Homological mirror symmetry for the quintic 3–fold

YUICHI NOHARA
KAZUSHI UEDA

We prove homological mirror symmetry for the quintic Calabi–Yau 3–fold. The proof follows that for the quartic surface by Seidel [16] closely, and uses a result of Sheridan [23]. In contrast to Sheridan's approach [22], our proof gives the compatibility of homological mirror symmetry for the projective space and its Calabi–Yau hypersurface.

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1 Introduction

Ever since the proposal by Kontsevich [9], homological mirror symmetry has been proved for elliptic curves (see Polishchuk and Zaslow [14], Polishchuk [13] and Seidel [19]), Abelian surfaces (see Fukaya [5], Kontsevich and Soibelman [11] and Abouzaid and Smith [1]) and quartic surfaces (see Seidel [16]). It has also been extended to other contexts such as Fano varieties (see Kontsevich [10]), varieties of general type (see Katzarkov [8]), and singularities (see Takahashi [24]), and various evidences have been accumulated in each cases.

The most part of the proof of homological mirror symmetry for the quartic surface by Seidel [16] works in any dimensions. Combined with the results of Sheridan [23], an expert reader will observe that one can prove homological mirror symmetry for the quintic 3–fold if one can show that

- the large complex structure limit monodromy of the pencil of quintic Calabi–Yau 3–folds is *negative* in the sense of Seidel [16, Definition 7.1], and
- the vanishing cycles of the pencil of quintic Calabi–Yau 3–folds are isomorphic in the Fukaya category to Lagrangian spheres constructed by Sheridan [23].

We prove these statements, and obtain the following:

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Theorem 1.1 *Let X_0 be a smooth quintic Calabi–Yau 3–fold in $\mathbb{P}^4_{\mathbb{C}}$ and Z_q^* be the mirror family. Then there is a continuous automorphism $\psi \in \operatorname{End}(\Lambda_{\mathbb{N}})^\times$ and an equivalence*

$$D^\pi \mathcal{F}(X_0) \cong \widehat{\psi}^* D^b \operatorname{coh} Z_q^* \tag{1}$$

of triangulated categories over $\Lambda_{\mathbb{Q}}$.

Here $\Lambda_{\mathbb{N}} = \mathbb{C}[\![q]\!]$ is the ring of formal power series in one variable and $\Lambda_{\mathbb{Q}}$ is its algebraic closure. The automorphism $\widehat{\psi}$ of $\Lambda_{\mathbb{Q}}$ is any lift of the automorphism ψ of $\Lambda_{\mathbb{N}}$, and the category $\widehat{\psi}^* D^b \operatorname{coh} Z_q^*$ is obtained from $D^b \operatorname{coh} Z_q^*$ by changing the $\Lambda_{\mathbb{Q}}$–module structure by $\widehat{\psi}$. The category $D^\pi \mathcal{F}(X_0)$ is the split-closed derived Fukaya category of X_0 consisting of rational Lagrangian branes. The symplectic structure of X_0 and hence the parameter q come from 5 times the Fubini–Study metric of the ambient projective space $\mathbb{P}^4_{\mathbb{C}}$. The mirror family $Z_q^* = [Y_q^* / \Gamma]$ is the quotient of the hypersurface

$$Y_q^* = \{[y_1 : \ldots : y_5] \in \mathbb{P}^4_{\Lambda_{\mathbb{Q}}} \mid y_1 \ldots y_5 + q(y_1^5 + \cdots + y_5^5) = 0\}$$

by the group

$$\Gamma = \{[\operatorname{diag}(a_1, \ldots, a_5)] \in PSL_5(\mathbb{C}) \mid a_1^5 = \cdots = a_5^5 = a_1 \cdots a_5 = 1\}. \tag{2}$$

Let $Z_q = [Y_q / \Gamma]$ be the quotient of the hypersurface Y_q of $\mathbb{P}^4_{\Lambda_{\mathbb{N}}}$ defined by the same equation as Y_q^* above. The equivalence (1) is obtained by combining the equivalences

$$D^\pi \mathcal{F}(X_0) \cong \widehat{\psi}^* D^\pi \mathcal{S}_q^* \cong \widehat{\psi}^* D^b \operatorname{coh} Z_q^*$$

for an A_∞–algebra $\mathcal{S}_q^* = \mathcal{S}_q \otimes_{\Lambda_{\mathbb{N}}} \Lambda_{\mathbb{Q}}$ as follows:

(1) The derived category $D^b \operatorname{coh} Z_q^*$ of coherent sheaves on Z_q^* has a split-generator, which extends to an object of $D^b \operatorname{coh} Z_q$. The quasi-isomorphism class of the endomorphism dg algebra $\mathcal{S}_q$ of this object is characterized by its cohomology algebra together with a couple of additional properties up to pull-back by $\operatorname{End}(\Lambda_{\mathbb{N}})^\times$.

(2) The Fukaya category $\mathcal{F}(X_0)$ contains 625 distinguished Lagrangian spheres. They are vanishing cycles for a pencil of quintic Calabi–Yau 3–folds, and a suitable combination of symplectic Dehn twists along them is isotopic to the *large complex structure limit monodromy*.

(3) The large complex structure limit monodromy has a crucial property of *negativity*, which enables one to show that the vanishing cycles split-generate the derived Fukaya category $D^\pi \mathcal{F}(X_0)$.

(4) The total morphism A_∞–algebra $\mathcal{F}_q$ of the vanishing cycles has the same cohomology algebra as $\mathcal{S}_q$ and satisfies the additional properties characterizing $\mathcal{S}_q$.

The condition that X_0 is a 3–fold is used in the proof that vanishing cycles split-generate the Fukaya category, cf. Remarks 3.6 and 3.9. Sheridan [22] proved homological mirror symmetry for Calabi–Yau hypersurfaces in projective spaces along the lines of Sheridan [23]. In contrast to Sheridan's approach, our proof is based on the relation between Sheridan's immersed Lagrangian sphere in a pair of pants and vanishing cycles on Calabi–Yau hypersurfaces, and gives the compatibility of homological mirror symmetry for the projective space and its Calabi–Yau hypersurface as in Remark 5.11.

This paper is organized as follows: Sections 2 and 3 have little claim in originality, and we include them for the readers' convenience. In Section 2, we recall the description of the derived category of coherent sheaves on Z_q^* due to Seidel [16]. In Section 3, we extend Seidel's discussion on the Fukaya category of the quartic surface to general projective Calabi–Yau hypersurfaces. Strictly speaking, the work of Fukaya, Oh, Ohta and Ono [6] that we rely on in this section gives not a full-fledged A_∞–category but an A_∞–algebra for a Lagrangian submanifold and an A_∞–bimodule for a pair of Lagrangian submanifolds. While there is apparently no essential difficulty in generalizing their work to construct an A_∞–category (for transversally intersecting sequence of Lagrangian submanifolds, one can regard it as a single immersed Lagrangian submanifold and use the work of Akaho and Joyce [2]), we do not attempt to settle this foundational issue in this paper. Sections 4 and 5 are at the heart of this paper. In Section 4, we prove the negativity of the large complex structure limit monodromy using ideas of Seidel [16] and Ruan [15]. In Section 5, we use ideas from Seidel [18] and Futaki and Ueda [7] to reduce Floer cohomology computations on vanishing cycles needed in Section 3 to a result of Sheridan [23].

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2 Derived category of coherent sheaves

Let V be an $(n+2)$–dimensional complex vector space spanned by $\{v_i\}_{i=1}^{n+2}$, and $\{y_i\}_{i=1}^{n+2}$ be the dual basis of $V^\vee$. The projective space $\mathbb{P}(V)$ has a full exceptional collection $\big(F_k = \Omega_{\mathbb{P}(V)}^{n+2-k}(n+2-k)[n+2-k]\big)_{k=1}^{n+2}$ by Beilinson [3]. The full dg subcategory of (the dg enhancement of) $D^b \operatorname{coh} \mathbb{P}(V)$ consisting of $(F_k)_{k=1}^{n+2}$ is quasi-isomorphic to the $\mathbb{Z}$–graded category $C_{n+2}^{\to}$ with $(n+2)$ objects $X_1, \dots, X_{n+2}$ and morphisms

$$\operatorname{Hom}_{C_{n+2}^{\to}}(X_j, X_k) = \begin{cases} \Lambda^{k-j} V & j \le k, \\ 0 & \text{otherwise.} \end{cases}$$

The differential is trivial, the composition is given by the wedge product, and the grading is such that V is homogeneous of degree one. One can equip $(F_k)_{k=1}^{n+2}$ with a $GL(V)$-linearization so that this quasi-isomorphism is $GL(V)$–equivariant. Let $\iota_0 \colon Y_0 \hookrightarrow \mathbb{P}(V)$ be the inclusion of the union of coordinate hyperplanes and set $E_{0,k} = \iota_0^* F_k$. The total morphism dg algebra $\bigoplus_{i,j=1}^{n+2} \hom(E_{0,i}, E_{0,j})$ of this collection will be denoted by $\mathcal{S}_{n+2}$.

Let C_{n+2} be the trivial extension category of $C_{n+2}^{\to}$ of degree n as defined by Seidel [16, Section 10a]. It is a category with the same object as $C_{n+2}^{\to}$. The morphisms are given by

$$\operatorname{Hom}_{C_{n+2}}(X_j, X_k) = \operatorname{Hom}_{C_{n+2}^{\to}}(X_j, X_k) \oplus \operatorname{Hom}_{C_{n+2}^{\to}}(X_k, X_j)^{\vee}[-n],$$

and the compositions are given by

$$(a, a^{\vee})(b, b^{\vee}) = (ab, a^{\vee}(b\,\cdot)) + (-1)^{\deg(a)(\deg(b)+\deg(b^{\vee}))} b^{\vee}(\cdot\, a).$$

From this definition, one can easily see that

$$\operatorname{Hom}_{C_{n+2}}(X_j, X_k) = \begin{cases} \Lambda^{k-j} V & j < k, \\ \Lambda^0 V \oplus \Lambda^{n+2} V[2] & j = k, \\ \Lambda^{k-j+n+2} V[2] & j > k. \end{cases}$$

The total morphism algebra Q_{n+2} of this category C_{n+2} admits the following description: Set $\gamma = \zeta_{n+2} \operatorname{id}_V$ for $\zeta_{n+2} = \exp(2\pi\sqrt{-1}/(n+2))$ and let $\Gamma_{n+2} = \langle \gamma \rangle \subset SL(V)$ be a cyclic subgroup of order $n+2$. The group algebra $R_{n+2} = \mathbb{C}\Gamma_{n+2}$ is a semisimple algebra of dimension $n+2$, whose primitive idempotents are given by

$$e_j = \frac{1}{n+2}(e + \zeta_{n+2}^{-j}\gamma + \cdots + \zeta_{n+2}^{-(n+1)j}\gamma^{n+1}) \in \mathbb{C}\Gamma_{n+2}.$$

Let $\Lambda V = \bigoplus_{i=0}^{n+2} \Lambda^i V$ be the exterior algebra equipped with the natural $\mathbb{Z}$–grading and $\widetilde{Q}_{n+2} = \Lambda V \rtimes \Gamma_{n+2}$ be the semidirect product. There is an R_{n+2}–algebra isomorphism between $\widetilde{Q}_{n+2}$ and Q_{n+2} sending $e_k \widetilde{Q}_{n+2} e_j$ to $\operatorname{Hom}_{C_{n+2}}(X_j, X_k)$. This isomorphism does not preserve the $\mathbb{Z}$–grading; Q_{n+2} is obtained from $\widetilde{Q}_{n+2}$ by assigning degree $\frac{n}{n+2}k$ to $\Lambda^k V \otimes \mathbb{C}\Gamma_{n+2}$ and adding $\frac{2}{n+2}(k-j)$ to the piece $e_k \widetilde{Q} e_j$.

Let H be a maximal torus of $SL(V)$ and T be its image in $PSL(V) = SL(V)/\Gamma_{n+2}$. The group T acts on Q_{n+2} by an automorphism of a graded R_{n+2}–algebra so that $[\mathrm{diag}(t_1, t_2, \ldots, t_{n+2})]$ sends $v \otimes e_i \in e_{i+1} Q_{n+2} e_i$ to $(\mathrm{diag}(1, t_2/t_1, \ldots, t_{n+2}/t_1) \cdot v) \otimes e_i$.

The dg algebra $\mathcal{S}_{n+2}$ is characterized by the following properties:

Lemma 2.1 (Seidel [16, Lemma 10.2]) *Assume that a T–equivariant A_∞–algebra $\mathcal{Q}_{n+2}$ over R_{n+2} satisfies the following properties:*

- *The cohomology algebra $H^*(\mathcal{Q}_{n+2})$ is T–equivariantly isomorphic to Q_{n+2} as an R_{n+2}–algebra.*
- *$\mathcal{Q}_{n+2}$ is not quasi-isomorphic to Q_{n+2}.*

Then one has a R_{n+2}–linear, T–equivariant quasi-isomorphism $\mathcal{Q}_{n+2} \xrightarrow{\sim} \mathcal{S}_{n+2}$.

