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HOMOMORPHISMS AND CONGRUENCES OF MEDIAL SEMIGROUPS WITH AN ASSOCIATE SUBGROUP

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Abstract. Let S be the model of a semigroup with an associate subgroup whose identity is a medial idempotent constructed by Blyth and Martins considered as a unary semigroup. For another such semigroup T, we construct all unary homomorphisms of S into T in terms of their parameters. On S we construct all unary congruences again directly from its parameters. This construction leads to a characterization of congruences in terms of kernels and traces. We describe the K, T, L, U and V relations on the lattice of all unary congruences on S, again in terms of parameters of S.

1. Introduction and Summary

The knowledge of homomorphisms and congruences of semigroups belonging to a particular class $\mathcal{C}$ is often essential for understanding their structure. This is even more so if there is a suitable structure theorem for semigroups in $\mathcal{C}$. In such a case, we attempt to construct all the homomorphisms and congruences on the model which greatly facilitates their manipulation.

Let S be a semigroup. For $s, t \in S$, t is an *associate* of s if $s = sts$; denote by $A(s)$ the set of all associates of s. Let $E(S)$ be the set of all idempotents of S and $C(S)$ the *core* of S, that is the subsemigroup of S generated by $E(S)$. For $z \in E(S)$, Green's $\mathcal{H}$-class H_z is an *associate subgroup* of S if for every $s \in S$, the set $A(s) \cap H_z$ is a singleton, say $\{s^*\}$; the mapping $s \to s^*$ is a *unary operation* on S. In such a case, we call z the *zenith* of S. Also, z is called a *medial idempotent* if for all $s \in C(S)$, we have $s = szs$, in which case we call S *medial.*

Under these hypotheses, a structure theorem for such semigroups was established in [1] in terms of an idempotent generated semigroup, a group and a single homomorphism. The purpose of this paper is to construct unary homomorphisms of,

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and congruences on, their model. Section 2 contains some special symbolism, the model of the semigroups from [1], and several lemmas giving certain properties of the model which will be used later. A description of unary homomorphisms between the models comprises Section 3. The main construction of unary congruences on the model can be found in Section 4. Unary congruences in terms of kernels and traces are briefly described in Section 5. Section 6 contains a characterization of the K, T, L, U and V relations on the lattice of unary congruences on the model.

2. Preparation

Let S be a semigroup. Then $\mathcal{A}(S)$ denotes the group of automorphisms of S. If $s, t \in S$ are such that $s = sts$ and $t = tst$, we write $s \in V(t)$. When G is a group, 1 denotes its identity. The remaining notation can be found in [5].

It was proved in [1] that a medial semigroup S with an associate subgroup G can be constructed by an action of G on a part of the core $C(S)$ of S. We state this construction below and establish a number of its properties. This will be needed in the main body of the paper.

Construction. *Let C be an idempotent generated semigroup with a medial idempotent w, G be a group and $\zeta : G \to \mathcal{A}(wCw)$ be a homomorphism with the notation*

$$\zeta : g \to \zeta_g \qquad (g \in G).$$

On the set

$$\{(x, g, a) \in Cw \times G \times wC \mid \zeta_g(aw) = wx\}$$

define a multiplication by

$$(x, g, a)(y, h, b) = (x\zeta_g(ay), gh, \zeta_{h^{-1}}(ay)b).$$

Denote the resulting groupoid by $[C, G; w, \zeta]$ and let $z = (w, 1, w)$.

Blyth and Martins [1] have shown that $[C, G; w, \zeta]$ is a semigroup with the associate subgroup H_z where z is a medial idempotent and conversely, that every semigroup S with an associate subgroup G whose identity z is medial is isomorphic to $[C(S), H_z; z, \zeta]$, where for $g \in H_z$,

$$\zeta_g : u \to gug^{-1} \qquad (u \in C(S)).$$

Following [2], on $S = [C, G; w, \zeta]$ we define a unary operation by letting

$$(x, g, a)^* = (w, g^{-1}, w).$$

We also set $c^* = w$ for all $c \in C$. This makes both S and C unary semigroups; for both we will consider unary homomorphisms and congruences. Note that groups are unary for both of these relative to the operation of inversion. For axioms characterizing the unary operation on S, see [2].

In this section we fix $S = [C, G, w, \zeta]$.

Recall that the natural partial order $\leq$ on a regular semigroup S is defined by

$$s \leq t \Longleftrightarrow s = et = tf \text{ for some } e, f \in E(S) \qquad (s, t \in S).$$

The next three lemmas state certain properties of S. The first one describes Green's relations and the natural partial order.

Lemma 2.1. *Let* $s = (x, g, a)$, $t = (y, h, b) \in S$.

(i) $ss^* = (x, 1, wx)$, $s^*s = (aw, 1, a)$.

(ii) $s \mathcal{L} t \Leftrightarrow a \mathcal{L} b$.

(iii) $s \mathcal{R} t \Leftrightarrow x \mathcal{R} y$.

(iv) $s \leq t \Longleftrightarrow x \leq y, \; g = h, \; a \leq b$.

Proof. (i) Trivial.

(ii) By ([2], Lemma 4.2(i)) and part (i), we have

$$\begin{aligned}
s \mathcal{L} t &\Longleftrightarrow s = st^*t, \quad t = ts^*s \\
&\Longleftrightarrow s = s(bw, 1, b), \quad t = t(aw, 1, a) \\
&\Longleftrightarrow (x, g, a) = (x\zeta_g(abw), g, ab), \quad (y, h, b) = (y\zeta_h(baw), h, ba) \\
&\Longleftrightarrow a = ab, \; b = ba \Longleftrightarrow a \mathcal{L} b.
\end{aligned}$$

(iii) Dual.

(iv) Using ([2], Lemma 4.6) and part (i), we get

$$\begin{aligned} s \le t \iff & s = ss^*t = ts^*s \\ \iff & (x,g,a) = (x,1,wx)(y,h,b) = (y,h,b)(aw,1,a) \\ \iff & (x,g,a) = (xw,h,\zeta_{h^{-1}}(wxy)b) = (y\zeta_h(baw),h,ba) \\ \iff & x = xy = y\zeta_h(bw)\zeta_h(aw),\ g = h,\ a = \zeta_{h^{-1}}(wx)\zeta_{h^{-1}}(wy)b = ba \\ \iff & x = xy = ywywx,\ g = h,\ a = awbwb = ba \\ \iff & x = xy = yx,\ g = h,\ a = ab = ba \\ \iff & x \le y,\ g = h,\ a \le b. \end{aligned}$$

□

We will generally use Lemma 2.1(i) without express mention. The second lemma identifies idempotents, the core and the associate subgroup of the model.

