Homology and topological full groups of étale groupoids on totally disconnected spaces *

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Abstract

For almost finite groupoids, we study how their homology groups reflect dynamical properties of their topological full groups. It is shown that two clopen subsets of the unit space has the same class in H_0 if and only if there exists an element in the topological full group which maps one to the other. It is also shown that a natural homomorphism, called the index map, from the topological full group to H_1 is surjective and any element of the kernel can be written as a product of four elements of finite order. In particular, the index map induces a homomorphism from H_1 to K_1 of the groupoid C^*-algebra. Explicit computations of homology groups of AF groupoids and étale groupoids arising from subshifts of finite type are also given.

1 Introduction

Étale groupoids play an important role in the theory of both topological dynamics and operator algebras. Among other things, their (co)homology theory and K-theory of the associated C^*-algebras have attracted significant interest. This paper analyses how the homology groups $H_*(G)$ reflect dynamical properties of the topological full group $[[G]]$ when the groupoid G has a compact and totally disconnected unit space $G^{(0)}$. The topological full group $[[G]]$ consists of all homeomorphisms on $G^{(0)}$ whose graph is 'contained' in the groupoid G as an open subset (Definition 2.3). It corresponds to a natural quotient of the group of unitary normalizers of $C(G^{(0)})$ in $C^*_r(G)$. (Proposition 5.6). With this correspondence, we also discuss connections between homology theory and K-theory of totally disconnected étale groupoids.

The AF groupoids ([24, 17, 14]) form one of the most important classes of étale groupoids on totally disconnected spaces and have already been classified completely up to isomorphism. The terminology AF comes from C^*-algebra theory and means approximately finite. In the present paper, we introduce a class of 'AF-like' groupoids, namely almost finite groupoids (Definition 6.2). Roughly speaking, a totally disconnected étale groupoid G is said to be almost finite if any compact subset of G is almost contained in an elementary subgroupoid. Clearly AF groupoids are almost finite. Any transformation groupoid arising from a free action of $\mathbb{Z}^N$ is shown to be almost finite (Lemma 6.3), but it

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is not known whether the same holds for other discrete amenable groups. For any almost finite groupoid G we first show that two G-full clopen subsets of $G^{(0)}$ has the same class in $H_0(G)$ if and only if one is mapped to the other by an element in $[[G]]$ (Theorem 6.12). The latter condition is equivalent to saying that the characteristic functions on the clopen sets are unitarily equivalent in $C^*_r(G)$ via a unitary normalizer of $C(G^{(0)})$. Next, we introduce a group homomorphism from $[[G]]$ to $H_1(G)$ and call it the index map. When G is almost finite, the index map is shown to be surjective (Theorem 7.5). Furthermore, we prove that any element of the kernel of the index map is a product of four elements of finite order (Theorem 7.13). In particular, if G is principal, then $H_1(G)$ is isomorphic to $[[G]]/[[G]]_0$, where $[[G]]_0$ is the subgroup generated by elements of finite order.

This paper is organized as follows. In Section 2 we collect notation, definitions and basic facts on étale groupoids. In Section 3 we recall the homology theory of étale groupoids, which was introduced by M. Crainic and I. Moerdijk [4]. We observe that homologically similar étale groupoids have isomorphic homology groups with constant coefficients (Proposition 3.5). A variant of the Lindon-Hochschild-Serre spectral sequence is also given. In Section 4 we introduce the notion of Kakutani equivalence for étale groupoids with compact and totally disconnected unit spaces and prove its elementary properties. Kakutani equivalent groupoids are shown to be homologically similar (Theorem 4.8). With the aid of the results of Section 3 and 4, we compute the homology groups of the AF groupoids (Theorem 4.10, 4.11) and the étale groupoids arising from subshifts of finite type (Theorem 4.14). Note that the homology groups agree with the K-groups of the associated groupoid C^*-algebras for these groupoids. In Section 5 we give a C^*-algebraic characterization of Kakutani equivalence (Theorem 5.4) by using a result of J. Renault [26]. Next, we study relationship between the topological full group $[[G]]$ and the unitary normalizers of $C(G^{(0)})$ in $C^*_r(G)$ and show a short exact sequence for them (Proposition 5.6). We also study the group of automorphisms of $C^*_r(G)$ preserving $C(G^{(0)})$ globally (Proposition 5.7). In Section 6 the definition of almost finite groupoids is given (Definition 6.2), and some basic properties are proved. Transformation groupoids arising from free $\mathbb{Z}^N$-actions are shown to be almost finite (Lemma 6.3). We also prove that H_0 of any minimal and almost finite groupoid is a simple, weakly unperforated, ordered abelian group with the Riesz interpolation property (Proposition 6.10). The main result of this section is Theorem 6.12, which says that two clopen subsets of the unit space with the same image in H_0 are mapped to each other by an element of the topological full group when the groupoid is almost finite. In Section 7 we investigate the index map $I : [[G]] \to H_1(G)$. For an almost finite groupoid G, Theorem 7.5 states that I is surjective and Theorem 7.13 determines the kernel of I. As a result, the existence of a natural homomorphism Φ_1 from $H_1(G)$ to $K_1(C^*_r(G))$ is shown (Corollary 7.15).

2 Preliminaries

The cardinality of a set A is written by $|A|$ and the characteristic function on A is written by 1_A. We say that a subset of a topological space is clopen if it is both closed and open. A topological space is said to be totally disconnected if its topology is generated by clopen subsets. By a Cantor set, we mean a compact, metrizable, totally disconnected space with no isolated points. It is known that any two such spaces are homeomorphic.

We say that a continuous map $f : X \to Y$ is étale, if it is a local homeomorphism, i.e. each $x \in X$ has an open neighborhood U such that $f(U)$ is open in Y and $f|U$ is a homeomorphism from U to $f(U)$. In this article, by an étale groupoid we mean a locally compact Hausdorff groupoid such that the range map is étale. We refer the reader to [24, 26] for the basic theory of étale groupoids. For an étale groupoid G, we let $G^{(0)}$ denote the unit space and let s and r denote the source and range maps. A subset $F \subset G^{(0)}$ is said to be G-full, if $r^{-1}(x) \cap s^{-1}(F)$ is not empty for any $x \in G^{(0)}$. For $x \in G^{(0)}$, $G(x) = r(Gx)$ is called the G-orbit of x. When every G-orbit is dense in $G^{(0)}$, G is said to be minimal. For an open subset $F \subset G^{(0)}$, the reduction of G to F is $r^{-1}(F) \cap s^{-1}(F)$ and denoted by $G|F$. The reduction $G|F$ is an étale subgroupoid of G in an obvious way. A subset $U \subset G$ is called a G-set, if $r|U, s|U$ are injective. For an open G-set U, we let τ_U denote the homeomorphism $r \circ (s|U)^{-1}$ from $s(U)$ to $r(U)$. The isotropy bundle is $G' = \{g \in G \mid r(g) = s(g)\}$. We say that G is principal, if $G' = G^{(0)}$. A principal étale groupoid G can be identified with $\{(r(g), s(g)) \in G^{(0)} \times G^{(0)} \mid g \in G\}$, which is an equivalence relation on $G^{(0)}$. Such an equivalence relation is called an étale equivalence relation (see [13, Definition 2.1]). When the interior of G' is $G^{(0)}$, we say that G is essentially principal (this is slightly different from Definition II.4.3 of [24], but the same as the definition given in [26]). For a second countable étale groupoid G, by [26, Proposition 3.1], G is essentially principal if and only if the set of points of $G^{(0)}$ with trivial isotropy is dense in $G^{(0)}$. For an étale groupoid G, we denote the reduced groupoid C^*-algebra of G by $C^*_r(G)$ and identify $C_0(G^{(0)})$ with a subalgebra of $C^*_r(G)$.

There are two important examples of étale groupoids. One is the class of transformation groupoids arising from actions of discrete groups.

Definition 2.1. Let $\varphi : \Gamma \curvearrowright X$ be an action of a countable discrete group Γ on a locally compact Hausdorff space X by homeomorphisms. We let $G_\varphi = \Gamma \times X$ and define the following groupoid structure: (γ, x) and (γ', x') are composable if and only if $x = \varphi^{\gamma'}(x')$, $(\gamma, \varphi^{\gamma'}(x')) \cdot (\gamma', x') = (\gamma\gamma', x')$ and $(\gamma, x)^{-1} = (\gamma^{-1}, \varphi^{\gamma}(x))$. Then G_φ is an étale groupoid and called the transformation groupoid arising from $\varphi : \Gamma \curvearrowright X$.

If the action φ is free (i.e. $\{\gamma \in \Gamma \mid \varphi^\gamma(x){=}x\} = \{e\}$ for all $x \in X$, where e denotes the neutral element), then G_φ is principal. The reduced groupoid C^*-algebra $C^*_r(G_\varphi)$ is naturally isomorphic to the reduced crossed product C^*-algebra $C_0(X) \rtimes_{r,\varphi} \Gamma$.

The other important class is AF groupoids ([24, Definition III.1.1], [14, Definition 3.7]).

Definition 2.2. Let G be a second countable étale groupoid whose unit space is compact and totally disconnected.

(1) We say that $K \subset G$ is an elementary subgroupoid if K is a compact open principal subgroupoid of G such that $K^{(0)} = G^{(0)}$.

(2) We say that G is an AF groupoid if it can be written as an increasing union of elementary subgroupoids.

An AF groupoid is principal by definition, and so it can be identified with an equivalence relation on the unit space. When G is an AF groupoid, the reduced groupoid C^*-algebra $C^*_r(G)$ is an AF algebra. W. Krieger [17] showed that two AF groupoids G_1 and G_2 are isomorphic if and only if $K_0(C^*_r(G_1))$ and $K_0(C^*_r(G_2))$ are isomorphic as ordered abelian groups with distinguished order units (see [27, Definition 1.1.8] for the definition of ordered abelian groups and order units). For classification of minimal AF groupoids up to orbit equivalence, we refer the reader to [12, 14].

We introduce the notion of full groups and topological full groups for étale groupoids.

Definition 2.3. Let G be an étale groupoid whose unit space $G^{(0)}$ is compact.

(1) The set of all $\gamma \in \mathrm{Homeo}(G^{(0)})$ such that for every $x \in G^{(0)}$ there exists $g \in G$ satisfying $r(g) = x$ and $s(g) = \gamma(x)$ is called the full group of G and denoted by $[G]$.

(2) The set of all $\gamma \in \mathrm{Homeo}(G^{(0)})$ for which there exists a compact open G-set U satisfying $\gamma = \tau_U$ is called the topological full group of G and denoted by $[[G]]$.

Obviously $[G]$ is a subgroup of $\mathrm{Homeo}(G^{(0)})$ and $[[G]]$ is a subgroup of $[G]$.

For a minimal homeomorphism φ on a Cantor set X, its full group $[\varphi]$ and topological full group $\tau[\varphi]$ were defined in [13]. One can check that $[\varphi]$ and $\tau[\varphi]$ are equal to $[G_\varphi]$ and $[[G_\varphi]]$ respectively, where $G_\varphi = \mathbb{Z} \times X$ is the transformation groupoid arising from φ. Moreover, for an étale equivalence relation on a compact metrizable and totally disconnected space, its topological full group was introduced in [21] and the above definition is an adaptation of it for a groupoid not necessarily principal. When G is the étale groupoid arising from a subshift of finite type, $[[G]]$ and its connection with C^*-algebras were studied by K. Matsumoto ([20]).

3 Homology theory for étale groupoids

We briefly recall homology theory for étale groupoids which was studied in [4]. In [4] homology groups are defined for sheaves on the unit space and discussed from various viewpoints by using methods of algebraic topology. Here, we restrict our attention to the case of constant coefficients and introduce homology groups in an elementary way, especially for people who are not familiar with algebraic topology.

3.1 Homology groups of étale groupoids

Let A be a topological abelian group. For a locally compact Hausdorff space X, we denote by $C_c(X, A)$ the set of A-valued continuous functions with compact support. When X is compact, we simply write $C(X, A)$. With pointwise addition, $C_c(X, A)$ is an abelian group. Let $\pi : X \to Y$ be an étale map between locally compact Hausdorff spaces. For $f \in C_c(X, A)$, we define a map $\pi_*(f) : Y \to A$ by

$$\pi_*(f)(y) = \sum_{\pi(x)=y} f(x).$$

It is not so hard to see that $\pi_*(f)$ belongs to $C_c(Y, A)$ and that π_* is a homomorphism from $C_c(X, A)$ to $C_c(Y, A)$. Besides, if $\pi' : Y \to Z$ is another étale map to a locally compact Hausdorff space Z, then one can check $(\pi' \circ \pi)_* = \pi'_* \circ \pi_*$ in a direct way.

Let G be an étale groupoid and let $G^{(0)}$ be the unit space. We let s and r denote the source and range maps. For $n \in \mathbb{N}$, we write $G^{(n)}$ for the space of composable strings of n elements in G, that is,

$$G^{(n)} = \{(g_1, g_2, \dots, g_n) \in G^n \mid s(g_i) = r(g_{i+1}) \text{ for all } i = 1, 2, \dots, n-1\}.$$

For $i = 0, 1, \dots, n$, we let $d_i : G^{(n)} \to G^{(n-1)}$ be a map defined by

$$d_i(g_1, g_2, \dots, g_n) = \begin{cases} (g_2, g_3, \dots, g_n) & i = 0 \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n) & 1 \leq i \leq n-1 \\ (g_1, g_2, \dots, g_{n-1}) & i = n. \end{cases}$$

Clearly d_i is étale. Let A be a topological abelian group. Define the homomorphisms $\delta_n : C_c(G^{(n)}, A) \to C_c(G^{(n-1)}, A)$ by

$$\delta_1 = s_* - r_* \qquad \text{and} \qquad \delta_n = \sum_{i=0}^{n} (-1)^i d_{i*}.$$

It is easy to see that

$$0 \xleftarrow{\delta_0} C_c(G^{(0)}, A) \xleftarrow{\delta_1} C_c(G^{(1)}, A) \xleftarrow{\delta_2} C_c(G^{(2)}, A) \xleftarrow{\delta_3} \dots$$

is a chain complex.

