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HOMOMORPHISMS OF COMMUTATIVE CANCELLATIVE SEMIGROUPS INTO NONNEGATIVE REAL NUMBERS

BY

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ABSTRACT. Let S be a commutative cancellative semigroup and T_0 be a cofinal subsemigroup of S. Let h_0 be a homomorphism of T_0 into the semigroup of nonnegative real numbers under addition. We prove that Kobayashi's condition [2] is necessary and sufficient for h_0 to be extended to S. Further, we find a necessary and sufficient condition in order that the extension be unique. Related to this, the "boundedness condition" is introduced. For further study, several examples are given.

1. **Introduction.** A commutative cancellative archimedean idempotent-free semigroup is called an $\mathfrak{N}$-semigroup. Kobayashi [2] proved the following:

THEOREM 1.1. *Let T_0 be a subsemigroup of an $\mathfrak{N}$-semigroup S and let h_0 be a homomorphism of T_0 into the semigroup $\mathbf{R}_+^0$ of nonnegative real numbers under addition. Then h_0 can be extended to a homomorphism of S into $\mathbf{R}_+^0$ if and only if the pair $\langle T_0, h_0\rangle$ satisfies the following condition: if $x, y \in T_0$ and $x \mid y$ (x divides y) in S, then $h_0(x) \leq h_0(y)$.*

One of the authors [4] has studied the homomorphisms of T_0 into $\mathbf{R}_+$ from the viewpoint of positive quasi-orders. In this paper, we treat the homomorphisms of T_0 into the nonnegative real numbers in the case when S is a commutative cancellative semigroup and T is its subsemigroup. Theorem 2.1 will be a straightforward generalization of the classical result that characters can be extended from a subgroup of an abelian group G to G itself. In §2, we will show that Theorem 1.1 holds if T_0 is cofinal in S. In §3, we will introduce a "boundedness condition" and discuss the relation between this condition and the extension of a homomorphism beyond a filter. In §4, we will give a few examples, which show that Theorem 2.1 does not necessarily hold if T_0 is not cofinal.

A subsemigroup U of a commutative semigroup S is called unitary in S if

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$x \in S$, $a \in U$ and $ax \in U$ imply $x \in U$. U is called cofinal in S if, for every $x \in S$, there is a $y \in S$ such that $xy \in U$. As is well known, see [1] or [5], a unitary cofinal subsemigroup U induces a group congruence ρ_U on S defined by $x \rho_U y$ if and only if $ax = by$ for some $a, b \in U$. We denote S/ρ_U by S/U. Furthermore the kernel of $S \longrightarrow S/U$ coincides with U.

Let T be a nonempty subsemigroup of S. The smallest unitary subsemigroup $\overline{T}$ of S containing the subsemigroup T is called the unitary closure of T in S. $\overline{T}$ is given by

$$\overline{T} = \{x \in S: \ xt \in T \text{ for some } t \in T\}.$$

A nonempty subsemigroup F of S is called a filter of S [3] if $x, y \in S$ and $xy \in F$ implies $x, y \in F$. The smallest filter $\widetilde{T}$ of S containing the subsemigroup T is called the filter closure of T in S. Then

$$\widetilde{T} = \{x \in S: xy \in T \text{ for some } y \in S\}.$$

(1.2) The following hold.
(1.2.1) $T \longrightarrow \overline{T}$ and $T \longrightarrow \widetilde{T}$ are closure mappings, that is, $T \subseteq \overline{T}$, $T \subseteq \widetilde{T}$.
$T_1 \subseteq T_2$ implies $\overline{T}_1 \subseteq \overline{T}_2$ and $\widetilde{T}_1 \subseteq \widetilde{T}_2$.
$\overline{\overline{T}} = \overline{T}$, $\widetilde{\widetilde{T}} = \widetilde{T}$.
(1.2.2) $\overline{T}$ is unitary in S, $\widetilde{T}$ is a filter in S, and T is cofinal in $\widetilde{T}$.
(1.2.3) $\widetilde{T} = \widetilde{\overline{T}} = \overline{\widetilde{T}}$.
(1.2.4) $\overline{T} \subseteq \widetilde{T}$ and $\overline{T}$ is unitary cofinal in $\widetilde{T}$.

Throughout this paper, $\mathbf{R}$ denotes the set of real numbers, R the set of rational numbers, $\mathbf{R}_+$ ($\mathbf{R}_-$) the set of positive (negative) real numbers; $\mathbf{R}_+^0$ ($\mathbf{R}_-^0$) the set of nonnegative (nonpositive) real numbers; Z_+ (Z_-) the set of positive (negative) integers and Z_+^0 (Z_-^0) the set of nonnegative (nonpositive) integers.

