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Positivity

Homomorphisms on Lattices of Continuous Functions

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1. Introduction

This paper deals with lattices of continuous functions and their homomorphisms, with emphasis on isomorphisms.

As usual, we write $C(X)$ for the lattice of all real-valued continuous functions on a topological space X with the order induced by that of $\mathbb{R}$, that is, $f \leq g$ meaning $f(x) \leq g(x)$ for all $x \in X$. The sublattice of bounded functions is denoted $C^*(X)$.

Until further notice X and Y will denote compact Hausdorff spaces. Suppose we are given an isomorphism $T : C(Y) \to C(X)$, that is, bijection satisfying $T(f \vee g) = Tf \vee Tg$ and (this is equivalent for bijections) $T(f \wedge g) = Tf \wedge Tg$. What can be said about T? In particular, how to represent it? We emphasize that T is not assumed to be linear.

As far as I know, these problems were first considered by Kaplansky in his venerable oldies [16] and [17]. In the former he showed that if the lattices $C(Y)$ and $C(X)$ are isomorphic, then X and Y are homeomorphic. The proof is of Stonian style and proceeds by duality (the points of X are identified as equivalence classes of prime ideals of $C(X)$, a prime ideal being the kernel of some homomorphism onto the lattice $\{0, 1\}$ and so on; see also [2, pp. 227–228] and [23, pp. 129–130]). The papers [14, 24] contain extensions to noncompact spaces.

The sequel [17] studies continuity properties of isomorphisms. For instance, it is proved that, referring to the usual sup norm topology, T is continuous if and only if it admits a representation as

$$Tf(x) = t(x, f(\tau(x))) \qquad (f \in C(Y), x \in X),$$

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where $\tau : X \to Y$ is a homeomorphism and $t : X \times \mathbb{R} \to \mathbb{R}$ is a continuous function (necessarily given by $t(x, c) = Tc(x)$). Moreover it is shown that, if X (and so Y) is metrizable, then every isomorphism between $C(Y)$ and $C(X)$ is continuous [17, p. 633]. Later on, Cater proved the same for X either sequentially compact or locally connected [6] – despite the criticism of the reviewer in [1].

In any case some restriction is necessary to get automatic continuity, as shown by Kaplansky himself in [17, p. 629]. Let us recall that if S is a completely regular space there is a compact space βS (its Stone-Čech compactification) containing it as a dense subspace in such a way that the Banach algebra $C^*(S)$ is isometrically isomorphic to $C(\beta S)$.

Let $\ell^\infty(\Gamma)$ be the space of all bounded functions $f : \Gamma \to \mathbb{R}$. By the preceding remark $\ell^\infty(\Gamma) = C(\beta\Gamma)$, where Γ is treated as a discrete set. Let $t : \Gamma \times \mathbb{R} \to \mathbb{R}$ be any function satisfying:

- For each $\gamma \in \Gamma$, the function $t(\gamma, \cdot)$ is an automorphism of $\mathbb{R}$ (in the 'default' lattice setting, that is, an increasing homeomorphism).
- ◦ A function $f : \Gamma \to \mathbb{R}$ is bounded if and only if so is $t(\cdot, f(\cdot))$.

(Such a function will be called **admissible** in all what follows.) Then we can define an automorphism of $\ell^\infty(\Gamma)$ taking

$$Tf(\gamma) = t(\gamma, f(\gamma)) \qquad (f \in \ell^\infty(\Gamma), \gamma \in \Gamma).$$

However, such a T is continuous if and only if the family $\{t(\gamma, \cdot) : \gamma \in \Gamma\}$ is equicontinuous on compact subsets of $\mathbb{R}$.

To be more concrete, following Ercan and Önal [9], let us define $\ell : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ by

$$\ell(x, c) = \begin{cases} c^x & \text{for } 0 \le c \le 1 \\ c & \text{otherwise.} \end{cases}$$

If f is a function defined on some subset of $(0, \infty)$ we put $Lf(x) = \ell(x, f(x))$, which has the same domain as f. If we consider $x \in \mathbb{N}$, then L becomes an automorphism of $\ell^\infty = C(\beta\mathbb{N})$. Considering $x \in (0, \infty)$ and $f \in C^*(0, \infty)$ we obtain an automorphism of $C^*(0, \infty) = C(\beta(0, \infty))$. Allowing f to be measurable and essentially bounded we extend L to an automorphism of $L^\infty(\lambda)$, where λ denotes Lebesgue measure on the Borel sets of the interval $(0, \infty)$. In any case L is discontinuous at $f = 1$ because $\|Lc - L1\|_\infty = 1$ for every $0 < c < 1$.

The presence of the Stone-Čech compactification in these examples is not accidental: actually Itô proved in [15, proof of Theorem 2] that if $C(X)$ admits a discontinuous automorphism, then X has a subset homeomorphic with $\beta\mathbb{N}$. Another remarkable result in [15, Theorem 1] states that $C(X)$ admits a continuous automorphism whose inverse is discontinuous if and only if $X = \beta S$ for some open F_σ proper subset $S \subset X$.