Definition 3.1. We let $H_n(G, A)$ be the homology groups of the chain complex above, i.e. $H_n(G, A) = \operatorname{Ker} \delta_n / \operatorname{Im} \delta_{n+1}$, and call them the homology groups of G with constant coefficients A. When $A = \mathbb{Z}$, we simply write $H_n(G) = H_n(G, \mathbb{Z})$. In addition, we define

$$H_0(G)^+ = \{[f] \in H_0(G) \mid f(u) \geq 0 \text{ for all } u \in G^{(0)}\},$$

where $[f]$ denotes the equivalence class of $f \in C_c(G^{(0)}, A)$.

Remark 3.2. The pair $(H_0(G), H_0(G)^+)$ is not necessarily an ordered abelian group in general, because $H_0(G)^+ \cap (-H_0(G)^+)$ may not equal $\{0\}$. In fact, when G is the étale groupoid arising from the full shift over N symbols, $H_0(G)^+ = H_0(G) \cong \mathbb{Z}/(N-1)\mathbb{Z}$. See Theorem 4.14.

Let $\varphi : \Gamma \curvearrowright X$ be an action of a discrete group Γ on a locally compact Hausdorff space X by homeomorphisms. With pointwise addition $C_c(X, A)$ is an abelian group, and Γ acts on it by translation. One can check that $H_n(G_\varphi, A)$ is canonically isomorphic to $H_n(\Gamma, C_c(X, A))$, the homology of Γ with coefficients in $C_c(X, A)$. Under the identification of $G^{(0)}$ with X, the image of δ_1 is equal to the subgroup of $C_c(X, A)$ generated by

$$\{f - f \circ \varphi^\gamma \mid f \in C_c(X, A),\ \gamma \in \Gamma\},$$

and $H_0(G_\varphi, A)$ is equal to the quotient of $C_c(X, A)$ by this subgroup.

Suppose that $G^{(0)}$ is compact, metrizable and totally disconnected. The canonical inclusion $\iota : C(G^{(0)}) \to C_r^*(G)$ induces a homomorphism $K_0(\iota) : K_0(C(G^{(0)})) \to K_0(C_r^*(G))$. The K_0-group of $C(G^{(0)})$ is naturally identified with $C(G^{(0)}, \mathbb{Z})$. If U is a compact open

G-set, then $u = 1_U$ is a partial isometry in $C^*_r(G)$ satisfying $u^*u = 1_{s(U)}$ and $uu^* = 1_{r(U)}$. Hence $(K_0(\iota)\circ\delta_1)(1_U)$ is zero. This means that the image of δ_1 is contained in the kernel of $K_0(\iota)$, because G has a countable base of compact open G-sets. It follows that we obtain a homomorphism $\Phi_0 : H_0(G) \to K_0(C^*_r(G))$ such that $\Phi_0([f]) = K_0(\iota)(f)$. It is natural to ask if the homomorphism Φ_0 is injective or not, but we do not know the answer even when G is the transformation groupoid arising from a free minimal action of $\mathbb{Z}^N$. In Section 7, we will show that there also exists a natural homomorphism $\Phi_1 : H_1(G) \to K_1(C^*_r(G))$ under the assumption that G is almost finite (Corollary 7.15).

Let $\varphi : \mathbb{Z}^N \curvearrowright X$ be an action of $\mathbb{Z}^N$ on a Cantor set X. As mentioned above, $H_*(G_\varphi)$ is isomorphic to the group homology $H_*(\mathbb{Z}^N, C(X,\mathbb{Z}))$, and hence to the group cohomology $H^*(\mathbb{Z}^N, C(X,\mathbb{Z}))$ by Poincaré duality. When $N = 1$, it is straightforward to check that $H_i(G_\varphi)$ is isomorphic to $K_i(C^*_r(G_\varphi))$ for $i = 0, 1$. It is natural to ask whether the isomorphisms

$$K_0(C^*_r(G_\varphi)) \cong \bigoplus_i H_{2i}(G_\varphi), \quad K_1(C^*_r(G_\varphi)) \cong \bigoplus_i H_{2i+1}(G_\varphi) \qquad (*)$$

hold for general N. As shown in [7], there exists a spectral sequence

$$E_2^{p,q} \Rightarrow K_{p+q+N}(C^*_r(G_\varphi)) \quad \text{with} \quad E_2^{p,q} = \begin{cases} H^p(\mathbb{Z}^N, C(X,\mathbb{Z})) & q \text{ is even} \\ 0 & q \text{ is odd,} \end{cases}$$

and if the (co)homology groups were always torsion-free, then the isomorphisms $(*)$ would follow from this spectral sequence. However, it turns out that there exists a free minimal $\mathbb{Z}^N$-action φ which contains torsion in its (co)homology ([9, 22]). Nevertheless, for certain classes of $\mathbb{Z}^N$-actions, it is known that the isomorphisms $(*)$ hold. We refer the reader to [2], [9] and [28] for detailed information. We also remark that the isomorphisms $(*)$ hold for AF groupoids and étale groupoids arising from subshifts of finite type (see Theorem 4.10, 4.11, 4.14).

3.2 Homological similarity

In this subsection we introduce the notion of homological similarity (Definition 3.4) and prove that homologically similar groupoids have isomorphic homology (Proposition 3.5). A variant of Lindon-Hochschild-Serre spectral sequence is also given (Theorem 3.8).

To begin with, we would like to consider functoriality of $H_n(G, A)$.

Definition 3.3. A map $\rho : G \to H$ between étale groupoids is called a homomorphism, if ρ is a continuous map satisfying

$$(g, g') \in G^{(2)} \Rightarrow (\rho(g), \rho(g')) \in H^{(2)} \text{ and } \rho(g)\rho(g') = \rho(gg').$$

We emphasize that continuity is already built in the definition.

Compactly supported cohomology of spaces is covariant along local homeomorphisms and contravariant along proper maps. Analogous properties hold for homology of étale groupoids. Let $\rho : G \to H$ be a homomorphism between étale groupoids. We let $\rho^{(0)}$ denote the restriction of ρ to $G^{(0)}$ and $\rho^{(n)}$ denote the restriction of the n-fold product $\rho \times \rho \times \cdots \times \rho$ to $G^{(n)}$. One can easily see that the following three conditions are equivalent.

(1) $\rho^{(0)}$ is étale (i.e. a local homeomorphism).

(2) ρ is étale.

(3) $\rho^{(n)}$ is étale for all $n \in \mathbb{N}$.

When ρ is étale, $\rho_*^{(n)} : C_c(G^{(n)}, A) \to C_c(H^{(n)}, A)$ are homomorphisms commuting with the boundary operators δ_n. It follows that we obtain a homomorphism

$$H_n(\rho) : H_n(G, A) \to H_n(H, A).$$

If ρ is proper, then one obtains a homomorphism from $C_c(H^{(n)}, A)$ to $C_c(G^{(n)}, A)$ by pullback, and hence a homomorphism

$$H_n^*(\rho) : H_n(H, A) \to H_n(G, A).$$

The following is a variant of (continuous) similarity introduced in [24]. See also [4, Proposition 3.8] and [8, 2.1.3].

Definition 3.4. Let G, H be étale groupoids.

(1) Two homomorphisms ρ, σ from G to H are said to be similar if there exists an continuous map $\theta : G^{(0)} \to H$ such that

$$\theta(r(g))\rho(g) = \sigma(g)\theta(s(g))$$

for all $g \in G$. Note that if ρ and σ are étale, then θ becomes automatically étale.

(2) The two groupoids G and H are said to be homologically similar if there exist étale homomorphisms $\rho : G \to H$ and $\sigma : H \to G$ such that $\sigma \circ \rho$ is similar to id_G and $\rho \circ \sigma$ is similar to id_H.

Proposition 3.5. *Let G, H be étale groupoids.*

(1) *If étale homomorphisms ρ, σ from G to H are similar, then $H_n(\rho) = H_n(\sigma)$.*

(2) *If G and H are homologically similar, then they have isomorphic homology with constant coefficients A. Moreover when $A = \mathbb{Z}$, the isomorphism maps $H_0(G)^+$ onto $H_0(H)^+$.*

Proof. It suffices to show (1). There exists an étale map $\theta : G^{(0)} \to H$ such that $\theta(r(g))\rho(g) = \sigma(g)\theta(s(g))$ for all $g \in G$. For each $n \in \mathbb{N} \cup \{0\}$ we construct a homomorphism $h_n : C_c(G^{(n)}, A) \to C_c(H^{(n+1)}, A)$ as follows. First we put $h_0 = \theta_*$. For $n \in \mathbb{N}$, we let $h_n = \sum_{j=0}^{n} (-1)^j k_{j*}$, where $k_j : G^{(n)} \to H^{(n+1)}$ is defined by

$$k_j(g_1, g_2, \dots, g_n) = \begin{cases} (\theta(r(g_1)), \rho(g_1), \rho(g_2), \dots, \rho(g_n)) & j = 0 \\ (\sigma(g_1), \dots, \sigma(g_j), \theta(s(g_j)), \rho(g_{j+1}), \dots, \rho(g_n)) & 1 \leq j \leq n-1 \\ (\sigma(g_1), \sigma(g_2), \dots, \sigma(g_n), \theta(s(g_n))) & j = n. \end{cases}$$

It is straightforward to verify $\delta_1 \circ h_0 = \rho_*^{(0)} - \sigma_*^{(0)}$ and

$$\delta_{n+1} \circ h_n + h_{n-1} \circ \delta_n = \rho_*^{(n)} - \sigma_*^{(n)}$$

for all $n \in \mathbb{N}$. Hence we get $H_n(\rho) = H_n(\sigma)$. □

Theorem 3.6. *Let G be an étale groupoid and let $F \subset G^{(0)}$ be an open G-full subset.*

(1) *If there exists a continuous map $\theta : G^{(0)} \to G$ such that $r(\theta(x)) = x$ and $s(\theta(x)) \in F$ for all $x \in G^{(0)}$, then G is homologically similar to $G|F$.*

(2) *Suppose that $G^{(0)}$ is σ-compact and totally disconnected. Then G is homologically similar to $G|F$.*

Proof. (1) Notice that θ is étale. Let $\rho : G \to G|F$ and $\sigma : G|F \to G$ be étale homomorphisms defined by $\rho(g) = \theta(r(g))^{-1} g\theta(s(g))$ and $\sigma(g) = g$. It is easy to see that $\rho \circ \sigma$ is similar to the identity on $G|F$ and that $\sigma \circ \rho$ is similar to the identity on G via the map θ. Hence G and $G|F$ are homologically similar.

(2) There exists a countable family of compact open G-sets $\{U_n\}_n$ such that $\{r(U_n)\}_n$ covers G and $s(U_n) \subset F$. Define compact open G-sets $V_1, V_2, \dots$ inductively by $V_1 = U_1$ and

$$V_n = U_n \setminus r^{-1}(r(V_1 \cup \dots \cup V_{n-1})).$$

We can define $\theta : G^{(0)} \to G$ by $\theta(x) = (r|V_n)^{-1}(x)$ for $x \in V_n$. Clearly θ satisfies the assumption of (1), and so the proof is completed. □

We recall from [24] the notion of skew products and semi-direct products of étale groupoids. Let G be an étale groupoid and let Γ be a countable discrete group. When $\rho : G \to \Gamma$ is a homomorphism, the skew product $G \times_\rho \Gamma$ is $G \times \Gamma$ with the following groupoid structure: (g, γ) and (g', γ') is composable if and only if g and g' are composable and $\gamma\rho(g) = \gamma'$, $(g, \gamma) \cdot (g', \gamma\rho(g)) = (gg', \gamma)$ and $(g, \gamma)^{-1} = (g^{-1}, \gamma\rho(g))$. We can define an action $\hat{\rho} : \Gamma \curvearrowright G \times_\rho \Gamma$ by $\hat{\rho}^\gamma(g', \gamma') = (g', \gamma\gamma')$.

When $\varphi : \Gamma \curvearrowright G$ is an action of Γ on G, the semi-direct product $G \rtimes_\varphi \Gamma$ is $G \times \Gamma$ with the following groupoid structure: (g, γ) and (g', γ') is composable if and only if g and $\varphi^\gamma(g')$ are composable, $(g, \gamma) \cdot (g', \gamma') = (g\varphi^\gamma(g'), \gamma\gamma')$ and $(g, \gamma)^{-1} = (\varphi^{\gamma^{-1}}(g^{-1}), \gamma^{-1})$. There exists a natural homomorphism $\tilde{\varphi} : G \rtimes_\varphi \Gamma \to \Gamma$ defined by $\tilde{\varphi}(g, \gamma) = \gamma$. The following proposition can be shown in a similar fashion to [24, I.1.8] by using Theorem 3.6 (1).

Proposition 3.7. *Let G be an étale groupoid and let Γ be a countable discrete group.*

(1) *When $\rho : G \to \Gamma$ is a homomorphism, $(G \times_\rho \Gamma) \rtimes_{\hat{\rho}} \Gamma$ is homologically similar to G.*

(2) *When $\varphi : \Gamma \curvearrowright G$ is an action, $(G \rtimes_\varphi \Gamma) \times_{\tilde{\varphi}} \Gamma$ is homologically similar to G.*

For skew products and semi-direct products, the following Lindon-Hochschild-Serre spectral sequences exist. This will be used later for a computation of the homology groups of étale groupoids arising from subshifts of finite type.

Theorem 3.8. *Let G be an étale groupoid and let Γ be a countable discrete group. Let A be a topological abelian group.*

(1) *Suppose that $\rho : G \to \Gamma$ is a homomorphism. Then there exists a spectral sequence:*

$$E^2_{p,q} = H_p(\Gamma, H_q(G \times_\rho \Gamma, A)) \Rightarrow H_{p+q}(G, A),$$

where $H_q(G \times_\rho \Gamma, A)$ is regarded as a Γ-module via the action $\hat{\rho} : \Gamma \curvearrowright G \times_\rho \Gamma$.