Sketch of proof The proof of the fact that these properties are satisfied by $\mathcal{S}_{n+2}$ is identical to Seidel [16, Section 10d]. The uniqueness comes from the Hochschild cohomology computations in [16, Section 10a]: The Hochschild cohomology of $\widetilde{Q}_{n+2}$ is given by

$$HH^{s+t}(\widetilde{Q}_{n+2}, \widetilde{Q}_{n+2})^t \cong \bigoplus_{\gamma \in \Gamma_{n+2}} \left(S^s (V^\gamma)^\vee \otimes \Lambda^{s+t-\operatorname{codim} V^\gamma}(V^\gamma) \otimes \Lambda^{\operatorname{codim} V^\gamma}(V/V^\gamma) \right)^{\Gamma_{n+2}},$$

where $SV = \bigoplus_{i=0}^\infty S^i V$ is the symmetric algebra of V (see [16, Proposition 4.2]). By the change of the grading from $\widetilde{Q}_{n+2}$ to Q_{n+2}, one obtains

$$HH^{s+t}(Q_{n+2}, Q_{n+2})^t \cong \bigoplus_{\gamma \in \Gamma_{n+2}} \left(S^s (V^\gamma)^\vee \otimes \Lambda^{s+\frac{n+2}{n}t-\operatorname{codim} V^\gamma}(V^\gamma) \otimes \Lambda^{\operatorname{codim} V^\gamma}(V/V^\gamma) \right)^{\Gamma_{n+2}}.$$

By passing to the T–invariant part, one obtains

$$\begin{aligned} (HH^2(Q_{n+2}, Q_{n+2})^{2-d})^T &= (S^d V^\vee \otimes \Lambda^{n+2-d} V)^H \\ &= \begin{cases} \mathbb{C} \cdot y_1 \cdots y_{n+2} & d = n+2, \\ 0 & \text{for all other } d > 2, \end{cases} \end{aligned} \tag{3}$$

so that $\mathcal{S}_{n+2}$ is determined by the above properties up to quasi-isomorphism [16, Lemma 3.2]. □

Let $\mathbb{P}_{\Lambda_{\mathbb{N}}} = \mathbb{P}(V \otimes_{\mathbb{C}} \Lambda_{\mathbb{N}})$ be the projective space over $\Lambda_{\mathbb{N}}$ and Y_q be the hypersurface defined by $q(y_1^{n+2} + \cdots + y_{n+2}^{n+2}) + y_1 \dots y_{n+2} = 0$. The geometric generic fiber of the family $Y_q \to \operatorname{Spec} \Lambda_{\mathbb{N}}$ is the smooth Calabi–Yau variety $Y_q^* = Y_q \times_{\Lambda_{\mathbb{N}}} \Lambda_{\mathbb{Q}}$ appearing in Section 1, and the special fiber is Y_0 above. The collection $E_{0,k}$ is the restriction of the collection $E_{q,k}$ on Y_q obtained from the Beilinson collection on $\mathbb{P}_{\Lambda_{\mathbb{N}}}$, and its restriction to Y_q^* split-generates $D^b \operatorname{coh} Y_q^*$ by [16, Lemma 5.4].

Let Γ be the abelian subgroup of $PSL_{n+2}(\mathbb{C})$ defined in (2). Each $E_{q,k}$ admits $(n+2)^n$ Γ–linearizations, so that one obtains $(n+2)^{n+1}$ objects of $D^b \operatorname{coh} Z_q = D^b \operatorname{coh}^\Gamma Y_q$, whose total morphism dg algebra will be denoted by $\mathcal{S}_q$. It is clear that their restriction to Z_q^* split-generates $D^b \operatorname{coh} Z_q^*$, so that one has the following:

Lemma 2.2 *There is an equivalence*

$$D^b \operatorname{coh} Z_q^* \cong D^\pi \mathcal{S}_q^*$$

of triangulated categories, where $\mathcal{S}_q^* = \mathcal{S}_q \otimes_{\Lambda_{\mathbb{N}}} \Lambda_{\mathbb{Q}}$.

We write the inverse image of $\Gamma \subset PSL(V)$ by the projection $SL(V) \to PSL(V)$ as $\widetilde{\Gamma}$, and set $Q = Q_{n+2} \rtimes \Gamma = \Lambda V \rtimes \widetilde{\Gamma}$. Then the cohomology algebra of $\mathcal{S}_q$ is given by $Q \otimes \Lambda_{\mathbb{N}}$, and the central fiber is $\mathcal{S}_0 = \mathcal{S}_{n+2} \rtimes \Gamma$. As explained in [16, Section 3], first order deformations of the dg (or A_∞–)algebra $\mathcal{S}_0$ are parametrized by the *truncated Hochschild cohomology* $HH^2(\mathcal{S}_0, \mathcal{S}_0)^{\le 0}$.

Lemma 2.3 (Seidel [16, Lemma 10.5]) *The truncated Hochschild cohomology of $\mathcal{S}_0$ satisfies*

$$HH^1(\mathcal{S}_0, \mathcal{S}_0)^{\le 0} = \mathbb{C}^{n+1}, \qquad HH^2(\mathcal{S}_0, \mathcal{S}_0)^{\le 0} = \mathbb{C}^{2n+3}.$$

Sketch of proof There is a spectral sequence leading to $HH^*(\mathcal{S}_0, \mathcal{S}_0)^{\le 0}$ such that

$$E_2^{s,t} = \begin{cases} HH^{s+t}(Q, Q)^t & t \le 0, \\ 0 & \text{otherwise.} \end{cases}$$

The isomorphism

$$HH^{s+t}(Q, Q)^t \cong \bigoplus_{\gamma \in \widetilde{\Gamma}} \left(S^s(V^\gamma)^\vee \otimes \Lambda^{s + \frac{n+2}{n} t - \operatorname{codim} V^\gamma}(V^\gamma) \otimes \Lambda^{\operatorname{codim} V^\gamma}(V/V^\gamma) \right)^{\widetilde{\Gamma}}$$

implies that $E_2^{s,t} = 0$ for $s < 0$ or $s + \frac{n+2}{n} t < 0$, which ensures the convergence of the spectral sequence. One can easily see that $E_2^{s,t}$ for $s + t \le 2$ is non-zero only if

$$(s,t) = (0,0), (1,0), (2,0), \text{ or } (n+2, -n).$$

The first nonzero differential is δ_{n+1}, which is the Schouten bracket with the order $n+2$ deformation class $y_1 \dots y_{n+2}$ from (3). In total degree $s+t=1$, we have the $\tilde{\Gamma}$–invariant part of $V^\vee \otimes V$, which is spanned by elements $y_k \otimes v_k$ satisfying

$$\delta_{n+1}^{1,0}(y_k \otimes v_k) = y_1 \dots y_{n+2}$$

for $k = 1, \dots, n+2$. In total degree $s+t=2$, we have

- $(S^2 V^\vee \otimes \Lambda^2 V)^{\tilde{\Gamma}}$ generated by $(n+2)(n+1)/2$ elements $y_j y_k \otimes v_j \wedge v_k$ satisfying

  $$\delta_{n+1}^{2,0}(y_j y_k \otimes v_j \wedge v_k) = (y_1 \dots y_{n+2}) y_k \otimes v_k - (y_1 \dots y_{n+2}) y_j \otimes v_j,$$

- $(S^{n+2} V^\vee)^{\tilde{\Gamma}}$ spanned by y_k^{n+2} together with $y_1 \dots y_{n+2}$.

The kernel of $\delta_{n+1}^{1,0}$ is spanned by

$$y_1 \otimes v_1 - y_2 \otimes v_2$$

and its $n+1$ cyclic permutations, which sum up to zero. The image of $\delta_{n+1}^{1,0}$ is spanned by $y_1 \dots y_{n+2}$. The kernel of $\delta_{n+1}^{2,0}$ is spanned by

$$y_1 y_2 \otimes v_1 \wedge v_2 + y_2 y_3 \otimes v_2 \wedge v_3 - y_1 y_3 \otimes v_1 \wedge v_3$$

and its $n+1$ cyclic permutations, which also sum up to zero. Differentials $\delta_k^{s,t}$ for $k > n+1$ and $s+t \le 2$ vanish, and one obtains the desired result. □

Unfortunately, the second truncated Hochschild cohomology group $HH^2(\mathcal{S}_0, \mathcal{S}_0)^{\le 0}$ has multiple dimensions, so that one needs additional structures to characterize $\mathcal{S}_q$ as a deformation of $\mathcal{S}_0$. The strategy adopted by Seidel is to use a $\mathbb{Z}/(n+2)\mathbb{Z}$–action coming from the cyclic permutation of the basis of V: Let U_{n+2} be an automorphism of $Q_{n+2} = \Lambda V \rtimes \Gamma_{n+2}$ as an R_{n+2}–algebra, which acts on the basis of V as $v_k \mapsto v_{k+1}$. This lifts to a $\mathbb{Z}/(n+2)\mathbb{Z}$–action on $\mathcal{S}_0 = \mathcal{S}_{n+2} \rtimes \Gamma$, and $\mathcal{S}_q$ is characterized as follows:

Proposition 2.4 (Seidel [16, Proposition 10.8]) *Let $\mathcal{Q}_q$ be a one-parameter deformation of $\mathcal{S}_0 = \mathcal{S}_{n+2} \rtimes \Gamma$, which is*

- *$\mathbb{Z}/(n+2)\mathbb{Z}$–equivariant, and*
- *non-trivial at first order.*

Then $\mathcal{Q}_q$ is quasi-isomorphic to $\psi^* \mathcal{S}_q$ for some $\psi \in \mathrm{End}(\Lambda_{\mathbb{N}})^\times$.

The proof that these conditions characterize $\mathcal{S}_q$ comes from the fact that the invariant part of the second truncated Hochschild cohomology of the central fiber $\mathcal{S}_0$ with respect to the cyclic group action induced by $\mathcal{U}_0$ is one-dimensional [16, Lemma 10.7];

$$HH^2(\mathcal{S}_0, \mathcal{S}_0)^{\leq 0, \mathbb{Z}/(n+2)\mathbb{Z}} \cong \mathbb{C} \cdot \left(y_1^{n+2} + \cdots + y_{n+2}^{n+2}\right).$$

The proof that these conditions are satisfied by $\mathcal{S}_q$ carries over verbatim from [16, Section 10d].

3 Fukaya categories

Let $X = \operatorname{Proj} \mathbb{C}[x_1, \dots, x_{n+2}]$ be an $(n+1)$–dimensional complex projective space and o_X be the anticanonical bundle on X. Let further h be a Hermitian metric on o_X such that the compatible unitary connection ∇ has the curvature $-2\pi\sqrt{-1}\omega_X$, where ω_X is $n+2$ times the Fubini–Study Kähler form on X. Any complex submanifold of X has a symplectic structure given by the restriction of ω_X. The restriction of (o_X, ∇) to any Lagrangian submanifold L has a vanishing curvature, and L is said to be *rational* if the monodromy group of this flat connection is finite. Note that this condition is equivalent to the existence of a flat multi-section λ_L of $o_X|_L$ which is of unit length everywhere.

Two sections $\sigma_{X,\infty} = x_1 \dots x_{n+2}$ and $\sigma_{X,0} = x_1^{n+2} + \cdots + x_{n+2}^{n+2}$ of o_X generate a pencil $\{X_z\}_{z \in \mathbb{P}^1_{\mathbb{C}}}$ of hypersurfaces

$$X_z = \{x \in X \mid \sigma_{X,0}(x) + z\sigma_{X,\infty}(x) = 0\},$$

such that X_0 is the Fermat hypersurface and X_∞ is the union of $n+2$ coordinate hyperplanes. The complement $M = X \setminus X_\infty$ is the big torus of X, which can naturally be identified as

$$M = \{x \in \mathbb{C}^{n+2} \mid x_1 \dots x_{n+2} \neq 0\}/\mathbb{C}^\times \cong \{x \in \mathbb{C}^{n+2} \mid x_1 \dots x_{n+2} = 1\}/\,\Gamma_{n+2}^*,$$

where $\Gamma_{n+2}^* = \left\{\zeta \operatorname{id}_{\mathbb{C}^{n+2}} \mid \zeta^{n+2} = 1\right\}$ is the kernel of the natural projection from $SL_{n+2}(\mathbb{C})$ to $PSL_{n+2}(\mathbb{C})$. The map

$$\pi_M = \sigma_{X,0}/\sigma_{X,\infty} \colon M \to \mathbb{C}$$

is a Lefschetz fibration, which has $n+2$ groups of $(n+2)^n$ critical points with identical critical values. The group $\Gamma^* = \operatorname{Hom}(\Gamma, \mathbb{C}^\times)$ of characters of the group Γ defined in (2) acts freely on M through a non-canonical isomorphism $\Gamma^* \cong \Gamma$ and the natural action of $\Gamma \subset PSL_{n+2}(\mathbb{C})$ on X. The quotient

$$\overline{M} = M/\,\Gamma^* = \{u = (u_1, \dots, u_{n+2}) \in \mathbb{C}^{n+2} \mid u_1 \dots u_{n+2} = 1\}$$

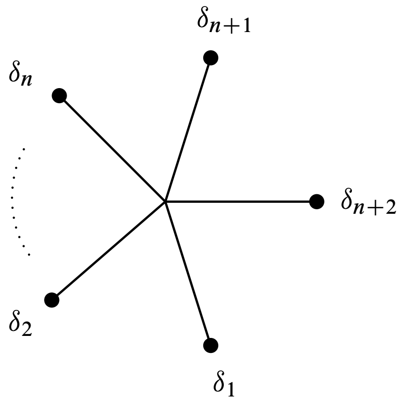

Figure 1: The distinguished set $(\delta_i)_{i=1}^{n+2}$ of vanishing paths

is another algebraic torus, where the natural projection $M \to \overline{M}$ is given by $u_k = x_k^{n+2}$. The map π_M is Γ^*–invariant and descends to the map $\pi_{\overline{M}}(u) = u_1 + \cdots + u_{n+2}$ from the quotient

$$\begin{tikzcd} M \arrow[rr, "\pi_M"] \arrow[dr] & & \mathbb{C} \arrow[dl, "\pi_{\overline{M}}"] \\ & \overline{M} & \end{tikzcd}$$

The map $\pi_{\overline{M}}\colon \overline{M} \to \mathbb{C}$ is the Landau–Ginzburg potential for the mirror of $\mathbb{P}^{n+1}$, which has $n+2$ critical points with critical values $\left\{(n+2)\zeta_{n+2}^{-i}\right\}_{i=1}^{n+2}$ where $\zeta_{n+2} = \exp\left[2\pi\sqrt{-1}/(n+2)\right]$. Choose the origin as the base point and take the distinguished set $(\delta_i)_{i=1}^{n+2}$ of vanishing paths $\delta_i\colon [0,1] \ni t \mapsto (n+2)\zeta_{n+2}^{-i}\, t \in \mathbb{C}$ as in Figure 1. The corresponding vanishing cycles in $\overline{M}_0 = \pi_{\overline{M}}^{-1}(0)$ will be denoted by V_i.