The plan of the paper is as follows. Section 2 contains generalities on homomorphisms. We give a simple proof that each homomorphism $T : C(Y) \to C(X)$ gives rise to a continuous mapping $\tau : X_T \to Y$, where X_T is an open subset of X depending on T, in such a way that

$$t(x, f(\tau(x))^-) \le Tf(x) \le t(x, f(\tau(x))^+) \qquad (f \in C(Y), x \in X_T),$$

with $t(x,c) = Tc(x)$. We keep this notation in all what follows.

Section 3 focuses on isomorphisms. In this case τ is a homeomorphism and the mappings τ and t completely determine T. This is shown through an explicit representation of T. Moreover, there is a dense G_δ subset of X where $Tf(x) = t(x, f(\tau(x)))$ holds for all $f \in C(Y)$. Also we give new examples of compact spaces X for which automorphisms of $C(X)$ are automatically continuous and we describe the automorphisms of $C^*(S) = C^*(\beta S)$ when S is a metric space, thus showing that the above example by Ercan and Önal may be regarded as a typical one. As an application, in Sect. 4 we give a nonlinear version of a recent result by Garrido and Jaramillo on uniformly continuous functions.

Section 5 is devoted to the 'von Neumann' algebra $L^\infty(\mu)$ – incidentally, this was the starting point of the research reported here. We obtain a very precise description of the automorphisms, at least when μ is a 'standard' measure. Sect. 6 describes the automorphisms of many function lattices appearing in analysis. This closing Section may be regarded as a small nonlinear complement to Chapter XV of [2].

Notations. Given sets A and B we write B^A for the set of all mappings from A to B, while B_A denotes the set of constant maps in B^A. In particular, $\mathbb{R}_A$ are the constant functions on A. Every $a \in A$ gives rise to an 'evaluation' map $\delta_a : B^A \to B$ (defined as $\delta_a(f) = f(a)$). Also, given a map $\tau : A \to B$ and a set C we write $\tau^* : C^B \to C^A$ for the composition on the right with τ, that is, $\tau^*(f) = f \circ \tau$. Similarly, $\tau_* : A^C \to B^C$ denotes composition on the left: $\tau_*(f) = \tau \circ f$. Finally, the identity on A is denoted by $\mathbf{1}_A$ while the characteristic function of A is 1_A.

2. Lattice Homomorphisms

In this Section we obtain some general results on homomorphisms between the lattices of continuous functions on compacta. Our results are closely related to those of Kaplansky [16, 17], Shirota [24] and Cater [6]. We remark, however, that these papers deal only with isomorphisms, but between more general sublattices of $C(X)$.

Let $T : C(Y) \to C(X)$ be a lattice homomorphism. We define $t : X \times \mathbb{R} \to \mathbb{R}$ by $t(x,c) = Tc(x)$. Two obvious properties of t are:

- For every real c the function $t(\cdot, c)$ is continuous.
- For fixed $x \in X$ the function $t(x, \cdot)$ is nondecreasing.

For each $x \in X$, set $O(x) = \{Tf(x) : f \in C(Y)\}$ and consider the (obviously open) set $X_T = \{x \in X : O(x) \text{ is not a single point}\}$. It is easily seen that

$$X \backslash X_T = \{x \in X : t(x,c) = t(x,d) \text{ for all } c, d \in \mathbb{R}\}.$$

Lemma 1. *With the above notations, given $x \in X_T$ there is a unique $y \in Y$ such that, for every $c \in \mathbb{R}$, $f(y) > c$ implies $Tf(x) \geq t(x,c)$ and $f(y) < c$ implies $Tf(x) \leq t(x,c)$.*

Proof. Fix $x \in X$ and $c \in \mathbb{R}$. We claim that there is some $y \in Y$ (possibly depending on c) such that $f(y) > c$ implies $Tf(x) \geq t(x,c)$. Otherwise, for every $y \in Y$ there is $f \in C(Y)$ such that $f(y) > c$ but $Tf(x) < t(x,c)$. By compactness there is a finite system $f_1, \dots, f_k$, with $f_1 \vee \dots \vee f_k \geq c$ while $Tf_i(x) < t(x,c)$. A contradiction.

Next we show that y is actually independent on c. Thus suppose there are c' and $y' \neq y$ such that $f(y') > c'$ implies $Tf(x) \geq t(x,c')$. There is no loss of generality in assuming $c' \geq c$. Then, given $d \in \mathbb{R}$ we may choose $f, f' \in C(Y)$ such that$f(y) > c$ and $f'(y') > c'$, but $f \wedge f' \leq d$. It follows that

$$t(x,d) = Td(x) \geq T(f \wedge f')(x) = Tf(x) \wedge Tf'(x) \geq t(x,c),$$

which is impossible since $x \in X_T$. The above argument shows that y is unique: just take $c' = c$. This establishes the first implication.

