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HOMOLOGY OF THE ZERO-SET OF A NILPOTENT VECTOR FIELD ON A FLAG MANIFOLD

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0. Introduction

0.1. Let X be a linear transformation of a finite-dimensional vector space V. The configuration of flags in V which are fixed by X has rather remarkable properties when X is unipotent. Though this case is especially interesting, the proper generality in which to study such configurations is in the theory of reductive algebraic groups, where their definition can be reformulated in the language of Borel subalgebras as follows.

Let G be a connected reductive group over $\mathbf{C}$, with Lie algebra $\mathfrak{g}$, and let $N \in \mathfrak{g}$ be a nilpotent element. Let $\mathscr{B}$ be the variety of all Borel subalgebras of $\mathfrak{g}$ and let

(a) $$\mathscr{B}_N = \{\mathfrak{b} \in \mathscr{B} \mid N \in \mathfrak{b}\}.$$

These varieties play an important role in representation theory, in particular in questions concerning characters of infinite dimensional representations of real semisimple groups and characters of complex representations of reductive groups over a finite field (see [Spr_1]).

The variety $\mathscr{B}_N$ has in general many irreducible components which may be singular and have a very complicated intersection pattern. (The reader is referred to [Spa_1] for a detailed discussion of the geometry of $\mathscr{B}_N$.)

One of our main results is that the integral homology of $\mathscr{B}_N$ is zero in odd degrees and is without torsion in even degrees. This answers a question of Springer (see [Spr_2]). (The vanishing of the *rational* homology in odd degrees of $\mathscr{B}_N$ has been proved earlier by Shoji [Sh] and Beynon-Spaltenstein [BS].)

We also show that all the homology of $\mathscr{B}_N$ comes from algebraic cycles. This result is new even over Q; it was known earlier only for GL_n and groups of low rank [Spa_2].

0.2. Let $s \in G$ be a semisimple element such that

(a) $$s \cdot N = qN \quad \text{for some } q \in \mathbf{C}^*.$$

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(Here we use the adjoint action of G and $\mathfrak{g}$ and the action of $\mathbf{C}^*$ on $\mathfrak{g}$ given by scalar multiplication.)

Following [L] we consider the variety

(b) $$\mathscr{B}_N^s = \{\mathfrak{b} \in \mathscr{B} \mid N \in \mathfrak{b}\,,\ s \cdot \mathfrak{b} = \mathfrak{b}\}$$

which reduces to $\mathscr{B}_N$ when $s = e$; this variety enters in a significant way in recent work on representations of affine Hecke algebras (see [G, KL]).

Our results on the homology of $\mathscr{B}_N$ are special cases of more general results which hold for any $\mathscr{B}_N^s$. In fact, in our proof, the case of $\mathscr{B}_N$ cannot be separated from the more general case $\mathscr{B}_N^s$. (The vanishing of the *rational* homology of $\mathscr{B}_N^s$ in odd degrees is proved in [KL, 4.1]].)

0.3. The proofs of our results are quite elementary in the sense that no intersection homology or reduction to characteristic p is used.

We shall now describe our method of analyzing the varieties $\mathscr{B}_N^s$.

We intersect $\mathscr{B}_N^s$ with the P-orbits on $\mathscr{B}$, where P is a certain parabolic subgroup of G canonically attached to N. Each of these intersections $\mathscr{V}$ is shown to be a vector bundle $\mathscr{V} \to \overline{\mathscr{V}}$ over a smooth projective variety $\overline{\mathscr{V}}$. We are then reduced to analyzing the varieties $\overline{\mathscr{V}}$. We can reduce ourselves to the case where N is distinguished (see §1.12) and $s = e$. In this case, the varieties $\mathscr{V} \subset \mathscr{B}_N$ are all pure of the same dimension, hence the closures of their connected components are precisely the irreducible components of $\mathscr{B}_N$. Moreover, the projective varieties $\overline{\mathscr{V}}$ can all be naturally imbedded in the flag manifold $\mathscr{F}$ of a Levi subgroup M of P; their images in $\mathscr{F}$ form a remarkable lattice of submanifolds of $\mathscr{F}$.

One of our observations is that for a certain natural prehomogeneous vector space V with respect to M, the previous lattice is isomorphic to the lattice of all subspaces of V which are stable under a fixed Borel subgroup of M and meet the open M-orbit in V. We show that for certain triples $\overline{\mathscr{V}}'' \subset \overline{\mathscr{V}} \subset \overline{\mathscr{V}}'$ of submanifolds in our lattice (each of codimension one in the next) the blow up of $\overline{\mathscr{V}}'$ along $\overline{\mathscr{V}}''$ is isomorphic to a $\mathbf{P}^1$-bundle over $\overline{\mathscr{V}}$. This gives many constraints for the homology of the varieties $\overline{\mathscr{V}}$, which, at least for exceptional groups, are sufficient to show that all the homology of $\overline{\mathscr{V}}$ comes from algebraic cycles and has no torsion.

In the classical groups, we follow a different approach which gives the following result: $\mathscr{B}_N$ can be partitioned into finitely many pieces isomorphic to affine space. This was proved earlier by Spaltenstein [Spa_1, Spa_2] for types A_n and E_6; we can also prove it for type F_4 and it is likely to be true also in types E_7, E_8.

1. Preliminaries

1.1. In this section we collect some material which will be needed in the later sections. We make the following conventions. All algebraic varieties are reduced and assumed to be over $\mathbf{C}$. All algebraic groups are assumed to be linear. If H

is an algebraic group, an H-module is always assumed to be a finite-dimensional $\mathbf{C}$-vector space with a given rational representation of H. Throughout this paper, $G, \mathfrak{g}, \mathscr{B}, \ldots$ are as in §0.1.

1.2. We recall a well-known result of Bialynicky-Birula. Let X be a smooth projective variety with an algebraic action of $\mathbf{C}^*$ denoted $(t, x) \to t \cdot x$. Then the fixed point set $X^{\mathbf{C}^*}$ is smooth. For each connected component Y of $X^{\mathbf{C}^*}$ we set $F_Y = \{x \in X \mid \lim_{t \to 0} t \cdot x \in Y\}$. Then we have a map $\pi_Y: F_Y \to Y$ given by $\pi_Y(x) = \lim_{t \to 0} t \cdot x$ and for each $y \in Y$, $\pi_Y^{-1}(y)$ is $\mathbf{C}^*$-stable. According to [BB] we have:

(a) There exists a vector bundle $\rho: E \to Y$ and an isomorphism $\psi: E \xrightarrow{\sim} F_Y$ such that $\rho = \pi_Y \cdot \psi$ and such that the $\mathbf{C}^*$-action on F_Y corresponds to a linear $\mathbf{C}^*$-action on E with strictly positive weights.

1.3. A finite partition of a variety X into subsets is said to be an *α-partition* if the subsets in the partition can be indexed $X_1, \ldots, X_n$ in such a way that $X_1 \cup X_2 \cup \cdots \cup X_i$ is closed in X for $i = 1, \ldots, k$. It is known that:

(a) the partition of X into the subsets F_Y in §1.2 is an α-partition.

1.4. Let X be a smooth projective variety with an action of a torus T. Then there exists an α-partition of X into subsets which are vector bundles over various connected components of the fixed point set X^T.

Indeed, we can choose a 1-parameter subgroup $\lambda: \mathbf{C}^* \to T$ such that the fixed point set X^T coincides with the fixed point set of $\mathbf{C}^*$ acting on X via λ. To this action of $\mathbf{C}^*$ we may apply §§1.2 and 1.3; the resulting partition of X has the required property.

1.5. Let $\rho: E \to Y$ be a vector bundle over a smooth variety Y, with a fiber preserving linear $\mathbf{C}^*$-action on E with strictly positive weights. Let $Z \subset E$ be a $\mathbf{C}^*$-stable smooth closed subvariety. Then $\pi(Z)$ is smooth and Z is a sub-bundle of E restricted to $\pi(Z)$.

This follows easily from [BH, Theorem 9.1].

1.6. If X is an algebraic variety, we denote by $A_k(X)$ the group generated by k-dimensional irreducible subvarieties modulo rational equivalence (see [Fu, 1.3]). Let $H_i(X)$ be the (Borel-Moore) integral homology of X; this is the singular homology of X if X is proper; it is the singular homology of $\overline{X}$ modulo $\overline{X} - X$ if X is not proper and $\overline{X}$ is a compactification of X. There is a canonical homomorphism ("cycle map", see [Fu, 19.1]):

$$\varphi_i: A_i(X) \to H_{2i}(X).$$

1.7. A variety X is said to have property (S) if

(a) $H_i(X) = 0$ for i odd, $H_i(X)$ has no torsion for i even,

(b) $\varphi_i: A_i(X) \xrightarrow{\sim} H_{2i}(X)$ for all i.

1.8. **Lemma.** *If X has an α-partition (see §1.3) into pieces which have property* (S), *then X has property* (S).