Let $\mathcal{F}_{n+2}$ be the A_∞–category whose set of objects is $\{V_i\}_{i=1}^{n+2}$ and whose spaces of morphisms are Lagrangian intersection Floer complexes. This is a full A_∞–subcategory of the Fukaya category $\mathcal{F}(\overline{M}_0)$ of the exact symplectic manifold $\overline{M}_0$. See Seidel [20] for the Fukaya category of an exact symplectic manifold, and Fukaya, Oh, Ohta and Ono [6] for that of a general symplectic manifold. We often regard the A_∞–category $\mathcal{F}_{n+2}$ with $n+2$ objects as an A_∞–algebra over the semisimple ring R_{n+2} of dimension $n+2$.

As explained in Section 5 below, the affine variety $\overline{M}_0$ is an $(n+2)$–fold cover of the n–dimensional pair of pants $\mathcal{P}^n$, and contains $n+2$ Lagrangian spheres $\{L_i\}_{i=1}^{n+2}$ whose projection to $\mathcal{P}^n$ is the Lagrangian immersion studied by Sheridan [23]. Let

$\mathcal{A}_{n+2}$ be the full A_∞–subcategory of $\mathcal{F}(\overline{M}_0)$ consisting of these Lagrangian spheres. The following proposition is proved in Section 5:

Proposition 3.1 *The Lagrangian submanifolds L_i and V_i are isomorphic in $\mathcal{F}(\overline{M}_0)$.*

The inclusion $\overline{M}_0 \subset \overline{M}$ induces an isomorphism $\pi_1(\overline{M}_0) \cong \pi_1(\overline{M})$ of the fundamental group. Let T be the torus dual to $\overline{M}$ so that $\pi_1(\overline{M}) \cong T^* := \operatorname{Hom}(T, \mathbb{C}^\times)$. One can equip $\mathcal{F}_{n+2}$ with a T–action by choosing lifts of V_i to the universal cover of $\overline{M}_0$. Let $\mathcal{F}_0$ be the Fukaya category of M_0 consisting of $N = (n+2)^{n+1}$ vanishing cycles $\{\widetilde{V}_i\}_{i=1}^N$ of π_M obtained by pulling-back $\{V_i\}_{i=1}^{n+2}$. The covering $M_0 \to \overline{M}_0$ comes from a surjective group homomorphism $\pi_1(\overline{M}_0) \to \Gamma^*$, which induces an inclusion $\Gamma \hookrightarrow T$ of the dual group. It follows from Seidel [16, Equation (8.13)] that $\mathcal{F}_0$ is quasi-isomorphic to $\mathcal{F}_{n+2} \rtimes \Gamma$, which in turn is quasi-isomorphic to $\mathcal{A}_{n+2} \rtimes \Gamma$ by Proposition 3.1.

The following proposition is due to Sheridan:

Proposition 3.2 (Sheridan [23, Proposition 5.15]) *$\mathcal{A}_{n+2}$ is T–equivariantly quasi-isomorphic to $\mathcal{S}_{n+2}$.*

Since $\mathcal{S}_0 = \mathcal{S}_{n+2} \rtimes \Gamma$, one obtains the following:

Corollary 3.3 *$\mathcal{F}_0$ is quasi-isomorphic to $\mathcal{S}_0$.*

The vanishing cycles $\{\widetilde{V}_i\}_{i=1}^N$ are Lagrangian submanifolds of the projective Calabi–Yau manifold X_0, which are rational since they are contractible in M. To show that they split-generate the Fukaya category of X_0, Seidel introduced the notion of *negativity* of a graded symplectic automorphism. Let $\mathfrak{L}_{X_0} \to X_0$ be the bundle of unoriented Lagrangian Grassmannians on the projective Calabi–Yau manifold X_0. The *phase function* $\alpha_{X_0}\colon \mathfrak{L}_{X_0} \to S^1$ is defined by

$$\alpha_{X_0}(\Lambda) = \frac{\eta_{X_0}(e_1 \wedge \ldots \wedge e_n)^2}{|\eta_{X_0}(e_1 \wedge \ldots \wedge e_n)|^2},$$

where $\Lambda = \operatorname{span}_{\mathbb{R}}\{e_1, \ldots, e_n\} \in \mathfrak{L}_{X_0,x}$ is a Lagrangian subspace of $T_x X_0$ and η_{X_0} is a holomorphic volume form on X_0. The *phase function* $\alpha_\phi\colon \mathfrak{L}_{X_0} \to S^1$ of a symplectic automorphism $\phi\colon X_0 \to X_0$ is defined by sending $\Lambda \in \mathfrak{L}_{X_0,x}$ to $\alpha_\phi(\Lambda) = \alpha_{X_0}(\phi_*(\Lambda))/\alpha_{X_0}(\Lambda)$, and a *graded symplectic automorphism* is a pair $\widetilde{\phi} = (\phi, \widetilde{\alpha}_\phi)$ of a symplectic automorphism ϕ and a lift $\widetilde{\alpha}_\phi\colon \mathfrak{L}_{X_0} \to \mathbb{R}$ of the phase function α_ϕ to the universal cover $\mathbb{R}$ of S^1. The group of graded symplectic automorphisms of X_0 will be denoted by $\widetilde{\mathrm{Aut}}(X_0)$. A graded symplectic automorphism $\widetilde{\phi} \in \widetilde{\mathrm{Aut}}(X_0)$ is *negative* if there is a positive integer d_0 such that $\widetilde{\alpha}_{\phi^{d_0}}(\Lambda) < 0$ for all $\Lambda \in \mathfrak{L}_{X_0}$.

The *phase function* $\alpha_L \colon L \to S^1$ of a Lagrangian submanifold $L \subset X_0$ is defined similarly by $\alpha_L(x) = \alpha_{X_0}(T_x L)$, and a *grading* of L is a lift $\widetilde{\alpha}_L \colon L \to \mathbb{R}$ of α_L to the universal cover of S^1. Let Λ_0 be the local subring of $\Lambda_{\mathbb{Q}}$ containing only non-negative powers of q, and Λ_+ be the maximal ideal of Λ_0. For a quintuple $L^\sharp = (L, \widetilde{\alpha}_L, \$_L, \lambda_L, J_L)$ consisting of a rational Lagrangian submanifold L, a grading $\widetilde{\alpha}_L$ on L, a spin structure $\$_L$ on L, a multi-section λ_L of $o_{X_0}|_L$, and a compatible almost complex structure J_L, one can endow the cohomology group $H^*(L; \Lambda_0)$ with the structure $\{\mathfrak{m}_k\}_{k=0}^\infty$ of a *filtered* A_∞*–algebra* (see Fukaya, Oh, Ohta and Ono [6, Definition 3.2.20]), which is well-defined up to isomorphism [6, Theorem A]. The map $\mathfrak{m}_0 \colon \Lambda_0 \to H^1(L; \Lambda_0)$ comes from holomorphic disks bounded by L, and measures the *anomaly* or *obstruction* to the definition of Floer cohomology. A solution $b \in H^1(L; \Lambda_+)$ to the *Maurer–Cartan equation*

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b, \cdots, b) = 0$$

is called a *bounding cochain*. A *rational Lagrangian brane* is a pair $L^\diamondsuit = (L^\sharp, b)$ of $L^\sharp$ and a bounding cochain $b \in H^1(L; \Lambda_+)$. For a pair $L_1^\diamondsuit = (L_1^\sharp, b_1)$ and $L_2^\diamondsuit = (L_2^\sharp, b_2)$ of rational Lagrangian branes, the *Floer cohomology* $HF\big(L_1^\diamondsuit, L_2^\diamondsuit; \Lambda_0\big)$ is well-defined up to isomorphism. The *Fukaya category* $\mathcal{F}(X_0)$ is an A_∞–category over $\Lambda_{\mathbb{Q}}$ whose objects are rational Lagrangian branes and whose spaces of morphisms are Lagrangian intersection Floer complexes.

Let $\mathcal{F}_q$ be the full A_∞–subcategory of $\mathcal{F}(X_0)$ consisting of vanishing cycles $\widetilde{V}_i$ equipped with the trivial complex line bundles, the canonical gradings and zero bounding cochains. Since the restrictions of (o_X, ∇) to vanishing cycles are trivial flat bundles, the category $\mathcal{F}_q$ is defined over $\Lambda_{\mathbb{N}}$.

Let η_M be the unique up to scalar holomorphic volume form on M which extends to a rational form on X with a simple pole along X_∞. This gives a holomorphic volume form η_M / dz on each fiber $M_z = \pi_M^{-1}(z)$, so that $\pi_M \colon M \to \mathbb{C}$ is a locally trivial fibration of graded symplectic manifolds outside the critical values. Let $\gamma_\infty \colon [0, 2\pi] \to \mathbb{C}$ be a circle of large radius $R \gg 0$ and $\widetilde{h}_{\gamma_\infty} \in \widetilde{\mathrm{Aut}}(M_R)$ be the monodromy along γ_∞. Since γ_∞ is homotopic to a product of paths around each critical values, one sees that $\widetilde{h}_{\gamma_\infty}$ is isotopic to a composition of Dehn twists along vanishing cycles. We prove the following in Section 4:

Proposition 3.4 (Seidel [16, Proposition 7.22]) *The graded symplectic automorphism $\widetilde{h}_{\gamma_\infty} \in \widetilde{\mathrm{Aut}}(M_R)$ is isotopic to a graded symplectic automorphism $\widetilde{\phi} \in \widetilde{\mathrm{Aut}}(M_R)$ whose* *extension to X_R has the following property: There is an arbitrary small neighborhood $W \subset X_R$ of the subset* $\operatorname{Sing}(X_\infty) \cap X_R$ *such that $\phi(W) = W$ and $\widetilde{\phi}|_{X_R \setminus W}$ is negative.*

Here $\operatorname{Sing}(X_\infty)$ is the singular locus of X_∞, which is the union of $(n-1)$–dimensional projective spaces.

Lemma 3.5 (Seidel [16, Lemma 9.2]) *If $n = 3$, then any rational Lagrangian brane is contained in split-closed derived category of $\mathcal{F}_q^* = \mathcal{F}_q \otimes_{\Lambda_{\mathbb{N}}} \Lambda_{\mathbb{Q}}$;*

$$D^\pi \mathcal{F}(X_0) \cong D^\pi \mathcal{F}_q^*.$$

The proof is identical to that of Seidel [16, Lemma 9.2], which is based on Seidel's long exact sequence [17] (see also [16, Section 9c] and Oh [12]).

Remark 3.6 (Seidel [16, Remark 9.3]) If $n = 3$, then the real dimension of the intersection $\operatorname{Sing}(X_\infty) \cap X_0$ is two, so that any Lagrangian submanifold can be made disjoint from a sufficiently small neighborhood W of $\operatorname{Sing}(X_\infty) \cap X_0$ by a generic perturbation. This is the only place where we use the condition $n = 3$, and one can show the equivalence (1) for any n with $D^\pi \mathcal{F}(X_0)$ replaced by the split-closure of Lagrangian branes which can be perturbed away from $\operatorname{Sing}(X_\infty) \cap X_0$.

A notable feature of Floer cohomologies over Λ_0 is their dependence on Hamiltonian isotopy: For a pair $\left(L_0^\sharp, L_1^\sharp\right)$ of Lagrangian submanifolds equipped with auxiliary choices, a symplectomorphism $\psi\colon X_0 \to X_0$ induces an isomorphism

$$\psi_*\colon \left(H^*(L_i^\sharp; \Lambda_0), \mathfrak{m}_k\right) \to \left(H^*(\psi(L_i^\sharp); \Lambda_0), \mathfrak{m}_k\right)$$

of filtered A_∞–algebras (see Fukaya, Oh, Ohta and Ono [6, Theorem A]), which induces a map ψ_* on the set of bounding cochains preserving the Floer cohomology over Λ_0 [6, Theorem G.3]:

$$HF\left((L_0^\sharp, b_0), (L_1^\sharp, b_1); \Lambda_0\right) \cong HF\left((\psi(L_0^\sharp), \psi_*(b_0)), (\psi(L_1^\sharp), \psi_*(b_1)); \Lambda_0\right).$$

On the other hand, if we move $L_0^\sharp$ and $L_1^\sharp$ by two distinct Hamiltonian isotopies ψ^0 and ψ^1, then the Floer cohomology over $\Lambda_{\mathbb{Q}}$ is preserved [6, Theorem G.4]

$$HF\left((L_0^\sharp, b_0), (L_1^\sharp, b_1); \Lambda_{\mathbb{Q}}\right) \cong HF\left((\psi^0(L_0^\sharp), \psi^0_*(b_0)), (\psi^1(L_1^\sharp), \psi^1_*(b_1)); \Lambda_{\mathbb{Q}}\right),$$

whereas the Floer cohomology over Λ_0 may not be preserved;

$$HF\left((L_0^\sharp, b_0), (L_1^\sharp, b_1); \Lambda_0\right) \ncong HF\left((\psi^0(L_0^\sharp), \psi^0_*(b_0)), (\psi^1(L_1^\sharp), \psi^1_*(b_1)); \Lambda_0\right).$$

See [6, Section 3.7.6] for a simple example where this occurs. This phenomenon is used by Seidel [16, Section 8g and 11a] to prove the following:

Proposition 3.7 (Seidel [16, Proposition 11.1]) *The A_∞–algebra $\mathcal{F}_q \otimes_{\Lambda_{\mathbb{N}}} \Lambda_{\mathbb{N}}/q^2\Lambda_{\mathbb{N}}$ is not quasi-isomorphic to the trivial deformation $\mathcal{F}_0 \otimes_{\mathbb{C}} \Lambda_{\mathbb{N}}/q^2\Lambda_{\mathbb{N}}$.*

To show this, Seidel takes a rational Lagrangian submanifold $L_{1/2}$ in X_z for sufficiently large z as follows:

(1) Consider a pencil $\{X_z\}_{z\in\mathbb{P}^1_{\mathbb{C}}}$ generated by two section $\sigma_{X,\infty} = x_1 \dots x_{n+2}$ and $\sigma_{X,0} = x_1^2(x_2^2 + x_3^2)x_4 \dots x_{n+1}$, whose general fiber is singular. Let $C = \{x_{n+2} = 0\}$ be an irreducible component of $X_\infty = \{x_1 \dots x_{n+2} = 0\} \subset X$, and $C_\infty = C \cap X_\infty$ be the intersection with other components. If we write $C_0 = X_0 \cap C$, then the set $C_0 \setminus C_\infty$ is the union of two $(n-1)$–planes $\{x_2 = \pm\sqrt{-1}x_3\}$.