By symmetry, there is $y' \in Y$ such that, for all c, $f(y') < c$ implies $Tf(x) \leq t(x,c)$. Let us show that $y' = y$. Take $c < c'$ such that $t(x,c) < t(x,c')$. Now, if $y' \neq y$, there is $f \in C(Y)$ with $f(y) < c$ and $f(y') > c'$ and so $Tf(x) \leq t(x,c)$ and $Tf(x) \geq t(x,c')$, which is impossible. $\square$

Lemma 2. *With the notations of Lemma 1, the map $\tau : X_T \to Y$ given by $\tau(x) = y$ is continuous.*

Proof. Suppose on the contrary there is a net (x_α) converging to $x \in X$ such that $\tau(x_\alpha)$ does not converge to $\tau(x)$. As Y is compact, passing to a subnet if necessary, we may assume there is a neighbourhood U of $\tau(x)$ such that $\tau(x_\alpha) \notin U$ for all α.

Fix $c, d \in \mathbb{R}$ and write $y = \tau(x), y_\alpha = \tau(x_\alpha)$. Then there is $f \in C(Y)$ such that $f(y) < c$ and $f(y_\alpha) > d$ for all α, as Y is completely regular. We have $Tf(x) \leq t(x,c)$ and $Tf(x_\alpha) \geq t(x_\alpha, d)$ for all α. By continuity we see that $Tf(x) \geq t(x,d)$ and so $t(x,d) \leq t(x,c)$. By symmetry we have $t(x,d) = t(x,c)$. $\square$

All this can be restated as follows:

Proposition 1. *Let $T : C(Y) \to C(X)$ be a homomorphism. Then there is a continuous mapping $\tau : X_T \to Y$ such that*

$$t(x, f(\tau(x))^-) \leq Tf(x) \leq t(x, f(\tau(x))^+) \qquad (f \in C(Y), x \in X_T),$$

where $t(x,c) = Tc(x)$ and $X_T = \{x \in X : t(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is not constant$\}$. If $x \notin X_T$, then $Tf(x)$ is the only value attained by $t(x,\cdot)$. $\square$

The above τ shall be referred to as the map **associated** to (or induced by) T.

We pause for one easy application. The following result shows that, in many respects, the behaviour of a homomorphism depends only on its action on the constant functions.

Corollary 1. *Let $T : C(Y) \to C(X)$ be a homomorphism.*

- *If T continuous on $\mathbb{R}_Y$ for the topology of pointwise convergence of $C(X)$, then $Tf(x) = t(x, f(\tau(x)))$ for all $f \in C(Y)$ and all $x \in X_T$.*
- *If T is linear on $\mathbb{R}_Y$, then it is a homomorphism of vector lattices and so $Tf = \omega \cdot \tau^*(f)$, where $\omega = T1$.*
- *If T preserves constants (that is, $Tc = c$ for every real c), then it is an algebra homomorphism and so $T = \tau^*$.* □

The third part of the preceding Corollary was proved long time ago by Mena and Roth in [21]. See [11] for a generalization. I take here the opportunity of thank the referee for this information.

Let us consider now the scalar-valued case $T : C(Y) \to \mathbb{R}$. This amounts to treat $\mathbb{R}$ as the lattice of continuous functions on a single point.

First, note that if T is not constant, it is **associated** to a unique point $y \in Y$ in the sense that, if we define $t(c) = Tc$, we have

$$t(f(y)^-) \leq Tf \leq t(f(y)^+) \qquad (f \in C(Y)).$$

This implies that, to some extent, $\delta_y f$ 'controls' Tf. Even if the exact value of T at f cannot be deduced from the value of f at y (see the example below), T has a local behaviour:

Proposition 2. *Let the homomorphism $T : C(Y) \to \mathbb{R}$ be associated to the point $y \in Y$. If f and g agree on a neighbourhood of y, then $Tf = Tg$.*

Proof. Replacing f and g by $f \wedge g$ and $f \vee g$ if necessary, we may assume $f \leq g$. It is clear from Lemma 1 that $Th \leq Tf$ provided $h(y) < f(y)$. Let A be an open neighborhood of y where f and g agree, and let $h' \geq 0$ be a continuous function vanishing outside A and such that $h'(y) = 1$. Take $h = g - h'$. We have $Th \leq Tf$. But $g = h \vee f$ and so $Tg = Th \vee Tf = Tf$. □

This can be used to improve [16, Lemma 6], assuming the underlying compact space to be R-separated.

Corollary 2. *Let $T : C(Y) \to C(X)$ be a homomorphism.*

(a) *If T is onto, then $X_T = X$ and $\tau : X \to Y$ embeds X as a closed subspace of Y.*

(b) *If T is injective, then $\tau(X_T)$ is dense in Y. If besides this $X_T = X$, then τ is onto.*

Proof. Part (a) clearly follows from Proposition 1.