(2) *Suppose that* $\varphi : \Gamma \curvearrowright G$ *is an action. Then there exists a spectral sequence:*

$$E^2_{p,q} = H_p(\Gamma, H_q(G, A)) \Rightarrow H_{p+q}(G \rtimes_\varphi \Gamma, A),$$

where $H_q(G, A)$ *is regarded as a* Γ*-module via the action* φ*.*

Proof. (1) is a special case of [4, Theorem 4.4]. (2) immediately follows from (1) and Proposition 3.7 (2). □

We remark that similar spectral sequences exist for cohomology of étale groupoids, too.

4 Kakutani equivalence

In this section we introduce the notion of Kakutani equivalence for étale groupoids whose unit spaces are compact and totally disconnected. We also compute the homology groups of AF groupoids and étale groupoids arising from subshifts of finite type.

4.1 Kakutani equivalence

Definition 4.1. Let G_i be an étale groupoid whose unit space is compact and totally disconnected for $i = 1, 2$. When there exists a G_i-full clopen subset $Y_i \subset G_i^{(0)}$ for $i = 1, 2$ and $G_1|Y_1$ is isomorphic to $G_2|Y_2$, we say that G_1 is Kakutani equivalent to G_2.

It will be proved in Lemma 4.5 that the Kakutani equivalence is really an equivalence relation.

Remark 4.2. In the case of transformation groupoids arising from $\mathbb{Z}$-actions, the Kakutani equivalence defined above is weaker than the Kakutani equivalence for $\mathbb{Z}$-actions introduced in [12]. Indeed, for minimal homeomorphisms φ_1, φ_2 on Cantor sets, the étale groupoids associated with them are Kakutani equivalent in the sense above if and only if φ_1 is Kakutani equivalent to either of φ_2 and φ_2^{-1} in the sense of [12, Definition 1.7]. See also [12, Theorem 2.4] and [3].

Let G be an étale groupoid whose unit space is compact and totally disconnected. For $f \in C(G^{(0)}, \mathbb{Z})$ with $f \geq 0$, we let

$$G_f = \{(g, i, j) \in G \times \mathbb{Z} \times \mathbb{Z} \mid 0 \leq i \leq f(r(g)),\ 0 \leq j \leq f(s(g))\}$$

and equip G_f with the induced topology from the product topology on $X \times \mathbb{Z} \times \mathbb{Z}$. The groupoid structure of G_f is given as follows:

$$G_f^{(0)} = \{(x, i, i) \mid x \in G^{(0)},\ 0 \leq i \leq f(x)\},$$

$(g, i, j)^{-1} = (g^{-1}, j, i)$, two elements (g, i, j) and (h, k, l) are composable if and only if $s(g) = r(h), j = k$ and the product is $(g, i, j)(h, j, l) = (gh, i, l)$. It is easy to see that G_f is an étale groupoid and the clopen subset $\{(x, 0, 0) \in G_f^{(0)} \mid x \in G^{(0)}\}$ of $G_f^{(0)}$ is G_f-full.

Lemma 4.3. *Let G be an étale groupoid whose unit space is compact and totally disconnected and let $Y \subset G^{(0)}$ be a G-full clopen subset. There exists $f \in C(Y,\mathbb{Z})$ and an isomorphism $\pi : (G|Y)_f \to G$ such that $\pi(g,0,0) = g$ for all $g \in G|Y$.*

Proof. We put $X = G^{(0)}$ for notational convenience. For any $x \in X \setminus Y$, there exists $g \in r^{-1}(x) \cap s^{-1}(Y)$, because Y is R-full. We can choose a compact open G-set U_x containing g so that $r(U_x) \subset X \setminus Y$ and $s(U_x) \subset Y$. The family of clopen subsets $\{r(U_x) \mid x \in X \setminus Y\}$ forms an open covering of $X \setminus Y$, and so we can find $x_1, x_2, \dots, x_n \in X \setminus Y$ such that $r(U_{x_1}) \cup r(U_{x_2}) \cup \dots \cup r(U_{x_n}) = X \setminus Y$. Define compact open G-sets $V_1, V_2, \dots, V_n$ inductively by

$$V_1 = U_{x_1} \quad \text{and} \quad V_k = U_{x_k} \setminus r^{-1}(r(V_1 \cup \dots \cup V_{k-1})).$$

Then $r(V_1), r(V_2), \dots, r(V_n)$ are mutually disjoint and their union is equal to $X \setminus Y$. For each subset $\lambda \subset \{1, 2, \dots, n\}$, we fix a bijection $\alpha_\lambda : \{k \in \mathbb{N} \mid k \leq |\lambda|\} \to \lambda$. For $y \in Y$, put $\lambda(y) = \{k \in \{1, 2, \dots, n\} \mid y \in s(V_k)\}$. We define $f \in C(Y,\mathbb{Z})$ by $f(y) = |\lambda(y)|$. Since each $s(V_k)$ is clopen, f is continuous. We further define $\theta : (G|Y)_f^{(0)} \to G$ by

$$\theta(y,i,i) = \begin{cases} y & \text{if } i = 0 \\ (s|V_l)^{-1}(y) & \text{otherwise,} \end{cases}$$

where $l = \alpha_{\lambda(y)}(i)$. It is not so hard to see that

$$\pi(g,i,j) = \theta(r(g),i,i) \cdot g \cdot \theta(s(g),j,j)^{-1}$$

gives an isomorphism from $(G|Y)_f$ to G. □

Lemma 4.4. *Let G be an étale groupoid whose unit space is compact and totally disconnected and let $Y, Y' \subset G^{(0)}$ be G-full clopen subsets. Then $G|Y$ and $G|Y'$ are Kakutani equivalent.*

Proof. By Lemma 4.3, there exists $f \in C(Y,\mathbb{Z})$ and an isomorphism $\pi : (G|Y)_f \to G$ such that $\pi(g,0,0) = g$ for all $g \in G|Y$. Define a clopen subset $Z \subset Y$ by

$$Z = \{y \in Y \mid \pi(y,k,k) \in Y' \text{ for some } k = 0, 1, \dots, f(y)\}.$$

Since Y' is G-full, we can see that Z is G-full. For each $z \in Z$, we let

$$g(z) = \min\{k \in \{0, 1, \dots, f(z)\} \mid \pi(z,k,k) \in Y'\}$$

and $U = \{\pi(z, g(z), 0) \mid z \in Z\}$. Then g is a continuous function on Z and U is a compact open G-set satisfying $s(U) = Z$ and $r(U) \subset Y'$. Clearly $Z' = r(U)$ is G-full, and $G|Z$ and $G|Z'$ are isomorphic. Hence $G|Y$ and $G|Y'$ are Kakutani equivalent. □

Lemma 4.5. *The Kakutani equivalence is an equivalence relation between étale groupoids whose unit spaces are compact and totally disconnected.*

Proof. It suffices to prove transitivity. Let G_i be an étale groupoid whose unit space is compact and totally disconnected for $i = 1, 2, 3$. Suppose that G_1 and G_2 are Kakutani equivalent and that G_2 and G_3 are Kakutani equivalent. We can find clopen subsets $Y_1 \subset G_1^{(0)}$, $Y_2, Y_2' \subset G_2^{(0)}$ and $Y_3 \subset G_3^{(0)}$ such that each of them are full, $G_1|Y_1$ is isomorphic to $G_2|Y_2$ and $G_2|Y_2'$ is isomorphic to $G_3|Y_3$. Let $\pi : G_2|Y_2 \to G_1|Y_1$ and $\pi' : G_2|Y_2' \to G_3|Y_3$ be isomorphisms. From Lemma 4.4, there exist G_2-full clopen subsets $Z \subset Y_2$ and $Z' \subset Y_2'$ such that $G_2|Z$ is isomorphic to $G_2|Z'$. Then $G_1|\pi(Z)$ is isomorphic to $G_3|\pi'(Z')$, and so G_1 and G_3 are Kakutani equivalent. □

Lemma 4.6. *Let G_i be an étale groupoid whose unit space is compact and totally disconnected for $i = 1, 2$. The following are equivalent.*

(1) *G_1 is Kakutani equivalent to G_2.*

(2) *There exist $f_i \in C(G_i^{(0)}, \mathbb{Z})$ such that $(G_1)_{f_1}$ is isomorphic to $(G_2)_{f_2}$.*

Proof. (1)⇒(2). By Lemma 4.3, we can find an étale groupoid G and $g_1, g_2 \in C(G^{(0)}, \mathbb{Z})$ such that G_{g_i} is isomorphic to G_i. Let $\pi : G_i \to G_{g_i}$ be the isomorphism. Put $g(x) = \max\{g_1(x), g_2(x)\}$. Define $f_i \in C(G_{g_i}^{(0)}, \mathbb{Z})$ by

$$f_i(x, k, k) = \begin{cases} g(x) - g_i(x) & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let $h_i = f_i \circ (\pi|G_i^{(0)})$. It is easy to see that $(G_i)_{h_i}$ is isomorphic to $(G_{g_i})_{f_i}$ and that $(G_{g_i})_{f_i}$ is isomorphic to G_g. Therefore we get (2).

(2)⇒(1). For $i = 1, 2$, we let $Y_i = \{(x, 0, 0) \in (G_i)_{f_i}^{(0)} \mid x \in G_i^{(0)}\}$. Then Y_i is $(G_i)_{f_i}$-full and $(G_i)_{f_i}|Y_i$ is isomorphic to G_i. It follows from Lemma 4.4 that G_1 and G_2 are Kakutani equivalent. □

From the lemma above, one can see that Kakutani equivalence is a generalization of bounded orbit injection equivalence introduced in [18, Definition 1.3] (see also Definition 5 and Theorem 6 of [19]).

Lemma 4.7. *Let G be an étale groupoid whose unit space is compact and totally disconnected. The following are equivalent.*

(1) *G is principal and compact.*

(2) *G is Kakutani equivalent to H such that $H = H^{(0)}$.*

Proof. This is immediate from [14, Lemma 3.4] and the definition of Kakutani equivalence. □

4.2 Examples of homology groups

Next, we turn to the consideration of homology of an étale groupoid G whose unit space is compact and totally disconnected.

Theorem 4.8. *Let G_i be an étale groupoid whose unit space is compact and totally disconnected for $i = 1, 2$. If G_1 and G_2 are Kakutani equivalent, then G_1 is homologically similar to G_2. In particular, $H_n(G_1, A)$ is isomorphic to $H_n(G_2, A)$ for any topological abelian group A. Moreover, there exists an isomorphism $\pi : H_0(G_1) \to H_0(G_2)$ such that $\pi(H_0(G_1)^+) = H_0(G_2)^+$.*

Proof. This follows from the definition of Kakutani equivalence, Theorem 3.6 and Proposition 3.5. □

Lemma 4.9. *Let G be a compact étale principal groupoid whose unit space is compact and totally disconnected. Let A be a topological abelian group. Then $H_n(G, A) = 0$ for $n \geq 1$.*

Proof. This follows from Lemma 4.7, Theorem 4.8 and the definition of homology groups. □

As for AF groupoids (i.e. AF equivalence relations), we have the following. For the definition of dimension groups, we refer the reader to [27, Section 1.4].

Theorem 4.10. (1) *For an AF groupoid G, there exists an isomorphism $\pi : H_0(G) \to K_0(C_r^*(G))$ such that $\pi(H_0(G)^+) = K_0(C_r^*(G))^+$ and $\pi([1_{G^{(0)}}]) = [1_{C_r^*(G)}]$. In particular, the triple $(H_0(G), H_0(G)^+, [1_{G^{(0)}}])$ is a dimension group with a distinguished order unit.*

(2) *Two AF groupoids G_1 and G_2 are isomorphic if and only if there exists an isomorphism $\pi : H_0(G_1) \to H_0(G_2)$ such that $\pi(H_0(G_1)^+) = H_0(G_2)^+$ and $\pi([1_{G_1^{(0)}}]) = [1_{G_2^{(0)}}]$.*

Proof. The first statement follows from [24, 16, 12, 14]. The second statement was proved in [17]. □

Theorem 4.11. *Let G be an AF groupoid and let A be a topological abelian group. Then $H_n(G, A) = 0$ for $n \geq 1$.*

Proof. The AF groupoid G is an increasing union of elementary subgroupoids. For any $f \in C_c(G^{(n)}, A)$, there exists an elementary subgroupoid $K \subset G$ such that $f \in C_c(K^{(n)}, A)$. By Lemma 4.9, $H_n(K, A) = 0$ for $n \geq 1$. Therefore $H_n(G, A) = 0$ for $n \geq 1$. □

Remark 4.12. Let G be an AF groupoid and let A be a topological abelian group. It is known that the cohomology group $H^n(G, A)$ with constant coefficients is zero for every $n \geq 2$ ([24, III.1.3]). Clearly $H^0(G, A)$ is equal to the set of continuous functions $f \in C(G^{(0)}, A)$ satisfying $f(r(g)) = f(s(g))$ for all $g \in G$. In particular, when G is minimal, $H^0(G, A)$ is isomorphic to A. When G is minimal and $A = \mathbb{Z}$, one can see that $H^1(G, \mathbb{Z})$ is always uncountable.

We now turn to a computation of homology groups of étale groupoids arising from subshifts of finite type. We refer the reader to [1, 25] for more details about these groupoids. Let σ be a one-sided subshift of finite type on a compact totally disconnected space X. We assume that σ is surjective. The étale groupoid G associated with σ is given by

$$G = \{(x, n, y) \in X \times \mathbb{Z} \times X \mid \exists k, l \in \mathbb{N},\ n = k - l,\ \sigma^k(x) = \sigma^l(y)\}.$$

Two elements (x,n,y) and (x',n',y') in G are composable if and only if $y=x'$, and the multiplication and the inverse are

$$(x,n,y)\cdot(y,n',y')=(x,n{+}n',y'),\quad (x,n,y)^{-1}=(y,-n,x).$$

The étale groupoid G has an open subgroupoid $H=\{(x,0,y)\in G\}$. It is well-known that $C^*_r(G)$ is isomorphic to the Cuntz-Krieger algebra introduced in [5] and that H is an AF groupoid. Moreover, there exists an automorphism π of $K_0(C^*_r(H))$ such that $K_0(C^*_r(G))\cong\operatorname{Coker}(\mathrm{id}-\pi)$ and $K_1(C^*_r(G))\cong\operatorname{Ker}(\mathrm{id}-\pi)$. It is also well-known that $K_0(C^*_r(G))$ is a finitely generated abelian group and $K_1(C^*_r(G))$ is isomorphic to the torsion-free part of $K_0(C^*_r(G))$.