Lemma 2.2.

(i) $E(S) = \{(x,1,a) \in S \mid x \in V(a)\}$.

(ii) $C(S) = \{(x,1,a) \in S\}$.

(iii) $H_z = \{(w,g,w) \mid g \in G\}$.

Proof. (i) For $(x,g,a) \in S$, we have

$$\begin{aligned} (x,g,a)(x,g,a) = (x,g,a) \iff & \big(x\zeta_g(ax), g^2, \zeta_{g^{-1}}(ax)a\big) = (x,g,a) \\ \iff & x\zeta_g(ax) = x,\ g^2 = g,\ \zeta_{g^{-1}}(ax)a = a \\ \iff & xax = x,\ g = 1,\ axa = a \\ \iff & x \in V(a),\ g = 1. \end{aligned}$$

(ii) Let $s \in C(S)$. We have $s = (sz)(zs)$ and so every element in $C(S)$ is a product of two idempotents. Let $(x,1,a),(y,1,b) \in E(S)$. By part (i), we have $x \in V(a)$ and $y \in V(b)$. Now $(x,1,a)(y,1,b) = (xay,1,ayb)$ and

$$C(S) \subseteq \{(x,1,a) \in S\}.$$

Conversely, let $(x,1,a) \in S$. Since

$$(x,1,wx)(aw,1,a) = (xwxaw,1,wxawa) = (xwx,1,awa) = (x,1,a)$$

where $(x,1,wx),(aw,1,a) \in E(S)$, we have that $(x,1,a) \in C(S)$.

(iii) For $g \in G$, we have

$$(w,g,w)\,z = (w,g,w) = z\,(w,g,w)\,,$$
$$(w,g,w)\left(w,g^{-1},w\right) = z = \left(w,g^{-1},w\right)(w,g,w)\,,$$

and $(w,g,w) \in H_z$.

Conversely, let $(x,g,a) \in H_z$. Then $(x,g,a)\ \mathcal{R}\ (w,1,w)$ and by Lemma 2.1(iii), $w\ \mathcal{R}\ x$ so that $x = xw = w$. Similarly, from $(w,1,w)\ \mathcal{L}\ (x,g,a)$ we conclude that $w = a$. Hence $(x,g,a) \in \{(w,h,w) \mid h \in G\}$. □

The following lemma establishes isomorphisms between $C(S)$ and C, and between H_z and G.

Lemma 2.3.
(i) The mappings

$$\sigma_S : (x,1,a) \longmapsto xa, \qquad \sigma : p \longmapsto (pw,1,wp)$$

are mutually inverse unary isomorphisms between $C\,(S)$ and C.

(ii) The mappings

$$\tau_S : (w,g,w) \longmapsto g, \qquad \tau : g \longmapsto (w,g,w)$$

are mutually inverse isomorphisms between H_z and G.

Proof. (i) Let $(x,1,a),(y,1,b) \in C(S)$. Since $ay \in wCw \subseteq E\,(C)$, we have

$$\sigma_S\left((x,1,a)\,(y,1,b)\right) = \sigma_S\,(xay,1,ayb) = xayayb = xayb = \sigma_S\,(x,1,a)\,\sigma_S\,(y,1,b)\,,$$

and clearly

$$\sigma_S((x,1,a)^*) = \sigma_S(z) = w = (\sigma_S(x,1,a))^*.$$

Moreover,

$$\sigma\sigma_S\,(x,1,a) = \sigma\,(xa) = (xaw,1,wxa) = (xwx,1,awa) = (x,1,a)$$

and, for every $p \in C$,

$$\sigma_S\sigma p = \sigma_S\,(pw,1,wp) = pwwp = p.$$

Clearly $\sigma_S(z) = w$.

(ii) Obvious. □

3. Homomorphisms

In this section $S = [C, G; w, \zeta]$ *and* $T = [D, H; v, \pi]$.

We prove first that unary homomorphisms of S into T can be expressed by means of unary homomorphisms of C into D and of G into H satisfying a certain compatibility condition. Recall the notation σ_S and τ_S from Lemma 2.3.

Theorem 3.1. *Let* $\varphi : C \to D$ *and* $\psi : G \to H$ *be unary homomorphisms such that*

$$\varphi\zeta_g = \pi_{\psi(g)}\varphi \qquad (g \in G).$$

Define

$$\chi : (x, g, a) \to (\varphi(x), \psi(g), \varphi(a)) \qquad ((x, g, a) \in S).$$

Then χ *is an unary homomorphism of* S *into* T*, to be denoted by* $[\varphi, \psi]$*. Moreover,* $[\varphi, \psi] = [\varphi', \psi']$ *if and only if* $\varphi = \varphi'$*,* $\psi = \psi'$*.*

Conversely, if $\chi : S \to T$ *is an unary homomorphism, then* $\chi = [\sigma_T\chi\sigma_S^{-1}, \tau_T\chi\tau_S^{-1}]$*.*

Proof. Direct part. Let $(x, g, a), (y, h, b) \in S$. Then

$$\begin{aligned}
\chi((x,g,a)(y,h,b)) &= \chi((x\zeta_g(ay), gh, \zeta_{h^{-1}}(ay)b)) \\
&= (\varphi(x\zeta_g(ay)), \psi(gh), \varphi(\zeta_{h^{-1}}(ay)b)) \\
&= (\varphi(x)\varphi\zeta_g(ay), \psi(g)\psi(h), \varphi\zeta_{h^{-1}}(ay)\varphi(b)) \\
&= (\varphi(x)\pi_{\psi(g)}\varphi(ay), \psi(g)\psi(h), \pi_{\psi(h)^{-1}}\varphi(ay)\varphi(b)) \\
&= (\varphi(x)\pi_{\psi(g)}(\varphi(a)\varphi(y)), \psi(g)\psi(h), \pi_{\psi(h)^{-1}}(\varphi(a)\varphi(y))\varphi(b)) \\
&= (\varphi(x), \psi(g), \varphi(a))(\varphi(y), \psi(h), \varphi(b)) \\
&= \chi(x, g, a)\chi(y, h, b).
\end{aligned}$$

Since also

$$\chi((x,g,a)^*) = \chi(w, g^{-1}, w) = (v, \psi(g)^{-1}, v) = (\varphi(x), \psi(g), \varphi(a))^* = (\chi(x, g, a))^*,$$

we conclude that χ is an unary homomorphism.