As for (b), suppose there is $y \in Y$ not in the closure of $\tau(X_T)$. Then there is $f \in C(Y)$ vanishing on some open set containing $\tau(X_T)$ with $f(y) = 1$. We show that $Tf = T0$. Of course, if $x \notin X_T$, then $Tf(x) = T0(x)$. If $x \in X_T$, then $Tf(x)$ is the value of the homomorphism $T^*\delta_x : C(Y) \to \mathbb{R}$ at f. Obviously $T^*\delta_x$ is associated to $\tau(x)$ and f and 0 agree on a neighbourhood of $\tau(x)$. Hence $Tf(x) = T0(x)$ and T is not injective □

Notice that (b) implies that if T is injective and $f, g \in C(Y)$ agree on some nonempty open $A \subset Y$, then Tf and Tg agree on $\tau^{-1}(A)$, which is a nonempty open subset of X.

We close the Section with the following.

Example 1. *A surjective homomorphism $T : C[0,1] \to \mathbb{R}$ associated with the point 0 and such that for each $c \in [-1,1]$ there is f such that $f(0) = 0$ and $Tf = c$.*

Proof. Let (t_n) be a sequence converging to 0 in $[0,1]$, with $t_n \neq 0$ for all n. Let U be a free ultrafilter on $\mathbb{N}$ and define $\gamma : C[0,1] \to [-\infty,\infty]$ by

$$\gamma(f) = \lim_{U(n)} nf(t_n).$$

Let $s : [-\infty,\infty] \to [-1,1]$ be any increasing homeomorphism and define

$$Tf = \begin{cases} f(0) - 1 & \text{if } f(0) < 0 \\ s(\gamma(f)) & \text{if } f(0) = 0 \\ f(0) + 1 & \text{if } f(0) > 0 \end{cases}$$

It is easily verified that T fulfills the required properties. □

A related example appears in [16, p. 620].

3. More on Isomorphisms

In this Section $T : C(Y) \to C(X)$ is assumed to be a lattice isomorphism, unless otherwise stated. The maps $\tau : X \to Y$ and $t : X \times \mathbb{R} \to \mathbb{R}$ have the same meaning as in Sect. 2. The following two results contain the main structural properties we can prove for a general automorphism. Proposition 3 is due to Kaplansky. It clearly follows from Corollary 2. However, we derive it straight from Lemmas 1 and 2.

The first part of Theorem 1 shows that an isomorphism depends only on the associated maps t and τ – which is not true for mere homomorphisms! Although this follows from abstract considerations, we prefer to make it explicit.

Part (b) proves that isomorphisms cannot be 'very pathological' in the sense that the set of those $x \in X$ where the functions $t(x, \cdot)$ are automorphisms of $\mathbb{R}$ is very large.

Proposition 3 (Kaplansky). *If $T : C(Y) \to C(X)$ is an isomorphism then $\tau : X \to Y$ is a homeomorphism.*

Proof. Let us show that the continuous mapping $Y \to X$ associated to T^{-1} is an inverse for τ. Define $s : Y \times \mathbb{R} \to \mathbb{R}$ by $s(y,c) = T^{-1}c(y)$. Fix $x \in X$ and let $y = \tau(x)$. Applying Lemma 1 to T^{-1} and y we get $x' \in X$ such that:

- $Tf(x) > t(x,c)$ implies $f(y) \geq c$,
- $Tf(x') < d$ implies $f(y) \leq s(y,d)$,

for every c, d and f. Note that the first implication here is just a rewording of the second one in Lemma 1. It only remains to see that $x' = x$. For if not, take $c = 1$ and let d so that $s(y, d) \leq 0$. Take f such that $Tf(x) > t(x, 0)$ and $Tf(x') < d$. Then $f(y)$ must be greater than 1 and smaller than 0. A contradiction. $\square$

Note that if $\tau : X \to Y$ is the homeomorphism induced by T, then the composition $T' = T \circ (\tau^{-1})^*$ is an automorphism of $C(X)$ whose associated homeomorphism is the identity on X. Moreover T' and T share any property preserved by composition with a linear isomorphism of algebras. This will be used to simplify some proofs.