The map $\rho:(x,n,y)\mapsto n$ is a homomorphism from G to $\mathbb{Z}$. We consider the skew product $G\times_\rho\mathbb{Z}$ and set $Y=G^{(0)}\times\{0\}\subset(G\times_\rho\mathbb{Z})^{(0)}$.

Lemma 4.13. *In the setting above, $G\times_\rho\mathbb{Z}$ is homologically similar to H.*

Proof. It is easy to see that Y is $(G\times_\rho\mathbb{Z})$-full. In addition, $(G\times_\rho\mathbb{Z})|Y$ is canonically isomorphic to H. By Theorem 3.6 (2), we get the conclusion. □

Theorem 4.14. *When G is the étale groupoid arising from a subshift of finite type, $H_0(G)\cong K_0(C^*_r(G))$, $H_1(G)\cong K_1(C^*_r(G))$ and $H_n(G)=0$ for all $n\geq 2$.*

Proof. It follows from Theorem 3.8 (1) that there exists a spectral sequence:

$$E^2_{p,q}=H_p(\mathbb{Z},H_q(G\times_\rho\mathbb{Z}))\Rightarrow H_{p+q}(G).$$

By the lemma above and Proposition 3.5, $H_q(G\times_\rho\mathbb{Z})$ is isomorphic to $H_q(H)$. This, together with Theorem 4.10 and Theorem 4.11, implies

$$H_q(G\times_\rho\mathbb{Z})\cong\begin{cases}K_0(C^*_r(H)) & q=0\\ 0 & q\geq 1.\end{cases}$$

Besides, the $\mathbb{Z}$-module structure on $H_0(G\times_\rho\mathbb{Z})\cong K_0(C^*_r(H))$ is given by the automorphism π. Hence one has

$$H_0(G)\cong H_0(\mathbb{Z},K_0(C^*_r(H)))\cong\operatorname{Coker}(\mathrm{id}-\pi)\cong K_0(C^*_r(G)),$$

$$H_1(G)\cong H_1(\mathbb{Z},K_0(C^*_r(H)))\cong\operatorname{Ker}(\mathrm{id}-\pi)\cong K_1(C^*_r(G))$$

and $H_n(G)=0$ for $n\geq 2$. □

5 Kakutani equivalence and C^*-algebras

In this section, we give a C^*-algebraic characterization of Kakutani equivalence. For an étale groupoid G, we denote the reduced groupoid C^*-algebra of G by $C^*_r(G)$ and identify $C_0(G^{(0)})$ with a subalgebra of $C^*_r(G)$. The following is an immediate consequence of (a special case of) Proposition 4.11 of [26].

Theorem 5.1. *For $i=1,2$, let G_i be an étale essentially principal second countable groupoid. The following are equivalent.*

(1) *G_1 and G_2 are isomorphic.*

(2) *There exists an isomorphism $\pi : C_r^*(G_1) \to C_r^*(G_2)$ such that $\pi(C_0(G_1^{(0)})) = C_0(G_2^{(0)})$.*

The following lemma is obvious from the definition of $C_r^*(G)$.

Lemma 5.2. *Let G be an étale groupoid whose unit space is compact and let $Y \subset G^{(0)}$ be a clopen subset. There exists a natural isomorphism $\pi : C_r^*(G|Y) \to 1_Y C_r^*(G) 1_Y$ such that $\pi(f) = f$ for every $f \in C(Y)$.*

An element in a C^*-algebra A is said to be full if it is not contained in any proper closed two-sided ideal in A. The following is an easy consequence of [24, Proposition II.4.5].

Lemma 5.3. *Let G be an étale groupoid whose unit space is compact and let $Y \subset G^{(0)}$ be a clopen subset. Then, Y is G-full if and only if 1_Y is a full projection in $C_r^*(G)$.*

Combining Theorem 5.1 with the above two lemmas, we get the following. This is a generalization of [18, Theorem 2.4].

Theorem 5.4. *Let G_i be an étale essentially principal second countable groupoid whose unit space is compact and totally disconnected for $i = 1, 2$. The following are equivalent.*

(1) *G_1 and G_2 are Kakutani equivalent.*

(2) *There exist two projections $p_1 \in C(G_1^{(0)})$, $p_2 \in C(G_2^{(0)})$ and an isomorphism π from $p_1 C_r^*(G_1) p_1$ to $p_2 C_r^*(G_2) p_2$ such that p_i is full in $C_r^*(G_i)$ and $\pi(p_1 C(G_1^{(0)})) = p_2 C(G_2^{(0)})$.*

We next consider relationship between $[[G]]$ and unitary normalizers of $C(G^{(0)})$ in $C_r^*(G)$. In what follows, an element in $C_r^*(G)$ is identified with a function in $C_0(G)$ ([24, II.4.2]). The following is a slight generalization of [24, II.4.10] and a special case of [26, Proposition 4.7]. We omit the proof.

Lemma 5.5. *Let G be an étale essentially principal second countable groupoid. Suppose that $v \in C_r^*(G)$ is a partial isometry satisfying $v^*v, vv^* \in C_0(G^{(0)})$ and $vC_0(G^{(0)})v^* = vv^*C_0(G^{(0)})$. Then there exists a compact open G-set $V \subset G$ such that*

$$V = \{g \in G \mid v(g) \neq 0\} = \{g \in G \mid |v(g)| = 1\}.$$

In addition, for any $f \in v^*vC_0(G^{(0)})$, one has $vfv^* = f \circ \tau_V^{-1}$.

For a unital C^*-algebra A, let $U(A)$ denote the unitary group of A and for a subalgebra $B \subset A$, let $N(B, A)$ denote the normalizer of B in $U(A)$, that is,

$$N(B, A) = \{u \in U(A) \mid uBu^* = B\}.$$

Let G be an étale groupoid whose unit space $G^{(0)}$ is compact. Clearly $U(C(G^{(0)}))$ is a subgroup of $N(C(G^{(0)}), C_r^*(G))$ and we let ι denote the inclusion map. An element $u \in N(C(G^{(0)}), C_r^*(G))$ induces an automorphism $f \mapsto ufu^*$ of $C(G^{(0)})$, and so there exists a homomorphism $\sigma : N(C(G^{(0)}), C_r^*(R)) \to \mathrm{Homeo}(G^{(0)})$ such that $ufu^* = f \circ \sigma(u)^{-1}$ for all $f \in C(G^{(0)})$. The following is a generalization of [23, Section 5], [29, Theorem 1] and [20, Theorem 1.2].

Proposition 5.6. *Suppose that G is an étale essentially principal second countable groupoid whose unit space is compact.*

(1) *The image of σ is contained in the topological full group $[[G]]$.*

(2) *The sequence*

$$1 \longrightarrow U(C(G^{(0)})) \xrightarrow{\iota} N(C(G^{(0)}), C_r^*(G)) \xrightarrow{\sigma} [[G]] \longrightarrow 1$$

is exact.

(3) *The homomorphism σ has a right inverse.*

Proof. (1) This follows from Lemma 5.5 and the definition of $[[G]]$.

(2) By definition, ι is injective. Since $C(G^{(0)})$ is abelian, the image of ι is contained in the kernel of σ. By [24, Proposition II.4.7], $C(G^{(0)})$ is a maximal abelian subalgebra. It follows that the kernel of σ is contained in the image of ι. The surjectivity of σ follows from (3).

(3) Take $\gamma \in [[G]]$. Since G is essentially principal, there exists a unique compact open G-set $U \subset G$ such that $\gamma = \tau_U$. We let $u \in C_c(G)$ be the characteristic function of U. One can see that u is a unitary in $C_r^*(G)$ satisfying $ufu^* = f \circ \gamma^{-1}$ for all $f \in C(G^{(0)})$. The map $\gamma \mapsto u$ gives a right inverse of σ. □

Let G be an étale essentially principal groupoid whose unit space is compact. We let $\mathrm{Aut}_{C(G^{(0)})}(C_r^*(G))$ denote the group of automorphisms of $C_r^*(G)$ preserving $C(G^{(0)})$ globally. Thus

$$\mathrm{Aut}_{C(G^{(0)})}(C_r^*(G)) = \{\alpha \in \mathrm{Aut}(C_r^*(G)) \mid \alpha(C(G^{(0)})) = C(G^{(0)})\}.$$

We let $\mathrm{Inn}_{C(G^{(0)})}(C_r^*(G))$ denote the subgroup of $\mathrm{Aut}_{C(G^{(0)})}(C_r^*(G))$ consisting of inner automorphisms. In other words,

$$\mathrm{Inn}_{C(G^{(0)})}(C_r^*(G)) = \{\mathrm{Ad}\, u \mid u \in N(C(G^{(0)}), C_r^*(G))\}.$$

Let $\mathrm{Out}_{C(G^{(0)})}(C_r^*(G))$ be the quotient group of $\mathrm{Aut}_{C(G^{(0)})}(C_r^*(G))$ by $\mathrm{Inn}_{C(G^{(0)})}(C_r^*(G))$. The automorphism group of G is denoted by $\mathrm{Aut}(G)$. For $\gamma = \tau_O \in [[G]]$, there exists an automorphism $\varphi_\gamma \in \mathrm{Aut}(G)$ satisfying

$$OgO^{-1} = \{\varphi_\gamma(g)\}$$

for all $g \in G$. We regard $[[G]]$ as a subgroup of $\mathrm{Aut}(G)$ via the identification of γ with φ_γ. Thanks to [26, Proposition 4.11], we can prove the following exact sequences for these automorphism groups, which generalize [13, Proposition 2.4], [30, Theorem 3] and [20, Theorem 1.3]. See [24, Definition I.1.12] for the definition of $Z^1(G, \mathbb{T})$, $B^1(G, \mathbb{T})$ and $H^1(G, \mathbb{T})$.

Proposition 5.7. *Let G be an étale essentially principal second countable groupoid whose unit space is compact.*

(1) *There exist short exact sequences:*

$$1 \longrightarrow Z^1(G, \mathbb{T}) \xrightarrow{j} \mathrm{Aut}_{C(G^{(0)})}(C_r^*(G)) \xrightarrow{\omega} \mathrm{Aut}(G) \longrightarrow 1,$$

$$1 \longrightarrow B^1(G, \mathbb{T}) \xrightarrow{j} \mathrm{Inn}_{C(G^{(0)})}(C_r^*(G)) \xrightarrow{\omega} [[G]] \longrightarrow 1,$$

$$1 \longrightarrow H^1(G, \mathbb{T}) \xrightarrow{j} \mathrm{Out}_{C(G^{(0)})}(C_r^*(G)) \xrightarrow{\omega} \mathrm{Aut}(G)/[[G]] \longrightarrow 1.$$

Moreover, they all split, that is, ω has a right inverse.

(2) *Suppose that G admits a covering by compact open G-sets O satisfying $r(O) = s(O) = G^{(0)}$. Then $\mathrm{Aut}(G)$ is naturally isomorphic to the normalizer $N([[G]])$ of $[[G]]$ in $\mathrm{Homeo}(G^{(0)})$.*

Proof. (1) It suffices to show the exactness of the first sequence, because the others are immediately obtained from the first one. Take $\alpha \in \mathrm{Aut}_{C(G^{(0)})}(C_r^*(G))$. Clearly α induces an automorphism of the Weyl groupoid of $(C_r^*(G), C(G^{(0)}))$ ([26, Definition 4.2]), which is canonically isomorphic to G by [26, Proposition 4.11]. Therefore there exists a homomorphism ω from $\mathrm{Aut}_{C(G^{(0)})}(C_r^*(G))$ to $\mathrm{Aut}(G)$. Evidently ω is surjective and has a right inverse. Take $\xi \in Z^1(G, \mathbb{T})$. We can define $j(\xi) \in \mathrm{Aut}_{C(G^{(0)})}(C_r^*(G))$ by setting

$$j(\xi)(f)(g) = \xi(g)f(g)$$

for $f \in C_r^*(G)$ and $g \in G$. Obviously j is an injective homomorphism and $\mathrm{Im}\, j$ is contained in $\mathrm{Ker}\, \omega$. It remains for us to show that $\mathrm{Ker}\, \omega$ is contained in $\mathrm{Im}\, j$. For $f \in C_c(G)$, we write $\mathrm{supp}(f) = \{g \in G \mid f(g) \neq 0\}$. Suppose $\alpha \in \mathrm{Ker}\, \omega$. We have $\alpha(f) = f$ for $f \in C(G^{(0)})$. Take $g \in G$. Choose $u \in C_c(G)$ so that $u(g) > 0$, $u(h) \geq 0$ for all $h \in G$ and $\mathrm{supp}(u)$ is an open G-set. Then $\alpha(u)(g)/u(g)$ is in $\mathbb{T}$, because

$$|u(g)|^2 = |u^*u(s(g))| = |\alpha(u^*u)(s(g))| = |\alpha(u)(g)|^2.$$

Let $v \in C_c(G)$ be a function which has the same properties as u. Then $O = \mathrm{supp}(u) \cap \mathrm{supp}(v)$ is an open G-set containing g. Let $w \in C(G^{(0)})$ be a positive element satisfying $w(s(g)) > 0$ and $\mathrm{supp}(w) \subset s(O)$. We have

$$(u(v^*v)^{1/2}w)(h) = u(h)((v^*v)^{1/2}w)(s(h)) = u(h)v(h)w(s(h))$$

for every $h \in G$. Since we also have the same equation for $v(u^*u)^{1/2}w$, we can conclude $u(v^*v)^{1/2}w = v(u^*u)^{1/2}w$. Accordingly, one obtains

$$\begin{aligned}
\alpha(u)(g)v(g)w(s(g)) &= (\alpha(u)(v^*v)^{1/2}w)(g) \\
&= \alpha(u(v^*v)^{1/2}w)(g) \\
&= \alpha(v(u^*u)^{1/2}w)(g) \\
&= (\alpha(v)(u^*u)^{1/2}w)(g) = \alpha(v)(g)u(g)w(s(g)).
\end{aligned}$$

It follows that the value $\alpha(u)(g)/u(g)$ does not depend on the choice of u. We write it by $\xi(g)$. From the definition, it is easy to verify that ξ belongs to $Z^1(G, \mathbb{T})$ and that α is equal to $j(\xi)$.