If S is a semigroup and if X is a subsemigroup of the additive group $\mathbf{R}$, then the notation $\mathrm{Hom}(S, X)$ denotes the semigroup of homomorphisms of S into X under the usual operation. Let X_1, X_2, Y_1 and Y_2 be commutative semigroups such that $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$. Let $h_1 \in \mathrm{Hom}(X_1, Y_1)$ and $h_2 \in \mathrm{Hom}(X_2, Y_2)$. If $h_2|X_1 = h_1$, we say that h_1 of $\mathrm{Hom}(X_1, Y_1)$ is extended to h_2 of $\mathrm{Hom}(X_2, Y_2)$; in particular, if $Y_1 = Y_2$, we say that h_1 of $\mathrm{Hom}(X_1, Y_1)$ is extended to X_2. If the extension h_2 of h_1 of $\mathrm{Hom}(X_1, Y_1)$ to X_2 is unique, we say that h_1 of $\mathrm{Hom}(X_1, Y_1)$ is uniquely extended to X_2. Let $h \in \mathrm{Hom}(S, \mathbf{R})$. h is called trivial if $h(x) = 0$ for all $x \in S$.

In this paper the binary operation in a commutative semigroup will be denoted by addition, i.e. $+$.

2. **Extensions from cofinal subsemigroups.** In this section, we will prove the following generalization of Theorem 1.1.

THEOREM 2.1. *Let T_0 be a cofinal subsemigroup of a commutative cancellative semigroup S and let h_0 be a homomorphism of T_0 into the additive semigroup $\mathbf{R}_+^0$ of nonnegative real numbers. Then h_0 can be extended to S if and only if*

$$(K) \qquad t_1 \in S + t_2 \text{ implies } h_0(t_1) \geqslant h_0(t_2) \text{ for all } t_1, t_2 \in T_0.$$

In this paper, the condition (K) will be called the K-condition. It is obvious that if h_0 can be extended to S then the K-condition must hold. We will prove sufficiency. Let X denote the set of pairs $\langle T, h\rangle$ where T is a subsemigroup of S containing T_0 and $h \in \operatorname{Hom}(T, \mathbf{R}_+^0)$ such that $h|T_0 = h_0$ and $\langle T, h\rangle$ satisfies the K-condition.

Let $[a]$ be the cyclic subsemigroup generated by a and let $[T, a]$ be the subsemigroup generated by T and a, i.e.,

$$[T, a] = T \cup (T + [a]) \cup [a].$$

LEMMA 2.2. *Let $\langle T, h\rangle \in X$ and suppose that $a \in S$ and $(T + [a]) \cap T \neq \varnothing$. Then there exists h': $[T, a] \longrightarrow \mathbf{R}_+^0$ such that $\langle [T, a], h'\rangle \in X$. Further, h' is unique.*

PROOF. There exist $t_1, t_2 \in T$, $N \in Z_+$ such that $t_1 = N \cdot a + t_2$. Then $h(t_1) \geqslant h(t_2)$ by the K-condition. Define h': $[T, a] \longrightarrow \mathbf{R}_+^0$ by

$$h'(t + n \cdot a) = h(t) + \frac{n}{N}[h(t_1) - h(t_2)], \qquad t \in T,\ n \in Z_+^0,$$

$$h'(na) = \frac{n}{N}[h(t_1) - h(t_2)], \qquad n \in Z_+.$$

First we show that h' is well defined: $t + n \cdot a = t' + n' \cdot a$, $t, t' \in T$, $n, n' \in Z_+$, implies $N \cdot t + Nn \cdot a + (n + n') \cdot t_2 = N \cdot t' + Nn' \cdot a + (n + n') \cdot t_2$, that is, $N \cdot t + n \cdot t_1 + n' \cdot t_2 = N \cdot t' + n' \cdot t_1 + n \cdot t_2$. This shows $h'(t + n \cdot a) = h'(t' + n' \cdot a)$, hence h' is well defined. From its definition, h' is clearly a homomorphism into $\mathbf{R}_+^0$, and $h'|T = h$. Assume $t + n \cdot a = s + t' + n' \cdot a$ for some $a \in S$. Then $N \cdot t + n \cdot t_1 + n' \cdot t_2 = N \cdot s + N \cdot t' + n' \cdot t_1 + n \cdot t_2$ which implies

$$N \cdot h(t) + n \cdot h(t_1) + n' \cdot h(t_2) \geqslant N \cdot h(t') + n' \cdot h(t_1) + n \cdot h(t_2)$$

by the K-condition. This gives $h'(t + n \cdot a) \geqslant h'(t' + n' \cdot a)$. Hence $\langle [T, a], h'\rangle \in X$. If h'' is any extension of h to a homomorphism of $[T, a]$ into $\mathbf{R}_+^0$, we must have

$$h(t_1) = h''(t_1) = N \cdot h''(a) + h''(t_2) = N \cdot h''(a) + h(t_2)$$

so that $h''(a) = N^{-1}[h(t_1) - h(t_2)] = h'(a)$. It follows that $h''(t + n \cdot a) = h'(t + n \cdot a)$ for all $t \in T$, all $n \in Z_+$, that is, $h'' = h'$. □

To consider the case when $(T + [a]) \cap T = \emptyset$, we need a lemma. From now on, t_1, t_2, t_3 and t_4 will denote arbitrary elements of T.