Assume that $[\varphi, \psi] = [\varphi', \psi']$ and let $x \in C$. Then

$$[\varphi, \psi](xw, 1, wx) = [\varphi', \psi'](xw, 1, wx)$$

which means that $\varphi(xw) = \varphi'(xw)$ and $\varphi(wx) = \varphi'(wx)$ and thus

$$\varphi(x) = \varphi(xwwx) = \varphi(xw)\varphi(wx) = \varphi'(xw)\varphi'(wx) = \varphi'(xwwx) = \varphi'(x).$$

Furthermore, if $g \in G$, then

$$[\varphi, \psi](w, g, w) = [\varphi', \psi'](w, g, w)$$

and so $\psi(g) = \psi'(g)$. Hence $\varphi = \varphi'$ and $\psi = \psi'$. Conversely, if $\varphi = \varphi'$ and $\psi = \psi'$, then trivially $[\varphi, \psi] = [\varphi', \psi']$.

Converse. Let $\chi : S \to T$ be an unary homomorphism, $(x, g, a) \in S$ and $\chi(x, g, a) = (y, h, b)$. Then

$$\begin{aligned} \sigma_T \chi \sigma_S^{-1}(x) &= \sigma_T \chi(x, 1, wx) = \sigma_T \chi((x, g, a)(x, g, a)^*) \\ &= \sigma_T(\chi(x, g, a)(\chi(x, g, a)^*) = \sigma_T((y, h, b)(y, h, b)^*) \\ &= \sigma_T(y, 1, wy) = yvy = y, \end{aligned}$$

and analogously $\sigma_T \chi \sigma_S^{-1}(a) = b$; further,

$$\begin{aligned} \tau_T \chi \tau_S^{-1}(g) &= \tau_T \chi(w, g, w) = \tau_T \chi((x, g, a)^{**}) = \tau_T((\chi(x, g, a))^{**}) \\ &= \tau_T((y, h, b)^{**}) = \tau_T(v, h, v) = h. \end{aligned}$$

Obviously $\sigma_T \chi \sigma_S^{-1}$ is an unary homomorphism. $\square$

It is convenient to introduce a new concept at this point.

Definition 3.2. *Let $\chi : S \to T$ be a semigroup homomorphism. We say that χ is* core pure *if $\chi(a) \in C(T)$ implies $a \in C(S)$.*

The next proposition establishes some properties that χ may may have relative to injectivity in terms of its restrictions to $C(S)$ and H_z, respectively.

Proposition 3.3. *Let $\chi = [\varphi, \psi] : S \to T$ with the notation as in Theorem 3.1.*

(i) $\chi|_{C(S)}$ is injective $\Longleftrightarrow$ φ is injective.

(ii) χ is core pure $\Longleftrightarrow$ $\chi\,|_{H_z}$ is injective $\Longleftrightarrow$ ψ is injective.

(iii) χ is injective $\Longleftrightarrow$ both φ and ψ are injective.

Proof. In each part we denote by A, B, ... the different statements.

(i) *A implies B.* Let $x, y \in C$ be such that $\varphi(x) = \varphi(y)$. Then

$$(xw, 1, wx), (yw, 1, wy) \in C(S)$$

and

$$\chi(xw, 1, wx) = (\varphi(xw), 1, \varphi(wx)) = (\varphi(yw), 1, \varphi(wy)) = \chi(yw, 1, wy).$$

The hypothesis implies that $(xw, 1, wx) = (yw, 1, wy)$ so that

$$x = (xw)(wx) = (yw)(wy) = y.$$

Therefore φ is injective.

B implies A. Let $(x, 1, a), (y, 1, b) \in C(S)$ be such that $\chi(x, 1, a) = \chi(y, 1, b)$. Then $\varphi(x) = \varphi(y)$ and $\varphi(a) = \varphi(b)$. The hypothesis implies that $x = y$ and $a = b$ and so $\chi\mid_{C(S)}$ is injective.

(ii) *A implies B.* Let $(w, g, w), (w, h, w) \in H_{(w,1,w)}$ be such that $\chi(w, g, w) = \chi(w, h, w)$. Then

$$\chi((w, g, w)(w, h^{-1}, w)) = (v, 1, v) \in C(T)$$

and the hypothesis implies that

$$(w, g, w)(w, h^{-1}, w) \in C(S).$$

Therefore $(w, g, w)(w, h^{-1}, w) = (w, 1, w)$ so that $(w, g, w) = (w, h, w)$.

B implies C. Let $g, h \in G$ be such that $\psi(g) = \psi(h)$. Then $\chi(w, g, w) = \chi(w, h, w)$ and thus $(w, g, w) = (w, h, w)$ which gives $g = h$.

C implies A. Let $(x, g, a) \in S$ be such that $\chi(x, g, a) \in C(T)$. Then $\psi(g) = 1$ and the hypothesis implies that $g = 1$. Hence $(x, g, a) \in C(S)$.

(iii) Straightforward. $\square$

For surjectivity , we do not have satisfactory equivalent statements, except, trivially, for an analogue of part (iii) of Proposition 3.3.

4. Congruences

In this section $S = [C, G; w, \zeta]$.

Let T be any semigroup. We denote by $\mathcal{C}(T)$ the lattice of all congruences on T. If T is a unary semigroup, $\mathcal{UC}(T)$ denotes the lattice of all unary congruences on T (recall that ρ is *unary* if $s\ \rho\ t$ implies that $s^*\ \rho\ t^*$ for all $s, t \in T$).