(2) Let $K_{1/2} = \{2|x_1| = |x_2| = \dots = |x_{n+2}|\} \subset C \setminus C_\infty$ be a Lagrangian n–torus in C, which is a fiber of the moment map for the torus action. The intersection $K_{1/2} \cap C_0$ consists of two $(n-1)$–tori.

(3) Take a Hamiltonian function H on C supported on a neighborhood of the two $(n-1)$–tori such that the corresponding Hamiltonian vector field points in opposite directions transversally to two $(n-1)$–tori. By flowing $K_{1/2}$ along the Hamiltonian vector field in both negative and positive time directions, one obtains a family $(K_r)_{r\in[0,1]}$ of Lagrangian submanifolds of $C \setminus C_\infty$.

(4) The Lagrangian submanifolds K_r for $r \neq 1/2$ are disjoint from C_0. They are exact Lagrangian submanifolds with respect to the one-form $\theta_{C\setminus C_0}$ obtained by pulling back the connection on o_X via $\sigma_{X,0}|_{C\setminus C_0}$.

(5) Now perform a generic perturbation of $\sigma_{X,0}$ so that a general member X_z of the pencil is smooth. One still has a Lagrangian submanifold $K_{1/2} \subset C \setminus C_\infty$ satisfying the following:
   - $K_{1/2} \cap C_0$ consists of two $(n-1)$–tori.
   - By flowing $K_{1/2}$ along a Hamiltonian vector field, one obtains a family $(K_r)_{r\in[0,1]}$ of Lagrangian submanifolds of $C \setminus C_\infty$.
   - K_r for $r \neq 1/2$ are disjoint from C_0. They are exact Lagrangian submanifolds of $C \setminus C_0$.

(6) By parallel transport along the graph
$$\widehat{X} = \{(y,x) \in \mathbb{C} \times X \mid \sigma_{X,\infty}(x) = y\sigma_{X,0}(x)\} \xrightarrow{y\text{–projection}} \mathbb{C}$$
of the pencil, one obtains a Lagrangian torus $L_{1/2}$ in X_z for sufficiently large $z = 1/y$, satisfying the following conditions:

- The intersection $Z = L_{1/2} \cap X_{z,\infty}$ of $L_{1/2} \cong (S^1)^n$ with the divisor $X_{z,\infty} = X_z \cap X_\infty$ at infinity is a smooth $(n-1)$–dimensional manifold disjoint from $\operatorname{Sing}(X_\infty) \cap X_z$. (In fact, it is a disjoint union of two $(n-1)$–tori; $Z = \{1/4, 3/4\} \times (S^1)^{n-1}$.)
- By flowing $L_{1/2}$ by a Hamiltonian vector field, one obtains a family $(L_r)_{r \in [0,1]}$ of Lagrangian submanifolds of X_z.
- L_r for any $r \in [0, 1]$ admits a grading.
- L_r for $r \neq 1/2$ are disjoint from $X_{z,\infty}$. They are exact Lagrangian submanifolds in the affine part $M_z = X_z \setminus X_{z,\infty}$ of X_z.

If the perturbation of $\sigma_{X,0}$ is generic, then there are no non-constant stable holomorphic disks in X_z bounded by L_r for $r \in [0, 1]$ with area less than 2. Indeed, such a disk cannot have a sphere component since a holomorphic sphere has area at least $n + 2$. If a holomorphic disk exists in X_z for all sufficiently large z, then Gromov compactness theorem gives a holomorphic disk in X_∞ bounded by K_r. This disk either have sphere components in irreducible components of X_∞ other than C, or passes through $C_\infty \cap C_0$. The former is impossible since sphere components have area at least $n + 2$, and the latter is impossible for a disk of area less than 2 since such disks have fixed intersection points with C_∞ by classification (see Cho [4, Theorem 10.1]) of holomorphic disks in C bounded by K_r.

The absence of holomorphic disks of area less than 2 shows that the Lagrangian submanifolds $L_0^\Diamond = \left(L_0^\sharp, 0\right)$ and $L_1^\Diamond = \left(L_1^\sharp, 0\right)$ equipped with auxiliary data and the zero bounding cochains give objects of the first order Fukaya category $D^\pi \mathcal{F}_q \otimes_{\Lambda_\mathbb{N}} \Lambda_\mathbb{N}/q^2\Lambda_\mathbb{N}$. Now the argument of Seidel [16, Section 8g] shows the following:

(1) The spaces $H^0\left(\hom_{\mathcal{F}_0}\left(L_i^\Diamond, L_j^\Diamond\right)\right)$ are one-dimensional for $0 \le i \le j \le 1$.

(2) The product

$$H^0\left(\hom_{\mathcal{F}_0}\left(L_1^\Diamond, L_0^\Diamond\right)\right) \otimes H^0\left(\hom_{\mathcal{F}_0}\left(L_0^\Diamond, L_1^\Diamond\right)\right) \to H^0\left(\hom_{\mathcal{F}_0}\left(L_0^\Diamond, L_0^\Diamond\right)\right)$$

vanishes.

(3) The map

$$\begin{gathered} H^0\left(\hom_{\mathcal{F}_q}\left(L_1^\Diamond, L_0^\Diamond\right) \otimes_{\Lambda_\mathbb{N}} \Lambda_\mathbb{N}/q^2\Lambda_\mathbb{N}\right) \otimes_\mathbb{C} H^0\left(\hom_{\mathcal{F}_q}\left(L_0^\Diamond, L_1^\Diamond\right) \otimes_{\Lambda_\mathbb{N}} \Lambda_\mathbb{N}/q^2\Lambda_\mathbb{N}\right) \\ \downarrow \\ H^0\left(\hom_{\mathcal{F}_q}\left(L_0^\Diamond, L_0^\Diamond\right) \otimes_{\Lambda_\mathbb{N}} \Lambda_\mathbb{N}/q^2\Lambda_\mathbb{N}\right) \end{gathered}$$

induced by $\mathfrak{m}_2^{\mathcal{F}_q}$ is non-zero.

The point is that L_0 and L_1 are exact Lagrangian submanifolds of M_z, which are not isomorphic in $\mathcal{F}(M_z)$, but are Hamiltonian isotopic in X_z, so that they are isomorphic in $D^\pi(\mathcal{F}_q \otimes_{\Lambda_{\mathbb{N}}} \Lambda_{\mathbb{Z}})$. Now [16, Lemma 3.9] concludes the proof of Proposition 3.7.

The symplectomorphism $\overline{\phi}_0\colon \overline{M}_0 \to \overline{M}_0$ sending $(u_1, \dots, u_{n+2})$ to $(u_2, \dots, u_{n+2}, u_1)$ lifts to a $\mathbb{Z}/(n+2)$–action on $\mathcal{F}_q$ just as in [16, Section 11b]. It follows that $\mathcal{F}_q$ satisfies all the properties characterizing $\mathcal{S}_q$ in Proposition 2.4, and one obtains the following;

Proposition 3.8 *$\mathcal{F}_q$ is quasi-isomorphic to $\psi^* \mathcal{S}_q$ for some $\psi \in \operatorname{End}(\Lambda_{\mathbb{N}})^\times$.*

Theorem 1.1 follows from Lemma 2.2, Lemma 3.5, and Proposition 3.8.

Remark 3.9 Since the Lagrangian torus used in the proof of Proposition 3.7 does not intersect with $\operatorname{Sing}(X_\infty)$, the proof of Proposition 3.7 (and hence Proposition 3.8) works for any n. Then the argument of Sheridan [22, Section 8.2], based on a split-generation criterion announced by Abouzaid, Fukaya, Oh, Ohta, and Ono, shows that $\{L_i\}_{i=1}^{n+2}$ split-generates $D^\pi \mathcal{F}(X_0)$ for any n.

4 Negativity of monodromy

In this section, we prove Proposition 3.4 by using local models of the quasi-Lefschetz pencil $\{X_z\}$ along the lines of Seidel [16, Section 7]. In the case where $\dim X_z \geq 3$, we need [16, Assumption 7.8] and a generalization of [16, Assumption 7.5].

Assumption 4.1 (Seidel [16, Assumption 7.8]) *Let $n \geq 2$ and $2 \leq k \leq n+1$.*

- *$Y \subset \mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ is an open ball around the origin equipped with the standard symplectic form ω_Y and the T^k–action*

$$\rho_s(y) = \left(e^{\sqrt{-1}s_1} y_1, \dots, e^{\sqrt{-1}s_k} y_k, y_{k+1}, \dots, y_{n+1}\right)$$

  with moment map $\mu\colon Y \to \mathbb{R}^k$. For any regular value $r \in \mathbb{R}^k$ of μ, the symplectic reduction $Y^{\mathrm{red}} = Y^{\mathrm{red},r} = \mu^{-1}(r)/T^k$ can be identified with an open subset in $\mathbb{C}^{n+1-k}$ equipped with the standard symplectic form.
- *J_Y is a complex structure on Y which is tamed by ω_Y. At the origin, it is ω_Y–compatible and T^k–invariant.*
- *$p\colon Y \to \mathbb{C}$ is a J_Y–holomorphic function with the following properties:*
  (i) $p(\rho_s(y)) = e^{\sqrt{-1}(s_1 + \dots + s_k)} p(y)$.
  (ii) *$\partial_{y_1} \dots \partial_{y_k} p$ is nonzero at $y = 0$.*

- η_Y *is a* J_Y*–complex volume form on* $Y \setminus p^{-1}(0)$ *such that* $p(y)\eta_Y$ *extends smoothly on* Y*, which is nonzero at* $y = 0$*.*

In this situation, the monodromy h_ζ satisfy the following:

Proposition 4.2 (Seidel [16, Lemma 7.16]) *For every* $d > 0$ *and* $\epsilon > 0$*, there exists* $\delta > 0$ *such that the following holds. For every* $y \in Y_\zeta = p^{-1}(\zeta)$ *with* $0 < \zeta < \delta$ *and* $\|y\| < \delta$*, and every Lagrangian subspace* $\Lambda^v \subset T_y Y_\zeta$*, the* d*–fold monodromy* h_ζ^d *is well-defined near* y*, and satisfies*

$$\widetilde{\alpha}_{h_\zeta^d}(\Lambda^v) \le -2d + n + 1 + \epsilon.$$

The other local model is the following:

Assumption 4.3 *Let* $n \ge 2$ *and* $2 \le k \le n+1$*.*

- $Y \subset \mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ *is an open ball around the origin equipped with the standard symplectic form* ω_Y *and the* T^k*–action*

$$\rho_s(y) = \left(e^{\sqrt{-1}s_1} y_1, \dots, e^{\sqrt{-1}s_k} y_k, y_{k+1}, \dots, y_{n+1}\right) \tag{4}$$

  with moment map $\mu\colon Y \to \mathbb{R}^k$*. For any regular value* $r \in \mathbb{R}^k$ *of* μ*, the symplectic reduction* $Y^{\mathrm{red}} = Y^{\mathrm{red},r} = \mu^{-1}(r)/T^k$ *can be identified with an open subset in* $\mathbb{C}^{n+1-k}$ *equipped with the standard symplectic form.*

- J_Y *is a complex structure on* Y *which is tamed by* ω_Y*. At the origin, it is* ω_Y*–compatible and* T^k*–invariant.*

- p *is a* J_Y*–meromorphic function on* Y *satisfying the following two conditions:*
  (i) $p(\rho_s(y)) = e^{\sqrt{-1}(-s_1+s_2+\cdots+s_k)} p(y)$.

  This implies that p *can be written as*

$$p(y) = \frac{y_2 \dots y_k}{y_1} q(|y_1|^2/2, \dots, |y_k|^2/2, y_{k+1}, \dots, y_{n+1})$$

  for some q*.*
  (ii) q *is a smooth function defined on* Y*,* $q(0) = 1$*, and* $q(y) \neq 0$ *for any* $y \in Y$*.*

- η_Y *is a* J_Y*–complex volume form on* $Y \setminus p^{-1}(0)$ *such that* $y_2 \dots y_k \eta_Y$ *extends smoothly on* Y*. It is normalized so that* $y_2 \dots y_k \eta_Y = dy_1 \wedge \cdots \wedge dy_{n+1}$ *at* $y = 0$*.*

In this setting, we will show the negativity of the monodromy in the following sense:

Proposition 4.4 (Seidel [16, Lemma 7.16]) *For any $d > 0$ and $\epsilon > 0$, there is $\delta_1 > \delta_2 > 0$ such that for $\zeta \in \mathbb{C}$ with $0 < |\zeta| < \delta_1$ and $y \in Y_\zeta$ with $\|y\| < \delta_1$ and $|y_1| > \delta_2$, the d–fold monodromy h_ζ^d is well-defined, and*

$$\widetilde{\alpha}_{h_\zeta^d}(\Lambda^v) \le -2d\frac{1}{1+|\zeta|^2/|y_3|^{2(k-1)}} + n + 1 + \epsilon$$

for all $\Lambda^v \in Y_\zeta$, provided $|y_2| \le |y_3| \le \cdots \le |y_k|$.

Note that
$$\frac{1}{1+|\zeta|^2/|y_3|^{2(k-1)}}$$
is uniformly bounded from above on the complement of a neighborhood of $y_2 = y_3 = 0$.