Let

$$A(a) = \{n^{-1}[h(t_2) - h(t_1)]: \ t_1 + n \cdot a \in t_2 + S\},$$
$$B(a) = \{n^{-1}[h(t_2) - h(t_1)]: \ t_2 \in t_1 + n \cdot a + S\},$$

where $a \in S$, $A(a)$ and $B(a)$ mean the sets depending on a. Note that $0 \in A(a)$ and hence $A(a) \neq \emptyset$.

LEMMA 2.3. *Let $\langle T, h\rangle \in \mathcal{X}$ and suppose that $(S + [a]) \cap T \neq \emptyset$. Then* $\operatorname{Sup} A(a) \leqslant \operatorname{Inf} B(a) < \infty$.

PROOF. Since $(S + [a]) \cap T \neq \emptyset$, there are $t_1, t_2 \in T$, $x \in S$ and $n \in Z_+$ such that $t_2 = t_1 + n \cdot a + x$. Hence $B(a) \neq \emptyset$ and $\operatorname{Inf} B(a) < \infty$. Suppose $t_1 + n_1 \cdot a = t_2 + s_1$ and $t_4 = t_3 + n_2 \cdot a + s_2$ where $n_1, n_2 \in Z_+$, $s_1, s_2 \in S$. Then $n_1 \cdot t_4 + n_2 \cdot t_1 = n_1 \cdot t_3 + n_2 \cdot t_2 + n_2 \cdot s_1 + n_1 \cdot s_2$. By the $\mathcal{K}$-condition, $n_1 \cdot h(t_4) + n_2 \cdot h(t_1) \geqslant n_1 \cdot h(t_3) + n_2 \cdot h(t_2)$. Hence $n_2^{-1}[h(t_4) - h(t_3)] \geqslant n_1^{-1}[h(t_2) - h(t_1)]$. Thus we get $\operatorname{Sup} A(a) \leqslant \operatorname{Inf} B(a)$. □

LEMMA 2.4. *Let $\langle T, h\rangle \in \mathcal{X}$ and suppose that $(S + [a]) \cap T \neq \emptyset$ but $(T + [a]) \cap T = \emptyset$. Then h can be extended to a homomorphism h': $[T, a] \longrightarrow \mathbf{R}_+^0$ and $\langle [T, a], h'\rangle \in \mathcal{X}$. The h' is determined by choosing $h'(a)$ such that $\operatorname{Sup} A(a) \leqslant h'(a) \leqslant \operatorname{Inf} B(a)$. Moreover, every extension h'' of h to $[T, a]$ such that $\langle [T, a], h''\rangle \in \mathcal{X}$ is obtained in this way.*

PROOF. Choose $b \in \mathbf{R}_+^0$ such that

$$\operatorname{Sup} A(a) \leqslant b \leqslant \operatorname{Inf} B(a). \tag{2.4.1}$$

Define

$$\begin{aligned} h'(t + n \cdot a) &= h(t) + n \cdot b \quad \text{for } t \in T,\ n \in Z_+^0, \\ h'(na) &= nb \qquad\qquad\quad \text{for } n \in Z_+. \end{aligned} \tag{2.4.2}$$

Since S is cancellative and $(T + [a]) \cap T = \emptyset$, every element of $T + [a]$ is uniquely expressed as $t + n \cdot a$ and hence h' is well defined. Then h' is clearly a homomorphism $[T, a] \longrightarrow \mathbf{R}_+^0$ and $h'|T = h$. Suppose that $t_1 + n_1 \cdot a = t_2 + n_2 \cdot a + s$, $n_1, n_2 \in Z_+^0$, $s \in S$. Then there are three possibilities: $n_1 = n_2$, $n_1 > n_2$ and $n_1 < n_2$. If $n_1 = n_2$, then, since S is cancellative, $t_1 = t_2 + s$, hence $h(t_1) \geqslant h(t_2)$ by the $\mathcal{K}$-condition. This implies $h'(t_1 + n_1 \cdot a) \geqslant h'(t_2 + n_2 \cdot a)$.

If $n_1 > n_2$, then $t_1 + (n_1 - n_2) \cdot a = t_2 + s$ and, by the choice of b,

$$h(t_2) - h(t_1) \leqslant (n_1 - n_2) \cdot b.$$

This implies

$$h'(t_2 + n_2 \cdot a) = h(t_2) + n_2 \cdot b \leqslant h(t_1) + n_1 \cdot b = h'(t_1 + n_1 \cdot a).$$

If $n_1 < n_2$, then $t_1 = t_2 + (n_2 - n_1) \cdot a + s$. By the choice of b,

$$(n_2 - n_1) \cdot b \leqslant h(t_1) - h(t_2).$$

This gives

$$h'(t_2 + n_2 \cdot a) = h(t_2) + n_2 \cdot b \leqslant h(t_1) + n_1 \cdot b = h'(t_1 + n_1 \cdot a).$$

Therefore $\langle [T, a], h' \rangle \in X$.

