categories,*

$$(13.7) \qquad D_{\mathrm{sing},Z}^\pi(W^{-1}(0)) \cong D_{\mathrm{sing}}^\pi(H).$$

The existence of a full and faithful functor $D_{\mathrm{sing},Z}^b(W^{-1}(0)) \to D_{\mathrm{sing}}^\pi(H)$ was proved in [42]. Essential surjectivity is proved in [61] for a somewhat different special case, but the method given there adapts to our situation. In fact, since we only need the statement for the split-closures, it is enough for us to consider complexes of sheaves with compactly supported cohomology, which removes the need for the most technical aspects of [13]. From (12.7) and Proposition 11.3, it follows that $D_{\mathrm{sing},Z}^\pi(W^{-1}(0)) \cong D^\pi(\mathcal{A} \rtimes Z)$, where $\mathcal{A}$ is as in Proposition 5.2. The conclusion is:

Corollary 13.2. $D_{\mathrm{sing}}^\pi(H) \cong D^\pi(\mathcal{A} \rtimes Z)$.

The right hand side is the same as in the description of the Fukaya category given in Corollary 10.2. Hence, the combination of the two results implies Theorem 1.1.

Acknowledgments

References

[1] M. Abouzaid. On the Fukaya categories of higher genus surfaces. *Adv. Math.*, 217:1192–1235, 2008. MR2383898 (2008k:53198)

[2] M. Akaho. Intersection theory for Lagrangian immersions. *Math. Res. Lett.*, 12:543–550, 2005. MR2155230 (2006h:53093)

[3] M. Akaho and D. Joyce. Immersed Lagrangian Floer theory. Preprint arXiv: 0803.0717.

[4] V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko. *Singularities of differentiable maps*. Birkhäuser, 1988. MR966191 (89g:58024)

[5] P. Aspinwall and S. Katz. Computation of superpotentials for D-branes. *Comm. Math. Phys.*, 264:227–253, 2006. MR2212222 (2007b:81178)

[6] M. Atiyah. Vector bundles over an elliptic curve. *Proc. London Math. Soc.* 7:414–452, 1957. MR0131423 (24:A1274)

[7] D. Auroux, L. Katzarkov and D. Orlov. Mirror symmetry for weighted projective planes and their noncommutative deformations. *Ann. of Math. (2)*, 167:867–943, 2008. MR2415388 (2009f:53142)

[8] D. Auroux. Mirror symmetry and T-duality in the complement of an anticanonical divisor. Preprint arXiv:0706.3207. MR2386535 (2009f:53141)

[9] P. Balmer and M. Schlichting. Idempotent completion of triangulated categories. *J. Algebra*, 236:819–834, 2001. MR1813503 (2002a:18013)

[10] V. Baranovsky and J. Pecharich. On equivalences of derived and singular categories. Preprint arXiv:0907.3717, 2009.

[11] A. Beilinson, V. Ginzburg, and W. Soergel. Koszul duality patterns in representation theory. *Journal of the Amer. Math. Soc.*, 9:473–527, 1996. MR1322847 (96k:17010)

[12] J. Birman. A normal form in the homeotopy group of a surface of genus 2, with applications to 3-manifolds. *Proceedings of the Amer. Math. Soc.*, 34:379–384, 1972. MR0295308 (45:4376)

[13] T. Bridgeland, A. King, and M. Reid. The McKay correspondence as an equivalence of derived categories. *Journal of the Amer. Math. Soc.*, 14:535–554, 2001. MR1824990 (2002f:14023)

[14] Yu. Chekanov. Differential algebras of Legendrian links. *Invent. Math.*, 150:441–483, 2002. MR1946550 (2003m:53153)

[15] C.-H. Cho. Holomorphic discs, spin structures, and the Floer cohomology of the Clifford torus. *Intern. Math. Res. Notices*, 35:1803–1843, 2004. MR2057871 (2005a:53148)

[16] C.-H. Cho and Y.-G. Oh. Floer cohomology and disc instantons of Lagrangian torus fibers in toric Fano manifolds. *Asian J. Math.*, 10:773–814, 2006. MR2282365 (2007k:53150)