Proof. Let $X_1, \dots, X_n$ be the pieces of the partition indexed so that $Y_j = X_1 \cup \dots \cup X_j$ is closed in X for all j. We show by induction on j that Y_j has property (S). This is clear for $j = 1$. Assume now that $j \geq 2$ and that Y_{j-1} is already known to have property (S). We have an exact sequence

$$\begin{array}{ccccc} H_{2i+1}(Y_{j-1}) & \to & H_{2i+1}(Y_j) & \to & H_{2i+1}(X_j) \\ \| & & & & \| \\ 0 & & & & 0 \end{array}$$

hence $H_{2i+1}(Y_j) = 0$. We have an exact sequence

$$0 \to H_{2i}(Y_{j-1}) \to H_{2i}(Y_j) \to H_{2i}(X_j) \to 0$$

(since $H_{2i-1}(X_j) = H_{2i+1}(Y_{j-1}) = 0$) in which $H_{2i}(Y_{j-1})$ and $H_{2i}(X_j)$ have no torsion. It follows that $H_{2i}(Y_j)$ has no torsion. We have a commutative diagram

$$\begin{array}{ccccccccc} & & A_i(Y_{j-1}) & \to & A_i(Y_j) & \to & A_i(X_j) & \to & 0 \\ & & \varphi\downarrow & & \varphi'\downarrow & & \varphi''\downarrow & & \\ 0 & \to & H_{2i}(Y_{j-1}) & \to & H_{2i}(Y_j) & \to & H_{2i}(H_j) & \to & 0 \end{array}$$

whose rows are exact (see [Fu, 1.8]) and the vertical arrows are as in §1.6. Moreover φ, φ'' are isomorphisms. It follows that φ' is an isomorphism.

1.9. **Lemma.** *Assume that $E \to X$ is a vector bundle and that X has property* (S). *Then E has property* (S).

Proof. This follows easily from [Fu, 1.9] and the analogous result for homology.

1.10. Since a point clearly has property (S) we see from Lemmas 1.8 and 1.9 that:

(a) If X admits an α-partition into affine spaces then X has property (S).

Using §1.4 and Lemmas 1.8 and 1.9 we see also that:

(b) If X is smooth projective with an action of a torus T such that X^T has property (S), then X has property (S).

1.11. Let P be a parabolic subgroup of G and let $\mathfrak{p}$ be its Lie algebra. The natural action of G on $\mathscr{B}$ restricts to an action of P on $\mathscr{B}$. The P-orbits on $\mathscr{B}$ form an α-partition of $\mathscr{B}$. Let L be a Levi subgroup of P with Lie algebra $\mathfrak{l}$. Let T be the identity component of the center of L. Then for each P-orbit $\mathscr{O}$, the T-fixed point set $\mathscr{O}^T$ is isomorphic to the variety of Borel subalgebras of $\mathfrak{l}$ under the map $\mathfrak{b} \to \mathfrak{b} \cap \mathfrak{l}$.

Let $\mathfrak{n}$ be the nilpotent radical of $\mathfrak{p}$, and let $\psi\colon \mathfrak{n} \times \mathscr{O}^T \to \mathscr{O}$ be defined by $\psi(Y, \mathfrak{b}) = \exp(Y)\mathfrak{b}$. There is a unique map $\pi\colon \mathscr{O} \to \mathscr{O}^T$ such that the diagram

$$\begin{array}{ccc} \mathfrak{n} \times \mathscr{O}^T & \xrightarrow{\ \psi\ } & \mathscr{O} \\ & {\scriptstyle \mathrm{pr}_2}\searrow \quad \nearrow{\scriptstyle \pi} & \\ & \mathscr{O}^T & \end{array}$$

is commutative; we may regard $\mathcal{O} \to \mathcal{O}^T$ as a vector bundle, quotient of the constant vector bundle $\mathrm{pr}_2 \colon \mathfrak{n} \times \mathcal{O}^T \to \mathcal{O}^T$.

Let $\lambda \colon \mathbf{C}^* \to T$ be a 1-parameter subgroup which acts on $\mathfrak{p}/\mathfrak{l}$ with strictly positive weights. Then $\mathbf{C}^*$ acts on $\mathcal{B}$ via λ. Its fixed point set on $\mathcal{B}$ coincides with $\mathcal{O}^T$. Each P-orbit $\mathcal{O}$ is $\mathbf{C}^*$-stable; moreover the action of $\mathbf{C}^*$ on $\mathcal{O}$ preserves the fibers of the vector bundle $\pi \colon \mathcal{O} \to \mathcal{O}^T$ and is linear with strictly positive weights on each fiber.

1.12. Let $N \in \mathfrak{g}$ be a nilpotent element. By the Jacobson-Morozov theorem, there exists a homomorphism of algebraic groups $\phi \colon \mathrm{SL}_2(\mathbf{C}) \to G$ such that $d\varphi\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = N$. For each $z \in \mathbf{C}^*$ let $D(z) = \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right)$. Let $\mathfrak{g}_i = \{x \in \mathfrak{g} \mid \varphi(D(z))x = z^i x,\ \forall z \in \mathbf{C}^*\}$. Then $N \in \mathfrak{g}_2$. We have $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ and $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for all i, j.

Let G_0 (resp. P) be the connected algebraic subgroup of G whose Lie algebra is $\mathfrak{g}_0$ (resp. $\bigoplus_{i \geq 0} \mathfrak{g}_i$). Note that $\mathfrak{g}_i$ is a G_0-module for each i.

Then P is a parabolic subgroup of G with Levi subgroup G_0. It is known that:

(a) P depends only on N (and not on the choice of φ).

(b) The P-orbit of N in $\bigoplus_{i \geq 2} \mathfrak{g}_i$ is dense.

(c) The G_0-orbit of N in $\mathfrak{g}_2$ is dense.

(d) If $(g, q) \in G \times \mathbf{C}^*$ satisfies $g \cdot N = qN$ then $g \in P$ (see [SS; KL, §2]).

We say that N is *distinguished* if it is not contained in any Levi subalgebra of a proper parabolic subalgebra of $\mathfrak{g}$. It is known [BC] that:

(e) If N is distinguished, then, with the previous notations, we have $\mathfrak{g}_i = 0$ for all odd i.

2. Prehomogeneous vector spaces

2.1. In this section, M denotes a connected algebraic group. An M-module V is said to be *prehomogeneous* if V contains a dense M-orbit V^0.

Let V be a prehomogeneous M-module, let $v \in V^0$ and let M_v be the stabilizer of v.

Given a closed subgroup H in M and an H-stable linear subspace U of V we construct a closed subvariety $X_U \subset M/H$ as follows: Set

$$M_U = \{g \in M \mid g^{-1} v \in U\}.$$

Then M_U is stable under right multiplication by H and we set $X_U = M_U/H$.

2.2. **Lemma.** (i) *X_U is smooth.*

(ii) *If $X_U \neq \varnothing$ (or, equivalently, $U \cap V^0 \neq \varnothing$) then*

$$\dim M/H - \dim X_U = \dim V/U.$$

(iii) *M_v acts transitively on the set of connected components of X_U.*

(iv) *If $U, U' \subset V$ are H-stable subspaces, then $X_U \cap X_{U'} = X_{U \cap U'}$.*

Proof. Consider the diagram

$$M/H \xleftarrow{\pi} M \xrightarrow{\varphi} V$$

where $\varphi(g) = g^{-1}v$ and π is the quotient map. Then $V^0 = \varphi(M)$ and φ is the composition of the quotient map $M \to M_v \backslash M$ with the open imbedding $i: M_v\backslash M \hookrightarrow V$ given by $i(M_v g) = g^{-1}v$. Clearly $U \cap V^0$ is either empty or a smooth closed subvariety of V^0 of dimension equal to $\dim U$. From this, (i) and (ii) follow immediately. Since $U \cap V^0$ is open in U, it is connected and isomorphic to $M_v\backslash M_U$, hence M_v acts transitively on the set of connected components of M_U and hence also of M_U/H. Statement (iv) is obvious.

2.3. *Remark.* If $U \subset V$ is a subspace and H is the stabilizer of U in M, a necessary condition for having $U \cap V^0 \neq \varnothing$ is that $\dim M/H \geq \dim V/U$. This follows from Lemma 2.2.(ii).

2.4. We can interpret the construction of X_U in the language of vector bundles. Consider the vector bundle $M \times_H V/U$ over M/H; this is a quotient of the vector bundle $M \times_H V$. The last vector bundle is isomorphic to the trivial bundle $(M/H) \times V$ by the map $(g, w) \bmod H \to (gH, gw)$. The element v gives a "constant" section of $(M/H)\times V$ and hence a section s_v of $M\times_H V/U$. By the method of proof of Lemma 2.2 we see that s_v is transversal to the zero section of $M \times_H V/U$ and X_U is the zero locus of s_v.

More generally, if $U_1 \subset U_2$ are H-stable subspaces of V, we have the exact sequence of vector bundles

$$0 \to M \times_H U_2/U_1 \to M \times_H V/U_1 \to M \times_H V/U_2 \to 0.$$

The section s_v of $M \times_H V/U_1$, restricted to X_{U_2}, lies in the sub-bundle $M \times_H U_2/U_1$ and $X_{U_1} \subset X_{U_2}$ is again the zero locus of this section.

2.5. In the special case that H is a Borel subgroup of M, we have first of all that the varieties X_U are projective. Furthermore for any H-stable subspace U in V, we can find a sequence $U = U_0 \subset U_1 \subset \cdots \subset U_h = V$ of H-stable subspaces such that for each i, $1 \leq i \leq h$, we have $\dim U_i/U_{i-1} = 1$, $M \times_H U_i/U_{i-1}$ is a line bundle over M/H, and $X_{U_{i-1}}$ is a hypersurface in X_{U_i}.