Assume that h'' is an extension of h to $[T, a]$ and that $t_1 + n_1 \cdot a = t_2 + s_1$ and $t_4 = t_3 + n_2 \cdot a + s_2$, $n_1, n_2 \in Z_+$, $s_1, s_2 \in S$. Using the assumption that h'' obeys the K-condition, $t_1 + n_1 a = t_2 + s$ gives $h''(t_1) + n_1 h''(a) \geqslant h''(t_2)$, so that $h''(a) \geqslant (h(t_2) - h(t_1))/n_1$, hence $h''(a) \geqslant \operatorname{Sup} A(a)$. Likewise we have $h''(a) \leqslant (h(t_4) - h(t_3))/n_2$, hence $h''(a) \leqslant \operatorname{Inf} B(a)$. By the former half of the lemma, $\langle [T, a], h'' \rangle \in X$. □

COROLLARY 2.5. *In Lemma* 2.4, *the extension h' is unique if and only if*

(2.5.1) $$\operatorname{Sup} A(a) = \operatorname{Inf} B(a).$$

PROOF OF SUFFICIENCY OF THEOREM 2.1. Define the partial order in X by $\langle T_1, h_1 \rangle \leq \langle T_2, h_2 \rangle$ if and only if $T_1 \subseteq T_2$ and h_2 is an extension of h_1 to T_2. Then it is easy to see that X satisfies the condition for Zorn's lemma and so X has maximal members. To show that any such maximal member has domain S, it suffices to show that if $\langle T, h \rangle \in X$ and $a \notin T$, then h can be extended to h': $[T, a] \longrightarrow \mathbf{R}_+^0$ such that $\langle [T, a], h' \rangle \in X$. Since T is cofinal, $(S + [a]) \cap T \neq \varnothing$, furthermore there are two possibilities: $(T + [a]) \cap T \neq \varnothing$ and $(T + [a]) \cap T = \varnothing$. Lemma 2.2 has dealt with the first case; Lemma 2.4 has done the second case. Thus the theorem has been proved. □

COROLLARY 2.6. *Let S be a commutative cancellative semigroup and T_0 a unitary cofinal subsemigroup of S. Then every homomorphism h of T_0 into $\mathbf{R}_+^0$ can be extended to S.*

PROOF. Every h satisfies the K-condition.

COROLLARY 2.7. *Let T_0 be an ideal of S. Then every homomorphism h of T_0 into $\mathbf{R}_+^0$ can be uniquely extended to S.*

PROOF. Lemma 2.2 is applied to this case since $(T_0 + [a]) \cap T_0 \neq \emptyset$ for each $a \in S$. The direct alternate proof of this corollary is left for the reader's exercise. □

Since every subsemigroup of a commutative archimedean semigroup is cofinal, Theorem 1.1 is a special case of Theorem 2.1.

THEOREM 2.8. *Let T be a cofinal subsemigroup of a commutative cancellative subsemigroup S, and let h: $T \longrightarrow \mathbf{R}_+^0$ be a homomorphism. Then h admits a unique extension to S if and only if, for each $a \in S$,* $\operatorname{Sup} A(a) = \operatorname{Inf} B(a)$.

PROOF. Assume h admits a unique extention to S. Then $\langle T, h\rangle$ satisfies the $\mathcal{K}$-condition. Suppose that h_1 and h_2 are distinct extensions such that $\langle [T, a], h_1\rangle$ and $\langle [T, a], h_2\rangle$ obey the $\mathcal{K}$-condition for some $a \notin T$. Then $(S + [a]) \cap T \neq \emptyset$ since T is cofinal in S; $(T + [a]) \cap T = \emptyset$ by Lemma 2.2. Now Lemma 2.4 shows that $\langle [T, a], h_1\rangle$ and $\langle [T, a], h_2\rangle$ are in $\mathcal{X}$. By Theorem 2.1, h_1 and h_2 can be extended to homomorphisms $h_1'\colon S \longrightarrow \mathbf{R}_+^0$ and $h_2'\colon S \longrightarrow \mathbf{R}_+^0$ respectively; but $h_1' \neq h_2'$. This contradicts the assumption. Therefore h admits a unique extension to $[T, a]$ for each $a \notin T$. If $(T + [a]) \cap T \neq \emptyset$, we can easily show that if $n \in Z_+$, $t_1, t_2 \in T$ and $t_1 + n \cdot a = t_2$, then

$$\operatorname{Sup} A(a) = \operatorname{Inf} B(a) = (h(t_2) - h(t_1))/n.$$

If $(T + [a]) \cap T = \emptyset$, then Corollary 2.5 shows $\operatorname{Sup} A(a) = \operatorname{Inf} B(a)$.