In a semigroup S with a unary operation $a \to a^*$, it is customary to call a multiplicative congruence ρ on S *normal* if for any $x, y, a \in S$, $x\ \rho\ y$ implies $axa^*\ \rho\ aya^*$. Applying this to $S = [C, G; w, \zeta]$ and restricting x and y to $C(S)$ and a to H_z, in view of the proof of ([1], Theorem 4), it is natural to introduce

Definition 4.1. *A congruence ξ on C is* normal *if for all $u, v \in wCw$ and $g \in G$, $u\ \xi\ v$ implies $\zeta_g(u)\ \xi\ \zeta_g(v)$. Denote by $\mathcal{NC}(C)$ the set of all normal congruences on C.*

Lemma 4.2. *The lattices $\mathcal{NC}(C)$ and $\mathcal{UC}(S)$ are complete sublattices of $\mathcal{C}(C)$ and $\mathcal{C}(S)$, respectively.*

We need one more basic concept.

Definition 4.3. *A normal congruence ξ on C and a congruence η on G are* linked *if for any $g, h \in G$ and $u \in C$, $g\ \eta\ h$ implies $\zeta_g(u)\ \xi\ \zeta_h(u)$. By saying that ξ and η are* linked *we will imply all these conditions.*

Lemma 4.4. *Let $\xi \in \mathcal{C}(C)$ and $\eta \in \mathcal{C}(G)$. Then ξ and η are linked if and only if*

$$u\ \xi\ v,\ g\ \eta\ h \Longrightarrow \zeta_g(wuw)\ \xi\ \zeta_h(wvw).$$

In order to characterize unary congruences on S, we will need the following notation.

Definition 4.5. *Let $\xi \in \mathcal{NC}(C)$ and $\eta \in \mathcal{C}(G)$ be linked. Define $\rho = [\xi, \eta]$ on S by*

$$(x, g, a)\ \rho\ (y, h, b) \Longleftrightarrow x\ \xi\ y,\ \ g\ \eta\ h,\ \ a\ \xi\ b.$$

It comes out directly from the definition that ρ is a relation having the following properties.

Lemma 4.6. $[\xi, \eta] \subseteq [\xi', \eta'] \Longleftrightarrow \xi \subseteq \xi',\ \eta \subseteq \eta'$.

Proof. Necessity. Let $u\ \xi\ v$. Then $(uw, 1, wu)\ [\xi, \eta]\ (vw, 1, wv)$. The hypothesis implies that $(uw, 1, wu)\ [\xi', \eta']\ (vw, 1, wv)$ so that $uw\ \xi'\ vw$ and $wu\ \xi'\ wv$. Therefore

$$u = uwwu\ \xi'\ vwwv = v.$$

Let $g\ \eta\ h$. Then $(w, g, w)\ [\xi, \eta]\ (w, h, w)$ and the hypothesis implies that

$$(w, g, w)\ [\xi', \eta']\ (w, h, w).$$

Hence $g\ \eta'\ h$.

Sufficiency. Let $(x, g, a)\ [\xi, \eta]\ (y, h, b)$. Then $x\ \xi\ y$, $g\ \eta\ h$, $a\ \xi\ b$ and the hypothesis implies that $x\ \xi'\ y$, $g\ \eta'\ h$ and $a\ \xi'\ b$. Thus $(x, g, a)\ [\xi', \eta']\ (y, h, b)$. ☐

Corollary 4.7. $[\xi, \eta] = [\xi', \eta'] \Longleftrightarrow \xi = \xi',\ \eta = \eta'$.

We are now in a position to establish the central result of this and the succeeding sections. Recall the notation σ_S and τ_S from Lemma 2.3.

Theorem 4.8. *If $\xi \in \mathcal{NC}(C)$ and $\eta \in \mathcal{C}(G)$ are linked, then $[\xi, \eta] \in \mathcal{UC}(S)$. Conversely, if $\rho \in \mathcal{UC}(S)$, then $\rho = [\xi_\rho, \eta_\rho]$, where ξ_ρ and η_ρ are given by*

$$u\ \xi_\rho\ v \Leftrightarrow \sigma_S^{-1}(u)\ \rho\ \sigma_S^{-1}(v) \qquad (u, v \in C),$$

$$g\ \eta_\rho\ h \Leftrightarrow \tau_S^{-1}(g)\ \rho\ \tau_S^{-1}(h) \qquad (g, h \in G).$$

Proof. Direct part. Let $\rho = [\xi, \eta]$. Clearly ρ is an equivalence relation. Let

$$s = (x, g, a),\ t = (y, h, b),\ s_1 = (x_1, g_1, a_1),\ t_1 = (y_1, h_1, b_1) \in S$$

be such that $s\ \rho\ t$ and $s_1\ \rho\ t_1$. Then

$$x\ \xi\ y,\ g\ \eta\ h,\ a\ \xi\ b,\ x_1\ \xi\ y_1,\ g_1\ \eta\ h_1,\ a_1\ \xi\ b_1.$$

First $a\ \xi\ b$ and $x_1\ \xi\ y_1$ give $ax_1\ \xi\ by_1$ so that $\zeta_g(ax_1)\ \xi\ \zeta_h(by_1)$. Since $x\ \xi\ y$, we get $x\zeta_g(ax_1)\ \xi\ y\zeta_h(by_1)$. In a similar way, we get $\zeta_{g_1^{-1}}(ax_1)a_1\ \xi\ \zeta_{h_1^{-1}}(by_1)b_1$. Hence

$$(x\zeta_g(ax_1), gg_1, \zeta_{g_1^{-1}}(ax_1)a_1)\ \rho\ (y\zeta_h(by_1), hh_1, \zeta_{h_1^{-1}}(by_1)b_1)$$

and thus $ss_1\ \rho\ tt_1$, and so ρ is a congruence. Note that ρ preserves the unary operation since $(x, g, a)^* = (w, g^{-1}, w)$. Therefore ρ is unary a congruence.

Converse. Let $s = (x, g, a), t = (y, h, b) \in S$ be such that

$$\sigma_S^{-1}(x)\ \rho\ \sigma_S^{-1}(y), \quad \tau_S^{-1}(g)\ \rho\ \tau_S^{-1}(h), \quad \sigma_S^{-1}(a)\ \rho\ \sigma_S^{-1}(b),$$

that is,

$$(x, 1, wx)\ \rho\ (y, 1, wy), \quad (w, g, w)\ \rho\ (w, h, w), \quad (aw, 1, a)\ \rho\ (bw, 1, b).$$

Then

$$s = (x, 1, wx)(w, g, w)(aw, 1, a)\ \rho\ (y, 1, wy)(w, h, w)(bw, 1, b) = t$$

and thus $[\xi_\rho, \eta_\rho] \subseteq \rho$.