(2) For $\gamma \in \mathrm{Aut}(G)$, it is easy to see that $\gamma|G^{(0)}$ is in the normalizer $N([[G]])$ of $[[G]]$. The map $q : \mathrm{Aut}(G) \to N([[G]])$ sending γ to $\gamma|G^{(0)}$ is clearly a homomorphism. Since G is essentially principal, one can see that q is injective. Let $h \in N([[G]])$. For $g \in G$, let O be a compact open G-set such that $r(O) = s(O) = G^{(0)}$ and $g \in O$. There exists a compact open G-set O' such that $h \circ \tau_O \circ h^{-1} = \tau_{O'}$, because h is in the normalizer of $[[G]]$. Set $g' = (r|O')^{-1}(h(r(g))$. Clearly $r(g') = h(r(g))$ and $s(g') = h(s(g))$. As G is essentially principal, we can conclude that g' does not depend on the choice of O. We define $\gamma : G \to G$ by letting $\gamma(g) = g'$. It is not so hard to see that γ is in $\mathrm{Aut}(G)$ and $q(\gamma) = h$, which means that q is surjective. Consequently, q is an isomorphism. □

Remark 5.8. Any transformation groupoid G_φ and any principal and totally disconnected G satisfy the hypothesis of Proposition 5.7 (2). When G is an étale groupoid arising from a subshift of finite type, $\mathrm{Aut}(G)$ is naturally isomorphic to $N([[G]])$ ([20, Theorem 1.3]).

6 Almost finite groupoids

In this section we introduce the notion of almost finite groupoids (Definition 6.2). Transformation groupoids arising from free actions of $\mathbb{Z}^N$ are shown to be almost finite (Lemma 6.3). Moreover, for two G-full clopen subsets, we prove that they have the same class in $H_0(G)$ if and only if there exists an element in $[[G]]$ which maps one to the other.

Throughout this section, we let G be a second countable étale groupoid whose unit space is compact and totally disconnected. The equivalence class of $f \in C(G^{(0)}, \mathbb{Z})$ in $H_0(G)$ is denoted by $[f]$. A probability measure μ on $G^{(0)}$ is said to be G-invariant if $\mu(r(U)) = \mu(s(U))$ holds for every open G-set U. The set of all G-invariant measures is denoted by $M(G)$. For $\mu \in M(G)$, we can define a homomorphism $\hat{\mu} : H_0(G) \to \mathbb{R}$ by

$$\hat{\mu}([f]) = \int f \, d\mu.$$

6.1 Almost finite groupoids

The following lemma will be used repeatedly later.

Lemma 6.1. *Suppose that G is compact and principal.*

(1) *If a clopen subset $U \subset G^{(0)}$ and $c > 0$ satisfy $|G(x) \cap U| < c|G(x)|$ for all $x \in G^{(0)}$, then $\mu(U) < c$ for all $\mu \in M(G)$.*

(2) *Let $U_1, U_2, \dots, U_n$ and O be clopen subsets of $G^{(0)}$ satisfying $\sum_{i=1}^n |G(x) \cap U_i| \leq |G(x) \cap O|$ for any $x \in G^{(0)}$. Then there exist compact open G-sets $C_1, C_2, \dots, C_n$ such that $r(C_i) = U_i$, $s(C_i) \subset O$ for all i and $s(C_i)$'s are mutually disjoint.*

(3) *Let U and V be clopen subsets of $G^{(0)}$ satisfying $|G(x) \cap U| = |G(x) \cap V|$ for any $x \in G^{(0)}$. Then there exists a compact open G-set C such that $r(C) = U$ and $s(C) = V$.*

(4) *For $f \in C(G^{(0)}, \mathbb{Z})$, $[f]$ is in $H_0(G)^+$ if and only if $\hat{\mu}([f]) \geq 0$ for every $\mu \in M(G)$.*

Proof. (1) is clear from the definition. (3) easily follows from (2). (4) can be proved in a similar fashion to (2). We show only (2). By Lemma 4.7, there exists a G-full clopen subset $Y \subset G^{(0)}$ such that $G|Y = Y$. It follows from Lemma 4.3 that there exist $f \in C(Y, \mathbb{Z})$ and an isomorphism π from $(G|Y)_f = Y_f$ to G such that $\pi(y, 0, 0) = y$ for all $y \in Y$. For each $(n+1)$-tuple $\Lambda = (\lambda_0, \lambda_1, \dots, \lambda_n)$ of finite subsets of $\mathbb{N}$ satisfying

$$|\lambda_0| \geq |\lambda_1| + |\lambda_2| + \cdots + |\lambda_n|,$$

we fix an n-tuple $\alpha_\Lambda = (\alpha_1, \alpha_2, \dots, \alpha_n)$ of injective maps $\alpha_i : \lambda_i \to \lambda_0$ such that $\alpha_i(\lambda_i) \cap \alpha_j(\lambda_j) = \emptyset$ for $i \neq j$. For $y \in Y$, we set

$$\lambda_0(y) = \{k \mid \pi(y, k, k) \in O\}, \quad \lambda_i(y) = \{k \mid \pi(y, k, k) \in U_i\} \quad \forall i = 1, 2, \dots, n$$

and $\Lambda(y) = (\lambda_0(y), \lambda_1(y), \dots, \lambda_n(y))$. Let $\alpha_{y,i}$ be the i-th summand of $\alpha_{\Lambda(y)}$. For $i = 1, 2, \dots, n$, we define $C_i \subset G$ by

$$\pi^{-1}(C_i) = \{(y, k, l) \in (G|Y)_f \mid k \in \lambda_i(y),\ \alpha_{y,i}(k) = l\}.$$

Then one can verify that C_i's are compact open G-sets which meet the requirement. □

Definition 6.2. We say that G is almost finite, if for any compact subset $C \subset G$ and $\varepsilon > 0$ there exists an elementary subgroupoid $K \subset G$ such that

$$\frac{|CKx \setminus Kx|}{|K(x)|} < \varepsilon$$

for all $x \in G^{(0)}$, where $K(x)$ stands for the K-orbit of x. We also remark that $|K(x)|$ equals $|Kx|$, because K is principal.

This definition may remind the reader of the Følner condition for amenable groups. While there is no direct relationship between them, it may be natural to expect that transformation groupoids arising from free actions of amenable groups are almost finite. Indeed, the next lemma shows that this is true at least for $\mathbb{Z}^N$. Notice that φ need not be minimal in the following statement.

Lemma 6.3. *When $\varphi : \mathbb{Z}^N \curvearrowright X$ is a free action of $\mathbb{Z}^N$ on a compact, metrizable and totally disconnected space X, the transformation groupoid G_φ is almost finite.*

Proof. We follow the arguments in [6] (see also [10, 11]). We regard $\mathbb{Z}^N$ as a subset of $\mathbb{R}^N$ and let $\|\cdot\|$ denote the Euclid norm on $\mathbb{R}^N$. Let ω be the volume of the closed unit ball of $\mathbb{R}^N$. We also equip $\mathbb{Z}^N$ with the lexicographic order. Namely, for $p = (p_1, p_2, \dots, p_N)$ and $q = (q_1, q_2, \dots, q_N)$ in $\mathbb{Z}^N$, p is less than q if there exists i such that $p_i < q_i$ and $p_j = q_j$ for all $j < i$.

Suppose that a compact subset $C \subset G_\varphi$ and $\varepsilon > 0$ are given. There exists $n \in \mathbb{N}$ such that C is contained in $\{(p, x) \in G_\varphi \mid \|p\| \leq n,\ x \in X\}$. We identify $x \in X$ with $(0, x) \in G_\varphi$. Choose m sufficiently large. By [19, Lemma 20] or [10, Proposition 4.4], we can construct a clopen subset $U \subset X$ such that $\bigcup_{\|p\| \leq m} \varphi^p(U) = X$ and $U \cap \varphi^p(U) = \emptyset$ for any p with $0 < \|p\| \leq m$. For each $x \in X$, we let

$$P(x) = \{p \in \mathbb{Z}^N \mid \varphi^p(x) \in U\}.$$

Then $P(x) \subset \mathbb{R}^N$ is m-separated and $(m+1)$-syndetic in the sense of [10]. Let $f(x) \in \mathbb{Z}^N$ be the minimum element of

$$\{p \in P(x) \mid \|p\| \leq \|q\| \quad \forall q \in P(x)\}$$

in the lexicographic order. Define $K \subset G_\varphi$ by

$$K = \{(p, x) \in G_\varphi \mid f(x) = p + f(\varphi^p(x))\}.$$

Then K is an elementary subgroupoid of G (see [6, 10]).

We would like to show that K meets the requirement. Fix $x \in U$ and consider the Voronoi tessellation with respect to $P(x)$. Let T be the Voronoi cell containing the origin, that is,

$$T = \{q \in \mathbb{R}^N \mid \|q\| \leq \|q - p\| \quad \forall p \in P(x)\}.$$

Then T is a convex polytope. Note that if $q \in \mathbb{Z}^N$ is in the interior of T, then (q, x) is in K. Since $P(x)$ is m-separated, T contains the closed ball of radius $m/2$ centred at the origin. Hence $|K(x)| \geq (m/2 - 2)^N$. On the other hand, T is contained in the closed ball of radius $m{+}1$ centred at the origin, because $P(x)$ is $(m{+}1)$-syndetic. It follows that the volume V of T is not greater than $(m{+}1)^N\omega$. Let B_1 be the set of $q \in T$ which is within distance $n{+}1$ from the boundary of T and let B_2 be the set of $q \in \mathbb{R}^N \setminus T$ which is within distance 1 from the boundary of T. If $q \in T \cap \mathbb{Z}^N$ is within distance n from the boundary of T, then the closed ball of radius $1/2$ centered at q is contained in $B_1 \cup B_2$. Since $B_1 \cup B_2$ is contained in

$$\left\{\theta q \in \mathbb{R}^N \mid 1 - \frac{2(n+1)}{m} \leq \theta \leq 1 + \frac{2}{m},\ q \in T\right\},$$

the volume of $B_1 \cup B_2$ is less than

$$\left\{\left(1 + \frac{2}{m}\right)^N - \left(1 - \frac{2(n+1)}{m}\right)^N\right\} V \leq \frac{((m+2)^N - (m-2n-2)^N)(m+1)^N\omega}{m^N}.$$

Hence

$$|CKx \setminus Kx| \leq \frac{((m+2)^N - (m-2n-2)^N)(m+1)^N\omega}{m^N} \times ((1/2)^N\omega)^{-1}.$$

As a consequence, by choosing m sufficiently large, we get

$$\frac{|CKx \setminus Kx|}{|K(x)|} < \varepsilon.$$

□

Remark 6.4. Let $\varphi : \mathbb{R}^N \curvearrowright \Omega$ be a free action of $\mathbb{R}^N$ on a compact, metrizable space Ω and let $X \subset \Omega$ be a flat Cantor transversal in the sense of [11, Definition 2.1]. As described in [11], we can construct an étale principal groupoid (i.e. étale equivalence relation) whose unit space is (homeomorphic to) X. In the same way as the lemma above, this groupoid is shown to be almost finite.

6.2 Basic facts about almost finite groupoids

In this subsection we collect several basic facts about almost finite groupoids.

Lemma 6.5. *If G is almost finite, then $M(G)$ is not empty.*

Proof. Take an increasing sequence of compact open subsets $C_1 \subset C_2 \subset \dots$ whose union is equal to G. For each $n \in \mathbb{N}$, there exists an elementary subgroupoid $K_n \subset G$ such that

$$\frac{|C_n K_n x \setminus K_n x|}{|K_n(x)|} < \frac{1}{n}$$

for all $x \in G^{(0)}$. Clearly $M(K_n)$ is not empty, and so we can choose $\mu_n \in M(K_n)$. By taking a subsequence if necessary, we may assume that μ_n converges to a probability measure μ. We would like to show that μ belongs to $M(G)$.

Let U be a compact open G-set. For sufficiently large $n \in \mathbb{N}$, C_n contains $U \cup U^{-1}$. For $x \in G^{(0)}$, one has

$$|K_n(x) \cap s(U \setminus K_n)| = |(U \setminus K_n)K_n x| = |UK_n x \setminus K_n x|,$$

which is less than $n^{-1}|K_n(x)|$ when C_n contains U. It follows from Lemma 6.1 (1) that $\mu_n(s(U \setminus K_n))$ is less than n^{-1}. Similarly, when C_n contains U^{-1}, one can see that $\mu_n(r(U \setminus K_n))$ is less than n^{-1}. Since μ_n is K_n-invariant, we have $\mu_n(r(U \cap K_n)) = \mu_n(s(U \cap K_n))$. Consequently, $|\mu_n(r(U)) - \mu_n(s(U))|$ is less than $2/n$ for sufficiently large n, which implies $\mu(r(U)) = \mu(s(U))$. Therefore μ is G-invariant. □

Remark 6.6. Let G' be the isotropy bundle. Take $\mu \in M(G)$ arbitrarily. In the proof above, $U \cap (G' \setminus G^{(0)})$ is contained in $U \setminus K_n$, because $G' \setminus G^{(0)}$ does not intersect with K_n. Hence $\mu(r(U \cap (G' \setminus G^{(0)})))$ is less than $1/n$, which implies $\mu(r(U \cap (G' \setminus G^{(0)}))) = 0$. Since G has a countable base of compact open G-sets, we have $\mu(r(G' \setminus G)) = 0$ for all $\mu \in M(G)$. Assume further that G is minimal. From Lemma 6.8 below, $\mu(U)$ is positive for any non-empty open subset $U \subset G^{(0)}$. It follows that $r(G' \setminus G^{(0)})$ contains no interior points, and so the set of points of $G^{(0)}$ with trivial isotropy is dense in $G^{(0)}$. Thus, if G is minimal and almost finite, then G is essentially principal.