Theorem 1. *Let $T : C(Y) \to C(X)$ be an isomorphism. Then:*

(a) *For every $f \in C(Y)$ and all $x \in X$ one has*

$$\begin{aligned} Tf(x) &= t(x, f(\tau(x))^+) \bigwedge \liminf_{z\to x} t(z, f(\tau(z))^+) \\ &= t(x, f(\tau(x))^-) \bigvee \limsup_{z\to x} t(z, f(\tau(z))^-). \end{aligned}$$

(b) *There is a G_δ dense subset $D \subset X$ such that, for every $x \in D$, the function $t(x, \cdot)$ is an automorphism of $\mathbb{R}$ and, in particular, one has:*

$$Tf(x) = t(x, f(\tau(x))) \qquad (f \in C(Y), x \in D).$$

Proof. We may assume $Y = X$ and $\tau = \mathbf{1}_X$. To prove the first part, we begin with the observation that, given arbitrary bounded functions a and c on X, some $b \in C(X)$ such that $a \leq b \leq c$ exists if and only if $\operatorname{upper} a \leq \operatorname{lower} c$, where $\operatorname{upper} a$, the upper regularization of a, defined by

$$\operatorname{upper} a(x) = a(x) \bigvee \limsup_{z\to x} a(z)$$

is the greatest upper semicontinuous function dominated by a, while $\operatorname{lower} c$, the lower regularization of c, is defined as

$$\operatorname{lower} c(x) = c(x) \bigwedge \liminf_{z\to x} c(z)$$

and is the smallest lower semicontinuous function dominating c. This follows from Hahn-Tong theorem; see Theorem 6.4.4 on p. 100 and the exercise 6.4.10 (D) on p. 102 in [23]. A moment's reflection suffices to realize that the 'interpolating' function b is unique if and only if $\operatorname{upper} a = \operatorname{lower} c$. The proof will be complete if we show that Tf is the unique continuous function between T_-f and T_+f, where $T_\pm f(x) = t(x, f(x)^\pm)$. Suppose $T_-f \leq g \leq T_+f$, with $g \in C(X)$. By Proposition 1 one has

$$T(f - \delta) \leq T_-f \leq g \leq T_+f \leq T(f + \delta) \qquad (\delta > 0.)$$

But any lattice isomorphism preserves arbitrary 'joins' and 'meets' and so

$$Tf = \bigvee_{\delta>0} T(f - \delta) \leq g \leq \bigwedge_{\delta>0} T(f + \delta) = Tf.$$

To prove (b), given $p, q \in \mathbb{Q}$, with $p < q$, consider the set $N(p,q) = \{x \in X : t(x,p) = t(x,q)\}$. By the remark after Corollary 2 this is a closed set with empty interior and so

$$N = \bigcup_{p<q} N(p,q) \qquad (p, q \in \mathbb{Q})$$

is of first category. Let D_1 be the complement of N in X. It is fairly obvious that $t(x,\cdot)$ is strictly increasing on $\mathbb{R}$ if and only if $x \in D_1$. Now define $s : X \times \mathbb{R} \to \mathbb{R}$ by $s(x,c) = T^{-1}c(x)$ and let D_2 the set of those x for which $s(x,\cdot)$ is stricty increasing on $\mathbb{R}$. It is clear that $D = D_1 \cap D_2$ is still a dense G_δ subset of X. Now, in view of Proposition 1, it suffices to show that $t(x,\cdot)$ is continuous (hence an automorphism of $\mathbb{R}$) for all $x \in D$.

Fix $x \in D$ and note that, given $c \in \mathbb{R}$, one has

$$s(x, t(x,c)) = c \tag{1}$$

provided $t(x,c)$ is a point of continuity of $s(x,\cdot)$. As $t(x,\cdot)$ is strictly increasing and $s(x,\cdot)$ is continuous outside a countable set we see that the set of those c where (1) holds is dense in $\mathbb{R}$, since the complement is at most countable. Thus the following result completes the proof. The momentary change of notation is quite natural. □

Lemma 3. *Let $t, s : \mathbb{R} \to \mathbb{R}$ be strictly increasing functions such that $s \circ t$ is the identity on some dense subset of the line. Then t is continuous.*

Proof. Put $S = \{c \in \mathbb{R} : s(t(c)) = c\}$. We show that for every real c one has $s(t(c^-)^-) = s(t(c^+)^+)$. Fix c and take sequences (p_n) and (q_n) in S such that $p_n \to c^-$ and $q_n \to c^+$. Then $t(p_n) \to t(c^-)^-$ and $t(q_n) \to t(c^+)^+$ and so $s(t(c^-)^-) = s(t(c^+)^+) = c$. □

We hasten to recall that the representation in part (b) of Theorem 1 is possible for all $x \in X$ only if T is continuous. In the following Lemma we gather results by Kaplansky, Cater and Itô. The easy proof is included for the sake of clarity.

Lemma 4. *Let $T : C(Y) \to C(X)$ be an isomorphism. The following are equivalent:*

(a) *T is continuous on $\mathbb{R}_Y$ for the topology of pointwise convergence in $C(X)$.*
(b) *T is continuous in the topology of pointwise convergence.*
(c) *$Tf(x) = t(x, f(\tau(x))$ for all f and x.*
(d) *T^{-1} is increasing.*
(e) *T is continuous in the norm topology.*

Proof. The implications (a) ⇒ (b) ⇒ (c) and (e) ⇒ (a) are trivial.