Let $s\ \rho\ t$. Then $s^*\ \rho\ t^*$ and hence

$$\begin{aligned}
\sigma_S^{-1}(x) = (x, 1, wx) = ss^*\ \rho\ tt^* = (y, 1, yw) = \sigma_S^{-1}(y),\\
\tau_S^{-1}(g) = (w, g, w) = s^{**}\ \rho\ t^{**} = (w, h, w) = \tau_S^{-1}(h),\\
\sigma_S^{-1}(a) = (aw, 1, a) = s^*s\ \rho\ t^*t = (bw, 1, b) = \sigma_S^{-1}(b).
\end{aligned}$$

Therefore $s\ [\xi_\rho, \eta_\rho]\ t$ and thus $\rho \subseteq [\xi_\rho, \eta_\rho]$. □

Corollary 4.9. *Let $\rho = [\xi, \eta] \in \mathcal{C}(S)$ and define*

$$s\ \rho_\xi\ t \Longleftrightarrow \sigma_S(s)\ \xi\ \sigma_S(t) \qquad (s, t \in C(S)),$$

$$s\ \rho_\eta\ t \Longleftrightarrow \tau_S(s)\ \eta\ \tau_S(t) \qquad (s, t \in H_z).$$

Then ρ is the unique unary congruence on S which extends the congruences ρ_ξ on $C(S)$ and ρ_η on H_z.

Proof. This follows easily from the converse in Theorem 4.8. □

Corollary 4.10. *Let $\rho = [\xi, \eta] \in \mathcal{C}(S)$. Then*

$$S/\rho \cong [C/\xi, G/\eta; w\xi, \zeta']$$

as unary semigroups where, for every $g \in G$,

$$\zeta'_{g\eta}(u\xi) = (\zeta_g(u))\,\xi \qquad (u \in wCw).$$

Let $\mathcal{LP}(S) = \{(\xi, \eta) \mid \xi \text{ and } \eta \text{ are linked}\}$ with coordinatewise inclusion.

Theorem 4.11. *With the notation of Theorem 4.8, the mappings*

$$\alpha : (\xi, \eta) \to [\xi, \eta], \qquad \beta : \rho \to (\xi_\rho, \eta_\rho)$$

are mutually inverse lattice isomorphisms between $\mathcal{LP}(S)$ and $\mathcal{UC}(S)$. Moreover, β is an isomorphism from $\mathcal{LP}(S)$ onto a subdirect product of $\mathcal{NC}(C)$ and $\mathcal{C}(G)$.

Proof. The fact that α and β preserve order follows easily from Lemma 4.6. By Corollary 4.7, we have $\alpha\beta = \iota_{\mathcal{C}(S)}$ and $\beta\alpha = \iota_{\mathcal{LP}(S)}$. Finally, by Theorem 4.8, we conclude that β is onto a subdirect product of $\mathcal{NC}(C)$ and $\mathcal{C}(G)$. ☐

Corollary 4.12. *Let $(\xi_i, \eta_i) \in \mathcal{LP}(S)$ for $i \in I$. Then*

$$\bigcap_{i\in I}[\xi_i, \eta_i] = [\bigcap_{i\in I}\xi_i, \bigcap_{i\in I}\eta_i], \qquad \bigvee_{i\in I}[\xi_i, \eta_i] = [\bigvee_{i\in I}\xi_i, \bigvee_{i\in I}\eta_i].$$

5. Kernels and traces

Let S be a regular semigroup and $\rho \in \mathcal{C}(S)$. The *kernel* of ρ is defined by

$$\ker\rho = \{s \in S \mid s\ \rho\ e \text{ for some } e \in E(S)\},$$

the *trace* of ρ by $\operatorname{tr}\rho = \rho|_{E(S)}$, and $(\ker\rho, \operatorname{tr}\rho)$ is the *congruence pair* of ρ.

For $S = [C, G; w, \zeta]$, we have seen that an unary congruence on S is given by a pair (ξ, η). We can transfer ξ and η to congruences on $C(S)$ and H_z by means of σ_S and τ_S, respectively. Hence our unary congruence ρ is given by the restrictions $\rho\mid_{C(S)}$ and $\rho\mid_{H_z}$. We may thus call $\rho\mid_{C(S)}$ the *core trace* of ρ and $\ker(\rho|_{H_z})$ the *restricted kernel* of ρ, in notation $\operatorname{ctr}\rho$ and $\operatorname{rker}\rho$, respectively. Abstractly, we let ξ be a normal congruence on C, K be a normal subgroup of G related by the condition:

$$g \in K,\ u \in wCw \Longrightarrow \zeta_g(u)\ \xi\ u.$$

We call (K, ξ) a *special congruence pair* and define a relation $\rho_{(K,\xi)}$ by

$$(x, g, a)\ \rho_{(K,\xi)}\ (y, h, b) \Longleftrightarrow x\ \xi\ y, \quad gh^{-1} \in K, \quad a\ \xi\ b.$$

As usual, see ([4],Theorem 2.13) and Theorem 4.8, one proves

Theorem 5.1. *If (K,ξ) is a special congruence pair for S, then $\rho_{(K,\xi)}$ is an unary congruence on S. Conversely, if ρ is an unary congruence on S, then* $(\mathrm{rker}\rho,\, \mathrm{ctr}\rho)$ *is a special congruence pair for S and $\rho = \rho_{(\mathrm{rker}\rho,\, \mathrm{ctr}\rho)}$.*

6. Relations K, T, L, U and V

Let T be an unary semigroup. If ρ is a relation on T, then $\rho^\natural$ denotes the congruence in T generated by ρ. For any relation P on $\mathcal{UC}(T)$, ρ_P (respectively ρ^P) denotes the least (respectively greatest) congruence on T which is P-related to ρ, if it exists. We denote by ϵ_X the equality relation on any nonempty set X.

In this section $S = [C, G; w, \zeta]$, $\rho = [\xi, \eta]$ and $\rho' = [\xi', \eta']$.