Lemma 6.7. *Suppose that G is almost finite. If two clopen subsets $U, V \subset G^{(0)}$ satisfy $\mu(U) < \mu(V)$ for all $\mu \in M(G)$, then there exists $\gamma \in [[G]]$ such that $\gamma(U) \subset V$. In fact, one can find such γ so that $\gamma^2 = \mathrm{id}$ and $\gamma(x) = x$ for $x \in G^{(0)} \setminus (U \cup \gamma(U))$.*

Proof. By removing $U \cap V$ if necessary, we may assume that U and V are disjoint. Let C_n and K_n be as in Lemma 6.5. Suppose that for each $n \in \mathbb{N}$ there exists $\mu_n \in M(K_n)$ such that $\mu_n(U) \geq \mu_n(V)$. By taking a subsequence if necessary, we may assume that μ_n converges to μ. By the proof of Lemma 6.5, we have $\mu \in M(G)$. This, together with $\mu(U) \geq \mu(V)$, contradicts the assumption. It follows that there exists $n \in \mathbb{N}$ such that $\mu(U) < \mu(V)$ for all $\mu \in M(K_n)$. Lemma 6.1 (2) applies and yields a compact open K_n-set C such that $r(C) = U$ and $s(C) \subset V$. Set $D = C \cup C^{-1} \cup (G^{(0)} \setminus (r(C) \cup s(C)))$. Then $\gamma = \tau_D$ is the desired element. □

Lemma 6.8. *Suppose that G is almost finite. For a clopen subset $U \subset G^{(0)}$, the following are equivalent.*

(1) *U is G-full.*

(2) *There exists $c > 0$ such that $\mu(U) > c$ for all $\mu \in M(G)$.*

(3) *$\mu(U) > 0$ for every $\mu \in M(G)$.*

In particular, if G is minimal, then $\mu(U) > 0$ for any non-empty clopen subset $U \subset G^{(0)}$ and $\mu \in M(G)$.

Proof. (1)⇒(2). Suppose that a clopen subset $U \subset G^{(0)}$ is G-full. By Lemma 4.3, there exists $f \in C(Y, \mathbb{Z})$ such that G and $(G|Y)_f$ are isomorphic. Put $n = \max\{f(x) \mid x \in U\}$. Then $1 = \mu(G^{(0)}) \leq (n+1)\mu(Y)$ for any $\mu \in M(G)$, which implies (2).

(2)⇒(3) is trivial.

(3)⇒(1). We need the hypothesis of almost finiteness for this implication. Suppose that U is not G-full. Let $V = r(s^{-1}(U))$. Then V is an open subset such that $U \subset V \neq G^{(0)}$ and $r(s^{-1}(V)) = V$. Let C_n and K_n be as in Lemma 6.5. Take $x \in G^{(0)} \setminus V$ and put

$$\mu_n = \frac{1}{|K_n(x)|} \sum_{y \in K_n(x)} \delta_y,$$

where δ_y is the Dirac measure on y. Then μ_n is K_n-invariant and $\mu_n(V) = 0$. By taking a subsequence if necessary, we may assume that μ_n converges to μ. From the proof of Lemma 6.5, μ is G-invariant. But $\mu(U) \leq \mu(V) = \lim \mu_n(V) = 0$, which contradicts (3). □

Lemma 6.9. *Suppose that G is almost finite and minimal. For $f \in C(G^{(0)}, \mathbb{Z})$, one has $[f] \in H_0(G)^+ \setminus \{0\}$ if and only if $\hat{\mu}([f]) > 0$ for every $\mu \in M(G)$.*

Proof. The 'only if' part easily follows from the lemma above. We prove the 'if' part. Let C_n and K_n be as in Lemma 6.5. In the same way as above, we can find $n \in \mathbb{N}$ such that $\int f \, d\mu > 0$ for every $\mu \in M(K_n)$. By Lemma 6.1 (4), $[f]$ is in $H_0(K_n)^+$, and hence in $H_0(G)^+$. □

For the terminologies about ordered abelian groups in the following statement, we refer the reader to [27].

Proposition 6.10. *Suppose that G is almost finite and minimal. Then $(H_0(G), H_0(G)^+)$ is a simple, weakly unperforated, ordered abelian group with the Riesz interpolation property.*

Proof. By virtue of Lemma 6.9, one can see that $(H_0(G), H_0(G)^+)$ is a simple, weakly unperforated, ordered abelian group. We would like to check the Riesz interpolation property. To this end, take $f_1, f_2, g_1, g_2 \in C(G^{(0)}, \mathbb{Z})$ satisfying $[f_i] \leq [g_j]$ for $i, j = 1, 2$. It suffices to find $h \in C(G^{(0)}, \mathbb{Z})$ such that $[f_i] \leq [h] \leq [g_j]$ for $i, j = 1, 2$. Clearly we may assume $[f_i] \neq [g_j]$ for $i, j = 1, 2$. Therefore $g_j - f_i$ is in $H_0(G)^+ \setminus \{0\}$ for any $i, j = 1, 2$. By Lemma 6.9, we get $\hat{\mu}([f_i]) < \hat{\mu}([g_j])$ for any $i, j = 1, 2$. Let C_n and K_n be as in Lemma 6.5. In the same way as above, we can find $n \in \mathbb{N}$ such that

$$\int f_i \, d\mu < \int g_j \, d\mu$$

for every $\mu \in M(K_n)$ and $i, j = 1, 2$. By Lemma 4.7, there exists a K_n-full clopen subset $Y \subset K_n^{(0)} = G^{(0)}$ such that $K_n|Y = Y$. Define $h \in C(G^{(0)}, \mathbb{Z})$ by

$$h(y) = \max \left\{ \sum_{x \in K_n(y)} f_1(x), \sum_{x \in K_n(y)} f_2(x) \right\}$$

for $y \in Y$ and $h(z) = 0$ for $z \notin Y$. It is not so hard to see

$$\int f_i \, d\mu \leq \int h \, d\mu \leq \int g_j \, d\mu$$

for every $\mu \in M(K_n)$, which implies $[f_i] \leq [h] \leq [g_j]$ in $H_0(K_n)$ by Lemma 6.1 (4). Hence we obtain the same inequalities in $H_0(G)$. □

6.3 Equivalence between clopen subsets

By using the lemmas above, we can prove the following two theorems concerning the 'equivalence' of clopen subsets under the action of the (topological) full groups.

Theorem 6.11. *Suppose that G is almost finite and minimal. For two clopen subsets $U, V \subset G^{(0)}$, the following are equivalent.*

(1) *There exists $\gamma \in [G]$ such that $\gamma(U) = V$.*

(2) *$\mu(U)$ equals $\mu(V)$ for all $\mu \in M(G)$.*

Moreover, γ in the first condition can be chosen so that $\gamma^2 = \mathrm{id}$ and $\gamma(x) = x$ for $x \in X \setminus (U \cup V)$.

Proof. Since G is minimal, $G^{(0)}$ is either a finite set or a Cantor set. If $G^{(0)}$ is a finite set, then the assertion is trivial, and so we may assume that $G^{(0)}$ is a Cantor set. Any $\mu \in M(G)$ has no atoms, because every G-orbit is an infinite set. By Lemma 6.8, $\mu(U)$ is positive for any non-empty clopen set U and $\mu \in M(G)$. Then the assertion follows from a simple generalization of the arguments in [15, Proposition 2.6] by using Lemma 6.7 instead of [15, Lemma 2.5]. See also [18, Theorem 3.20] and its proof. □

Theorem 6.12. *Suppose that G is almost finite. For two G-full clopen subsets $U, V \subset G^{(0)}$, the following are equivalent.*

(1) *There exists $\gamma \in [[G]]$ such that $\gamma(U) = V$.*

(2) *$[1_U]$ equals $[1_V]$ in $H_0(G)$.*

Moreover, γ in the first condition can be chosen so that $\gamma^2 = \mathrm{id}$ and $\gamma(x) = x$ for $x \in X \setminus (U \cup V)$.

Proof. Recall that $H_0(G)$ is the quotient of $C(G^{(0)}, \mathbb{Z})$ by $\operatorname{Im} \delta_1$, where $\delta_1 : C_c(G, \mathbb{Z}) \to C(G^{(0)}, \mathbb{Z})$ is given by

$$\delta_1(f)(x) = s_*(f)(x) - r_*(f)(x) = \sum_{g \in s^{-1}(x)} f(g) - \sum_{g \in r^{-1}(x)} f(g).$$

Suppose that there exists $\gamma \in [[G]]$ such that $\gamma(U) = V$. Let O be a compact open G-set such that $\gamma = \tau_O$. It is easy to see $\delta_1(1_{O\cap r^{-1}(V)}) = 1_U - 1_V$, which implies $[1_U] = [1_V]$ in $H_0(G)$.

We would like to show the other implication (2)⇒(1). Clearly we may assume that U and V are disjoint. Suppose that there exists $f \in C_c(G, \mathbb{Z})$ such that $\delta_1(f) = 1_U - 1_V$. For a compact open G-set C, one has $\delta_1(1_{C^{-1}}) = -1_C$. Since G has a base of compact open G-sets, we may assume that there exist compact open G-sets $C_1, C_2, \dots, C_n$ such that $f = 1_{C_1} + 1_{C_2} + \cdots + 1_{C_n}$. By Lemma 6.8, there exists $\varepsilon > 0$ such that $\mu(U) > \varepsilon$ and $\mu(V) > \varepsilon$ for all $\mu \in M(G)$. Almost finiteness of G yields an elementary subgroupoid $K \subset G$ such that

$$\frac{|C_i Kx \setminus Kx|}{|K(x)|} < \frac{\varepsilon}{n} \quad \text{and} \quad \frac{|C_i^{-1} Kx \setminus Kx|}{|K(x)|} < \frac{\varepsilon}{n}$$

for all $x \in G^{(0)}$ and $i = 1, 2, \dots, n$. Moreover, by the proof of Lemma 6.5, we may further assume

$$\frac{|U \cap K(x)|}{|K(x)|} > \varepsilon \quad \text{and} \quad \frac{|V \cap K(x)|}{|K(x)|} > \varepsilon$$

for all $x \in G^{(0)}$. It follows from $|C_i Kx \setminus Kx| = |s(C_i \setminus K) \cap K(x)|$ that

$$|U \cap K(x)| > \sum_{i=1}^{n} |s(C_i \setminus K) \cap K(x)|$$

for all $x \in G^{(0)}$. Likewise we have

$$|V \cap K(x)| > \sum_{i=1}^{n} |r(C_i \setminus K) \cap K(x)|$$

for all $x \in G^{(0)}$. By Lemma 6.1 (2), there exist compact open K-sets $A_1, A_2, \dots, A_n$ such that

$$r(A_i) = s(C_i \setminus K), \quad s(A_i) \subset U \quad \forall i = 1, 2, \dots, n$$

and $s(A_i)$'s are mutually disjoint. Similarly there exists compact open K-sets $B_1, B_2, \dots, B_n$ such that

$$s(B_i) = r(C_i \setminus K), \quad r(B_i) \subset V \quad \forall i = 1, 2, \dots, n$$

and $r(B_i)$'s are mutually disjoint. Then

$$D = \bigcup_{i=1}^{n} B_i (C_i \setminus K) A_i$$

is a compact open G-set such that $r(D) \subset V$ and $s(D) \subset U$. Moreover, for any $x \in G^{(0)}$,

$$\begin{aligned}\sum_{y \in K(x)} 1_{U \setminus s(D)}(y) &= \sum_{y \in K(x)} 1_U(y) - 1_{s(D)}(y) \\ &= \sum_{y \in K(x)} \left(1_U(y) - \sum_{i=1}^{n} 1_{s(C_i \setminus K)}(y) \right) \\ &= \sum_{y \in K(x)} \left(1_U(y) - \sum_{i=1}^{n} s_*(1_{C_i \setminus K})(y) \right) \\ &= \sum_{y \in K(x)} \left(1_V(y) - \sum_{i=1}^{n} r_*(1_{C_i \setminus K})(y) \right) \\ &= \sum_{y \in K(x)} \left(1_V(y) - \sum_{i=1}^{n} 1_{r(C_i \setminus K)}(y) \right) = \sum_{y \in K(x)} 1_{V \setminus r(D)}(y),\end{aligned}$$

and so one can find a compact open K-set E such that $s(E) = U \setminus s(D)$ and $r(E) = V \setminus r(D)$ by Lemma 6.1 (3). Hence $F = D \cup E$ is a compact open G-set satisfying $s(F) = U$ and $r(F) = V$. Define a compact open G-set O by $O = F \cup F^{-1} \cup (G^{(0)} \setminus (U \cup V))$. Then $\gamma = \tau_O \in [[G]]$ is a desired element. □

Remark 6.13. In the light of Proposition 5.6, the two conditions of the theorem above are also equivalent to

(3) There exists $w \in N(C(G^{(0)}), C_r^*(G))$ such that $w 1_U w^* = 1_V$.

We do not know when this is equivalent to the condition that the two projections 1_U and 1_V have the same class in $K_0(C_r^*(G))$.

We also remark that a special case of Theorem 6.12 is implicitly contained in the proof of [18, Theorem 3.16].

7 The index map

In this section, we introduce a group homomorphism, called the index map, from $[[G]]$ to $H_1(G)$. When G is almost finite, it will be shown that the index map is surjective (Theorem 7.5) and that any element in the kernel of the index map can be written as a product of four elements of finite order (Theorem 7.13).