Let J'_Y be the constant complex structure on Y which coincides with J_Y at the origin, and let η'_Y be the constant J'_Y–complex volume form given by

$$\eta'_Y = dy_1 \wedge \frac{dy_2}{y_2} \wedge \cdots \wedge \frac{dy_k}{y_k} \wedge \eta'_{Y^{\mathrm{red}}}$$

for some $\eta'_{Y^{\mathrm{red}}}$. The phase functions corresponding to η_Y and η'_Y are denoted by α_Y and α'_Y respectively. The proof of the following lemma is parallel to that in [16]:

Lemma 4.5 (Seidel [16, Lemma 7.12]) *For any $\epsilon > 0$, there exists $\delta > 0$ such that if $\|y\| < \delta$ and $p(y) \neq 0$ then*

$$\left| \frac{1}{2\pi} \arg(\alpha_Y(\Lambda)/\alpha'_Y(\Lambda)) \right| < \epsilon$$

for all $\Lambda \in \mathfrak{L}_{Y,y}$.

Let $H(y) = -\frac{1}{2}|p(y)|^2$ and consider its Hamiltonian vector field X and flow ϕ_t. For a regular value r of μ, the induced function, Hamiltonian vector field, and its flow on Y^{red} are denoted by

$$H^{\mathrm{red}}(y^{\mathrm{red}}) = -2^{k-3}\frac{r_2 \ldots r_k}{r_1} q(r_1, \ldots, r_k, y_{k+1}, \ldots, y_{n+1}),$$

X^{red}, and ϕ_t^{red} respectively. We write the complex structure on Y^{red} induced from J'_Y as $J'_{Y^{\mathrm{red}}}$. Then $\eta'_{Y^{\mathrm{red}}}$ gives a $J'_{Y^{\mathrm{red}}}$–complex volume form on Y^{red}. Let $\alpha'_{Y^{\mathrm{red}}}$ be the phase function corresponding to $\eta'_{Y^{\mathrm{red}}}$. The proof of the following lemma is the same as in [16]:

Lemma 4.6 (Seidel [16, Lemma 7.13]) *For any $\epsilon > 0$, there is $\delta > 0$ such that for $\|r\| < \delta$, $r_2 \dots r_k / r_1 < \delta$, $\|y^{\mathrm{red}}\| < \delta$, and $|t| < \delta r_1 / r_2 \dots r_k$, ϕ_t^{red} is well-defined and*

$$\left| \widetilde{\alpha}'_{\phi_t^{\mathrm{red}}}(\Lambda^{\mathrm{red}}) \right| < \epsilon$$

for any Lagrangian subspace Λ^{red}.

Now we prove the following:

Lemma 4.7 (Seidel [16, Lemma 7.14]) *For any $\epsilon > 0$, there is $\delta_1 > \delta_2 > 0$ such that if $\|y\| < \delta_1$, $|y_1| > \delta_2$, $0 < |p(y)| < \delta_1$ and $|t| < \delta_1 |p(y)|^{-2}$, then ϕ_t is well-defined and satisfies*

$$\left| \widetilde{\alpha}'_{\phi_t}(\Lambda) - \frac{2t}{2\pi} \left(1 + \frac{|y_1|^2}{|y_2|^2} + \cdots + \frac{|y_1|^2}{|y_k|^2} \right)^{-1} \right| < n + 1 + \epsilon$$

for any $\Lambda \in \mathfrak{L}_{Y,y}$.

Proof The proof of well-definedness of ϕ_t is parallel to [16]. Note that the condition $|y_1| > \delta_2$ is preserved under the flow since ϕ_t is T^k–equivariant. Let $H' = -\frac{1}{2}|y_2 \dots y_k / y_1|^2$ and

$$X' = -\sqrt{-1} \left(\frac{1}{|y_1|^2} + \cdots + \frac{1}{|y_k|^2} \right)^{-1} \left(-\frac{y_1}{|y_1|^2}, \frac{y_2}{|y_2|^2}, \dots, \frac{y_k}{|y_k|^2}, 0, \dots, 0 \right)$$

be its Hamiltonian vector field. Then $H(y) = H'(y) r(y)$ for some smooth function $r(y) = 1 + O(\|y\|)$. By direct computation, we have

$$\begin{aligned} \|dH'\| &\le C \left| \frac{y_2 \dots y_k}{y_1} \right|^2 \left(\frac{1}{|y_1|^2} + \cdots + \frac{1}{|y_k|^2} \right) \\ &\le C \left| \frac{y_2 \dots y_k}{y_1} \right|^2 \frac{k \|y\|^{2(k-1)}}{|y_1 \dots y_k|^2} \\ &= C \frac{k \|y\|^{2(k-1)}}{|y_1|^4}, \end{aligned}$$

which is bounded if $\|y\| < \delta_1$ and $|y_1| > \delta_2$. Then

$$\|dH - dH'\| \le |r - 1| \|dH'\| + |H'| \|dr\| \le C(\|y\| + |H'|),$$

and this implies that $\|dH - dH'\|$ is small if $|H|$ is also sufficiently small. Hence we obtain

$$\|X - X'\| < \epsilon \tag{5}$$

for small δ_1. Take a Lagrangian subspace Λ^{red} in $T_{y^{\mathrm{red}}} Y^{\mathrm{red}}$ and consider a Lagrangian subspace given by

$$\Lambda = \sqrt{-1}\, y_1 \mathbb{R} \oplus \cdots \oplus \sqrt{-1}\, y_k \mathbb{R} \oplus \Lambda^{\mathrm{red}} \subset T_y Y.$$

Then we have

$$\alpha'_Y(\Lambda) = (-1)^k \frac{y_1^2}{|y_1|^2} \cdot \alpha'_{Y^{\mathrm{red}}}(\Lambda^{\mathrm{red}}),$$

and hence

$$\begin{aligned}
\widetilde{\alpha}'_{\phi_t}(\Lambda) &= \frac{1}{2\pi} \int_0^t X \arg(\alpha'_Y((D\phi_\tau(\Lambda)) d\tau \\
&= \frac{1}{2\pi} \int_0^t X' \arg \frac{y_1^2}{|y_1|^2} d\tau + \frac{1}{2\pi} \int_0^t (X - X') \arg \frac{y_1^2}{|y_1|^2} d\tau \\
&\quad + \frac{1}{2\pi} \int_0^t X^{\mathrm{red}} \arg(\alpha'_{Y^{\mathrm{red}}}((D\phi_\tau^{\mathrm{red}}(\Lambda^{\mathrm{red}})) d\tau.
\end{aligned}$$

The third term is small from Lemma 4.6. The second term is bounded by

$$\frac{1}{2\pi} \int_0^t \|X - X'\| \left\| D \arg \frac{y_1^2}{|y_1|^2} \right\| d\tau,$$

which is also small from (5) and the fact that

$$\left\| D \arg \frac{y_1^2}{|y_1|^2} \right\| \le C \|X\| = C \|dH\|$$

is uniformly bounded. Since $|y_1|^2$ is preserved under the flow, the first term is

$$\begin{aligned}
&\frac{1}{2\pi} \int_0^t X' \arg \frac{y_1^2}{|y_1|^2} d\tau \\
&\quad = \frac{1}{2\pi} \left(\frac{1}{|y_1|^2} + \cdots + \frac{1}{|y_k|^2} \right)^{-1} \int_0^t \frac{1}{|y_1|^2} \sqrt{-1}\, y_1 \partial_{y_1} \arg \frac{y_1^2}{|y_1|^2} d\tau \\
&\quad = \frac{1}{2\pi} \left(\frac{1}{|y_1|^2} + \cdots + \frac{1}{|y_k|^2} \right)^{-1} \frac{2t}{|y_1|^2} \\
&\quad = \frac{2t}{2\pi} \left(1 + \frac{|y_1|^2}{|y_1|^2} + \cdots + \frac{|y_1|^2}{|y_k|^2} \right)^{-1}.
\end{aligned}$$

Then we obtain

$$\left| \widetilde{\alpha}'_{\phi_t}(\Lambda) - \frac{2t}{2\pi} \left(1 + \frac{|y_1|^2}{|y_2|^2} + \cdots + \frac{|y_1|^2}{|y_k|^2} \right)^{-1} \right| < \epsilon.$$

For arbitrary Lagrangian subspace Λ_1, the desired bound for $\widetilde{\alpha}'_{\phi_t}(\Lambda_1)$ is obtained from this and the fact that

$$|\widetilde{\alpha}'_{\phi_t}(\Lambda_1) - \widetilde{\alpha}'_{\phi_i}(\Lambda)| < n+1$$

(see [16, Lemma 6.11]). □

Let Z be the horizontal lift of $-\sqrt{-1}\zeta\partial_\zeta$, and ψ_t be its flow. Then there is a positive function f such that $Z = fX$, and hence $\psi_t(y) = \phi_{g_t(y)}(y)$ for

$$g_t(y) = \int_0^t f(\psi_\tau(y))d\tau.$$

By the same argument as in [16], we have:

Lemma 4.8 (Seidel [16, Lemma 7.15]) *For any $d > 0$ and $\epsilon > 0$, there is $\delta > 0$ such that for $\zeta \in \mathbb{C}$ with $0 < |\zeta| < \delta$ and $y \in Y_\zeta = p^{-1}(\zeta)$ with $\|y\| < \delta$, the d–fold monodromy h_ζ^d is well-defined, $\epsilon/|\zeta|^2 > 2\pi d$, and satisfies*

$$g_{2\pi d}(y) \le \epsilon/|\zeta|^2.$$

Proof of Proposition 4.4 Let $\eta_{Y_\zeta} = \eta_Y/(d\zeta/\zeta^2)$ be a complex volume form on Y_ζ, and α_{Y_ζ} be the corresponding phase function. Take $\Lambda \in \mathfrak{L}_{Y,y}$ such that $Dp(\Lambda) = a\mathbb{R}$ for $a \in U(1)$, and set $\Lambda^v = \Lambda \cap \ker Dp \in \mathfrak{L}_{Y_\zeta,y}$. Then

$$\alpha_{Y_\zeta}(\Lambda^v) = \frac{\zeta^4}{a^2|\zeta|^4}\alpha_Y(\Lambda). \tag{6}$$

We consider a Lagrangian subspace $\Lambda^v \in \mathfrak{L}_{Y_\zeta,y}$ such that $Dp(\Lambda^v) = \sqrt{-1}\zeta\mathbb{R}$, and containing the tangent space of the torus action on Y_ζ. Then Λ^v has the form

$$\Lambda^v = (\sqrt{-1}y_1\mathbb{R} \oplus \cdots \oplus \sqrt{-1}y_k\mathbb{R} \oplus \Lambda^{\mathrm{red}}) \cap \ker Dp.$$

Let $\Lambda = \Lambda^v \oplus Z_y\mathbb{R} \in \mathfrak{L}_{Y,y}$. Since Z is the horizontal lift of $-\sqrt{-1}\zeta\partial_\zeta \in T_\zeta(\sqrt{-1}\zeta\mathbb{R})$, $Z_{\psi_t(y)}$ is contained in $D\psi_t(\Lambda)$, and hence we have

$$D(\psi_t|_{Y_\zeta})(\Lambda^v) = D\psi_t(\Lambda) \cap \ker(Dp).$$

From this and (6) we have

$$\alpha_{\psi_t|_{Y_\zeta}}(\Lambda^v) = e^{-2t}\alpha_{\psi_t}(\Lambda).$$

Combining this with Lemma 4.5 and 4.7, we obtain

$$
\begin{aligned}
\widetilde{\alpha}_{h_\zeta^d}(\Lambda^v) &= \widetilde{\alpha}_{g_{2\pi d}(y)}(\Lambda) - 2d \\
&\le \widetilde{\alpha}'_{g_{2\pi d}(y)}(\Lambda) - 2d + \epsilon \\
&\le 2d\left(\left(1 + \frac{|y_1|^2}{|y_2|^2} + \cdots + \frac{|y_1|^2}{|y_k|^2}\right)^{-1} - 1\right) + \epsilon \\
&= -2d \frac{\frac{1}{|y_2|^2} + \cdots + \frac{1}{|y_k|^2}}{\frac{1}{|y_1|^2} + \frac{1}{|y_2|^2} + \cdots + \frac{1}{|y_k|^2}} + \epsilon \\
&\le -2d \frac{1}{1 + |\zeta|^2/|y_3|^{2(k-1)}} + \epsilon
\end{aligned}
$$

if $|y_2| \le |y_3| \le \cdots \le |y_k|$. □

Now we discuss gluing of the local models. Let $X = \mathbb{P}_{\mathbb{C}}^{n+1}$ equipped with the standard complex structure J_X, the Kähler form ω_X and the anticanonical bundle $o_X = \mathcal{K}_X^{-1} = \mathcal{O}(n+2)$ as in Section 3. For $\sigma_{X,\infty} = x_1 \cdots x_{n+2}$ and a generic section $\sigma_{X,0} \in H^0(\mathbb{P}_{\mathbb{C}}^{n+1}, \mathcal{O}(n+2))$, we consider a pencil of Calabi–Yau hypersurfaces defined by

$$X_z = \{\sigma_{X,0} - z\sigma_{X,\infty} = 0\} = p_X^{-1}(1/z),$$

where $p_X = \sigma_{X,\infty}/\sigma_{X,0}$. Let $C_i = \{x_i = 0\} \cong \mathbb{P}_{\mathbb{C}}^n$, $i = 1, \dots, n+2$ be the irreducible components of X_∞ and set $C_0 = X_0$. We assume that $\sigma_{X,0}$ is generic so that the divisor $X_0 \cup X_\infty$ is normal crossing. For $I \subset \{0, 1, \dots, n+2\}$, we write $C_I = \bigcap_{i \in I} C_i$ and $C_I^\circ = C_I \setminus \bigcup_{J \supsetneq I} C_J$. We will deform ω_X in such a way that it satisfies Assumption 4.1 (resp. Assumption 4.3) near C_I with $0 \notin I$ (resp. $0 \in I$).