Recall that the *kernel relation* K is defined by

$$\rho\ K\ \rho' \Longleftrightarrow \ker\rho = \ker\rho' \qquad (\rho, \rho' \in \mathcal{C}(S)).$$

We first find an expression for $\ker\rho$ in the present setting.

Lemma 6.1. *Let $\rho \in \mathcal{UC}(S)$. Then*

$$\ker\rho = \{(x, g, a) \in S \mid wx\ \xi\ ax\ \xi\ aw,\ g \in \ker\eta\}.$$

Proof. For $(x, g, a) \in S$ we have

$$\begin{aligned}
(x, g, a)\ \rho\ (x, g, a)^2 &\Longleftrightarrow x\ \xi\ x\zeta_g(ax),\ g\ \eta\ g^2,\ a\ \xi\ \zeta_{g^{-1}}(ax)a \\
&\Longleftrightarrow \zeta_{g^{-1}}(wx)\ \xi\ \zeta_{g^{-1}}(wx)ax,\ g \in \ker\eta,\ \zeta_g(aw)\ \xi\ ax\zeta_g(aw) \\
&\Longleftrightarrow aw\ \xi\ ax,\ g \in \ker\eta,\ wx\ \xi\ ax.
\end{aligned}$$

□

A description of the K-relation follows.

Theorem 6.2. *We have*

$$\rho\ K\ \rho' \Longleftrightarrow \xi\ K\ \xi',\ \eta = \eta'.$$

Proof. Assume that $\rho\ K\ \rho'$ and let $u \in \ker\xi$. Then for $s = (uw, 1, wu)$, we have $s\ \rho\ s^2$ so that $s \in \ker\rho$. The hypothesis implies that $s \in \ker\rho'$ whence $s\ \rho'\ s^2$ and thus $uw\ \xi'\ u^2w$ and $wu\ \xi'\ wu^2$. Hence

$$u = (uw)(wu)\ \xi'\ (u^2w)(wu^2) = u^2$$

and $u \in \ker \xi'$. Therefore $\ker \xi \subseteq \ker \xi'$ and equality follows by symmetry, so that $\xi \ K \ \xi'$.

Next let $g, h \in G$ be such that $g \ \eta \ h$. Then $(w, g, w) \ \rho \ (w, h, w)$ whence

$$(w, gh^{-1}, w) \in \ker \rho.$$

Thus $(w, gh^{-1}, w) \in \ker \rho'$ so that $gh^{-1} \in \ker \eta'$. But then $g \ \eta' \ h$. Therefore $\eta \subseteq \eta'$ and equality follows by symmetry.

Conversely, suppose that $\xi \ K \ \xi'$ and $\eta = \eta'$, and let $s = (x, g, a) \in \ker \rho$. By Lemma 6.1, we have

$$aw \ \xi \ ax \ \xi \ wx, \ \ g \in \ker \eta.$$

Then $g \in \ker \eta'$ so that, by the linking condition,

$$wx = \zeta_g(aw) \ \xi' \ \zeta_1(aw) = aw.$$

Moreover, from $xa = x(wx)a \ \xi \ x(ax)a$ we have that $xa \in \ker \xi$ and the hypothesis implies that $xa \in \ker \xi'$. Thus $xa \ \xi' \ (xa)^2$ and then

$$aw = (aw)aw \ \xi' \ w(xa)w \ \xi' \ (wx)ax(aw) \ \xi' \ awaxwx = ax.$$

Hence $s \in \ker \rho'$ and thus $\ker \rho \subseteq \ker \rho'$. The opposite inclusion follows by symmetry. Therefore $\rho \ K \ \rho'$. $\square$

According to ([3], Theorem 3.2),

$$\rho_K = \{(x, x^2) \mid x \in \ker \rho\}^\natural,$$

and ρ^K is the principal congruence on $\ker \rho$, that is

$$s \ \rho^K \ t \iff (xsy \in \ker \rho \Leftrightarrow xty \in \ker \rho \text{ for all } x, y \in S^1).$$

It does not seem that in our case either ρ_K or ρ^K can be expressed in a reasonable way by means of ξ and η. As a consequence, for the bounds for V in Corollary 6.7, all we can say that, in general for relations P and Q on $\mathcal{UC}(S)$, whose classes are intervals, we have

$$\rho_{P \cap Q} = \rho_P \vee \rho_Q, \qquad \rho^{P \cap Q} = \rho^P \cap \rho^Q.$$

Recall that the *trace relation* T is defined by

$$\rho \ T \ \rho' \iff \operatorname{tr} \rho = \operatorname{tr} \rho' \qquad (\rho, \rho' \in \mathcal{C}(S)).$$

In order to characterize the T-relation we first state

Lemma 6.3. *Let $\xi \in \mathcal{NC}(C)$. Define*

$$g \ \xi_{\max} \ h \iff \zeta_g(x) \ \xi \ \zeta_h(x) \text{ for all } x \in wCw \qquad (g, h \in G).$$

Then $\xi_{\max}$ is the greatest congruence on G linked to ξ.

We are now in a position to establish

Theorem 6.4. *We have*

$$\rho\ T\ \rho' \Longleftrightarrow \xi = \xi'; \qquad \rho_T = [\xi, \epsilon_G], \qquad \rho^T = [\xi, \xi_{\max}].$$

Proof. Assume that $\rho\ T\ \rho'$ and let $u, v \in C$ be such that $u\ \xi\ v$. Then

$$(uw, 1, wu)\ \rho\ (vw, 1, wv)$$

where $(uw, 1, wu), (vw, 1, wv) \in E(S)$. The hypothesis implies that

$$(uw, 1, wu)\ \rho'\ (vw, 1, wv).$$

Thus $uw\ \xi'\ vw$ and $wu\ \xi'\ wv$ and hence

$$u = (uw)(wu)\ \xi'\ (vw)(wv) = v.$$

Therefore $\xi \subseteq \xi'$ and the opposite inclusion follows by symmetry.

Assume now that $\xi = \xi'$ and let $(x, 1, a), (y, 1, b) \in E(S)$ be such that

$$(x, 1, a)\ \rho\ (y, 1, b).$$

Then $x\ \xi\ y$ and $a\ \xi\ b$. Hence $(x, 1, a)\ \rho'\ (y, 1, b)$ and thus $\operatorname{tr}\rho \subseteq \operatorname{tr}\rho'$. The opposite inclusion follows by symmetry. Therefore $\rho\ T\ \rho'$.