We show (c) ⇒ (d). Of course (d) means that if $f(x) > g(x)$ for all x, then $T^{-1}f(y) > T^{-1}g(y)$. Writing $h' = T^{-1}h$ for $h = f, g$ this can be restated as: if $f'(y) \le g'(y)$ for some $y \in Y$, then $Tf'(x) \le Tg'(x)$ for some $x \in X$, which is clearly implied by (c).

The implication (d) ⇒ (e) is obvious since the sets:

$$\{h \in C(X) : f < h < g\} \qquad (f, g \in C(X))$$

form a base for the norm topology of $C(X)$. □

Corollary 3. *Suppose every (nonempty) zero set has nonempty interior in X. Then every automorphism of $C(X)$ is continuous.*

Proof. Corollary 2 (b) implies that automorphisms of $C(X)$ are increasing. $\square$

The most typical example where Corollary 3 applies is $\mathbb{N}^* = \beta\mathbb{N}\backslash\mathbb{N}$, the growth of the integers in its Stone-Čech compactification. It is well-known that $C(\mathbb{N}^*) = \ell_\infty/c_0$, as Banach algebras. Suppose $f \in C(\mathbb{N}^*)$ vanishes at some $x \in \mathbb{N}^*$ and let $f' \in C(\beta\mathbb{N})$ be any extension. Clearly, there is an infinite $M \subset \mathbb{N}$ such that $f'(n) \to 0$ as $n \to \infty$ in M. Hence $f' - 1_M f'$ is another extension of f vanishing on M and so f is zero on $\mathrm{cl}_{\beta\mathbb{N}} M \cap \mathbb{N}^*$ an open subset of $\mathbb{N}^*$.

Another important class of lattices whose automorphisms are automatically continuous are the (Banach space) ultraproducts of any countable family of $C(X)$. We will refrain from entering into further details here. All you need to get the continuity of lattice automorphims in ultraproducts out from Corollary 3 is in [4].

In the following result, we borrow an idea from Cater's [6, Proof of Theorem IV] to obtain a sharp description of the action of T at certain points of X, even if such description is impossible on the whole of X. The proof, however, develops an argument sketched by Kaplansky in [17, Note added in proof on p. 633].

Lemma 5. *Let $T : C(Y) \to C(X)$ be an isomorphism and let $x \in X$ be a point satisfying one of the following conditions:*

(a) *there is a sequence (x_n) converging to x, with $x_n \neq x$ for all n.*

(b) *X is locally connected at x.*

Then $t(x, \cdot)$ is an automorphism of $\mathbb{R}$ and $Tf(x) = t(x, f(\tau(x)))$ for all $f \in C(Y)$.

With the notations of Theorem 1, this means that some points must be in D irrespective of the isomorphism T (this should be compared to Example 2 below). The proof requires the construction of a function having certain oscillation property:

Lemma 6. *Let $x \in X$ satisfy one of the conditions in Lemma 5. Suppose $f, g \in C(X)$ agree at x, but $\{z \in X : f(z) = g(z)\}$ is not a neighborhood of x. Then there is $h \in C(X)$ such that $\{z : h(z) > f(z)\}$ and $\{z : h(z) < g(z)\}$ meet every neighbourhood of x.*

Proof of Lemma 6. In the first case, define $h : \{x_n\} \cup \{x\} \to \mathbb{R}$ taking $h(x) = f(x) = g(x), h(x_n) = f(x_n) + 1/n$ for even n and $h(x_n) = g(x_n) - 1/n$ for odd n, and extend by Tiezte. As for the second one, we may assume $f \geq g = 0$. Set $h(z) = 2f(z)\cos(1/f(z))$, with $h(z) = 0$ if $f(z) = 0$.

Proof of Lemma 5. We may assume $\tau = \mathbf{1}_X$. Suppose $f(x) = g(x)$. We must show that $Tf(x) = Tg(x)$. If f and g agree on a neighborhood of x, this follows from Proposition 2. Otherwise, take h as in Lemma 6. It follows from Proposition 1 that $\{z : Th(z) \geq Tf(z)\}$ and $\{z : Th(z) \leq Tg(z)\}$ meet every neighbourhood of x. Hence $Tf(x) \leq Th(x) \leq Tg(x)$ and, by symmetry, $Tf(x) = Tg(x)$.

Therefore $Tf(x) = t(x, f(x))$ and (by surjectivity) $t(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is onto whence continuous. To see that it is an automorphism, simply note that the function $s(x, \cdot) : \mathbb{R} \to \mathbb{R}$ given by $s(x, c) = T^{-1}c(x)$ is a continuous inverse for $t(x, \cdot)$.

Now, we present some examples in the light of the above result. As we mentioned in the Introduction, discontinuous automorphisms appear in connection with Stone-Čech compactifications. The following result describes the automorphisms of $C(\beta S)$ for metrizable S, showing that, to some extent, they have a simple behaviour.