Proposition 4.9 *For each I, there exists a tubular neighborhood U_I of C_I in $\mathbb{P}_{\mathbb{C}}^{n+1}$ and a fibration structure $\pi_I \colon U_I \to C_I$ such that for each $p \in C_I$ the tangent space $T_p \pi_I^{-1}(p)$ of the fiber is a complex subspaces in $T_p X$. Moreover π_I and π_J are compatible if $I \subset J$.*

See Ruan [15, Proposition 7.1] for the definition of the compatibility. This proposition is a weaker version of [15, Proposition 7.1] in the sense that each fiber $\pi_I^{-1}(p)$ is required to be holomorphic only at $p \in C_I$.

Proof For each I we take a tubular neighborhood U_I of C_I, and consider an open covering $\{V_\alpha\}_{\alpha \in A}$ of $\bigcup_I U_I$ satisfying

- for each $\alpha \in A$, there exists a unique subset I_α in $\{0, 1, \dots, n+1\}$ such that $V_\alpha \cap C_{I_\alpha} \ne \varnothing$ and $V_\alpha \cap C_J = \varnothing$ for all J with $|J| > |I_\alpha|$,

- V_α is a tubular neighborhood of $V_\alpha \cap C_{I_\alpha}$, and
- for each α, there exits a unique $J_\alpha \supset I_\alpha$ such that if $V_{\alpha'}$ intersects V_α and $|I_{\alpha'}| > |I_\alpha|$ then $I_\alpha \subset I_{\alpha'} \subset J_\alpha$.

We take holomorphic coordinates $(w_\alpha, z_\alpha) = (w_\alpha^1 \dots, w_\alpha^{n+1-|I_\alpha|}, z_\alpha^1, \dots, z_\alpha^{|I_\alpha|})$ on V_α such that C_{I_α} is given by $z_\alpha = 0$ and w_α gives a coordinate on $C_{I_\alpha} \cap V_\alpha$, and satisfying the following property: the projection $\pi_\alpha\colon V_\alpha \to C_{I_\alpha}$, $(w_\alpha, z_\alpha) \mapsto w_\alpha$ is compatible with π_J for each $J \supset I_\alpha$. Let $\{\rho_\alpha\}_{\alpha \in A}$ be a partition of unity associated to $\{V_\alpha\}$.

Fix $p \in C_I^\circ$, and set $A_p := \{\alpha \in A \mid p \in V_\alpha\}$. Note that $I_\alpha \supset I$ for any $\alpha \in A_p$. Take $\alpha_0 \in A$ such that $V_{\alpha_0} \cap V_\alpha \neq \varnothing$ for $\alpha \in A_p$ and $I_{\alpha_0} = J_\alpha$ is maximal. Rename the coordinates on V_α, $\alpha \in A_p$ so that the projection $\pi'_\alpha\colon V_\alpha \to C_I$ is given by $(w'_\alpha, z'_\alpha) \mapsto w'_\alpha$. Let

$$\mathrm{pr}\colon TV_{\alpha_0}|_{C_I} = \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial w'^i_{\alpha_0}} \right\} \oplus \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z'^j_{\alpha_0}} \right\} \longrightarrow \operatorname{Ker} d\pi'_{\alpha_0} = \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z'^j_{\alpha_0}} \right\}$$

be the projection. After a coordinate change which is linear in z'_α, we assume that $\mathrm{pr}(\partial/\partial z'^j_\alpha) = \partial/\partial z'^j_{\alpha_0}$ for each j. Define

$$E_{I,p} = \mathrm{span}_{\mathbb{C}} \left\{ \sum_\alpha \rho_\alpha(p) \frac{\partial}{\partial z'^j_\alpha} \,\middle|\, j = 1, \dots, |I| \right\}.$$

Then $E_I = \bigcup_{p \in C_I} E_{I,p} \subset TX|_{C_I}$ is a complex subbundle which gives a splitting of $TX|_{C_I} \to \mathcal{N}_{C_I/X} = TX|_{C_I}/TC_I$. After shrinking U_I if necessary, we obtain a fibration $\pi_I\colon U_I \to C_I$ such that $T_p\pi_I^{-1}(p) = E_{I,p}$. □

Set $U_I^\circ = \pi_I^{-1}(C_I^\circ)$. We prove a weaker version of [15, Theorem 7.1].

Proposition 4.10 *There exists a Kähler form ω'_X in the class $[\omega_X]$ such that*

(i) *it tames J_X, and compatible with J_X on $\bigcup_I C_I$,*

(ii) *$\omega'_X = \omega_X$ outside a neighborhood of $\mathrm{Sing}(X_0 \cup X_\infty) = \bigcup_{|I| \geq 2} C_I$,*

(iii) *C_i's intersect orthogonally, and*

(iv) *each fiber of $\pi_I\colon U_I \to C_I$ is orthogonal to C_I.*

Proof It is shown by Seidel [17, Lemma 1.7] and Ruan [15, Lemma 4.3] that ω_X can be modified locally so that it is standard near the lowest dimensional stratum $\bigcup_{|I|=n+1} C_I$. We deform the symplectic form inductively to obtain ω'_X

Fix $I \subset \{0, 1, \dots, n+1\}$ and take a distance function $r\colon X \to \mathbb{R}_{\geq 0}$ from C_I, i.e., $C_I = r^{-1}(0)$. Fix a local trivialization of $o_X|_{U_I}$ by a section which has unit pointwise norm and parallel in the radial direction of the fibers of π_I, and let θ_X denote the connection 1–form. Then we have $\theta_X - \pi_I^*(\theta_X|_{TC_I}) = O(r)$.

Let $\pi\colon NC_I \to C_I$ be the symplectic normal bundle, i.e., $N_p C_I \subset T_p X$ is the orthogonal complement of $T_p C_I$ with respect to the symplectic form. Let ω_N be the induced symplectic form on the fibers of NC_I. From the symplectic neighborhood theorem, a neighborhood of C_I is symplectomorphic to a neighborhood of the zero section of NC_I equipped with the symplectic form $\pi^*(\omega_X|_{C_I}) + \omega_N$. Identifying NC_I with E_I, we obtain a symplectic form ω_{U_I} on U_I satisfying (i) and (iv). Note that ω_{U_I} and ω_X coincide only on TC_I in general. Let θ_{U_I} be a connection 1–form on $o_X|_{U_I}$ such that $d\theta_{U_I} = \omega_{U_I}$ and $\theta_{U_I}|_{TC_I} = \theta_X|_{TC_I}$. We define $\eta = \theta_X - \theta_{U_I}$. Then $\eta = 0$ on C_I. Fix a constant $\delta > 0$ such that $\{r \leq \delta\} \subset U_I$ and take $C > 0$ satisfying

$$\begin{cases} C^{-1}\omega_X \leq t\omega_{U_I} + (1-t)\omega_X \leq C\omega_X, & t \in [0,1], \\ \|\eta\| \leq Cr, \\ \|dr\| \leq C \end{cases}$$

on $\{r \leq \delta\}$. Let $h\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a smooth function satisfying

- $\lim_{s \to -\infty} h(s) = 1$,
- $h(s) = 0$ for $s \geq \log \delta$, and
- $-1/(2C^3) \leq h'(s) \leq 0$,

and set $f = h(\log r)$. We define

$$\theta' = \theta_X - f\eta = f\theta_{U_I} + (1-f)\theta_X$$

and

$$\begin{aligned} \omega' := d\theta' &= f\omega_{U_I} + (1-f)\omega_X - df \wedge \eta \\ &= f\omega_{U_I} + (1-f)\omega_X - h' dr \wedge \frac{\eta}{r}. \end{aligned}$$

Then ω' is compatible with J_X along C_I and the fibers of π_I intersect C_I orthogonally. From the choice of h, we have

$$\|df \wedge \eta\| \leq \frac{1}{2C^3} \cdot C \cdot C = \frac{1}{2C},$$

which implies that ω' tames J_X, and hence it is non-degenerate.

By applying the argument in Seidel [17, Lemma 1.7] or Ruan [15, Lemma 4.3] to each fiber of π_I, we can modify ω' to make $\omega'|_{\pi_I^{-1}(p)}$ standard at each $p \in C_I$, which means that C_J's intersect orthogonally along C_I. □

Next we construct local torus actions. Set $\mathcal{L}_i = \mathcal{O}(1) = \mathcal{O}(C_i)$ for $i = 1, \dots, n+2$ and $\mathcal{L}_0 = \mathcal{O}(n+2) = \mathcal{O}(C_0)$. Note that the normal bundle of C_I is given by

$$\mathcal{N}_{C_I/X} = \bigoplus_{i \in I} \mathcal{L}_i|_{C_I}.$$

For each $I = \{i_1 < \cdots < i_k\} \subset \{0, 1, \dots, n+2\}$, we define a T^k–action on U_I° as follows. First we consider the case $0 \notin I$. We may assume $\left(\prod_{j \notin I \cup \{0\}} x_j\right) / \sigma_{X,0} \neq 0$ on U_I° (after making U_I smaller if necessary). Then

$$\bullet \otimes \frac{\prod_{j \notin I \cup \{0\}} x_j}{\sigma_{X,0}} \colon \mathcal{L}_{i_k}|_{U_I^\circ} \longrightarrow \mathcal{L}_{i_k} \otimes \mathcal{L}_0^{-1} \otimes \bigotimes_{j \notin I \cup \{0\}} \mathcal{L}_j \Bigg|_{U_I^\circ} \cong \mathcal{O}(1-k)|_{U_I^\circ}$$

is an isomorphism, and thus we have

$$\mathcal{N}_{C_I/X}|_{C_I^\circ} \cong \mathcal{N}_I|_{C_I^\circ},$$

where

$$\begin{aligned} \mathcal{N}_I :=& \mathcal{L}_{i_1} \oplus \cdots \oplus \mathcal{L}_{i_{k-1}} \oplus \left(\mathcal{L}_{i_k} \otimes \mathcal{L}_0^{-1} \otimes \bigotimes_{j \notin I \cup \{0\}} \mathcal{L}_j \right) \\ \cong& \underbrace{\mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1)}_{k-1} \oplus \mathcal{O}(1-k). \end{aligned}$$

We identify U_I° with a neighborhood of the zero section of $\mathcal{N}_I|_{C_I^\circ}$ by a map $\nu_I \colon U_I^\circ \to \mathcal{N}_I|_{C_I^\circ}$ obtained by combining

$$\left(x_{i_1}, \dots, x_{i_{k-1}}, \frac{x_{i_k} \prod_{j \notin I \cup \{0\}} x_j}{\sigma_{X,0}} \right) \colon U_I^\circ \longrightarrow \mathcal{N}_I$$

with parallel transport along the fibers of $\pi_I \colon U_I^\circ \to C_I^\circ$. The torus action on U_I° is defined to be the pull back the natural T^k–action on $\mathcal{N}_I|_{C_I^\circ}$. By construction,

$$\begin{tikzcd} U_i^\circ \arrow[rr, "\nu_I"] \arrow[dr, "p_X"'] & & \mathcal{N}_I|_{C_I^\circ} \arrow[dl] \\ & \mathbb{C} & \end{tikzcd} \tag{7}$$

is commutative, where the right arrow is the natural map

$$\mathcal{N}_I = \mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1) \oplus \mathcal{O}(1-k) \longrightarrow \mathbb{C}, \quad (\zeta_1, \ldots, \zeta_k) \longmapsto \zeta_1 \ldots \zeta_k.$$

Hence $p_X = \sigma_{X,\infty}/\sigma_{X,0}$ is T^k–equivalent on U_I°:

$$p_X(\rho_{I,s}(x)) = e^{\sqrt{-1}(s_1+\cdots+s_k)} p_X(x).$$

Next we consider the case where $i_1 = 0 \in I$. In this case we set

$$\begin{aligned}\mathcal{N}_I :=& \mathcal{L}_{i_1} \oplus \cdots \oplus \mathcal{L}_{i_{k-1}} \oplus \Big(\mathcal{L}_{i_k} \otimes \bigotimes_{j \notin I} \mathcal{L}_j\Big) \\ \cong& \mathcal{O}(n+2) \oplus \underbrace{\mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1)}_{k-2} \oplus \mathcal{O}(n+4-k).\end{aligned}$$

Assuming $\prod_{j \notin I} x_j \neq 0$ on U_I°, we have an isomorphism

$$\bigoplus_{i \in I} \mathcal{L}_i|_{U_I^\circ} \longrightarrow \mathcal{N}_I|_{U_I^\circ}.$$

By using

$$\Big(\sigma_{X,0}, x_{i_2}, \ldots, x_{i_{k-1}}, x_{i_k} \prod_{j \notin I} x_j\Big) \colon U_I^\circ \longrightarrow \mathcal{N}_I,$$

we have a map $\nu_I \colon U_I^\circ \to \mathcal{N}_I|_{C_I^\circ}$ identifying U_I° with a neighborhood the zero section, which gives a T^k–action on U_I° as above. We also have a similar commutative diagram (7) where the right arrow in this case is

$$\mathcal{O}(n+2) \oplus \mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1) \oplus \mathcal{O}(n+4-k) \longrightarrow \mathbb{C}, \quad (\zeta_1, \ldots, \zeta_k) \longmapsto \frac{\zeta_2 \ldots \zeta_k}{\zeta_1}.$$

This means that p_X is T^k–equivariant on U_I°:

$$p_X(\rho_{I,s}(x)) = e^{\sqrt{-1}(-s_1+s_2+\cdots+s_k)} p_X(x).$$

We can easily check the compatibility of the above torus actions. For example, we consider the case where $I = \{0, 1, \ldots, k-1\} \supset J = \{1, \ldots, l\}$. Take coordinates $(w_1, \ldots, w_{n+1})$ around a point in C_I such that $(w_1, \ldots, w_k)$ gives fiber coordinates of π_I corresponding to

$$(\sigma_{X,0}, x_1, \ldots, x_{k-2}, x_{k-1} \cdots x_{n+2}) \colon U_I \to \mathcal{N}_I.$$

Then the torus action is given by

$$(w_1, \ldots, w_n) \longmapsto \big(e^{\sqrt{-1}s_1} w_1, \ldots, e^{\sqrt{-1}s_k} w_k, w_{k+1}, \ldots, w_{n+1}\big).$$

On the other hand, since $\nu_J\colon U_J^\circ \to \mathcal{N}_J|_{C_J^\circ}$ is obtained from

$$\left(x_1, \dots, x_{l-1}, \frac{x_l \dots x_{n+2}}{\sigma_{X,0}}\right)\colon U_J^\circ \longrightarrow \mathcal{N}_J,$$

ν_J restricted to $U_I^\circ \cap U_J^\circ \subset U_J^\circ$ is given by

$$\nu_J(w_1, \dots, w_{n+1}) = \left(w_2, \dots, w_l, \frac{w_{l+1} \dots w_k}{w_1}\right).$$

This means that the torus action induced from ρ_J is given by

$$(w_1, \dots, w_{n+1}) \longmapsto \left(w_1, e^{\sqrt{-1}s_2} w_2, \dots, e^{\sqrt{-1}s_{l+1}} w_{l+1}, w_{l+2}, \dots, w_{n+1}\right).$$

(Note that $(w_1, w_{l+2}, \dots, w_{n+1})$ is a coordinate on the base $C_J \cap U_I$.) Other cases can be checked in similar ways.