Throughout this section, we let G be a second countable étale essentially principal groupoid whose unit space is compact and totally disconnected. For $f \in C_c(G, \mathbb{Z})$, we denote its equivalence class in $H_1(G)$ by $[f]$.

Definition 7.1. For $\gamma \in [[G]]$, a compact open G-set U satisfying $\gamma = \tau_U$ uniquely exists, because G is essentially principal. It is easy to see that 1_U is in $\operatorname{Ker} \delta_1$. We define a map $I : [[G]] \to H_1(G)$ by $I(\gamma) = [1_U]$ and call it the index map.

Remark 7.2. When G arises from a minimal homeomorphism on a Cantor set, $H_1(G)$ is $\mathbb{Z}$ and the above definition agrees with that in [13, Section 5]. In this case, the index map can be understood through the Fredholm index of certain Fredholm operators.

Lemma 7.3. (1) *If $U, U' \subset G$ are compact open G-sets satisfying $s(U) = r(U')$, then*

$$O = \{(g, g') \in G^{(2)} \mid g \in U,\ g' \in U'\}$$

is a compact open subset of $G^{(2)}$ and $\delta_2(1_O) = 1_U - 1_{UU'} + 1_{U'}$.

(2) *The index map $I : [[G]] \to H_1(G)$ is a homomorphism.*

(3) *$[1_U] = 0$ for any clopen subset $U \subset G^{(0)}$.*

(4) *$[1_U + 1_{U^{-1}}] = 0$ for any compact open G-set $U \subset G$.*

Proof. (1) follows from a straightforward computation. (2), (3) and (4) are direct consequences of (1). □

7.1 Surjectivity of the index map

In this subsection we prove that the index map $I : [[G]] \to H_1(G)$ is surjective when G is almost finite (Theorem 7.5). To this end, we need the following lemma.

Lemma 7.4. *Let K be an elementary subgroupoid of G and let $Y \subset G^{(0)}$ be a K-full clopen subset such that $K|Y = Y$. Suppose that $f \in C_c(G, \mathbb{Z})$ is in $\operatorname{Ker}\delta_1$. Then, the function $\tilde{f} \in C_c(G, \mathbb{Z})$ defined by*

$$\tilde{f}(g) = \begin{cases} \sum_{g_1, g_2 \in K} f(g_1 g g_2) & g \in G|Y \\ 0 & \textit{otherwise} \end{cases}$$

is also in $\operatorname{Ker}\delta_1$ and $[f] = [\tilde{f}]$ in $H_1(G)$.

Proof. Put $f_0 = s_*(f) = r_*(f) \in C(G^{(0)}, \mathbb{Z})$. Define $k, \bar{k} \in C(K, \mathbb{Z})$ by

$$k(g) = \begin{cases} f_0(s(g)) & r(g) \in Y \\ 0 & r(g) \notin Y \end{cases}$$

and $\bar{k}(g) = k(g^{-1})$ for $g \in K$. By Lemma 7.3 (4), $[k + \bar{k}] = 0$ in $H_1(K)$ and hence in $H_1(G)$. We define $h_1 \in C_c(G^{(2)}, \mathbb{Z})$ by

$$h_1(g, g') = \begin{cases} f(g') & g \in K,\ r(g) \in Y \\ 0 & \text{otherwise.} \end{cases}$$

Letting $d_i : G^{(2)} \to G$ $(i = 0, 1, 2)$ be the maps introduced in Section 3, we have

$$\begin{aligned} \delta_2(h_1)(g) &= d_{0*}(h_1)(g) - d_{1*}(h_1)(g) + d_{2*}(h_1)(g) \\ &= \sum_{g_0 \in G} h_1(g_0, g) - d_{1*}(h_1)(g) + \sum_{g_0 \in G} h_1(g, g_0) \\ &= f(g) - d_{1*}(h_1)(g) + k(g). \end{aligned}$$

Define $h_2 \in C_c(G^{(2)}, \mathbb{Z})$ by

$$h_2(g, g') = \begin{cases} d_{1*}(h_1)(g) & r(g) \in Y,\ g' \in K,\ s(g') \in Y \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} d_{0*}(h_2)(g) &= \sum_{g_0 \in G} h_2(g_0, g) \\ &= \begin{cases} \sum_{r(g_0)\in Y} d_{1*}(h_1)(g_0) & g \in K,\ s(g) \in Y \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} s_*(f)(r(g)) & g \in K,\ s(g) \in Y \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} f_0(r(g)) & g \in K,\ s(g) \in Y \\ 0 & \text{otherwise} \end{cases} \\ &= \bar{k}(g). \end{aligned}$$

Moreover, it is easy to see $d_{1*}(h_2) = \tilde{f}$ and $d_{2*}(h_2) = d_{1*}(h_1)$. Hence

$$\begin{aligned} \delta_2(h_1) + \delta_2(h_2) &= (f - d_{1*}(h_1) + k) + (\bar{k} - \tilde{f} + d_{1*}(h_1)) \\ &= f + k + \bar{k} - \tilde{f}, \end{aligned}$$

and so $\tilde{f}$ is in $\operatorname{Ker}\delta_1$ and $[f] = [\tilde{f}]$ in $H_1(G)$. □

Theorem 7.5. *When G is almost finite, the index map I is surjective.*

Proof. Take $f \in C_c(G, \mathbb{Z})$ such that $\delta_1(f) = 0$. We will show that there exists $\gamma \in [[G]]$ satisfying $I(\gamma) = [f]$. By Lemma 7.3 (4), we may assume $f(g) \geq 0$ for all $g \in G$. Since G has a base of compact open G-sets, there exist compact open G-sets $C_1, C_2, \dots, C_n$ such that $f = 1_{C_1} + 1_{C_2} + \cdots + 1_{C_n}$. Almost finiteness of G yields an elementary subgroupoid $K \subset G$ such that

$$\frac{|C_i K x \setminus Kx|}{|K(x)|} < \frac{1}{n} \quad \text{and} \quad \frac{|C_i^{-1} K x \setminus Kx|}{|K(x)|} < \frac{1}{n}$$

for all $x \in G^{(0)}$ and $i = 1, 2, \dots, n$. For $x \in G^{(0)}$, let $E_x = \{g \in G \mid g \notin K,\ s(g) \in K(x)\}$. Then

$$\sum_{g \in E_x} f(g) = \sum_{i=1}^{n} \sum_{g \in E_x} 1_{C_i}(g) = \sum_{i=1}^{n} |C_i K x \setminus Kx| < \sum_{i=1}^{n} n^{-1} |K(x)| = |K(x)|$$

for any $x \in G^{(0)}$. Likewise, we have $\sum_{g \in E_x} f(g^{-1}) < |K(x)|$.

By Lemma 4.7, there exists a K-full clopen subset $Y \subset K^{(0)} = G^{(0)}$ such that $K|Y = Y$. Let $\tilde{f}$ be as in the preceding lemma. Define $f_0 \in C(G^{(0)}, \mathbb{Z})$ by $f_0 = \tilde{f}|G^{(0)}$. By Lemma 7.3 (3), $[\tilde{f}] = [\tilde{f} - f_0]$ in $H_1(G)$. Since G has a base of compact open G-sets, we may assume that there exist compact open G-sets $D_1, D_2, \dots, D_m \subset G \setminus G^{(0)}$ such that

$$\tilde{f} - f_0 = 1_{D_1} + 1_{D_2} + \cdots + 1_{D_m}.$$

Notice that $r(D_i)$ and $s(D_i)$ are contained in Y and that D_i does not intersect with K. For any $y \in Y$ one has

$$\begin{aligned}\sum_{i=1}^{m} |s(D_i) \cap K(y)| &= \sum_{i=1}^{m} 1_{s(D_i)}(y) = \sum_{i=1}^{m} s_*(1_{D_i})(y) = s_*(\tilde{f} - f_0)(y) \\ &= \sum_{s(g)=y,\ g \notin K} \tilde{f}(g) = \sum_{g \in E_y} f(g) < |K(y)|.\end{aligned}$$

It follows from Lemma 6.1 (2) that there exist compact open K-sets $A_1, A_2, \ldots, A_m$ such that $r(A_i) = s(D_i)$ for all $i = 1, 2, \ldots, m$ and $s(A_i)$'s are mutually disjoint. In a similar way, we also have

$$\sum_{i=1}^{m} |r(D_i) \cap K(y)| < |K(y)|,$$

and so there exist compact open K-sets $B_1, B_2, \ldots, B_m$ such that $s(B_i) = r(D_i)$ for all $i = 1, 2, \ldots, m$ and $r(B_i)$'s are mutually disjoint. Besides, from $s_*(\tilde{f} - f_0) = r_*(\tilde{f} - f_0)$, we get

$$\sum_{i=1}^{m} 1_{r(A_i)} = \sum_{i=1}^{m} 1_{s(D_i)} = \sum_{i=1}^{m} 1_{r(D_i)} = \sum_{i=1}^{m} 1_{s(B_i)},$$

which implies

$$\begin{aligned}\left| \bigcup_{i=1}^{m} s(A_i) \cap K(x) \right| &= \sum_{i=1}^{m} |s(A_i) \cap K(x)| = \sum_{i=1}^{m} |r(A_i) \cap K(x)| \\ &= \sum_{i=1}^{m} |s(B_i) \cap K(x)| = \sum_{i=1}^{m} |r(B_i) \cap K(x)| = \left| \bigcup_{i=1}^{m} r(B_i) \cap K(x) \right|\end{aligned}$$

for $x \in G^{(0)}$, because A_i and B_i are K-sets. Hence, by Lemma 6.1 (3), we may replace A_i and assume

$$\bigcup_{i=1}^{m} s(A_i) = \bigcup_{i=1}^{m} r(B_i).$$

Then

$$E = \bigcup_{i=1}^{m} B_i D_i A_i$$

is a compact open G-set satisfying $s(E) = r(E)$. Let $k = \sum_{i=1}^{m} 1_{A_i} + 1_{B_i} \in C(K, \mathbb{Z})$. It is easy to verify that k is in $\operatorname{Ker} \delta_1$. Therefore, by Lemma 4.9, $[k]$ is zero in $H_1(K)$ and hence in $H_1(G)$. By Lemma 7.3 (1), $1_{B_i} - 1_{B_i D_i} + 1_{D_i}$ and $1_{B_i D_i} - 1_{B_i D_i A_i} + 1_{A_i}$ are zero in $H_1(G)$ for all $i = 1, 2, \ldots, m$. Consequently,

$$[f] = [\tilde{f}] = [\tilde{f} - f_0] = [1_{D_1} + \cdots + 1_{D_m}] = [1_{D_1} + \cdots + 1_{D_m}] + [k] = [1_E]$$

in $H_1(G)$. Let $F = E \cup (G^{(0)} \setminus s(E))$. Then F is a compact open G-set satisfying $r(F) = s(F) = G^{(0)}$ and $[1_F] = [1_E]$. Thus $\gamma = \tau_F$ is in $[[G]]$ and $I(\gamma) = [f]$. □

7.2 Kernel of the index map

Next, we would like to determine the kernel of the index map.

Definition 7.6. (1) We say that $\gamma \in \mathrm{Homeo}(G^{(0)})$ is elementary, if γ is of finite order and $\{x \in G^{(0)} \mid \gamma^k(x) = x\}$ is clopen for any $k \in \mathbb{N}$.

(2) We let $[[G]]_0$ denote the subgroup of $[[G]]$ which is generated by all elementary homeomorphisms in $[[G]]$. Evidently $[[G]]_0$ is a normal subgroup of $[[G]]$.

Lemma 7.7. (1) *When G is principal, $\gamma \in [[G]]$ is elementary if and only if γ is of finite order.*

(2) *$\gamma \in [[G]]$ is elementary if and only if there exists an elementary subgroupoid $K \subset G$ such that $\gamma \in [[K]]$.*

(3) *If $\gamma \in [[G]]$ is elementary, then $I(\gamma) = 0$. In particular, $\mathrm{Ker}\, I$ contains $[[G]]_0$.*

Proof. (1) This is clear from the definition.

(2) The 'if' part follows from [21, Proposition 3.2] and its proof. Let us show the 'only if' part. Suppose that $\gamma = \tau_U \in [[G]]$ is elementary. There exists $n \in \mathbb{N}$ such that $\gamma^n = \mathrm{id}$. Then $K = (U \cup G^{(0)})^n$ is a compact open subgroupoid of G. Since the fixed points of γ^k form a clopen set for any $k \in \mathbb{N}$, K is principal. It follows from $U \subset K$ that γ belongs to $[[K]]$.

(3) This readily follows from (2) and Lemma 4.9. □

Remark 7.8. Even if $\gamma \in [[G]]$ is of finite order, $I(\gamma)$ is not necessarily zero. Let $\varphi : \mathbb{Z}/N\mathbb{Z} \curvearrowright X$ be an action of $\mathbb{Z}/N\mathbb{Z}$ on a Cantor set X by homeomorphisms and let G_φ be the transformation groupoid arising from φ. The generator γ of φ is clearly in $[[G_\varphi]]$ and of finite order. It is well-known that $H_1(G_\varphi) \cong H_1(\mathbb{Z}/N\mathbb{Z}, C(X,\mathbb{Z}))$ is isomorphic to

$$\{f \in C(X,\mathbb{Z}) \mid f = f \circ \gamma\}/\{f + f \circ \gamma + \cdots + f \circ \gamma^{N-1} \mid f \in C(X,\mathbb{Z})\}.$$

Hence, when φ is not free, $I(\gamma)$ is not zero in $H_1(G_\varphi)$.

Remark 7.9. In Corollary 7.16, it will be shown that $[[G]]/[[G]]_0$ is isomorphic to $H_1(G)$ via the index map, when G is almost finite and principal. This, however, does not mean that $H_1(G)$ is always torsion free. Indeed, it was shown in [9, Section 6.4] that the dual canonical D_6 tiling contains 2-torsions in its H_1-group, and so there exists a free action φ of $\mathbb{Z}^3$ on a Cantor set by homeomorphisms such that $H_1(G_\varphi)$ contains 2-torsions. Note that G_φ is almost finite by Lemma 6.3.