The congruences ξ and ϵ_G are obviously linked and we obtain $[\xi, \epsilon_G]\ T\ [\xi, \eta]$. By Lemma 4.6, we get that $\rho_T = [\xi, \epsilon_G]$. By Lemma 6.3, the congruences ξ and $\xi_{\max}$ are linked. In view of Lemma 4.6, we conclude that $\rho^T = [\xi, \xi_{\max}]$. $\square$

Recall that the *L-relation* is defined by

$$\rho\ L\ \rho' \Longleftrightarrow \rho|_{eSe} = \rho'|_{eSe} \text{ for all } e \in E(S).$$

The desired characterization follows.

Theorem 6.5. *We have*

$$\rho\ L\ \rho' \Longleftrightarrow \xi\ L\ \xi',\ \eta = \eta'; \qquad \rho_L = [(\xi\ |_{wCw})^\natural, \eta], \qquad \rho^L = [\xi^L, \eta].$$

Proof. Assume that $\rho\ L\ \rho'$ and let $e \in E(C)$ and $u, v \in eCe$ be such that $u\ \xi\ v$. For $f = (ew, 1, we)$, we get $f \in E(S)$ and

$$(uw, 1, wu), (vw, 1, wv) \in fSf$$

with $(uw, 1, wu)\ \rho\ (vw, 1, wv)$. The hypothesis implies that

$$(uw, 1, wu)\ \rho'\ (vw, 1, wv),$$

whence

$$u = (uw)(wu)\ \xi'\ (vw)(wv) = v.$$

Hence $\rho|_{eCe} \subseteq \rho'|_{eCe}$ and the opposite inclusion follows by symmetry. Therefore $\xi\ L\ \xi'$.

Let $g, h \in G$ be such that $g\ \eta\ h$. For $s = (w, g, w)$, $t = (w, h, w) \in zSz$, we obtain $s\ \rho\ t$. By hypothesis, we get $s\ \rho'\ t$ whence $g\ \eta'\ h$. Hence $\eta \subseteq \eta'$ and the equality prevails by symmetry.

Conversely, suppose that $\xi\ L\ \xi'$ and $\eta = \eta'$. Let $e = (u, 1, c) \in E(S)$ and $s = (x, g, a)$, $t = (y, h, b) \in eSe$. Then $x = ucx$, $y = ucy$, $a = auc$ and $b = buc$. Let $s\ \rho\ t$. Then $x\ \xi\ y,\ g\ \eta\ h,\ a\ \xi\ b$ and thus $wx\ \xi\ wy,\ g\ \eta\ h,\ aw\ \xi\ bw$. The hypothesis implies that $wx\ \xi'\ wy,\ g\ \eta'\ h,\ aw\ \xi'\ bw$ so that

$$ucwx\ \xi'\ ucwy,\ g\ \eta'\ h,\ awuc\ \xi'\ bwuc.$$

Hence $x\ \xi'\ y,\ g\ \eta'\ h,\ a\ \xi'\ b$ and $s\ \rho'\ t$. Hence $\rho|_{eSe} \subseteq \rho'|_{eSe}$ and the equality prevails by symmetry.

First note that $(\xi\ |_{wCw})^\natural$ is normal since, for $u, v \in wCw$, we have

$$u\ (\xi\ |_{wCw})^\natural\ v \Longleftrightarrow u\ \xi\ v \Longrightarrow \zeta_g(u)\ \xi\ \zeta_g(v) \Longleftrightarrow \zeta_g(u)\ (\xi\ |_{wCw})^\natural\ \zeta_g(v).$$

If $g\ \eta\ h$, then $\zeta_g(u)\ \xi\ \zeta_h(u)$ for every $u \in wCw$. Since $\zeta_g(u), \zeta_h(u) \in wCw$, we get that $\zeta_g(u)\ (\xi\ |_{wCw})^\natural\ \zeta_h(u)$. Hence $(\xi|_{wCw})^\natural$ and η are linked. In order to prove that $[(\xi\ |_{wCw})^\natural, \eta]\ L\ [\xi, \eta]$, let $e \in E(C)$ and $u, v \in eCe$ be such that $u\ \xi\ v$. Then $wuw\ \xi\ wvw$ so that $wuw\ \xi\ |_{wCw}\ wvw$. Hence $wuw\ (\xi\ |_{wCw})^\natural\ wevew$ and

$$u = ewuwe\ (\xi\ |_{wCw})^\natural\ ewvwe = v.$$

Therefore $\xi\ |_{eCe} \subseteq (\xi\ |_{wCw})^\natural\ |_{eCe}$ and equality follows by symmetry.

In order to prove that $\rho^L = [\xi^L, \eta]$, we will use the fact that $\xi|_{wCw} = \xi^L|_{wCw}$ several times. Let $u, v \in wCw$ be such that $u\ \xi^L\ v$. Since $\xi^L|_{wCw} = \xi|_{wCw}$, we have $u\ \xi\ v$ so that $\zeta_g(u)\ \xi\ \zeta_g(v)$. Again, since $\zeta_g(u), \zeta_g(v) \in wCw$, we have that $\zeta_g(u)\ \xi^L\ \zeta_g(v)$. Hence ξ^L is normal. Let $g\ \eta\ h$. For $u \in wCw$, we have $\zeta_g(u)\ \xi\ \zeta_h(u)$ and thus $\zeta_g(u)\ \xi^L\ \zeta_h(u)$. Therefore ξ^L and η are linked. Moreover, since $\xi^L\ L\ \xi$, we have that $[\xi^L, \eta]\ L\ [\xi, \eta]$. Finally, let $[\xi', \eta'] \in \mathcal{C}(S)$ be such that $[\xi', \eta']\ L\ [\xi, \eta]$. Then $\xi'\ L\ \xi$ and $\eta = \eta'$. Hence $\xi' \subseteq \xi^L$ and, by Lemma 4.6, we get $[\xi', \eta'] \subseteq [\xi^L, \eta]$.