Conversely, suppose $\operatorname{Sup} A(a) = \operatorname{Inf} B(a)$ for every $a \in S$. If $t_2 = t_1 + s$, $t_1, t_2 \in T$, $s \in S$, then $2t_2 \in 2t_1 + s + S$, which implies

$$\operatorname{Inf} B(s) \leqslant 2[h(t_2) - h(t_1)].$$

As $t_1 + 2 \cdot s \in t_2 + S$, $\operatorname{Sup} A(s) \geqslant \frac{1}{2}[h(t_2) - h(t_1)]$. Hence

$$\tfrac{1}{2}[h(t_2) - h(t_1)] \leqslant 2[h(t_2) - h(t_1)].$$

It follows that $h(t_2) \geqslant h(t_1)$. Hence h satisfies the $\mathcal{K}$-condition, and so h is extended to S. By Lemma 2.2 and Corollary 2.5, the extension is unique since $\operatorname{Sup} A(a) = \operatorname{Inf} B(a)$ for each $a \in S$. □

3. Boundedness condition. In Lemma 2.3, we see that the set A is bounded. In light of this, we will introduce the boundedness condition ($\mathcal{B}$-condition). In this section, we assume that S is a commutative cancellative semigroup and let $P = S\backslash F$ where P is a prime ideal, $P \neq \emptyset$, and F is a filter [3], $F \neq \emptyset$. Let $a \in P$. The subsemigroup of S generated by a and F is denoted by $\mathbf{P}_F(a)$ or $\mathbf{P}(a)$ if F is fixed. We define the relation ρ on P as follows: $c \,\rho\, d$ if and only if $m \cdot c + s = n \cdot d + t$ for some $s, t \in F$ and some $m, n \in Z_+$. Then ρ is an equivalence relation on P and each ρ-class is a subsemigroup of P; i.e., ρ has the following properties:

(3.1.1) $x \rho y$ implies $x \rho x + y$ for all $x, y \in P$.

(3.1.2) $x \rho m \cdot x + t$ for all $t \in F$ and all $m \in Z_+$.

Let $Q(a)$ denote the ρ-class containing $a \in P$. Let $\mathrm{U}_F(a)$ or $\mathrm{U}(a) = [Q(a), F]$, i.e., the subsemigroup of S generated by $Q(a)$ and F. By (3.1.1) and (3.1.2), we see $Q(a) + F \subseteq Q(a)$. If $s, t \in F$ and $b \in Q(a)$, then $k \cdot b + s = l \cdot b + t$, $(k, l \in Z_+^0)$, implies $k = l$ and $s = t$. In fact, if $k > l$, $(k - l) \cdot b + s = t$ by cancellation, but this is impossible since F is a filter and $b \in P$. Hence $k \leqslant l$. Likewise $k \geqslant l$. Therefore, $k = l$, and hence $s = t$ by cancellation. Thus we have

(3.2) Each element of $Q(a) + F$ has a unique expression as the sum of an element of $Q(a)$ and an element of F.

As defined in §1, $\overline{X}$ denotes the unitary closure of X and $\widetilde{X}$ denotes the filter closure of X.

LEMMA 3.3.

$$\text{(3.3.1)} \qquad \mathrm{U}(a) = \{x \in S\colon\ m \cdot x \in \overline{\mathrm{P}(a)} \textit{ for some } m \in Z_+\}$$

and

$$\text{(3.3.2)} \qquad \overline{\mathrm{P}(a)} \subseteq \mathrm{U}(a) = \overline{\mathrm{U}(a)} \subseteq \widetilde{\mathrm{P}(a)} = \widetilde{\mathrm{U}(a)}.$$

PROOF. (3.3.1) If $x \in F$, then $x \in \mathrm{P}(a) \subseteq \overline{\mathrm{P}(a)}$. If $x \in Q(a)$, then $m \cdot x + s = n \cdot a + t$ for some $s, t \in F$, some $m, n \in Z_+$; hence $m \cdot x \in \overline{\mathrm{P}(a)}$. Therefore U(a) is contained in the set at the right-hand side. To prove the other direction, let $m \cdot x \in \overline{\mathrm{P}(a)}$. By definition, $n \cdot a + s + m \cdot x = l \cdot a + t$ for some $s, t \in F$, and some $n, l \in Z_+^0$. Suppose $n > l$. Then $a + z = t$ for some $z \in S$. This contradicts $a \in P$. Hence $n \leqslant l$. If $n = l$, then $x \in F$. If $n < l$, then $m \cdot x + s = (l - n) \cdot a + t$ which implies $x \in Q(a)$, hence $x \in \mathrm{U}(a)$. Thus we have (3.3.1).