Corollary 4. *Let S be metrizable space, τ a homeomorphism of S and $t : S \times \mathbb{R} \to \mathbb{R}$ a continuous admissible function. Then the map defined by*

$$Tf(x) = t(x, f(\tau(x))) \qquad (x \in S, f \in C^*(S)),$$

is an automorphism of $C^*(S)$. All automorphisms of $C^*(S)$ arise in this way.

Proof. Everything in the first part is obvious but that T is onto. Since $C^*(S) = C(\beta S)$ and the G_δ points in βS are exactly those of S everything in the second part follows from Lemma 5, but the continuity of t. The following remark ends the proof. □

Lemma 7. *Let $t : S \times \mathbb{R} \to \mathbb{R}$ be admissible, where S is a metric space (or even a normal space). We define $s : S \times \mathbb{R} \to \mathbb{R}$ taking $s(x, d) = c$ if and only if $t(x, c) = d$. The following statements are equivalent:*

(a) *The function $x \mapsto t(x, f(x))$ is in $C^*(S)$ if f is.*
(b) *t is continuous.*
(c) *t has closed graph.*
(d) *s has closed graph.*
(e) *s is continuous.*
(f) *The function $x \mapsto t(x, f(x))$ is in $C^*(S)$ only if f is.*

Proof. It is clear that the equivalences (d) ⇔ (e) ⇔ (f) hold if and only if (a) ⇔ (b) ⇔ (c) holds. That (c) ⇔ (d) is trivial, hence it suffices to prove that the first three conditions are equivalent. Clearly, (b) implies both (a) and (c), while (a) ⇒ (b) is as in [17, Lemma 2].

And, finally, we prove (c) ⇒ (b). In fact, we show that every locally bounded real-valued function with closed graph defined on a metric space is continuous. Let $u : M \to \mathbb{R}$ be such a function. The graph of u is the set

$$G(u) = \{(x, c) \in M \times \mathbb{R} : c = u(x)\}.$$

Let (x_n) converge to x in M. We show that $u(x_n) \to u(x)$ as $n \to \infty$. Indeed, if u^* is a cluster point of the sequence $(u(x_n))$, then (x, u^*) belongs to (the closure and therefore to) $G(u)$ and so $u^* = u(x)$. □

Another somewhat surprising consequence is that each $C(X)$ embeds as a subalgebra (complemented by a homomorphism of algebras) in a larger $C(Y)$ having only continuous automorphisms. Just consider $Y = X \times [0, 1]$, the natural proyection $\pi : Y \to X$ and the embedding $\imath : X \to Y$ given by $\imath(x) = (x, 0)$. The maps $\pi^* : C(X) \to C(Y)$ and $\imath^* : C(Y) \to C(X)$ are homomorphisms of algebras, the first one is injective and, moreover, $\imath^* \circ \pi^* = \mathbf{1}_{C(X)}$. Needless to say, every point of Y satisfies the first hypothesis in Lemma 5.

4. Uniformly Continuous Functions

In this Section we present a nonlinear counterpart of a recent result of Maribel Garrido and Jesús Jaramillo concerning lattices of uniformly continuous functions. As in [12] we consider only metric spaces. We abandon our earlier convention that X and Y denote compact spaces and, given a metric space X, we write $U^*(X)$ for the lattice of all bounded uniformly continuous funtions on X.

As it is remarked in [12], each metric space (X, d) admits a compactification κX such that the Banach algebras $U^*(X)$ and $C(\kappa X)$ are isometrically isomorphic in the obvious way. What is more important [12, Lemma 1]: if X is complete, then the points of κX having countable neighborhood bases are exactly those of X.

Suppose we are given a lattice isomorphism $T : U^*(Y) \to U^*(X)$, where X and Y are complete metric spaces. Then T induces a homeomorphism $\tau : \kappa X \to \kappa Y$ and, by the preceding remark, τ maps X onto Y. On the other hand composition with τ preserves uniform continuity and so, by an old result of Efremovich, τ is uniformly continuous and, in fact, a uniform homeomorphism. We then have that $U^*(Y)$ and $U^*(X)$ are lattice isomorphic if and only if X and Y are uniformly homeomorphic. Our aim here is to represent T. In this line we have the following nonlinear companion of [12, Corollary 3].

Theorem 2. *Let Y and X be complete metric spaces, $\tau : X \to Y$ a uniform homeomorphism and $t : X \times \mathbb{R} \to \mathbb{R}$ an admissible function satisfying the condition:*

- *If (x_n) and (y_n) are sequences in X with $0 < d(x_n, y_n) \to 0$ and $(c_n), (d_n)$ are bounded in $\mathbb{R}$ one has $c_n - d_n \to 0 \iff t(x_n, c_n) - t(y_n, d_n) \to 0$.*

Then the mapping $T : U^*(Y) \to U^*(X)$ defined by

$$Tf(x) = t(x, f(\tau(x))) \qquad (x \in X) \tag{2}$$

is an isomorphism. And, conversely, all isomorphisms arise in this way.