In order to prove Theorem 7.13, we need a series of lemmas.

Lemma 7.10. *Suppose that G is almost finite. For any $\gamma \in [[G]]$, there exist an elementary homeomorphism $\gamma_0 \in [[G]]$ and a clopen subset $V \subset G^{(0)}$ such that $\gamma_0\gamma(x) = x$ for any $x \in V$ and $\mu(V) \geq 1/2$ for any $\mu \in M(G)$.*

Proof. Take a compact open G-set U satisfying $\gamma = \tau_U$. Since G is almost finite, there exists an elementary subgroupoid $K \subset G$ such that

$$|UKx \setminus Kx| < 2^{-1}|K(x)|$$

for all $x \in G^{(0)}$. Let $V = s(U \cap K)$. Then

$$|K(x) \cap V| = |K(x)| - |UKx \setminus Kx| \geq 2^{-1}|K(x)|.$$

By Lemma 6.1 (1), we have $\mu(V) \geq 2^{-1}$ for all $\mu \in M(K)$ and hence for all $\mu \in M(G)$. Moreover, one also has

$$|K(x) \cap s(U \setminus K)| = |K(x) \setminus s(U \cap K)| = |K(x) \setminus r(U \cap K)| = |K(x) \cap r(U \setminus K)|$$

for all $x \in G^{(0)}$. It follows from Lemma 6.1 (3) that there exists a compact open K-set W such that $s(W) = r(U \setminus K)$ and $r(W) = s(U \setminus K)$. Then $O = W \cup (U^{-1} \cap K)$ is a compact open K-set satisfying $s(O) = r(O) = G^{(0)}$, and so $\gamma_0 = \tau_O$ is elementary by Lemma 7.7 (2). Clearly $\gamma_0\gamma(x) = x$ for $x \in V$, which completes the proof. □

Lemma 7.11. *Suppose that G is almost finite. Let $V \subset G^{(0)}$ be a clopen subset and let $\gamma \in \operatorname{Ker} I$. Suppose that $\gamma(x) = x$ for any $x \in V$ and $\mu(V) \geq 1/2$ for any $\mu \in M(G)$. Then, there exist an elementary subgroupoid $K \subset G$ and $\tau_U \in [[G]]$ such that $\tau_U\gamma^{-1}$ is elementary and*

$$\sum_{g_1,g_2 \in K} 1_U(g_1 g g_2) = \sum_{g_1,g_2 \in K} 1_U(g_1 g^{-1} g_2)$$

holds for all $g \in G$.

Proof. Let $\gamma = \tau_O$. Since $I(\gamma) = 0$, 1_O is in $\operatorname{Im}\delta_2$. It follows from Lemma 7.3 (1) that there exist compact open G-sets $A_1, \dots, A_n$, $B_1, \dots, B_n$, $C_1, \dots, C_m$ and $D_1, \dots, D_m$ such that $s(A_i) = r(B_i)$, $s(C_j) = r(D_j)$ and

$$1_O = \left(\sum_{i=1}^{n} 1_{A_i} - 1_{A_iB_i} + 1_{B_i}\right) - \left(\sum_{j=1}^{m} 1_{C_j} - 1_{C_jD_j} + 1_{D_j}\right).$$

Set

$$E = \bigcup_{i=1}^{n}(A_i \cup B_i \cup A_iB_i) \cup \bigcup_{j=1}^{m}(C_j \cup D_j \cup C_jD_j).$$

Almost finiteness of G yields an elementary subgroupoid $K \subset G$ such that

$$|(E \cup E^{-1})Kx \setminus Kx| < \frac{1}{18(n+m)}|K(x)|$$

for all $x \in G^{(0)}$. Since $\mu(V)$ is not less than 1/2 for every $\mu \in M(G)$, by the proof of Lemma 6.5, we may further assume

$$|V \cap K(x)| > 3^{-1}|K(x)|$$

for all $x \in G^{(0)}$. Define compact open G-sets A'_i, B'_i, C'_j, D'_j by

$$A'_i = A_i \setminus ((A_i \cap K)r(B_i \cap K)), \quad B'_i = B_i \setminus (s(A_i \cap K)(B_i \cap K)),$$

$$C'_j = C_j \setminus ((C_j \cap K)r(D_j \cap K)), \quad D'_j = D_j \setminus (s(C_j \cap K)(D_j \cap K)).$$

Then $s(A'_i) = r(B'_i)$, $s(C'_j) = r(D'_j)$ and

$$1_O = \left(\sum_{i=1}^{n} 1_{A'_i} - 1_{A'_iB'_i} + 1_{B'_i}\right) - \left(\sum_{j=1}^{m} 1_{C'_j} - 1_{C'_jD'_j} + 1_{D'_j}\right) + k$$

for some $k \in C(K, \mathbb{Z})$. In addition, since

$$(A_i \cap K)r(B_i \cap K) = (A_i \cap K) \cap (A_iB_i \cap K)B_i^{-1},$$

we have

$$\begin{aligned}
|r(A'_i) \cap K(x)| &\leq |r(A_i \setminus (A_i \cap K)) \cap K(x)| + |r(A_i \setminus (A_iB_i \cap K)B_i^{-1}) \cap K(x)| \\
&= |A_i^{-1}Kx \setminus Kx| + |(A_iB_i)^{-1}Kx \setminus Kx| \\
&< \frac{1}{9(n+m)}|K(x)|
\end{aligned}$$

for any $x \in G^{(0)}$. Similar estimates can be obtained for $s(A'_i)$, $s(B'_i)$, $r(C'_j)$, $s(C'_j)$ and $s(D'_j)$. By Lemma 6.1 (2), we can find compact open K-sets $P_{k,i}$ ($k = 1, 2, 3$, $i = 1, 2, \ldots, n$) and $Q_{l,j}$ ($l = 1, 2, 3$, $j = 1, 2, \ldots, m$) such that

$$s(P_{1,i}) = r(A'_i), \quad s(P_{2,i}) = s(A'_i), \quad s(P_{3,i}) = s(B'_i)$$

$$s(Q_{1,j}) = r(C'_j), \quad s(Q_{2,j}) = s(C'_j), \quad s(Q_{3,j}) = s(D'_j)$$

and the ranges of $P_{k,i}$'s and $Q_{l,j}$'s are mutually disjoint and contained in V. Define compact open G-sets A''_i, B''_i, C''_j, D''_j by

$$A''_i = P_{1,i}A'_iP_{2,i}^{-1}, \quad B''_i = P_{2,i}B'_iP_{3,i}^{-1}, \quad C''_j = Q_{1,j}C'_jQ_{2,j}^{-1}, \quad D''_i = Q_{2,j}D'_iQ_{3,j}^{-1}.$$

Then

$$F = \bigcup_{i=1}^{n}(A''_i \cup B''_i \cup (A''_iB''_i)^{-1})^{-1} \cup \bigcup_{j=1}^{m}(C''_j \cup D''_j \cup (C''_jD''_j)^{-1})$$

is a compact open G-set satisfying $s(F) = r(F) = F^3 \subset V$. Moreover, τ_F and τ_F^2 have no fixed points. Set $F_0 = G^{(0)} \setminus s(F)$ and $\widetilde{F} = F \cup F_0$. Then $\tau_{\widetilde{F}}$ is an elementary homeomorphism in $[[G]]$. Define a compact open subset $U \subset G$ by $U = F \cup OF_0$. Since $s(F)$ is contained in V, U is a G-set and $\tau_{\tilde{F}}\tau_O = \tau_U$. Furthermore,

$$\begin{aligned}
1_U &= 1_F + 1_{OF_0} = 1_F + 1_O - 1_{s(F)} \\
&= 1_F + \left(\sum_{i=1}^{n} 1_{A'_i} - 1_{A'_iB'_i} + 1_{B'_i}\right) - \left(\sum_{j=1}^{m} 1_{C'_j} - 1_{C'_jD'_j} + 1_{D'_j}\right) + k - 1_{s(F)}.
\end{aligned}$$

It is not so hard to check

$$\sum_{g_1,g_2 \in K} 1_U(g_1gg_2) = \sum_{g_1,g_2 \in K} 1_U(g_1g^{-1}g_2).$$

□

Lemma 7.12. *Let K be an elementary subgroupoid of G and let $\gamma{=}\tau_U \in [[G]]$. If*

$$\sum_{g_1,g_2\in K} 1_U(g_1gg_2) = \sum_{g_1,g_2\in K} 1_U(g_1g^{-1}g_2)$$

holds for all $g \in G$, then there exists $\gamma_0 \in [[G]]$ such that $\gamma_0^2 \in [[K]]$ and $\gamma_0\gamma \in [[K]]$.

Proof. By Lemma 4.7, there exists a K-full clopen subset $Y \subset K^{(0)} = G^{(0)}$ such that $K|Y = Y$. If C is a compact open $G|Y$-set, then

$$KCK = \{g_1gg_2 \in G \mid g_1g_2, \in K,\ g \in C\}$$

is a compact open subset of G. Since $G|Y$ is written as a disjoint union of compact open $G|Y$-sets, there exist mutually disjoint compact open $G|Y$-sets $C_1, C_2, \ldots, C_n$ such that $U \cup U^{-1}$ is contained in $\bigcup_i KC_iK$. Note that KC_iK's are also mutually disjoint, because of $K|Y = Y$. Define a (possibly empty) compact open G-set $D_{i,j}$ by

$$D_{i,j} = U \cap KC_iK \cap KC_j^{-1}K = U \cap K(C_i \cap C_j^{-1})K,$$

so that $1_U = \sum_{i,j} 1_{D_{i,j}}$. Take $i, j \in \{1, 2, \ldots, n\}$ and $y \in Y$ arbitrarily. If $r(C_i \cap C_j^{-1})$ does not contain y, then clearly

$$r(D_{i,j}) \cap K(y) = \emptyset = s(D_{j,i}) \cap K(y).$$

Suppose that $r(C_i \cap C_j^{-1})$ contains y. There exists a unique element $g \in C_i \cap C_j^{-1}$ such that $r(g) = y$ and one has

$$\begin{aligned} |r(D_{i,j}) \cap K(y)| &= |\{g_1gg_2 \in U \mid g_1, g_2 \in K\}| \\ &= \sum_{g_1,g_2\in K} 1_U(g_1gg_2) \\ &= \sum_{g_1,g_2\in K} 1_U(g_1g^{-1}g_2) \\ &= |\{g_1g^{-1}g_2 \in U \mid g_1, g_2 \in K\}| = |s(D_{j,i}) \cap K(y)|. \end{aligned}$$

It follows from Lemma 6.1 (3) that there exists a compact open K-set $A_{i,j}$ such that $s(A_{i,j}) = r(D_{i,j})$ and $r(A_{i,j}) = s(D_{j,i})$. Set

$$B = \bigcup_{i,j=1}^{n} D_{j,i}A_{i,j}.$$

It is easy to check that B is a compact open G-set satisfying $r(B) = s(B) = G^{(0)}$. Furthermore, one can see that $D_{j,i}A_{i,j}D_{i,j}$ is a compact open K-set for any i, j. Let $\gamma_0 = \tau_B$. Then we obtain $\gamma_0^2 \in [[K]]$ and $\gamma_0\gamma \in [[K]]$. □

From Lemma 7.10, 7.11, 7.12 and Lemma 7.7, we deduce the following theorem.

Theorem 7.13. *Suppose that G is almost finite. Suppose that $\gamma \in [[G]]$ is in the kernel of the index map. Then there exist $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in [[G]]$ such that $\gamma = \gamma_1\gamma_2\gamma_3\gamma_4$ and $\gamma_1, \gamma_2^2, \gamma_3, \gamma_4$ are elementary. In particular, γ is written as a product of four elements in $[[G]]$ of finite order.*

Remark 7.14. The theorem above is a generalization of [21, Lemma 4.1], in which it was shown that any $\gamma \in [[G_\varphi]] \cap \operatorname{Ker} I$ can be written as a product of two elementary homeomorphisms when φ is a minimal free action of $\mathbb{Z}$ on a Cantor set.

7.3 Conclusions

We conclude this paper with the following immediate consequences of Theorem 7.5 and Theorem 7.13.

Corollary 7.15. *Suppose that G is almost finite. Then there exists a homomorphism $\Phi_1 : H_1(G) \to K_1(C_r^*(G))$ such that $\Phi_1(I(\gamma))$ is equal to the K_1-class of $\rho(\gamma)$ for $\gamma \in [[G]]$, where $\rho : [[G]] \to N(C(G^{(0)}), C_r^*(G))$ is the homomorphism described in Proposition 5.6 (3).*

Proof. When $I(\gamma)$ is zero, by Theorem 7.13, γ is a product of four homeomorphisms of finite order. If a unitary in a C^*-algebra is of finite order, then its K_1-class is zero. Therefore the K_1-class of $\rho(\gamma)$ is zero for $\gamma \in \operatorname{Ker} I$. Since the index map $I : [[G]] \to H_1(G)$ is surjective by Theorem 7.5, we can define a homomorphism $\Phi_1 : H_1(G) \to K_1(C_r^*(G))$ by letting $\Phi_1(I(\gamma))$ be the K_1-class of $\rho(\gamma)$. □

The corollary above says that $H_1(G)$ corresponds to a subgroup of $K_1(C_r^*(G))$ generated by unitary normalizers of $C(G^{(0)})$. We do not know whether the homomorphism Φ_1 is injective or not and whether the range of Φ_1 is a direct summand of $K_1(C_r^*(G))$ or not.

When G is principal, combining Theorem 7.5 and Theorem 7.13, we obtain the following corollary.

Corollary 7.16. *Suppose that G is almost finite and principal. Then the kernel of the index map is equal to $[[G]]_0$, and the quotient group $[[G]]/[[G]]_0$ is isomorphic to $H_1(G)$ via the index map.*

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