(3.3.2) It immediately follows from (3.3.1) and the definition that $\overline{\mathrm{P}(a)} \subseteq \mathrm{U}(a) \subseteq \widetilde{\mathrm{P}(a)}$. Taking their filter closures, we get $\widetilde{\mathrm{P}(a)} = \widetilde{\mathrm{U}(a)}$ by (1.2.3). It remains to show $\overline{\mathrm{U}(a)} \subseteq \mathrm{U}(a)$. Let $x \in \overline{\mathrm{U}(a)}$. Then $b = c + x$ for some $b, c \in \mathrm{U}(a)$. By (3.3.1), we can choose $m \in Z_+$ such that $m \cdot b, m \cdot c \in \overline{\mathrm{P}(a)}$. Since $m \cdot b = m \cdot c + m \cdot x$ and $\overline{\mathrm{P}(a)}$ is unitary by (1.2.2), we see that $m \cdot x \in \overline{\mathrm{P}(a)}$. So $x \in \mathrm{U}(a)$. Therefore $\overline{\mathrm{U}(a)} \subseteq \mathrm{U}(a)$. This completes the proof. □

Let T be a subsemigroup of a commutative cancellative semigroup S and let $h \in \mathrm{Hom}\langle T, \mathbf{R}_+^0\rangle$. We say that $\langle T, h\rangle$ satisfies the $\mathcal{B}$-condition (boundedness condition) in S if, for each $a \in S$, there is an $M \in \mathbf{R}_+^0$ such that

$$(\mathcal{B}) \quad x, y \in T,\ m \in Z_+^0 \text{ and } y + m \cdot a \in x + S \text{ implies } h(x) - h(y) \leqslant m \cdot M.$$

Here M is required to be independent of x, y and m. The notation $0 \cdot a + y$ expresses y itself, and hence the $\mathcal{B}$-condition implies the $\mathcal{K}$-condition. The $\mathcal{B}$-condition is equivalent to the combination of the $\mathcal{K}$-condition and the following:

($\mathcal{B}'$) For each $a \in S$, the set $\{m^{-1}[h(x) - h(y)]: \ x, y \in T, m \in Z_+, y + m \cdot a \in x + S\}$ is bounded.

LEMMA 3.4. *The following are equivalent*:

(3.4.1) *$\langle T, h\rangle$ satisfies the $\mathcal{K}$-condition in S.*

(3.4.2) *h is extended to $\bar{h} \in$* $\mathrm{Hom}(\bar{T}, \mathbf{R}_+^0)$.

(3.4.3) *h is extended to $\tilde{h} \in$* $\mathrm{Hom}(\tilde{T}, \mathbf{R}_+^0)$.

PROOF. (3.4.1) ⇒ (3.4.2). This follows from Theorem 2.1 since T is cofinal in $\bar{T}$.

(3.4.2) ⇒ (3.4.3). Since $\bar{T}$ is unitary cofinal in $\tilde{T}$ by (1.2.4), $\bar{h}$ can be extended to $\tilde{h} \in \mathrm{Hom}(\tilde{T}, \mathbf{R}_+^0)$ by Corollary 2.6, and hence h is extended to $\tilde{T}$.

(3.4.3) ⇒ (3.4.1). This is obvious from the definition of $\tilde{T}$. □

LEMMA 3.5. *Let T be a filter of S, $T \neq S$, and let $h \in$* $\mathrm{Hom}(T, \mathbf{R}_+^0)$. *Let $a \in S \backslash T$ and $r \in \mathbf{R}_+^0$. Define h_r: $\mathbf{P}_T(a) \longrightarrow \mathbf{R}_+^0$ by*

$$h_r(x) = \begin{cases} m \cdot r + h(s) & \text{if } x = m \cdot a + s \text{ where } m \in Z_+^0, s \in T, \\ m \cdot r & \text{if } x = m \cdot a \text{ where } m \in Z_+. \end{cases}$$

Every extension of h to $\mathbf{P}_T(a)$ is obtained as h_r for some $r \in \mathbf{R}_+^0$.

PROOF. Since the expression of x is unique, h_r is well defined. The proof of the lemma is easy. □

THEOREM 3.6. *Let T be a filter of a commutative cancellative semigroup S, $T \neq S$, and let $h \in$* $\mathrm{Hom}(T, \mathbf{R}_+^0)$. *Then the following are equivalent*:

(3.6.1) *$\langle T, h\rangle$ satisfies the $\mathcal{B}$-condition.*

(3.6.2) *h can be extended to $\widetilde{\mathbf{U}_T(a)}$ for each $a \in S\backslash T$.*

(3.6.3) *For each $a \in S\backslash T$, $\langle \mathbf{P}_T(a), h_r\rangle$ satisfies the $\mathcal{K}$-condition in S for some $r \in \mathbf{R}_+$.*