2.6. Returning now to the setup in §2.1, let $M_{\bar{v}}$ be the stabilizer in M of the line through v. Let $\chi: M_{\bar{v}} \to \mathbf{C}^*$ be the character through which $M_{\bar{v}}$ acts on that line. Let $T \subset M_{\bar{v}}$ be a maximal torus; we denote the restriction of χ to T again by χ. Let L be the centralizer of T in M. Given an H-stable subspace $U \subset V$ and an L-orbit $\mathscr{O}$ on the set $(M/H)^T$ of T-fixed points on M/H, we want to describe the intersection $X_U \cap \mathscr{O}$. Let $g_0H \in X_U \cap \mathscr{O}$ and let $C = L \cap g_0Hg_0^{-1}$ be the stablizer of g_0H in L. The set $X_U \cap \mathscr{O}$

can be described as L_U/C where $L_U = \{l \in L \mid l^{-1}v \in g_0U\}$. Consider the subspaces $V^\chi \supset (g_0U)^\chi$ where, for any T-module W, we denote by W^χ the χ-eigenspace. Note that V^χ is L-stable, $(g_0U)^\chi$ is C-stable and $v \in V^\chi$. Note that $T \cap M_v$ is the unique maximal torus of $L_v = L \cap M_v$ and it is exactly the kernel of $L \to \mathrm{GL}(V^\chi)$. It follows that L acts on V^χ via its quotient $\overline{L} = L/T \cap M_v$ and that $\overline{L}_v = \{\overline{l} \in \overline{L} \mid \overline{l}v = v\}$ is such that $\overline{L}_v^0$ is unipotent.

We claim that $Lv = \overline{L}v$ is dense in V^χ. Indeed let $\mathfrak{m} = \mathrm{Lie}\, M$, $\mathfrak{l} = \mathrm{Lie}\, L$. We have $\mathfrak{l} = \mathfrak{m}^T$. Since $\mathfrak{m} \cdot v = V$ we get $\mathfrak{l}v = \mathfrak{m}^T v = V^\chi$ and our claim follows.

We see therefore that:

(a) V^χ is a prehomogeneous L-module (or $\overline{L}$-module), v is in the open L-orbit in V^χ and $X_U \cap \mathscr{O}$ is the subvariety of L/C associated to the C-stable subspace $(g_0U)^\chi \subset V^\chi$ according to §1.1 (for L, V^χ, C instead of M, V, H).

Of special interest is the case where H is a Borel subgroup of M. In this case, $(M/H)^T$ is a finite union of L-orbits and C is a Borel subgroup of L. Applying §1.4 to the T-action on X_U which is smooth, projective (Lemma 2.2(i), §2.5) we see that:

(b) X_U admits an α-partition into varieties which are vector bundles over the various components of the varieties $X_U \cap \mathscr{O}$ (which have been described in (a)).

This allows us to reduce certain questions for (M, V) (for example the question of whether the varieties X_U have the property (S)) to the special case where M_v^0 is unipotent.

2.7. Let M, V, H, v be as in §2.1. We assume for the remainder of this section that H is a Borel subgroup of M. Let Γ be the set of all H-stable subspaces of V. For each $U \in \Gamma$ let P_U be the stabilizer of U in M; this is a parabolic subgroup containing H.

We want to consider Γ as the set of vertices of a graph whose edges are the pairs U, U' with the following three properties:

(i) $U \subset U'$,
(ii) $\dim U'/U = 1$.
(iii) There exists a parabolic subgroup $P \supset H$ of semisimple rank 1 and $U'' \in \Gamma$ such that $U'' \subset U$, $P \subset P_{U''}$, $P \not\subset P_U$, $P \subset P_{U'}$ and $\dim U/U'' = 1$.

Note that in (iii), U'' is necessarily equal to $\bigcap_{g \in P} gU$. Let Γ^* be the set of all $U \in \Gamma$ such that $U \cap V^0 \neq \varnothing$.

2.8. Assume that the prehomogeneous M-module V arises in the context of §1.12 from a nilpotent element N in a simple Lie algebra $\mathfrak{g}$, that is $V = \mathfrak{g}_2$, $M = G_0$ (see §1.12(c)). Let $H \subset M$ be as in §2.7 and let $v = N$. A maximal torus of H acts on V with distinct weights (which are certain roots of

G). Hence the set Γ of H-invariant subspaces of V is finite. If furthermore, $\mathfrak{g}$ is simple of type A, D or E, the definition (§2.7) of an edge of Γ can be simplified as follows.

(a) Let $U \subset U'$ be in Γ such that $\dim U'/U = 1$ and such that there exists a parabolic subgroup P of M containing H, of semisimple rank 1 with $P \not\subset P_U$, $P \subset P_{U'}$.

We claim that if U, U' are as in (a) then automatically there exists a P-stable subspace $U'' \subset U$ such that $\dim U/U'' = 1$. (Hence condition (a) defines the edges of Γ.)

Let R be the (solvable) radical of P and let K be a semisimple subgroup of type A_1 of P. The K-module V is a sum of K-stable subspaces of dimension 1 and 2. This follows from the following fact: if α is a simple root and β is a positive root of a simple Lie algebra of type A, D or E then $\beta - \alpha$ and $\beta + \alpha$ cannot both be roots.

Consider the P-module $(U')^*$ (= dual of U'). The hyperplane U of U' corresponds to a line $D \subset (U')^*$ which is B-stable but not P-stable. The P-submodule Y of $(U')^*$ generated by D is irreducible and distinct of D, hence of dimension ≥ 2. It is also irreducible as a K-module since R acts trivially on it; hence it has dimension 2. It follows that the annihilator of Y is a P-stable subspace $U'' \subset U'$ of codimension 2, contained in U, as claimed.

2.9. Returning to the setup of §2.7, we define for any $U \in \Gamma$ two numbers:

$$\delta(U) = \dim(M/P_U) - \dim(V/U),$$
$$\nu(U) = \dim(M/H) - \dim(V/U) = \delta(U) + \dim P_U/H.$$

Notice that:

(a) the condition $\delta(U) < 0$ implies that the subvariety X_U of M/H is empty, while the condition $\delta(U) = 0$ implies that X_U is the preimage in M/H of a finite number of points in M/P_U under the natural projection.

This follows from Lemma 2.2(ii) and Remark 2.3.

2.10. We say that the M-module V is *good* if for any $U \in \Gamma$ one has either $U \subset U'$ for some $U' \in \Gamma$ with $\delta(U') < 0$, or U lies in the same connected component of Γ as some $U' \in \Gamma$ with $\delta(U') \leq 0$.

2.11. **Lemma.** *Let $U \subset U'$ be an edge of Γ and let U'' and P be as in* §2.7(iii). *Note that $X_{U''}$ has codimension 2 in $X_{U'}$; let $Bl_{X_{U''}}(X_{U'})$ be the blow up of $X_{U'}$ along $X_{U''}$. Let $Z = \{(gH, g'H) \in M/H \times M/H \mid g^{-1}v \in U,\ g^{-1}g' \in P\}$. Then*

(i) $\mathrm{pr}_1 : Z \to X_U$ *is a* $\mathbf{P}^1$*-bundle.*

(ii) *There is a canonical isomorphism* $Z \xrightarrow{\approx} Bl_{X_{U''}}(X_{U'})$.

Proof. Let $\widetilde{F}$ be the rank 2 vector bundle over G/H defined as $M \times_H U'/U''$ and let F be its restriction to $X_{U'}$. The fiber of $\widetilde{F}$ at $g'H \in M/H$ is

$g'U'/g'U''$. Let s_v be the section of F which attaches to gH the image of v in gU'/gU''. By §2.4, the zero locus of s_v is exactly $X_{U''}$. Hence $Bl_{X_{U''}}(X_{U'})$ can be identified with the subspace of the projective bundle $P(F)$ consisting of all lines in F which contain the image of the section s_v. Let $P(\widetilde{F})$ be the projective bundle of $\widetilde{F}$. Let $\widetilde{Z} = \{(gH, g'H) \in M/H \times M/H \mid g^{-1}g' \in P\}$. The map $\widetilde{Z} \to P(\widetilde{F})$ which associates to $(gH, g'H)$ the line $gU/g'U''$ in $g'U'/g'U''$ is an isomorphism. It restricts to an isomorphism $\{(gH, g'H) \in M/H \times M/H | g'^{-1} v \in U', g^{-1}g' \in P\} \xrightarrow{\sim} P(F)$ which, in turn, restricts to an isomorphism as in (ii). (We regard $Bl_{X_{U''}}(X_{U'})$ as a subvariety of $P(F)$, as above.) Statement (i) is obvious.

2.12. **Proposition.** *If the M-module V is good (see §2.10), then X_U has property* (S) *(see §1.7) for any $U \in \Gamma$.*

Proof. Consider for any $U \in \Gamma$ and any integer $n \geq 0$ the following statement:

$$(\mathrm{S}_n) \qquad \begin{cases} H_{2i+1}(X_U) = 0 & \text{if } 2i+1 \geq 2\nu(U) - 2n, \\ H_{2i}(X_U) \text{ has no torsion if } 2i \geq 2\nu(U) - 2n, \\ \varphi_i \colon A_i(X_U) \to H_{2i}(X_U) \text{ is an isomorphism if } i \geq \nu(U) - n. \end{cases}$$

We shall prove (S_n) by induction on n. This is trivial for $n = 0$ (see Lemma 2.2(ii)).