[17] O. Cornea and F. Lalonde. Cluster homology: an overview of the construction and results. *Electron. Res. Announc. Amer. Math. Soc.*, 12:1–12 (electronic), 2006. MR2200949 (2006m:53139)

[18] A. Craw and M. Reid. How to calculate A-Hilb $\mathbb{C}^3$. In *Geometry of toric varieties*, volume 6 of *Sémin. Congr.*, pages 129–154. Soc. Math. France, 2002. MR2075608 (2005d:14004)

[19] T. Dyckerhoff. Compact generators in categories of matrix factorizations. Preprint arXiv:0904.4713.

[20] A. Efimov. Homological mirror symmetry for curves of higher genus. Preprint arXiv:0907.3903, 2009.

[21] D. Eisenbud. Homological algebra on a complete intersection, with an application to group representations. *Trans. Amer. Math. Soc.*, 260:35–64, 1980. MR570778 (82d:13013)

[22] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. Lagrangian Floer theory on compact toric manifolds. I. Preprint arXiv:0802.1703.

[23] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. Lagrangian Floer theory on compact toric manifolds. II: Bulk deformations. Preprint arXiv:0810.5654.

[24] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. Lagrangian intersection Floer theory - anomaly and obstruction. In press.

[25] Kenji Fukaya. Mirror symmetry of abelian varieties and multi-theta functions. *J. Algebraic Geom.*, 11:393–512, 2002. MR1894935 (2003m:14059)

[26] B. Fang, C.-C. Liu, D. Treumann and E. Zaslow. The coherent-constructible correspondence and Homological Mirror Symmetry for toric varieties. Preprint arXiv:0901.4276, 2009.

[27] W. Goldman and J. Millson. The deformation theory of the fundamental group of compact Kähler manifolds. *IHES Publ. Math.*, 67:43–96, 1988. MR972343 (90b:32041)

[28] K. Hori and C. Vafa. Mirror symmetry. Preprint hep-th/0002222, 2000.

[29] P. Horja. Derived category automorphisms from mirror symmetry. *Duke Math. J.* 127:1–34, 2005. MR2126495 (2006a:14023)

[30] G. Hochschild, B. Kostant, and A. Rosenberg. Differential forms on regular affine algebras. *Trans. Amer. Math. Soc.*, 102:383–408, 1962. MR0142598 (26:167)

[31] D. Huybrechts and R. Thomas. $\mathbb{P}$-objects and autoequivalences of derived categories. *Math. Res. Lett.* 13:87–98, 2006. MR2200048 (2006j:14015)

[32] L. Katzarkov. Birational geometry and homological mirror symmetry. In *Real and complex singularities*, pages 176–206. World Sci. Publ., 2007. MR2336686 (2008g:14062)

[33] A. Kapustin, L. Katzarkov, D. Olov and M. Yotov. Homological Mirror Symmetry for manifolds of general type. Preprint, 2009.

[34] M. Kontsevich. Deformation quantization of Poisson manifolds I. *Lett. Math. Phys.*, 66:157–216, 2003. MR2062626 (2005i:53122)

[35] M. Kontsevich. Homological algebra of mirror symmetry. In *Proceedings of the International Congress of Mathematicians (Zürich, 1994)*, pages 120–139. Birkhäuser, 1995. MR1403918 (97f:32040)

[36] M. Kontsevich and Y. Soibelman. Homological mirror symmetry and torus fibrations. In *Symplectic geometry and mirror symmetry*, pages 203–263. World Scientific, 2001. MR1882331 (2003c:32025)

[37] T. Pantev, L. Katzarkov, M. Kontsevich. Hodge theoretic aspects of mirror symmetry. Preprint arXiv:0806.0107. MR2483750 (2009j:14052)

[38] B. Keller. Introduction to A-infinity algebras and modules. *Homology Homotopy Appl. (electronic)*, 3:1–35, 2001. MR1854636 (2004a:18008a)

[39] T. Lada and M. Markl. Strongly homotopy Lie algebras. *Comm. Algebra*, 23:2147–2161, 1995. MR1327129 (96d:16039)

[40] M. Markl. Transferring A_∞ (strongly homotopy associative) structures. *Rend. Circ. Mat. Palermo (2) Suppl.*, 79:139–151, 2006. MR2287133 (2007j:18016)

[41] D. McDuff. The structure of rational and ruled symplectic 4-manifolds. *Journ. Amer. Math. Soc.*, 3:679–712, 1990. Erratum: same journal, 5:987–988, 1992. MR1049697 (91k:58042)

[42] S. Mehrotra. *Triangulated categories of singularities, matrix factorizations and LG models*. PhD thesis, Univ. of Pennsylvania, 2005.