PROOF. (3.6.1) ⇒ (3.6.2). Choose $r \in \mathbf{R}_+$ such that

$$r \geqslant \mathrm{Sup}\,\{m^{-1}[h(x) - h(y)]: \ y + m \cdot a \in x + S\}$$

and then define $\bar{\bar{h}}$: $\mathbf{U}_T(a) \longrightarrow \mathbf{R}_+^0$ by

$$\bar{\bar{h}}(b) = \begin{cases} h(b) & \text{if } b \in T, \\ \dfrac{m \cdot r + h(s) - h(t)}{n} & \text{if } b \in Q(a) \text{ and } n \cdot b + t = m \cdot a + s \\ & \text{for some } s,\ t \in T. \end{cases}$$

By the choice of r, $\bar{\bar{h}}(b) \geq 0$ for all $b \in \mathbf{U}_T(a)$. To show $\bar{\bar{h}}$ is well defined, let $n \cdot b + t = m \cdot a + s$ and $n_1 \cdot b + t_1 = m_1 \cdot a + s_1$ where $s, t, s_1, t_1 \in T$. Then we have $(mn_1) \cdot a + n_1 \cdot s + n \cdot t_1 = (m_1 n) \cdot a + n \cdot s_1 + n_1 \cdot t$ which implies $mn_1 = m_1 n$ and $n_1 \cdot s + n \cdot t_1 = n \cdot s_1 + n_1 \cdot t$ since T is a filter. Then it follows that $\bar{\bar{h}}$ is well defined. Next we show that $\bar{\bar{h}}$ is a homomorphism. If $b, c \in Q(a)$, then $n \cdot b + t = m \cdot a + s$, $k \cdot c + u = l \cdot a + v$ for some $t, s, u, v \in T$, $n, m, k, l \in Z_+$; so $(nk) \cdot (b + c) + k \cdot t + n \cdot u = (mk + ln) \cdot a + k \cdot s + n \cdot v$ which implies $\bar{\bar{h}}(b) + \bar{\bar{h}}(c) = \bar{\bar{h}}(b + c)$. If $b \in Q(a)$ and $c \in T$, then $n \cdot b + t = m \cdot a + s$ and $n \cdot (b + c) + t = m \cdot a + s + n \cdot c$, so the same result follows, and we see $\bar{\bar{h}}(b) + \bar{\bar{h}}(c) = \bar{\bar{h}}(b + c)$ for all $b, c \in \mathbf{U}_T(a)$. Since $\mathbf{U}_T(a)$ is unitary cofinal in $\widetilde{\mathbf{U}_T(a)}$ by Lemma 3.3 and (1.2.4), $\bar{\bar{h}}$ can be extended to $\tilde{h} \in \operatorname{Hom}(\widetilde{\mathbf{U}_T(a)}, \mathbf{R}_+^0)$ by Corollary 2.6.

(3.6.2) ⇒ (3.6.1). Let $x, y \in T$ and assume $b + x = m \cdot a + y$ for some $b \in S$. Hence $b \in Q(a)$. By assumption, h is extended to $\tilde{h} \in \operatorname{Hom}(\widetilde{\mathbf{U}_T(a)}, \mathbf{R}_+^0)$, and $m \cdot \tilde{h}(a) + h(y) - h(x) = \tilde{h}(b) \geq 0$ which implies the conclusion.

(3.6.2) ⇒ (3.6.3). By Lemma 3.3, $\widetilde{\mathbf{U}_T(a)} = \widetilde{\mathbf{P}_T(a)}$. Let $\tilde{h}$ be the extension of h to $\widetilde{\mathbf{U}_T(a)}$. Then $\tilde{h} | \mathbf{P}_T(a) = h_r$ for some $r \in \mathbf{R}_+^0$ by Lemma 3.5. By Lemma 3.4, $\langle P_T(a), h_r \rangle$ satisfies the K-condition.

(3.6.3) ⇒ (3.6.2). Again use Lemma 3.3 and Lemma 3.4. □

4. **Examples.** Examples 4.1, 4.2 and 4.3 show that the K-condition does not imply the B-condition; Theorem 2.1 is not true in general if T_0 is not cofinal.

EXAMPLE 4.1. A commutative cancellative idempotent-free semigroup S is defined by

$$S = \{(x, y)\colon\ y \in Z_+ \text{ if } x = 0, y \in Z \text{ if } x \in Z_+\}$$

in which the operation is

$$(x, y) + (z, u) = (x + z, y + u).$$

Let $T_0 = \{(0, y)\colon\ y \in Z_+\}$. T_0 is not cofinal in S. Define $h_0 \in \operatorname{Hom}(T_0, \mathbf{R}_+^0)$ by $h_0(0, y) = y$. Suppose h_0 is extended to $h \in \operatorname{Hom}(S, \mathbf{R}_+^0)$. Let $x_0 \in Z_+$ be fixed and let $\lambda = h(x_0, 0)$, $\lambda \in \mathbf{R}_+^0$. Choose $y \in Z_+$ such that $y > \lambda$. Then $h(x_0, -y) + h(0, y) = h(x_0, 0) = \lambda$, hence $h(x_0, -y) = \lambda - y \geq 0$. This is a contradiction. Therefore h_0 cannot be extended to any element of $\operatorname{Hom}(S, \mathbf{R}_+^0)$.