We can therefore assume that $n \geq 1$ and that (S_{n-1}) is already proved for all $U \in \Gamma$. If $U \in \Gamma$ is contained in $U' \in \Gamma$ such that $\delta(U') < 0$ then X_U is empty (see §2.9(a)) and (S_n) holds trivially for U. If $\delta(U) \leq 0$ then by §2.9(a), X_U is a finite union of copies of the flag manifold P_U/H, hence (S_n) holds for U. (Note that a flag manifold P_U/H has property (S) by Bruhat decomposition and §1.10(a).) Since Γ is good we see that in every connected component of Γ there is some U for which (S_n) holds. We are therefore reduced to proving the following result.

Lemma. *Assume that $n \geq 1$ and that (S_{n-1}) holds for all $U_1 \in \Gamma$. Let $U \subset U'$ be an edge of Γ. Then (S_n) holds for U if and only if (S_n) holds for U'.*

Proof. Let P and U'' be as in §2.7(iii) and let Z be as in Lemma 2.11. Using Lemma 2.11 and [Fu, 6.7, 3.3] we see that there are natural isomorphisms

$$\text{(a)} \qquad \begin{cases} A_i(Z) \xrightarrow{\sim} A_{i-1}(X_{U''}) \oplus A_i(X_{U'}), \\ A_i(Z) \xrightarrow{\sim} A_{i-1}(X_U) \oplus A_i(X_U). \end{cases}$$

Similarly, we have natural isomorphisms

$$\begin{cases} H_j(Z) \xrightarrow{\sim} H_{j-2}(X_{U''}) \oplus H_j(X_{U'}), \\ H_j(Z) \xrightarrow{\sim} H_{j-2}(X_U) \oplus H_j(X_U), \end{cases}$$

which are compatible with (a) under the maps $\varphi\colon A(\ ) \to H(\ )$ of §1.6. Hence we obtain isomorphisms

$$\text{(b)} \qquad \begin{cases} A_{i-1}(X_U) \oplus A_i(X_U) \xrightarrow{\sim} A_{i-1}(X_{U''}) \oplus A_i(X_{U'}), \\ H_{j-2}(X_U) \oplus H_j(X_U) \xrightarrow{\sim} H_{j-2}(X_{U''}) \oplus H_j(X_{U'}), \end{cases}$$

which are compatible with the maps $\varphi: A(\ ) \to H(\ )$ of §1.6.

If $j-2 \geq 2\nu(U) - 2n$ (or, equivalently, $j \geq 2\nu(U') - 2n$) we have $j \geq 2\nu(U) - 2(n-1)$, $j-2 \geq 2\nu(U'') - 2(n-1)$, hence, by (S_{n-1}), $H_j(X_U)$ and $H_{j-2}(X_{U''})$ are zero for odd j and have no torsion for even j. Hence from (b) it follows that the two groups $H_{j-2}(X_U)$ and $H_j(X_{U'})$ are isomorphic for odd j and have isomorphic torsion subgroups for even j.

If $i-1 \geq \nu(U) - n$ (or, equivalently, $i \geq \nu(U') - n$) we consider the commutative diagram

$$\text{(c)}\qquad \begin{array}{ccc} A_{i-1}(X_U) \oplus A_i(X_U) & \xrightarrow{\sim} & A_{i-1}(X_{U''}) \oplus A_i(X_{U'}) \\ \downarrow{\scriptstyle \varphi_{i-1}\oplus\varphi_i} & & \downarrow{\scriptstyle \varphi''_{i-1}\oplus\varphi'_i} \\ H_{2i-2}(X_U) \oplus H_{2i}(X_U) & \xrightarrow{\sim} & H_{2i-2}(X_{U''}) \oplus H_{2i}(X_{U'}) \end{array}$$

where $\varphi_{i-1}, \varphi_i, \varphi''_{i-1}, \varphi'_i$ are as in §1.6, and the horizontal arrows are as in (b). From (S_{n-1}), it follows that φ_i and φ''_{i-1} are isomorphisms. From (c) it then follows that φ_{i-1} is an isomorphism if and only if φ'_i is an isomorphism. The lemma is proved.

2.13. Let $I = M_\nu/M_\nu^0$. Note that M_ν acts by left translation on M/H leaving stable each of the subvarieties X_U ($U \in \Gamma$). This induces an action of I on $H_i(X_U)$. By Lemma 2.2(ii), I acts transitively on the set of connected components of X_U (when $X_U \neq \emptyset$); we denote by I_U the stabilizer in I of some connected component of X_U. (This is defined only up to conjugacy and only when $X_U \neq \emptyset$.) From Lemma 2.11 we deduce

(a) If $U \subset U'$ is an edge of Γ then either both X_U and $X_{U'}$ are empty or both are nonempty and $I_U, I_{U'}$ are conjugate in I. In particular, Γ^* (see §2.7) is a union of connected components of Γ.

Assume now that we are given a set of representatives $U_1, U_2, \dots, U_t$ for the connected components of Γ^* such that $\delta(U_j) = 0$ ($1 \leq j \leq t$). We also assume that I_{U_j} is known for $1 \leq j \leq t$. If this information is given, we can determine inductively the structure of the I-module $H_{2i}(X_U) \otimes \mathbf{C}$ for any i and any $U \in \Gamma$, as follows.

If $i = \nu(U)$ we have

$$H_{2i}(X_U) \otimes \mathbf{C} = \begin{cases} \operatorname{Ind}_{I_U}^I(\mathbf{C}) & \text{if } U \in \Gamma^* \\ 0 & \text{otherwise.} \end{cases}$$

(This follows from (a) and §2.9(a).) Assume now that the I-modules $H_{2i}(X_U) \otimes \mathbf{C}$ are known for all U for $i \geq \nu(U) - (n-1)$ for some $n \geq 1$. We wish to determine the I-modules $H_{2i}(X_U) \otimes \mathbf{C}$ for $i = \nu(U) - n$. If $U \subset U'$ is an edge of Γ and P, U'' are as in §2.7(iii), we have

$$\begin{aligned} &H_{2(\nu(U)-n)}(X_U) \otimes \mathbf{C} - H_{2(\nu(U')-n)}(X_{U'}) \otimes \mathbf{C} \\ &\quad = H_{2(\nu(U'')-(n-1))}(X_{U''}) \otimes \mathbf{C} - H_{2(\nu(U)-(n-1))}(X_U) \otimes \mathbf{C} \end{aligned}$$

as virtual I-modules. By our inductive hypothesis, the right-hand side of this equality is known; hence its left-hand side is known. Hence $H_{2(\nu(U)-n)}(X_U) \otimes \mathbf{C}$

is known if and only if $H_{2(\nu(U')-n)}(X_{U'}) \otimes \mathbf{C}$ is known. Hence it is enough to describe $H_{2(\nu(U_j)-n)}(X_{U_j}) \otimes \mathbf{C}$ for $1 \le j \le t$.

This is $\mathrm{Ind}_{I_{U_j}}^{I}(\mathbf{C}) \otimes H_{2n}(P_{U_j}/H)$ (see §2.9(a)). Here $H_{2n}(P_{U_j}/H)$ has trivial action of I. This completes our inductive description of the I-modules $H_{2i}(X_U) \otimes \mathbf{C}$.

We see, in particular, that:

(a) $H_{2i}(X_U) \otimes \mathbf{C}$ is a $\mathbf{Z}$-linear combination of I-modules of form $\mathrm{Ind}_{I_{U'}}^{I}(\mathbf{C})$ for various $U' \in \Gamma^*$.

3. The varieties $\mathscr{B}_N, \mathscr{B}_N^s$

3.1. Let N be a nilpotent element in $\mathfrak{g}$ and let $\mathscr{B}_N \subset \mathscr{B}$ be as in §0.1(a). Let P be a parabolic subgroup of G such that the closure $\overline{P \cdot N} \subset \mathfrak{g}$ is a linear subspace V.

3.2. **Proposition.** *The intersection $\mathscr{B}_{N,\mathscr{O}}$ of $\mathscr{B}_N$ with any P-orbit $\mathscr{O}$ on $\mathscr{B}$ is smooth.*

Proof. Let $\mathfrak{b} \in \mathscr{B}_N$ and let $\mathscr{O} \subset \mathscr{B}$ be the P-orbit of $\mathfrak{b}$ so that $\mathscr{B}_{N,\mathscr{O}} = \mathscr{B}_N \cap P \cdot \mathfrak{b}$. Let B be the Borel subgroup of G with Lie algebra $\mathfrak{b}$. The stabilizer of $\mathfrak{b}$ in P is $H = B \cap P$ and $P \cdot \mathfrak{b} \cong P/H$. We have $p\mathfrak{b} \in \mathscr{B}_N$, $p \in P$, if and only if $p^{-1}N \in \mathfrak{b}$. Hence $\mathscr{B}_{N,\mathscr{O}} = \mathscr{B}_N \cap p \cdot \mathfrak{b}$ is as a subvariety of P/H exactly like the one analyzed in §2.1 relative to the P-prehomogeneous space $V = \overline{P \cdot N}$, and the subspace $U = \mathfrak{b} \cap V$, so it is smooth of pure dimension $\dim P/H - \dim V/\mathfrak{b} \cap V$ (see Lemma 2.2).