By using the same argument as in Seidel [16, Lemma 7.20], we have

Proposition 4.11 *There exists a Kähler form ω_X'' in the class $[\omega_X]$ satisfying the conditions in Proposition 4.10, and $\omega_X''|_{U_I^\circ}$ is invariant under the torus action ρ_I for each I.*

We fix $x \in C_I^\circ$ with $|I| = k$ and take a neighborhood $U_x \subset U_I^\circ$ of x. Let $Y \subset \mathbb{C}^{n+1}$ be a small ball around the origin with the standard symplectic structure ω_Y and the T^k–action (4). Take a T^k–equivariant Darboux coordinate $\varphi\colon (U_x, \omega_X'') \to (Y, \omega_Y)$, and define $J_Y = (\varphi^{-1})^* J_X$, $p = (\varphi^{-1})^* p_X$, $\eta_Y = C(\varphi^{-1})^* \sigma_{X,\infty}^{-1}$, where C is a constant. Then $(Y, \omega_Y, J_Y, \eta_Y, p)$ satisfies Assumption 4.1 if $0 \notin I$, or Assumption 4.3 if $0 \in I$ for a suitable choice of C. Now we can follow the argument of [16, Proposition 7.22] to complete the proof of Proposition 3.4.

5 Sheridan's Lagrangian as a vanishing cycle

An n–dimensional pair of pants is defined by

$$\mathcal{P}^n = \left\{[z_1 : \dots : z_{n+2}] \in \mathbb{P}_{\mathbb{C}}^{n+1} \mid z_1 + \dots + z_{n+2} = 0,\ z_i \neq 0,\ i = 1, \dots, n+2\right\},$$

equipped with the restriction of the Fubini–Study Kähler form on $\mathbb{P}_{\mathbb{C}}^{n+1}$. It is the intersection of the hyperplane $H = \{z_1 + \dots + z_{n+2} = 0\}$ with the big torus T of $\mathbb{P}_{\mathbb{C}}^{n+1}$. Sheridan [23] perturbs the standard double cover $S^n \to H_{\mathbb{R}}$ of the real projective space $H_{\mathbb{R}} \cong \mathbb{P}_{\mathbb{R}}^n$ by the n–sphere slightly to obtain an exact Lagrangian immersion

$i\colon S^n \to \mathcal{P}^n$. The real part $\mathcal{P}^n \cap H_{\mathbb{R}}$ of the pair of pants consists of $2^{n+1}-1$ connected components U_K parametrized by proper subsets $K \subset \{1, 2, \dots, n+2\}$ as

$$U_K = \{[z_1 : \cdots : z_{n+2}] \in \mathcal{P}^n \cap H_{\mathbb{R}} \mid z_i/z_j < 0 \text{ if and only if } i \in K \text{ and } j \in K^c\}.$$

Note that the set $\{1, \dots, n+2\}$ has $2^{n+2}-2$ proper subsets, and one has $U_K = U_{K^c}$. The inverse images of the connected component U_K by the double cover $S^n \to H_{\mathbb{R}}$ are the cells $W_{K,K^c,\varnothing}$ and $W_{K^c,K,\varnothing}$ of the dual cellular decomposition in [23, Definition 2.6].

The map $p_{\overline{M}}\colon \overline{M} \to T$ sending $(u_1, \dots, u_{n+1}, u_{n+2} = 1/u_1 \cdots u_{n+1})$ to $[z_1 : \cdots : z_{n+1} : 1]$ for $z_i = u_i \cdot u_1 \cdots u_{n+1}$, $i = 1, \dots, n+1$ is a principal Γ^*_{n+2}–bundle, where the action of $\zeta \cdot \mathrm{id}_{\mathbb{C}^{n+2}} \in \Gamma^*_{n+2}$ sends $(u_1, \dots, u_{n+2})$ to $(\zeta u_1, \dots, \zeta u_{n+2})$. The inverse map is given by $u_1^{n+2} = z_1^{n+1}/z_2 \cdots z_{n+1}$ and $u_i = u_1 \cdot z_i/z_1$ for $i = 2, \dots, n+1$. The restriction $p_{\overline{M}_0}\colon \overline{M}_0 \to \mathcal{P}^n$ turns $\overline{M}_0$ into a principal Γ^*_{n+2}–bundle over the pair of pants. One has

$$z_1 = -(1 + z_2 + \cdots + z_{n+1})$$

on $\mathcal{P}^n$, so that $u_1^{n+2} = (-1)^{n+1} f(z_2, \dots, z_{n+1})$ where

$$f(z_2, \dots, z_{n+1}) = \frac{(1 + z_2 + \cdots + z_{n+1})^{n+1}}{z_2 \cdots z_{n+1}}. \tag{8}$$

The pull-back of Sheridan's Lagrangian immersion by $p_{\overline{M}_0}$ is the union of $n+2$ embedded Lagrangian spheres $\{L_i\}_{i=1}^{n+2}$ in $\overline{M}_0$.

Recall that the *coamoeba* of a subset of a torus $(\mathbb{C}^\times)^{n+1}$ is its image by the argument map $\mathrm{Arg}\colon (\mathbb{C}^\times)^{n+1} \to \mathbb{R}^{n+1}/2\pi\mathbb{Z}^{n+1}$. Let Z be the zonotope in $\mathbb{R}^{n+1}$ defined as the Minkowski sum of $\pi e_1, \dots, \pi e_{n+1}, -\pi e_1 - \cdots - \pi e_{n+1}$, where $\{e_i\}_{i=1}^{n+1}$ is the standard basis of $\mathbb{R}^{n+1}$. The projection $\overline{Z}$ of Z to $\mathbb{R}^{n+1}/2\pi\mathbb{Z}^{n+1}$ is the closure of the complement $(\mathbb{R}^{n+1}/2\pi\mathbb{Z}^{n+1}) \setminus \mathrm{Arg}(\mathcal{P}^n)$ of the coamoeba of the pair of pants [23, Proposition 2.1], and the argument projection of the immersed Lagrangian sphere is close to the boundary of the zonotope by construction [23, Section 2.2]. The coamoeba of $\overline{M}_0$ and the projections of Lagrangian spheres L_i are obtained from those for $\mathcal{P}^n$ as the pull-back by the $(n+2)$–fold cover

$$\begin{array}{ccc} \mathbb{R}^{n+1}/2\pi\mathbb{Z}^{n+1} & \to & \mathbb{R}^{n+1}/2\pi\mathbb{Z}^{n+1} \\ \Psi & & \Psi \\ e_i & \mapsto & e_i + \sum_{j=1}^{n+1} e_j \end{array} \tag{9}$$

induced by $p_{\overline{M}}\colon \overline{M} \to T$. It is elementary to see that none of the pull-backs of the zonotope $\overline{Z}$ by the map (9) has self-intersections. It follows that the argument projection of L_i does not have self-intersections either, which in turn implies that L_i itself does not have self-intersections, so that L_i is not only immersed but embedded. We choose the numbering on these embedded Lagrangian spheres so that the argument projection of L_i is close to the boundary of the zonotope centered at $[\frac{2\pi}{n+2}(i,\dots,i)] \in \mathbb{R}^{n+1}/2\pi\mathbb{Z}^{n+1}$.

When $n=1$, the coamoeba of $\overline{M}_0$ is the union of the interiors and the vertices of six triangles shown in Figure 2(a). The projection of L_3 is also shown as a solid loop in Figure 2(a). The zonotope $\overline{Z}$ in this case is a hexagon, whose pull-backs by the three-to-one map (9) are three hexagons constituting the complement of the coamoeba. Although the zonotope $\overline{Z}$ has self-intersections at its vertices, none of its pull-backs has self-intersections as seen in Figure 2(a). The coamoeba of $\overline{M}_0$ for $n=2$ is a four-fold cover of the coamoeba of $\mathcal{P}^2$ shown in [23, Figure 2(b)].

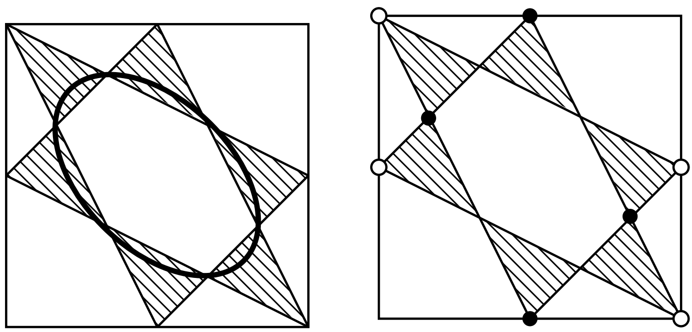

Figure 2: (a) The coamoeba (b) The cut and the thimble

Let $\varpi\colon \overline{M}_0 \to \mathbb{C}^\times$ be the projection sending $(u_1,\cdots,u_{n+2})$ to u_1.

Lemma 5.1 *The critical values of ϖ are given by $(n+2)$ solutions to the equation*

$$u_1^{n+2} = (-1)^{n+1}(n+1)^{n+1}. \tag{10}$$

Proof The defining equation of $\overline{M}_0$ in $\overline{M} = \operatorname{Spec}\mathbb{C}[u_1^{\pm 1},\dots,u_{n+1}^{\pm 1}]$ is given by

$$\sum_{i=1}^{n+1} u_i \cdot u_1 \cdots u_{n+1} + 1 = 0. \tag{11}$$

By equating the partial derivatives by $u_2,\dots,u_{n+1}$ with zero, one obtains the linear equations

$$u_i + \sum_{j=1}^{n+1} u_j = 0, \qquad i = 2,\dots,n+1,$$

whose solution is given by $u_2 = \cdots = u_{n+1} = -u_1/(n+1)$. By substituting this into (11), one obtains the desired equation (10). □

Note that the connected component

$$U_1 = U_{\{2,\dots,n+2\}} = \{[z_1 : z_2 : \cdots : z_{n+1} : 1] \in \mathcal{P}^n \mid (z_2, \dots, z_{n+1}) \in (\mathbb{R}^{>0})^n\}$$

of the real part of the pair of pants can naturally be identified with $(\mathbb{R}^{>0})^n$.

Lemma 5.2 *The function*

$$f(z_2, \dots, z_{n+1}) = \frac{(1 + z_2 + \cdots + z_{n+1})^{n+1}}{z_2 \cdots z_{n+1}}$$

has a unique non-degenerate critical point in $U_1 \cong (\mathbb{R}^{>0})^n$ with the critical value $(n+1)^{n+1}$.

Proof The partial derivatives are given by

$$\frac{\partial f}{\partial z_2} = ((n+1)z_2 - (1 + z_2 + \cdots + z_{n+1}))\frac{(1 + z_2 + \cdots + z_{n+1})^n}{z_2^2 z_3 \cdots z_{n+1}}$$

and similarly for $z_3, \dots, z_{n+1}$. By equating them with zero, one obtains the equations

$$(n+1)z_i - (1 + z_2 + \cdots + z_{n+1}) = 1, \qquad i = 2, \dots, n+1$$

whose solution is given by $z_2 = \cdots = z_{n+1} = 1$ with the critical value $(n+1)^{n+1}$. □

As an immediate corollary, one has:

Corollary 5.3 *The inverse image of $f\colon U_1 \to \mathbb{R}$ at $t \in \mathbb{R}$ is*

- *empty if $t < (n+1)^{n+1}$,*
- *one point if $t = (n+1)^{n+1}$, and*
- *diffeomorphic to S^{n-1} if $t > (n+1)^{n+1}$.*

Recall that f is introduced in (8) to study the inverse image of the map $p\colon \overline{M}_0 \to \mathcal{P}^n$.