[43] S. Mukai. Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves. *Nagoya Math. J.* 81:153–175, 1981. MR607081 (82f:14036)

[44] I. Nakamura. Hilbert schemes of abelian group orbits. *J. Algebraic Geom.*, 10:757–779, 2001. MR1838978 (2002d:14006)

[45] Y.-G. Oh. Floer cohomology of Lagrangian intersections and pseudo-holomorphic discs I. *Comm. Pure Appl. Math.*, 46:949–994, 1993. MR1223659 (95d:58029a)

[46] D. Orlov. D-branes of type B in Landau-Ginzburg models and mirror symmetry for the genus two curve. Conference talk, IAS Princeton, 3/12/2008.

[47] D. Orlov. Formal completions and idempotent completions of triangulated categories of singularities. Preprint arXiv:0901.1859.

[48] D. Orlov. Triangulated categories of singularities and D-branes in Landau-Ginzburg models. *Proc. Steklov Inst. Math.*, 246:227–248, 2004. MR2101296 (2006i:81173)

[49] A. Polishchuk and E. Zaslow. Categorical mirror symmetry: the elliptic curve. *Adv. Theor. Math. Phys.*, 2:443–470, 1998. MR1633036 (99j:14034)

[50] A. Polishchuk. Massey and Fukaya products on elliptic curves. *Adv. Theor. Math. Phys.*, 4:1187–1207, 2000. MR1894854 (2003h:14058)

[51] P. Rossi. Gromov-Witten theory of orbicurves, the space of tri-polynomials and Symplectic Field Theory of Seifert fibrations. Preprint arXiv:0808.2626.

[52] M. Schwarz. A quantum cup-length estimate for symplectic fixed points. *Invent. Math.*, 133:353–397, 1998. MR1632778 (99e:58041)

[53] P. Seidel. Python code for computing A_∞-deformations, 2008. Can be downloaded from the author's homepage, `http://math.mit.edu/~seidel`.

[54] P. Seidel. Homological mirror symmetry for the quartic surface. Preprint math.SG/0310414, 2003.

[55] P. Seidel. A long exact sequence for symplectic Floer cohomology. *Topology*, 42:1003–1063, 2003. MR1978046 (2004d:53105)

[56] P. Seidel. *Fukaya categories and Picard-Lefschetz theory*. European Math. Soc., 2008. MR2441780 (2009f:53143)

[57] P. Seidel and R. Thomas. *Braid group actions on derived categories of coherent sheaves. Duke Math. Jour.*, 108:37–108, 2001. MR1831820 (2002e:14030)

[58] A. Subotic. Ph.D. thesis, MIT, in preparation.

[59] A. Takahashi. Weighted projective lines associated to regular systems of weights of dual type. Preprint arXiv:0711.3907.

[60] J.-C. Tougeron. Idéaux des fonctions différentiables. *Ann. Inst. Fourier*, 18:177–240, 1970. MR0240826 (39:2171)

[61] A. Quintero Velez. McKay correspondence for Landau-Ginzburg models. Preprint arXiv:0711.4774.

[62] K. Wehrheim and C. Woodward. Functoriality for Lagrangian correspondences in Floer theory. Preprint arXiv:0708.2851.

[63] C. A. Weibel. *An introduction to homological algebra*. Cambridge Univ. Press, 1994. MR1269324 (95f:18001)

Massachusetts Institute of Technology, 77 Massachusetts Avenue, Room 2-270, Cambridge, Massachusetts 02139

E-mail address: seidel@math.mit.edu