3.3. **Corollary.** *If N is a Richardson element in $\mathfrak{p} = \mathrm{Lie}\, P$, then the intersections of $\mathscr{B}_N$ with the P-orbits on $\mathscr{B}$ are pure of dimension $\dim P/B_1$ (where B_1 is a Borel subgroup of P). In particular, the closures of their connected components are the irreducible components of $\mathscr{B}_N$.*

Proof. In this case $V = \mathfrak{n}$ is the nilpotent radical of $\mathfrak{p}$ and $\mathrm{Lie}\, H = \mathfrak{b} \cap \mathfrak{p}$ (notations of Proposition 3.2). It is known that $(\mathfrak{b} \cap \mathfrak{p}) + \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{p}$. Hence

$$\begin{aligned}\dim P/H - \dim V/\mathfrak{b} \cap V &= \dim \mathfrak{p}/\mathfrak{b} \cap \mathfrak{p} - \dim \mathfrak{n}/\mathfrak{b} \cap \mathfrak{n} \\ &= \dim(\mathfrak{p} + \mathfrak{b})/(\mathfrak{n} + \mathfrak{b}) = \dim \mathfrak{p}/(\mathfrak{b} \cap \mathfrak{p}) + \mathfrak{n},\end{aligned}$$

as required.

3.4. Returning to an arbitrary nilpotent element $N \in \mathfrak{g}$, we consider the canonical parabolic subgroup P attached to N in §1.12 (see §1.12(a)). According to §1.12(b), $\overline{P \cdot N}$ is a linear subspace of $\mathfrak{g}$.

Hence Proposition 3.2 is applicable and we see that the intersections $\mathscr{B}_{N,\mathscr{O}}$ of $\mathscr{B}_N$ with the various P-orbits $\mathscr{O}$ on $\mathscr{B}$ are smooth. They form an α-partition of $\mathscr{B}_N$ (see §§1.3, 1.11).

Now let $(s, q) \in G \times \mathbf{C}^*$ be a semisimple element such that $sN = qN$. Then $s \in P$ (see §1.12(d)), hence $s: \mathscr{B} \to \mathscr{B}$ leaves stable each P-orbit $\mathscr{O}$. From $sN = qN$ we see that $s: \mathscr{B} \to \mathscr{B}$ also leaves stable $\mathscr{B}_N$, hence s leaves stable each $\mathscr{B}_{N,\mathscr{O}}$. Since $\mathscr{B}_{N,\mathscr{O}}$ is smooth and s is semisimple, the fixed point set $\mathscr{B}^s_{N,\mathscr{O}}$ of $s: \mathscr{B}_{N,\mathscr{O}} \to \mathscr{B}_{N,\mathscr{O}}$ is smooth. Clearly, $\bigcup_{\mathscr{O}} \mathscr{B}^s_{N,\mathscr{O}} = \mathscr{B}^s_N$ (see §0.2(b)), hence

(a) The subsets $\mathscr{B}^s_{N,\mathscr{O}}$ (for various P-orbits $\mathscr{O}$ on $\mathscr{B}$) form an α-partition $\mathscr{B}^s_N$ into smooth varieties.

According to [KL, 2.4(g)] there exists a homomorphism of algebraic groups $\varphi: \mathrm{SL}_2(\mathbf{C}) \to G$ such that

(b)
$$\begin{cases} d\varphi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = N, \\ s\varphi(\gamma)s^{-1} = \varphi(D(q^{1/2})\gamma D(q^{-1/2})) \quad \text{for all } \gamma \in \mathrm{SL}_2(\mathbf{C}) \end{cases}$$

(notations of §1.12), where $q^{1/2}$ is a fixed square root of q.

We define $\mathfrak{g}_i, G_0$ in terms of such a φ, as in §1.12. Let $\lambda: \mathbf{C}^* \to G$ be the 1-parameter group $\lambda(z) = \varphi(D(z))$, $z \in \mathbf{C}^*$, and let $\mathscr{D}$ be its image in G. From (b) we see that

(c)
$$s \in Z(\mathscr{D}).$$

By §1.11, each $\mathscr{O}$ is naturally a vector bundle over the $\mathscr{D}$-fixed point set $\mathscr{O}^{\mathscr{D}}$ and $\mathbf{C}^*$ acts (via λ) on this vector bundle linearly on each fiber with strictly positive weights. Now $\mathscr{B}^s_{N,\mathscr{O}}$ is a closed, smooth subvariety of $\mathscr{O}$ (see (a)) stable under the $\mathbf{C}^*$-action (see (b), (c)). From §1.5 it follows that

(d) $\mathscr{B}^s_{N,\mathscr{O}}$ is a vector bundle over the $\mathscr{D}$-fixed point set $\mathscr{B}^{s,\mathscr{D}}_{N,\mathscr{O}}$, which is a smooth projective variety (hence a union of connected components of $\mathscr{B}^{s,\mathscr{D}}_N$).

(The fact that $\mathscr{B}^{s,\mathscr{D}}_{N,\mathscr{O}}$ is projective follows from the fact that it is the intersection of two projective varieties: $\mathscr{B}^s_N$ and $\mathscr{O}^{\mathscr{D}}$.)

3.5. Let $s_1 = sD(q^{-1/2})$. Then s_1 commutes with $\varphi(\mathrm{SL}_2(\mathbf{C}))$ (see §3.4(b)), and $s_1 N = N$. Hence φ can be regarded as a homomorphism $\varphi: \mathrm{SL}_2(\mathbf{C}) \to Z^0(s_1)$ and N is in the Lie algebra $\mathfrak{z}$ of $Z^0(s_1)$.

Let $\mathscr{B}^{s_1}$ be the fixed point set of $s_1: \mathscr{B} \to \mathscr{B}$ and let $\overline{\mathscr{B}}$ be the variety of Borel subalgebras of $\mathfrak{z}$. The map $\mathscr{B}^{s_1} \to \overline{\mathscr{B}}$ defined by $\mathfrak{b} \to \mathfrak{b} \cap \mathfrak{z}$ is an isomorphism when restricted to any connected component of $\mathscr{B}^{s_1}$. It defines a map $\mathscr{B}^{s_1}_N \to \overline{\mathscr{B}}_n$ (the last space is defined as in §0.1(a) replacing $\mathfrak{g}$ by $\mathfrak{z}$) which becomes an isomorphism when restricted to any connected component of $\mathscr{B}^{s_1}_N$. (Recall that $\overline{\mathscr{B}}_N$ is connected.) These isomorphisms are compatible with the action of $\mathscr{D}$, hence taking $\mathscr{D}$-fixed points we get a map $\mathscr{B}^{s_1,\mathscr{D}}_N \to \overline{\mathscr{B}}^{\mathscr{D}}_N$ which maps each connected component of $\mathscr{B}^{s_1,\mathscr{D}}_N$ isomorphically to a connected component of $\overline{\mathscr{B}}_N^{\mathscr{D}}$. On the other hand we have clearly $\mathscr{B}_N^{s_1,\mathscr{D}} = \mathscr{B}_N^{s,\mathscr{D}}$. From §3.4(a),(d) we now see that:

(a) There exists an α-partition of $\mathscr{B}_N^s$ into pieces which are vector bundles over various connected components of $\overline{\mathscr{B}}_N^{\mathscr{D}}$.

3.6. Let T be a maximal torus in the connected stabilizer of N in G. We can choose T so that $M = Z(T)$ contains the image of φ. Then N is a distinguished nilpotent element of $\mathfrak{m} = \operatorname{Lie} M$. Let $\hat{\mathscr{B}}$ be the variety of all Borel subalgebras of $\mathfrak{m}$ and let $\mathscr{B}^T$ be the fixed point set of T on $\mathscr{B}$. The map $\mathscr{B}^T \to \hat{\mathscr{B}}$ defined by $\mathfrak{b} \to \mathfrak{b} \cap \mathfrak{m}$ restricted to any connected component of $\mathscr{B}^T$ is an isomorphism. It defines a map $\mathscr{B}_N^T \to \hat{\mathscr{B}}_N$ where $\mathscr{B}_N^T = \mathscr{B}^T \cap \mathscr{B}_N$ and $\hat{\mathscr{B}}_N$ is defined as in §0.1(a) replacing $\mathfrak{g}$ by $\mathfrak{m}$. This map becomes an isomorphism when restricted to any connected component of $\mathscr{B}_N^T$. These isomorphisms are compatible with the action of $\mathscr{D}$, hence taking $\mathscr{D}$-fixed points we get a map $\mathscr{B}_N^{T,\mathscr{D}} \to \hat{\mathscr{B}}_N^{\mathscr{D}}$ which maps each connected component of $\mathscr{B}_N^{T,\mathscr{D}}$ isomorphically onto a connected component of $\hat{\mathscr{B}}_N^{\mathscr{D}}$.

On the other hand, $\mathscr{B}_N^{T,\mathscr{D}}$ can also be considered as the set of T-fixed points on $\mathscr{B}_N^{\mathscr{D}}$. Note that

(a) $\mathscr{B}_N^{\mathscr{D}}$ is smooth, projective. (We have $\mathscr{B}_N^{\mathscr{D}} = \bigcup_{\mathscr{O}} \mathscr{B}_{N,\mathscr{O}}^{\mathscr{D}}$ where each $\mathscr{B}_{N,\mathscr{O}}^{\mathscr{D}}$ is smooth, projective by §3.4(a) with $s = e$; these pieces do not meet each other, hence $\mathscr{B}_N^{\mathscr{D}}$ is smooth, projective. The fact that $\mathscr{B}_N^{\mathscr{D}}$ is smooth was conjectured in [L] and first proved by Ginzburg in a quite different way.)