Let $\xi' \in \mathcal{C}(C)$ be such that ξ' and η are linked and $\xi'\ L\ \xi$. Then $\xi'\ |_{wCw} = \xi\ |_{wCw}$ so that $(\xi\ |_{wCw})^\natural \subseteq \xi'$. Therefore, by Lemma 4.6, we conclude that

$$\rho_L = [(\xi\ |_{wCw})^\natural, \eta]. \qquad \square$$

The *U-relation* was defined in [4] by

$$\rho\ U\ \rho' \Longleftrightarrow (\rho \cap \leq) = (\rho' \cap \leq) \qquad (\rho, \rho' \in \mathcal{C}(S)).$$

In our case the wanted characterization is

Theorem 6.6. *We have*

$$\rho\ U\ \rho' \Longleftrightarrow \xi\ U\ \xi'; \qquad \rho_U = [\lambda, \epsilon_G], \qquad \rho^U = [\theta, \theta_{\max}],$$

where

$$\lambda = \cap\{\gamma \in \mathcal{NC}(C) \mid (\xi \cap \leq) \subseteq \gamma\},$$
$$\theta = \vee\{\gamma \in \mathcal{NC}(C) \mid (\gamma \cap \leq) = (\xi \cap \leq)\}.$$

Proof. We will use Lemma 2.1(iv) without express mention.

Assume that $\rho\ U\ \rho'$ and let $u, v \in C$ be such that $u\ (\xi \cap \leq)\ v$. We have $uw\ \xi\ vw$ and $u = ev = vf$ for some $e, f \in E(C)$. Then

$$e(vw) = uw = vfw = vw(vfw)$$

where $vfw \in E(C)$ so that $uw\ (\xi \cap \leq)\ vw$; analogously $wu\ (\xi \cap \leq)\ wv$. Now

$$(uw, 1, wu)\ (\rho \cap \leq)\ (vw, 1, wv)$$

which by hypothesis yields

$$(uw, 1, wu)\ (\rho' \cap \leq)\ (vw, 1, wv)$$

and thus $uw\ \xi'\ vw$ and $wu\ \xi'\ wv$. Hence

$$u = uwwu\ \xi'\ vwwv = v$$

which proves that $u\ (\xi' \cap \leq)\ v$. Thus $(\xi \cap \leq) \subseteq (\xi' \cap \leq)$ and equality follows by symmetry. Therefore $\xi\ U\ \xi'$.

Conversely suppose that $\xi\ U\ \xi'$ and let $(x, g, a)\ (\rho \cap \leq)\ (y, h, b)$. Then

$$x\ (\xi \cap \leq)\ y,\ \ g = h,\ \ a\ (\xi \cap \leq)\ b$$

and thus

$$x\ (\xi' \cap \leq)\ y,\ \ g = h,\ \ a\ (\xi' \cap \leq)\ b$$

so that $(x, g, a)\ (\rho' \cap \leq)\ (y, h, b)$. Therefore $(\rho \cap \leq) \subseteq (\rho' \cap \leq)$ and equality follows by symmetry. Consequently $\rho\ U\ \rho'$.

Let $u, v \in C$. If $u\ (\xi \cap \leq)\ v$ and $\gamma \in \mathcal{NC}(C)$ is such that $(\xi \cap \leq) \subseteq \gamma$, then trivially $u\ (\gamma \cap \leq)\ v$.

Conversely, assume that for every $\gamma \in \mathcal{NC}(C)$ such that $(\xi \cap \leq) \subseteq \gamma$, we have $u\ (\gamma \cap \leq)\ v$. For $\gamma = \xi$, we trivially have that $\xi \in \mathcal{NC}(C)$ and $(\xi \cap \leq) \subseteq \xi$. Thus

$u\ (\xi \cap \leq)\ v$ and $u\ \xi\ v$.

First $\lambda \in \mathcal{NC}(C)$ by Lemma 4.2 and hence λ and ϵ_G are linked. It suffices to prove that

$$(\rho \cap \leq) = ([\lambda, \epsilon_G] \cap \leq). \tag{2}$$

Since

$$\begin{aligned}(x, g, a)\ ([\lambda, \epsilon_G] \cap \leq)\ (y, h, b) \iff \quad & g = h \text{ and for every } \gamma \in \mathcal{NC}(C) \text{ such that} \\ & (\xi \cap \leq) \subseteq \gamma,\ \text{ we have } x\ (\gamma \cap \leq)\ y,\ a\ (\gamma \cap \leq)\ b\end{aligned}$$

and

$$(x, g, a)\ (\rho \cap \leq)\ (y, h, b) \iff x\ (\xi \cap \leq)\ y,\ g = h,\ a\ (\xi \cap \leq)\ b,$$

relation (2) holds. Therefore $\rho_U = [\lambda, \epsilon_G]$.

Finally, by Lemmas 4.2 and 6.3, $\theta \in \mathcal{NC}(C)$ and $\theta_{\max}$ are linked and $\rho^U = [\theta, \theta_{\max}]$. $\square$

Recall that the *V-relation* was defined in [4] by $V = K \cap U$. From Theorems 6.2 and 6.6 we deduce

Corollary 6.7. *We have*

$$\rho\ V\ \rho' \iff \xi\ V\ \xi',\ \eta = \eta'.$$

We can express the above relations in a uniform way as follows.

Corollary 6.8 (to Theorems 6.2, 6.4, 6.5, 6.6 and Corollary 6.7)**.** *For $P \in \{K, T, L, U, V\}$, we have*

$$\rho\ P\ \rho' \iff \xi\ P\ \xi', \eta\ P\ \eta'.$$

Proof. $P = K$. This follows directly from Theorem 6.2.

$P = T$. By Theorem 6.4, we have

$$\rho\ T\ \rho' \iff \xi\ = \xi'.$$

Since obviously $\eta\ T\ \eta'$, the assertion follows.

$P = L$. By Theorem 6.5, we have

$$\rho\ L\ \rho' \iff \xi\ L\ \xi', \eta = \eta'.$$

Clearly on the group G, $\eta\ L\ \eta'$ implies $\eta = \eta'$, and the claim follows.

$P = U$. By Theorem 6.6, we have

$$\rho\ U\ \rho' \iff \xi\ U\ \xi'.$$

The natural partial order on G is trivial. Hence $\eta \; U \; \eta'$ which proves the assertion.

$P = V$. This follows directly from Corollary 6.7. $\square$

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