Let $(b, c) \in S\backslash T_0$ be fixed. Take arbitrarily $x, y, m \in Z_+$. Let $p = mb$, and $q = mc + y - x$. Then $p \in Z_+$, $q \in Z$ and $(0, x) + (p, q) = m \cdot (b, c) + (0, y)$. Since $(x - y)/m = (h_0(0, x) - h_0(0, y))/m$ can be arbitrarily large, $\langle T_0, h_0\rangle$ does not satisfy the $\mathcal{B}$-condition. Since T_0 is a filter of S, the $\mathcal{K}$-condition is satisfied by $\langle T_0, h_0\rangle$.

EXAMPLE 4.2. Let $S = \{(x, y): x \in Z_+^0, y \in Z, y \geqslant 1 - x^2\}$ and define T_0 and h_0 by

$$T_0 = \{(0, y): y \in Z_+\}, \qquad h_0(0, y) = y.$$

T_0 is not cofinal but is a filter in S. Suppose that h_0 can be extended to a homomorphism h of S into $\mathbf{R}_+$. For each $n \in Z_+^0$, let

$$\varphi(n) = h_0(n, 1 - n^2).$$

Then $\varphi(1) + \varphi(n - 1) - \varphi(n) = h_0(0, 2n - 1) = 2n - 1$ for each $n \in Z_+$. From this recurrence relation, we have

$$n\varphi(1) + \varphi(0) - \varphi(n) = \sum_{i=1}^{n} (2i - 1) = n(n + 1) - n = n^2.$$

Since $\varphi(0) = h(0, 1) = 1$, it follows that $n\varphi(1) - \varphi(n) = n^2 - 1$. By the assumption $\varphi(n) \geqslant 0$ for all $n \in Z_+$, we have

$$n\varphi(1) \geqslant n^2 - 1 \quad \text{for all } n \in Z_+$$

hence $\varphi(1) \geqslant n - 1/n$ for all $n \in Z_+$. This is impossible. It follows that h_0 cannot be extended to an element of $\operatorname{Hom}(S, \mathbf{R}_+^0)$. We show that the $\mathcal{B}$-condition is not satisfied. Let $m \in Z_+$, $m > 1$, and choose $y, z \in Z_+$ such that $z - y = m^2 - 1$. Then

$$(0, z) + (m, 1 - m^2) = m \cdot (1, 0) + (0, y)$$

but $(z - y)/m = m - 1/m$ can be taken arbitrarily large.

EXAMPLE 4.3. Let π be the transcendental real number and let $a = \pi/4$. Then $0 < \pi/4 < 1$, and a is transcendental over the field R of rational numbers. If $a_k = a^k$ $(k = 1, 2, \ldots)$ (a^k is the usual kth power of a), then $1, a_1, a_2, \ldots$ are linearly independent over R and $0 < a_k < 1$ $(k = 1, 2, \ldots)$. Let T_0 be the additive semigroup of $\mathbf{R}_+$ generated by $a_1, \ldots, a_k, \ldots$. T_0 is actually a free commutative semigroup over $a_1, \ldots, a_k, \ldots$. Let $b_k = 1 - a_k > 0$ $(k = 1, 2, \ldots)$, and let S be the subsemigroup of $\mathbf{R}_+$ generated by T and $b_1, b_2, \ldots, b_k \ldots$. T_0 is a filter of S. Define $h_0: T_0 \longrightarrow \mathbf{R}_+$ by the homomorphism given by $h_0(a_k) = k$. Since T_0 is free, h_0 is well defined. Then h_0 cannot be extended to $h \in \operatorname{Hom}(S, \mathbf{R}_+^0)$. For suppose h_0 is extended to h. Then $1 = a_k + b_k \in S$, which implies $h(1) = h(a_k + b_k) \geqslant h(a_k) = k$ for all k. This is a contradiction. Finally we show that $\langle T_0, h_0\rangle$ does not satisfy the $\mathcal{B}$-condition. Let $m \in Z_+$ and $a_k \in T$. As $a_k|1$, $a_k|m$ and so $a_k|(m + a_i)$ in S for all $m \in Z_+$, all a_i, $a_k \in T$. Then

$$(h_0(a_k) - h_0(a_i))/m = (k - i)/m$$

is not bounded.

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ADDENDUM. The assumption of cancellation does not restrict our discussion by the following reason. Let S be a commutative semigroup, S_0 the greatest cancellative homomorphic image of S, and g_0: $S \longrightarrow S_0$ the homomorphism. If f is a homomorphism of S_0 into $\mathbf{R}_+^0$, then $h = fg_0$ is a homomorphism of S into $\mathbf{R}_+^0$. Every homomorphism h of S into $\mathbf{R}_+^0$ can be obtained in this manner. Accordingly the results in this paper are extended to the case in which cancellation is not assumed.

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