Now using §1.4 for the action of T on $\mathscr{B}_N^{\mathscr{D}}$ we see that

(b) $\mathscr{B}_N^{\mathscr{D}}$ admits an α-partition whose pieces are vector bundles over the various connected components of $\hat{\mathscr{B}}_N^{\mathscr{D}}$.

3.7. We fix a Borel subgroup B_0 of G_0 with Lie algebra $\mathfrak{b}_0$. For each P-orbit $\mathscr{O}$ on $\mathscr{B}$ there is a unique $\mathfrak{b}_{\mathscr{O}} \in \mathscr{O}^{\mathscr{D}}$ such that $\mathfrak{b}_{\mathscr{O}} \cap \mathfrak{g}_0 = \mathfrak{b}_0$. The intersection $U_{\mathscr{O}} = \mathfrak{b}_{\mathscr{O}} \cap \mathfrak{g}_2$ is clearly a $\mathfrak{b}_0$-stable (hence B_0-stable) subspace of $\mathfrak{g}_2$. Hence the subvariety $X_{U_{\mathscr{O}}} \subset G_0/B_0$ is well defined as in §2.1 (for $M = G_0$, $V = \mathfrak{g}_2$, $H = B_0$, $v = N$); note that the G_0-module $\mathfrak{g}_2$ is prehomogeneous by §1.12(c).

From the definitions it is clear that:

(a) The map $X_{U_{\mathscr{O}}} \to \mathscr{B}_{N,\mathscr{O}}^{\mathscr{D}}$ defined by $gB_0 \to g \cdot \mathfrak{b}_{\mathscr{O}}$ is an isomorphism. Hence for any $\mathscr{O}$, $\mathscr{B}_{N,\mathscr{O}}^{\mathscr{D}}$ is of the form X_U for some B_0-stable subspace U of the prehomogeneous G_0-module $\mathfrak{g}_2$.

The converse is true for certain N:

(b) If N is such that $\mathfrak{g}_i = 0$ for i odd (in particular, if N is distinguished, see §1.12(e)) then for any B_0-stable subspace U of $\mathfrak{g}_2$ there exists a P-orbit $\mathscr{O}$ on $\mathscr{B}$ such that $U = U_{\mathscr{O}}$.

This follows immediately from the theorem in the appendix.

3.8. Assume that E is an irreducible G-module of dimension ≥ 2 such that in the corresponding $\mathfrak{g}$-module, N acts as a nilpotent transformation $\tilde{N}$

with Jordan blocks of distinct sizes. Then the image of $\mathscr{D}$ in $\mathrm{GL}(E)$ acts on $\ker(\tilde{N}: E \to E)$ with distinct weights, hence there are only finitely many $\mathscr{D}$-stable lines $L \subset E$ such that $\tilde{N}L = 0$. Now G has a closed orbit on $P(E)$ of dimension > 0. The stabilizers in G of the points of this orbit form a conjugacy class $\mathscr{P}$ of proper parabolic subgroups of G. It follows that the set $\mathscr{P}_N^{\mathscr{D}} = \{Q \in \mathscr{P} \mid \mathscr{D} \subset Q, N \in \mathrm{Lie}\, Q\}$ is finite.

Let σ be an element of D such that $\mathscr{B}^{\mathscr{D}} = \mathscr{B}^{\sigma}$ (= fixed point set of $\sigma: \mathscr{B} \to \mathscr{B}$).

We have a natural map $\mathscr{B}_N^{\sigma} = \mathscr{B}_N^{\mathscr{D}} \to \mathscr{P}_N^{\mathscr{D}}$ which associates to $\mathfrak{b}$ the unique $Q \in \mathscr{P}$ such that $\mathfrak{b} \subset \mathrm{Lie}\, Q$. Taking the fibers of this map we find a partition of $\mathscr{B}_N^{\sigma} = \mathscr{B}_N^{\mathscr{D}}$ into finitely many pieces which are both open and closed. Consider the piece corresponding to some $Q \in \mathscr{P}_N^{\mathscr{D}}$. It is clearly isomorphic to $(\mathscr{B}_{\overline{Q}})_{\overline{N}}^{\bar{\sigma}}$, the variety of Borel subalgebras of $\mathrm{Lie}\, \overline{Q}$ ($\overline{Q} = Q$ modulo its radical) containing $\overline{N}$ (= image of N) and which are fixed by the action of $\bar{\sigma} \in \overline{Q}$ (the image of $\sigma \in Q$ in $\overline{Q}$).

We now state one of our main results.

3.9. **Theorem.** *Let $N \in \mathfrak{g}$ be a nilpotent element and let $(s, q) \in G \times \mathbf{C}^*$ be a semisimple element such that $s \cdot N = qN$. Then, in general, $\mathscr{B}_N^s$ satisfies property* (S) (*see* §1.7). *If G is a classical group, then $\mathscr{B}_N^s$ admits an α-partition* (*see* §1.3) *into subvarieties which are affine spaces.*

Proof. The theorem is trivial when G is a torus. Hence we may assume that G is not a torus and that the theorem is already proved for G replaced by a group of strictly smaller dimension.

We preserve the notations of §3.4.

From Lemmas 1.8 and 1.9 and §3.5(a) we see that it is enough to prove the statement of the theorem for G, with $\mathscr{B}_N^s$ replaced by $\mathscr{B}_N^{\mathscr{D}}$. (This is in fact a special case of the theorem since $\mathscr{B}_N^{\mathscr{D}} = \mathscr{B}_N^{s'}$ for a suitable element $s' \in \mathscr{D}$.) Using Lemmas 1.8 and 1.9 and §3.6(b) we see that we can further assume that N is distinguished. We can also assume that G is almost simple, simply connected. If $G = \mathrm{SL}_n(\mathbf{C})$, then N is regular, $\mathscr{B}_N$ is a point so there is nothing to prove.

Assume that G is $\mathrm{Sp}_{2n}(\mathbf{C})$ $(n \geq 2)$ or $\mathrm{Spin}_n(\mathbf{C})$ $(n \geq 7)$. We define a G-module E as follows: it is the standard representation of $\mathrm{Sp}_{2n}(\mathbf{C})$, or it is the standard representation of $\mathrm{SO}_n(\mathbf{C})$ lifted to $\mathrm{Spin}_n(\mathbf{C})$. From the classification of distinguished nilpotent elements [BC], we see that E satisfies the assumption of §3.8. Using the induction hypothesis for $\overline{Q}$ (notations of §3.8), we see that $\mathscr{B}_N^{\mathscr{D}}$ admits an α-partition into subvarieties which are affine spaces; in particular, it satisfies property (S) (see §1.10(a)).

Next assume that G is simply connected of type G_2, F_4, E_6, E_7 or E_8. Using §3.7(a) and Proposition 2.12 we see that it is enough to show that the G_0-module $\mathfrak{g}_2$ is good (see §2.10). This can be verified case-by-case, using the classification [BC] of distinguished nilpotent classes. This verification is very long, but completely mechanical. One first has to make a list of all B_0-subspaces U of $\mathfrak{g}_2$ (a finite set, by §2.8), then one determines P_U (see §2.7) for each U, and then one applies the definition of the graph Γ (see §2.7) (which simplifies, as in §2.8, in type E_n). Note that in each case, G_0 is of type A and $\mathfrak{g}_2$ is an explicitly known representation of G_0.

We omit further details. This completes the proof of the theorem.

4. Examples arising from exceptional groups

4.1. In this section we shall consider some examples of the prehomogeneous vector spaces $(G_0, \mathfrak{g}_2)$ associated as in §1.12 to a distinguished nilpotent element N in the Lie algebra $\mathfrak{g}$ of a simply connected exceptional group G.

The list of such distinguished classes can be found in [BC]; up to G-conjugacy there are two such elements in type G_2, four in F_4, three in E_6, six in E_7 and 11 in E_8. In each case, the derived group of G_0 is a product of SL_n's $(n \leq 5)$ and its representation on $\mathfrak{g}_2$ can be described explicitly. Recall that the graph Γ associated to $(G_0, \mathfrak{g}_2)$ is finite (see §2.8). As we have already mentioned in the proof of Theorem 3.9 the G_0-module $\mathfrak{g}_2$ is good.

Furthermore the following can be verified in each case.

(a) If $U \in \Gamma$, then we have $U \notin \Gamma^*$ if and only if U is contained in some $U' \in \Gamma$ which is in the same connected component in Γ as some U'' with $\delta(U'') < 0$.

(b) If $U, U' \in \Gamma^*$ then U, U' lie in the same connected component of Γ if and only if $I_U, I_{U'}$ are conjugate in I (see §2.13).

(c) The number of connected components of Γ^* is equal to the number of conjugacy classes in I.

(d) I is isomorphic to one of the symmetric groups S_n $(1 \leq n \leq 5)$.

(e) If $I \approx S_2$, then for $U \in \Gamma^*$, I_U is either $\{e\}$ or S_2.

If $I \approx S_3$, then for $U \in \Gamma^*$, I_U is either $\{e\}$ or S_2 or S_3 except for the nilpotent class

$$\begin{matrix} 0\,0\,2\,0\,0\,0\,2 \\ 0 \end{matrix}$$

in E_8 in which case I_U is either S_3 or a cyclic group of order 2 or 3. (In this last case there is a unique $U \in \Gamma^*$ such that I_U is of order 3.)