Corollary 5.4 *The inverse image $p^{-1}(U_1)$ consists of $n+2$ connected components U_ζ indexed by solutions to the equation $\zeta^{n+2} = (-1)^{n+1}(n+1)^{n+1}$ by the condition that $\zeta \in \varpi(U_\zeta)$.*

One obtains an explicit description of Lefschetz thimbles:

Lemma 5.5 *U_ζ is the Lefschetz thimble for $\varpi\colon \overline{M}_0 \to \mathbb{C}^\times$ above the half line $\ell\colon [0,\infty) \to \mathbb{C}^\times$ on the x_1–plane given by $\ell(t) = t\zeta + \zeta$.*

Proof The restriction of ϖ to U_ζ has a unique critical point at $(x_1, \dots, x_{n+1}) = \frac{\zeta}{n+1}(n+1, -1, \dots, -1)$. For $x = (x_1, \dots, x_{n+1}) \in U_\zeta$ outside the critical point, the fiber $\mathcal{V}_{x_1} = U_\zeta \cap \varpi^{-1}(x_1)$ is diffeomorphic to S^{n-1} by Corollary 5.3, and it suffices to show that the orthogonal complement of $T_x \mathcal{V}_{x_1}$ in $T_x U_\zeta$ is orthogonal to $T_x \varpi^{-1}(x_1)$ with respect to the Kähler metric g of $\overline{M}_0$. Let $X \in T_x U_\zeta$ be a tangent vector orthogonal to $T_x \mathcal{V}_{x_1}$. Then it is also orthogonal to $T_x \varpi^{-1}(x_1)$ since any element in $T_x \varpi^{-1}(x_1)$ can be written as zY for $z \in \mathbb{C}$ and $Y \in T_x \mathcal{V}_{x_1}$, so that $g(zY, X) = zg(Y, X) = 0$. $\square$

The following simple lemma is a key to the proof of Proposition 3.1:

Lemma 5.6 *U_ζ for $\arg \zeta \neq \pm \frac{n+1}{n+2}\pi$ does not intersect L_{n+2}.*

Proof The map $\mathbb{R}^{n+1}/2\pi\mathbb{Z}^{n+1} \to \mathbb{R}^{n+1}/2\pi\mathbb{Z}^{n+1}$ induced from the map $p\colon \overline{M} \to T$ is given on coordinate vectors by $e_i \mapsto e_i + \sum_{j=1}^{n+1} e_j$. The inverse map is given by $e_i \mapsto f_i = e_i - \frac{1}{n+2}\sum_{j=1}^{n+1} e_j$, so that the argument projection of L_{n+2} is close to the boundary of the zonotope Z_{n+2} generated by $\pi f_1, \dots, \pi f_{n+1}, -\pi f_1 - \cdots - \pi f_{n+1}$. The argument projection of U_ζ consists of just one point $(\arg(\zeta), \arg(\zeta) + \pi, \dots, \arg(\zeta) + \pi)$, which is disjoint from Z_{n+2} if $\arg \zeta \neq \pm \frac{n+1}{n+2}\pi$. $\square$

The $n = 1$ case is shown in Figure 2(b). Black dots are images of U_ζ for $\zeta = \sqrt[3]{4}$, $\sqrt[3]{4}\exp(2\pi\sqrt{-1}/3)$, $\sqrt[3]{4}\exp(4\pi\sqrt{-1}/3)$, and white dots are images of $\overline{M}_0 \setminus E$ defined below. One can see that L_3 is contained in E and disjoint from $U_{\sqrt[3]{4}}$.

Now we use symplectic Picard–Lefschetz theory developed by Seidel [20]. Put $S = \mathbb{C}^\times \setminus (-\infty, 0)$ and let $E = \varpi^{-1}(S)$ be an open submanifold of $\overline{M}_0$. Note that both V_{n+2} and L_{n+2} are contained in E. The restriction $\varpi_E\colon E \to S$ of ϖ to E is an exact symplectic Lefschetz fibration, in the sense that all the critical points are non-degenerate with distinct critical values. Although ϖ_E does not fit in the framework of Seidel [20, Section III] where the total space of a fibration is assumed to be a compact manifold with corners, one can apply the whole machinery of [20] by using the tameness of ϖ_E (i.e., the gradient of $\|\varpi_E\|$ is bounded from below outside of a compact set by a positive number) as in Seidel [21, Section 6]. Let $\mathcal{F}(\varpi_E)$ be the Fukaya category of the Lefschetz fibration in the sense of Seidel [20, Definition 18.12]. It is the $\mathbb{Z}/2\mathbb{Z}$–invariant part of the Fukaya category of the double cover $\widetilde{E} \to E$ branched along $\varpi_E^{-1}(*)$, where $* \in S$ is a regular value of ϖ_E. Different base points $* \in S$ lead to symplectomorphic double covers, so that the quasi-equivalence class of $\mathcal{F}(\varpi_E)$ is independent of this choice. We choose $*$ to be a sufficiently large real number. Let $(\gamma_1, \dots, \gamma_{n+2})$ be a distinguished set of vanishing paths chosen as in

Figure 3(a). The pull-backs of the corresponding Lefschetz thimbles in E by the double cover $\widetilde{E} \to E$ will be denoted by $(\widetilde{\Delta}_1, \dots, \widetilde{\Delta}_{n+2})$, which are called type (B) Lagrangian submanifolds by Seidel [20, Section 18a]. On the other hand, the pull-back of a closed Lagrangian submanifold of E, which is disjoint from the branch locus, is a Lagrangian submanifold of $\widetilde{E}$ consisting of two copies of the original Lagrangian submanifold. It also gives rise to an object of $\mathcal{F}(\varpi_E)$, which is called a type (U) Lagrangian submanifold by Seidel. The letters (B) and (U) stand for 'branched' and 'unbranched' respectively.

Theorem 5.7 (Seidel [20, Propositions 18.13, 18.14, and 18.17])

- $(\widetilde{\Delta}_1, \dots, \widetilde{\Delta}_{n+2})$ *is an exceptional collection in* $\mathcal{F}(\varpi_E)$*.*
- *There is a cohomologically full and faithful* A_∞*–functor* $\mathcal{F}(E) \to \mathcal{F}(\varpi_E)$*.*
- *The essential image of* $\mathcal{F}(E)$ *is contained in the full triangulated subcategory generated by* $(\widetilde{\Delta}_1, \dots, \widetilde{\Delta}_{n+2})$*.*

We abuse the notation and use the same symbol L_{n+2} for the corresponding object in $\mathcal{F}(\varpi_E)$. The following lemma is a consequence of Lemma 5.6:

Lemma 5.8 *One has* $\operatorname{Hom}^*_{\mathcal{F}(\varpi_E)}(\widetilde{\Delta}_i, L_{n+2}) = 0$ *for* $i \neq 1, n+2$*.*

Proof For $2 \le i \le n+1$, move $* \in S$ continuously from the positive real axis to

$$*' = \exp[(-n-3+2i)\pi\sqrt{-1}/(n+2)] \cdot *$$

and move the distinguished set $(\gamma_1, \dots, \gamma_{n+2})$ of vanishing paths in Figure 3(a) to $(\gamma'_1, \dots, \gamma'_{n+2})$ in Figure 3(b) accordingly. The corresponding double covers $\widetilde{E}$ and $\widetilde{E}'$ are related by a Hamiltonian isotopy sending type (B) Lagrangian submanifolds $(\widetilde{\Delta}_1, \dots, \widetilde{\Delta}_{n+2})$ of $\widetilde{E}$ to type (B) Lagrangian submanifolds $(\widetilde{\Delta}'_1, \dots, \widetilde{\Delta}'_{n+2})$ of $\widetilde{E}'$. It follows from Lemma 5.6 that the type (U) Lagrangian submanifold of $\widetilde{E}'$ associated with L_{n+2} does not intersect with $\widetilde{\Delta}'_i$. This shows that $\operatorname{Hom}^*_{\mathcal{F}(\varpi_{E'})}(\widetilde{\Delta}'_i, L_{n+2}) = 0$, which implies $\operatorname{Hom}^*_{\mathcal{F}(\varpi_E)}(\widetilde{\Delta}_i, L_{n+2}) = 0$ by Hamiltonian isotopy invariance of the Floer cohomology. □

It follows that L_{n+2} belongs to the triangulated subcategory generated by the exceptional collection $(\widetilde{\Delta}_1, \widetilde{\Delta}_{n+2})$. Since L_{n+2} is exact, the Floer cohomology of L_{n+2} with itself is isomorphic to the classical cohomology of L_{n+2}.

Lemma 5.9 (Seidel [18, Lemma 7]) *Let* $\mathcal{T}$ *be a triangulated category with a full exceptional collection* $(\mathcal{E}, \mathcal{F})$ *such that* $\operatorname{Hom}^*(\mathcal{E}, \mathcal{F}) \cong H^*(S^{n-1}; \mathbb{C})$*, and* L *be an object of* $\mathcal{T}$ *such that* $\operatorname{Hom}^*(L, L) \cong H^*(S^n; \mathbb{C})$*. Then* L *is isomorphic to the mapping cone* $\operatorname{Cone}(\mathcal{E} \to \mathcal{F})$ *over a non-trivial element in* $\operatorname{Hom}^0(\mathcal{E}, \mathcal{F}) \cong \mathbb{C}$ *up to shift.*

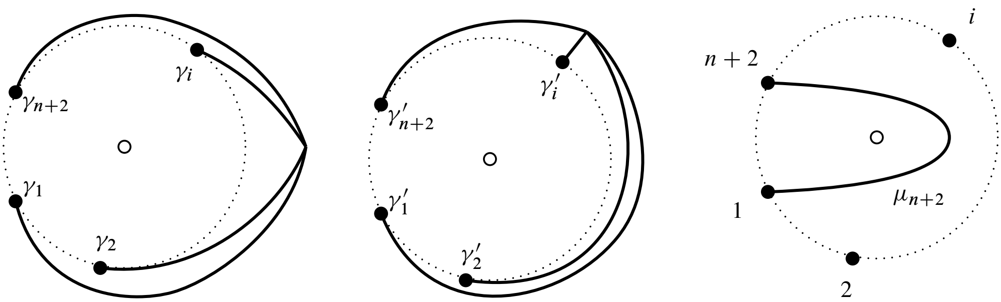

Figure 3: (a) A distinguished set of vanishing paths. (b) Another distinguished set of vanishing paths. (c) The matching path

This shows that L_{n+2} is isomorphic to $\mathrm{Cone}(\widetilde{\Delta}_1 \to \widetilde{\Delta}_{n+2})$ in $D^\pi \mathcal{F}(\varpi_E)$ up to shift. On the other hand, it is shown by Futaki and Ueda [7, Section 5] that V_{n+2} is isomorphic to the matching cycle associated with the matching path μ_{n+2} shown in Figure 3(c) (see [7, Figure 5.2]). Here, a *matching path* is a path on the base of a Lefschetz fibration between two critical values, together with additional structures which enables one to construct a Lagrangian sphere (called the *matching cycle*) in the total space by arranging vanishing cycles along the path (see Seidel [20, Section 16g]). Since the matching path μ_{n+2} does not intersect γ_i for $i \neq 1, n+2$, the vanishing cycle V_{n+2} is also orthogonal to $\widetilde{\Delta}_2, \dots, \widetilde{\Delta}_{n+1}$ in $D^\pi \mathcal{F}(\varpi_E)$. It follows that L_{n+2} equipped with a suitable grading is isomorphic to V_{n+2} in $\mathcal{F}(E)$. Note that any holomorphic disk in $\overline{M}_0$ bounded by $L_{n+2} \cup V_{n+2}$ is contained in E, since any such disk projects by ϖ to a disk in S. This shows that the isomorphism $L_{n+2} \xrightarrow{\sim} V_{n+2}$ in $\mathcal{F}(E)$ extends to an isomorphism in $\mathcal{F}(\overline{M}_0)$, and the following proposition is proved:

Proposition 5.10 *L_{n+2} and V_{n+2} are isomorphic in $\mathcal{F}(\overline{M}_0)$.*

Proposition 3.1 follows from Proposition 5.10 by the Γ^*_{n+2}–action, which is simply transitive on both $\{V_i\}_{i=1}^{n+2}$ and $\{L_i\}_{i=1}^{n+2}$.

Remark 5.11 Let $\mathcal{F}^{\to}$ be the directed subcategory of $\mathcal{F}(M_0)$ consisting of the distinguished basis $(\widetilde{V}_i)_{i=1}^N$ of vanishing cycles of the exact Lefschetz fibration $\pi_M \colon M \to \mathbb{C}$;

$$\hom_{\mathcal{F}^{\to}}(\widetilde{V}_i, \widetilde{V}_j) = \begin{cases} \mathbb{C} \cdot \mathrm{id}_{\widetilde{V}_i} & i = j, \\ \hom_{\mathcal{F}(M_0)}(\widetilde{V}_i, \widetilde{V}_j) & i < j, \\ 0 & \text{otherwise.} \end{cases}$$

It is also isomorphic to the directed subcategory of $\mathcal{F}(X_0)$, since the compositions $\mathfrak{m}_2$ are the same on $\mathcal{F}(M_0)$ and $\mathcal{F}(X_0)$, and higher A_∞–operations $\mathfrak{m}_k$ for $k \geq 3$

vanish on the directed subcategories. Symplectic Picard–Lefschetz theory developed by Seidel [20, Theorem 18.24] gives an equivalence

$$D^b \mathcal{F}^{\to} \cong D^b \mathcal{F}(\pi_M)$$

with the Fukaya category of the Lefschetz fibration π_M. This provides a commutative diagram

$$\begin{array}{ccc} \mathcal{F}^{\to} & \hookrightarrow & \mathcal{F}_q \\ \wr\| & & \wr\| \\ C^{\to}_{n+2} \rtimes \Gamma & \hookrightarrow & \psi^* \mathcal{S}_q \end{array}$$

of A_∞–categories, where horizontal arrows are embeddings of directed subcategories. Combined with the equivalences

$$D^b \mathcal{F}^{\to} \cong D^b \mathcal{F}(\pi_M), \qquad D^\pi(\mathcal{F}_q \otimes_{\Lambda_{\mathbb{N}}} \Lambda_{\mathbb{Q}}) \cong D^\pi \mathcal{F}(X_0),$$
$$D^b(C^{\to}_{n+2} \rtimes \Gamma) \cong D^b \operatorname{coh}[\mathbb{P}^n_{\mathbb{C}} / \Gamma] \quad \text{and} \quad D^\pi(\mathcal{S}_q \otimes_{\Lambda_{\mathbb{N}}} \Lambda_{\mathbb{Q}})) \cong D^b \operatorname{coh} Z_q^*,$$

this gives the compatibility of homological mirror symmetry

$$D^b \mathcal{F}(\pi_M) \cong D^b \operatorname{coh}[\mathbb{P}^n_{\mathbb{C}} / \Gamma]$$

for the ambient space and homological mirror symmetry

$$D^\pi \mathcal{F}(X_0) \cong \widehat{\psi}^* D^b \operatorname{coh} Z_q^*$$

for its Calabi–Yau hypersurface.

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Faculty of Education, Kagawa University
1-1 Saiwai-cho, Takamatsu 760-8522, Japan

Department of Mathematics, Osaka University
Graduate School of Science, Machikaneyama 1-1, Toyonaka 560-0043, Japan

nohara@ed.kagawa-u.ac.jp, kazushi@math.sci.osaka-u.ac.jp

http://www.math.sci.osaka-u.ac.jp/~kazushi/

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