If $I \approx S_4$, then, for $U \in \Gamma^*$, I_U is either S_2 or $S_2 \times S_2$ or S_3 or D_8 or S_4.

If $I \approx S_5$, then for $U \in \Gamma^*$, I_U is either S_2 or $S_2 \times S_2$ or S_3 or D_8 or $S_3 \times S_2$ or S_4 or S_5.

(Here, D_8 denotes the dihedral group of order 8 imbedded in S_4 or S_5 in the standard way.)

In the rest of this section we shall discuss the cases where $I \approx S_4$ or S_5; we shall omit the other cases.

4.2. Assume now that G is of type F_4 and that N is a distinguished nilpotent element in $\mathfrak{g}$ with Dynkin diagram 0200. In this case we can identify $\mathfrak{g}_2$ with $\mathrm{Hom}(E, \mathrm{Hom}(S^2V, \mathbf{C}))$ where E is a 2-dimensional $\mathbf{C}$-vector space, V is a 3-dimensional $\mathbf{C}$-vector space, and G^0 is an extension of $\mathrm{SL}(E) \times \mathrm{SL}(V)$ by a one-dimensional torus with the natural action.

We can think of N as the most general family (pencil) of symmetric bilinear forms $Q_e(\ ,\ )$ on V indexed by vectors $e \in E$ depending linearly on e. Using N, we identify a vector $e \in E$ with the corresponding form $Q_e(\ ,\ )$.

There are exactly four lines L_1, L_2, L_3, L_4 in V which are isotropic for all of the forms $Q_e(\ ,\ )$. There are exactly three lines D_a, D_b, D_c in E consisting of degenerate forms. The radicals of these degenerate forms give us three lines R_a, R_b, R_c in V. We can choose notations so that $R_a = (L_1 + L_2) \cap (L_3 + L_4)$, $R_b = (L_1 + L_3) \cap (L_2 + L_4)$, $R_c = (L_1 + L_4) \cap (L_2 + L_3)$. The group I permutes naturally the four lines L_i; this gives an isomorphism of I with S_4.

One verifies that Γ^* consists of nine subspaces U_i $(1 \le i \le 9)$. We shall regard the corresponding varieties X_{U_i} (§2.1) as subvarieties of the flag manifold

$$\mathscr{F} = \{(E_1 \subset E_2 = E,\ V_1 \subset V_2 \subset V_3 = V)\}$$

of G_0. (Here $\dim E_i = i$, $\dim V_j = j$.)

X_{U_1} is the full flag manifold $\mathscr{F}$.

X_{U_2} is the subset of $\mathscr{F}$ defined by the equation $Q_e(V_1, V_1) = 0$ for all $e \in E_1$.

X_{U_3} is the subset of $\mathscr{F}$ defined by the equation $Q_e(V_1, V_2) = 0$ for all $e \in E_1$.

X_{U_4} is defined by the equation $Q_e(V_1, V_1) = 0$ for all $e \in E$.

$X_{U_5} = X_{U_3} \cap X_{U_4}$.

X_{U_6} is defined by the equation $Q_e(V_1, V_3) = 0$ for all $e \in E_1$.

X_{U_7} is defined by the equation $Q_e(V_2, V_2) = 0$ for all $e \in E_1$.

$X_{U_8} = X_{U_6} \cap X_{U_7}$.

$X_{U_9} = X_{U_4} \cap X_{U_7}$.

We now give a geometric description of the varieties X_{U_i} $(2 \le i \le 9)$. X_{U_2} is a $\mathbf{P}^1$-bundle over the variety obtained from $P(V)$ by blowing up the four points $[L_i]$. X_{U_3} is obtained from $P(V)$ by blowing up the seven points $[L_1], [L_2], [L_3], [L_4], [R_a], [R_b], [R_c]$. X_{U_4} is isomorphic to four copies of $\mathbf{P}^1 \times \mathbf{P}^1$; its natural projection to $P(V)$ consists of the four points $[L_i]$. X_{U_5} consists of four copies of $\mathbf{P}^1$. X_{U_6} consists of three copies of $\mathbf{P}^1$; the natural projection to $P(V)$ (resp. $P(E)$) consists of the three points $[R_a]$, $[R_b], [R_c]$ (resp. $[D_a], [D_b], [D_c]$). X_{U_7} consists of six copies of $\mathbf{P}^1$. X_{U_8} consists of six points; X_{U_9} consists of 12 points. The graph Γ^* is:

$$U_1 - U_2 - U_3,\quad U_4 - U_5,\quad U_6,\quad U_7 - U_8,\quad U_9.$$

4.3. Assume now that G is of type E_8 and that N is a distinguished nilpotent element in $\mathfrak{g}$ with Dynkin diagram

$$\begin{matrix} 0\,0\,0\,2\,0\,0\,0 \\ 0 \end{matrix}.$$

In this case we can identify the pair $(G^0, \mathfrak{g}_2)$ with

$$(S(\mathrm{GL}(E) \times \mathrm{GL}(V)), \mathrm{Hom}(E, \mathrm{Hom}(\Lambda^2 V, \mathbf{C}))),$$

where E is a 4-dimensional $\mathbf{C}$-vector space and V is a 5-dimensional $\mathbf{C}$-vector space; the action is the obvious one.

We can think of N as the most general family of alternating bilinear forms $\omega_e(\ ,\ )$ on V, indexed by vectors $e \in E$, depending linearly on e. Using N we identify a vector $e \in E$ with the corresponding form $\omega_e(\ ,\ )$.

There are exactly five lines D_1, D_2, D_3, D_4, D_5 in E whose nonzero vectors are forms of rank 2. (This can be seen using the fact that the Grassmannian $G_3(\mathbf{C}^5)$ imbedded in $P(\bigwedge^2 \mathbf{C}^5)$ by the Plücker imbedding, has degree 5.) The radicals of these forms of rank 2 give us five 3-dimensional subspaces H_1, H_2, H_3, H_4, H_5 in V.

The group I permutes naturally the five lines D_i; this gives an isomorphism of I with S_5.

The graph Γ^* has 502 vertices. Of these, 12 have $I_U = S_2$, 71 have $I_U = S_2 \times S_2$, 40 have $I_U = S_3$, 8 have $I_U = D_8$, 121 have $I_U = S_3 \times S_2$, 98 have $I_U = S_4$ and 152 have $I_U = S_5$.

Giving a Borel subgroup B_0 of G_0 is the same as giving complete flags $E_1^0 \subset E_2^0 \subset E_3^0 \subset E_4^0 = E$ and $V_1^0 \subset V_2^0 \subset V_3^0 \subset V_4^0 \subset V_5^0 = V$. A B_0-invariant subspace of $\mathfrak{g}_2$ is an intersection of subspaces of form

$$U^{ijk} = \mathrm{Ann}(E_i^0 \otimes V_j^0 \wedge V_k^0) \subset \mathfrak{g}_2, \qquad 1 \le i \le 4,\ 1 \le j \le k \le 5.$$

The corresponding variety X_U, regarded as a subvariety of the flag manifold

$$\mathscr{F} = \{(E_1 \subset E_2 \subset E_3 \subset E_4 = E,\ V_1 \subset V_2 \subset V_3 \subset V_4 \subset V_5 = V)\}$$

of G_0, is the intersection

$$\bigcap_{(i,j,k)\in A} X_{U^{ijk}}$$

where $U = \bigcap_{(i,j,k)\in A} U^{ijk}$ for some set of indices A, and $X_{U^{ijk}} \subset \mathscr{F}$ is defined by the equation $\omega_e(v, v') = 0$ for every $e \in E_i$, $v \in V_j$, $v' \in V_k$.

We shall give enough information on the varieties X_U so as to be able to compute their Betti numbers (even equivariant ones) by the method indicated in §2.13. We shall therefore exhibit in each connected component of Γ^* a subspace U such that $\delta(u) = 0$.

Let $U_1 = \mathfrak{g}_2$, $U_2 = U^{135}$, $U_3 = U^{215}$, $U_4 = U^{422} \cap U^{224}$, $U_5 = U_2 \cap U_3$, $U_6 = U^{314} \cap U^{333}$, $U_7 = U_6 \cap U^{125}$.

By §2.9(a), X_{U_i} is the inverse image (under the natural projection) of a 0-dimensional subvariety Y_{U_i} of a partial flag manifold.

X_{U_1} is the full flag manifold $\mathscr{F}$, so Y_{U_1} is a point.

Y_{U_2} is the set of all (E_1, V_3) such that V_3 is in the radical of each form in E_1. This consists of five points (D_i, H_i).

Y_{U_3} is the set of all (E_2, V_1) such that V_1 is in the radical of each form in E_2. This consists of ten points $(D_i + D_j, H_i \cap H_j)$, $i < j$.

Y_{U_4} is the set of all $(E_2, V_2 \subset V_4)$ such that $\omega_e(V_2, V_2) = 0$ for all $e \in E$ and $\omega_e(V_2, V_4) = 0$ for all $e \in E_2$. This consists of 15 points

$$((D_i + D_j + D_k) \cap (D_i + D_l + D_m), (H_j \cap H_k) + (H_l \cap H_m) \subset H_i + (H_j \cap H_k) + (H_l \cap H_m)),$$

where i, j, k, l, m are all distinct.

Y_{U_5} is the set of all $(E_1 \subset E_2, V_1 \subset V_3)$ such that V_3 is in the radical of each form in E_1 and V_1 is in the radical of each form in E_2. This consists of 20 points

$$(D_i \subset D_i + D_j, H_i \cap H_j \subset H_i), \qquad i \neq j.$$

Y_{U_6} is the set of all $(E_3, V_1 \subset V_3 \subset V_4)$ such that $\omega_e(V_1, V_4) = \omega_e(V_3, V_3) = 0$ for all $e \in E_3$. This consists of 30 points

$$(D_i + D_j + D_k, H_j \cap H_k \subset (H_i \cap H_j) + (H_i \cap H_k) + (H_j \cap H_k) \subset H_i + (H_j \cap H_k)), \qquad i, j, k \text{ distinct}.$$

Y_{U_7} is the set of all $(E_1 \subset E_3, V_1 \subset V_2 \subset V_3 \subset V_4)$ such that $\omega_e(V_1, V_4) = \omega_e(V_3, V_3) = 0$ for all $e \in E_3$ and V_2 is in the radical of each form in E_1. This consists of 60 points

$$(D_j \subset D_i + D_j + D_k, H_j \cap H_k \subset (H_j \cap H_k) + (H_i \cap H_j) \subset (H_i \cap H_j) + (H_i \cap H_k) + (H_j \cap H_k) \subset H_i + (H_j \cap H_k)).$$

Now using §2.13(a) we see that for any $U \in \Gamma^*$, the S_5-module $H_{2i}(X_U) \otimes \mathbf{C}$ does not contain the sign representation of S_5. (It follows that the S_5-module $H_{2i}(\mathscr{B}_N)$ does not contain the sign representation of S_5.) (This was proved earlier, by a less elementary method, in [BS].)

Appendix. A result on graded semisimple Lie algebras

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbf{C}$ and let $\mathfrak{g} = \bigoplus_{i \in 2\mathbf{Z}} \mathfrak{g}_i$ be a decomposition such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for all i, j.

Let $(\, , \,)$ be the Killing form on $\mathfrak{g}$. Then $(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ unless $i + j = 0$. Hence $(\, , \,)$ defines a nonsingular pairing $\mathfrak{g}_{-i} \times \mathfrak{g}_i \to \mathbf{C}$. Also $\mathfrak{g}_0$ is reductive.

Theorem. *Let $\mathfrak{b}_0$ be a Borel subalgebra of $\mathfrak{g}_0$ and let $\mathfrak{b}_2$ be a subspace of $\mathfrak{g}_2$ such that $[\mathfrak{b}_0, \mathfrak{b}_2] \subset \mathfrak{b}_2$. Define $\mathfrak{b}_{2i} \subset \mathfrak{g}_{2i}$ for $i \geq 2$ by the inductive formula $\mathfrak{b}_{2i} = [\mathfrak{b}_2, \mathfrak{b}_{2i-2}]$. Define $\mathfrak{b}_{2i}$ for $i \leq -1$ by $\mathfrak{b}_{2i} = \{x \in \mathfrak{g}_{2i} \mid (x, \mathfrak{b}_{-2i}) = 0\}$. Then $\mathfrak{b} = \bigoplus_{i \in 2\mathbf{Z}} \mathfrak{b}_i$ is a Borel subalgebra of $\mathfrak{g}$.*

We first prove

Lemma 1. $[\mathfrak{b}_i, \mathfrak{b}_j] \subset \mathfrak{b}_{i+j}$ *for* $i, j \geq 0$.

Proof. Assume that $i = 0$. If $j = 0$ or 2 then clearly $[\mathfrak{b}_0, \mathfrak{b}_j] \subset \mathfrak{b}_j$. If $j \geq 4$ we can assume that $[\mathfrak{b}_0, \mathfrak{b}_{j-2}] \subset \mathfrak{b}_{j-2}$ is already known and then

$$\begin{aligned}[\mathfrak{b}_0, \mathfrak{b}_j] = [\mathfrak{b}_0, [\mathfrak{b}_2, \mathfrak{b}_{j-2}]] \subset [[\mathfrak{b}_0, \mathfrak{b}_2], \mathfrak{b}_{j-2}] \\ + [[\mathfrak{b}_0, \mathfrak{b}_{j-2}], \mathfrak{b}_2] \subset [\mathfrak{b}_2, \mathfrak{b}_{j-2}] \subset \mathfrak{b}_j.\end{aligned}$$

For $i = 2$ the lemma is true from definitions. We now assume that $i \geq 4$ and that $[\mathfrak{b}_{i-2}, \mathfrak{b}_j] \subset \mathfrak{b}_{i+j-2}$ is already known for all $j \geq 0$. We have

$$\begin{aligned}[\mathfrak{b}_i, \mathfrak{b}_j] &= [[\mathfrak{b}_2, \mathfrak{b}_{i-2}], \mathfrak{b}_j] \subset [\mathfrak{b}_{i-2}, [b_2, \mathfrak{b}_j]] + [\mathfrak{b}_2, [\mathfrak{b}_{i-2}, \mathfrak{b}_j]] \\ &\subset [\mathfrak{b}_{i-2}, \mathfrak{b}_{j+2}] + [\mathfrak{b}_2, \mathfrak{b}_{i+j-2}] \\ &\subset \mathfrak{b}_{i+j} + \mathfrak{b}_{i+j} \subset \mathfrak{b}_{i+j}.\end{aligned}$$

The lemma is proved.

Lemma 2. $[\mathfrak{b}_{-i}, \mathfrak{b}_j] \subset \mathfrak{b}_{j-i}$ *for* $i, j \geq 0$.

Proof. In the case where $i = 0$, this follows from Lemma 1. Hence we can assume that $i \geq 2$. Assuming $j \leq i$, we have, using Lemma 1:

$$([\mathfrak{b}_{-i}, \mathfrak{b}_j], \mathfrak{b}_{i-j}) = (\mathfrak{b}_{-i}, [\mathfrak{b}_j, \mathfrak{b}_{i-j}]) \subset (\mathfrak{b}_{-i}, \mathfrak{b}_i) = 0,$$

hence,

$$\text{(a)} \qquad \begin{cases} [\mathfrak{b}_{-i}, \mathfrak{b}_j] \subset \mathfrak{b}_{j-i} & (\text{if } j < i), \\ [\mathfrak{b}_{-i}, \mathfrak{b}_j] \subset \mathfrak{b}'_0 \subset \mathfrak{b}_0 & (\text{if } j = i). \end{cases}$$

(Here $\mathfrak{b}'_0 = \{x \in \mathfrak{g}_0 \mid (x, \mathfrak{b}_0) = 0\}$.)

We prove the lemma by induction on j. If $j = 0$ or 2 then $i \geq j$, which has been considered already. Hence we can assume that $j \geq 4$ and that $[\mathfrak{b}_{-i}, \mathfrak{b}_{j-2}] \subset \mathfrak{b}_{j-i-2}$ is already known for all $i \geq 0$. We have (for $i \geq 2$):

$$\begin{aligned}[\mathfrak{b}_{-i}, \mathfrak{b}_j] &= [\mathfrak{b}_{-i}, [\mathfrak{b}_2, \mathfrak{b}_{j-2}]] \subset [[\mathfrak{b}_{-i}, \mathfrak{b}_2], \mathfrak{b}_{j-2}] + [[\mathfrak{b}_{-i}, \mathfrak{b}_{j-2}], \mathfrak{b}_2] \\ &\subset [\mathfrak{b}_{-i+2}, \mathfrak{b}_{j-2}] + [\mathfrak{b}_{-i+j-2}, \mathfrak{b}_2] \\ &\subset \mathfrak{b}_{j-i} + \mathfrak{b}_{j-i} = \mathfrak{b}_{j-i}.\end{aligned}$$

(Note that $[\mathfrak{b}_{-i+j-2}, \mathfrak{b}_2] \subset \mathfrak{b}_{j-i}$ holds by (a) if $j - 1 \leq 0$ and by Lemma 1 if $j - i \geq 2$.) The lemma is proved.

Lemma 3. $[\mathfrak{b}_{-i}, \mathfrak{b}_{-j}] \subset \mathfrak{b}_{-i-j}$ *for* $i, j \geq 0$.

Proof. By Lemma 2 we can assume that $i > 0$, $j > 0$. We have

$$\begin{aligned}([\mathfrak{b}_{-i}, \mathfrak{b}_{-j}], \mathfrak{b}_{i+j}) &= ([\mathfrak{b}_{-i}, \mathfrak{b}_{i+j}], \mathfrak{b}_{-j}), \\ &\subset (\mathfrak{b}_j, \mathfrak{b}_{-j}) \quad \text{by Lemma 2} \\ &= 0,\end{aligned}$$

and the lemma follows.

Proof of the Theorem. Let

$$\mathfrak{b}'_i = \begin{cases} \mathfrak{b}_i, & i \neq 0, \\ \mathfrak{b}'_0, & i = 0, \end{cases}$$

and let $\mathfrak{b}' = \bigoplus_i \mathfrak{b}'_i$. From Lemmas 1, 2, 3 and from (a) above, we see that $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{b}'$; in particular, $\mathfrak{b}$ is a subalgebra. It is clear that $(\mathfrak{b}', \mathfrak{b}) = 0$, and from Cartan's criterion it follows that $\mathfrak{b}$ is solvable. We have $\dim \mathfrak{b} = \dim(\mathfrak{b}_0 \oplus \bigoplus_{i>0} \mathfrak{g}_i)$ and the last space is a Borel subalgebra. It follows that $\mathfrak{b}$ is a Borel subalgebra.

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