Proof. The first part is nearly obvious, taking into account that f is uniformly continuous if and only if $d(x_n, y_n) \to 0$ implies $f(x_n) - f(y_n) \to 0$. Thus, assuming $\tau = \mathbf{1}_X$ and taking T is as in (2) we get an endomorphism of $U^*(X)$. In fact we only need the implication '$\Rightarrow$' here. The implication '$\Leftarrow$' means that if s is the admissible inverse of t, then we can obtain the inverse of T taking $Sf(x) = s(x, f(x))$.

To prove the converse, suppose $T : U^*(Y) \to U^*(X)$ is an isomorphism. By the preceding remarks T induces a homeomorphism $\tau : \kappa X \to \kappa Y$ mapping X onto Y, as a uniform homeomorphism and since the points of X have countable neighbourhood bases in κX we can apply Lemma 5 (a) to get (2), where $t(x, c) = Tc(x)$. Obviously t is admissible. The proof will be complete if we show that every admissible function $t : X \times \mathbb{R} \to \mathbb{R}$ such that $t(\cdot, f(\cdot))$ belongs to $U^*(X)$ for all $f \in U^*(X)$

satisfies the above condition with '$\Rightarrow$' replacing '$\Leftrightarrow$'. The implication '$\Leftarrow$' follows from the same argument applied to s.

Indeed let us assume $(x_n), (y_n)$ are sequences in X, with $0 \neq d(x_n, y_n) \to 0$ and $(s_n), (t_n)$ are bounded sequences with $c_n - d_n \to 0$. We must show that $t(x_n, c_n) - t(y_n, d_n) \to 0$.

But a sequence converges to ℓ if and only if every subsequence admits a further subsequence converging to ℓ. Hence it suffices to see that some subsequence of $t(x_n, c_n) - t(y_n, d_n)$ converges to zero.

Thus, without loss of generality we assume (c_n) and (d_n) converge, say to c. On the other hand, either

(a) (x_n) has a convergent subsequence, or
(b) (x_n) has a 'discrete' subsequence.

In case (a) we may assume (x_n) converges, say to x. We will prove that $t(x_n, c_n) \to t(x, c)$ and the result follows by symmetry. Again it suffices to find a subsequence converging to $t(x, c)$. Now, by passing to a subsequence if necessary, we may assume either

(a.1) $x_n = x$ for all n, or
(a.2) $x_n \neq x_m \neq x$ for every $n \neq m$.

In the first case we have $t(x_n, c_n) \to t(x, c)$ by continuity in the second variable and we are done. Assuming (a.2), the function sending x_n into c_n and x into c is uniformly continuous on the compact set $\{x_n\} \cup \{x\}$ and so, there is $f \in U^*(X) = C(\kappa X)$ such that $c_n = f(x_n)$ and $c = f(x)$. We have

$$t(x_n, c_n) = t(x_n, f(x_n)) \to t(x, f(x)) = t(x, c),$$

which completes the proof on the assumption (a).

As for (b) we may assume $d(x_n, x_m) \geq r > 0$ for $n \neq m$ and $d(x_n, y_n) < r/2$. We can define a function on $S = \{x_n, y_n : n \in \mathbb{N}\}$ sending x_n into c_n and y_n into d_n. Clearly, that function is uniformly continuous on S and so it extends as a continuous function to the closure of S in κX and, by normality, to the whole of κX. Thus, there is $f \in U^*(X)$ such that $c_n = f(x_n)$ and $d_n = f(y_n)$ for all n. Since the function $x \mapsto t(x, f(x))$ is uniformly continuous and $d(x_n, y_n) \to 0$ we have

$$t(x_n, c_n) - t(y_n, d_n) = t(x_n, f(x_n)) - t(y_n, f(y_n)) \to 0,$$

which completes the proof. □

The condition '$d(x_n, y_n) > 0$' cannot be removed in the preceding Theorem. Indeed, $U^*(\mathbb{N}) = \ell^\infty$ and if T is an automorphism of ℓ^∞, then t satisfies the above condition removing '$d(x_n, y_n) > 0$' and with '$\Rightarrow$' replacing '$\Leftrightarrow$' if and only if T is continuous.

Actually one can remove '$d(x_n, y_n) > 0$' in Theorem 2 if (and only if) $\mathbb{N}$ does not appear as a direct summand of X in the 'uniform' category. Notice that $\mathbb{N}$ is such a direct summand if and only if there is $\varepsilon > 0$ and an infinite subset $S \subset X$ such that $d(s, x) \geq \varepsilon$ whenever $s \in S$ and $x \in X$, with $x \neq s$. Please check this. In that case, the relevant condition on t is easily seen to be equivalent to the uniform continuity of t **and** of s on every set of the form $X \times [a, b]$. I thank Javier Cabello Sánchez for this observation.

5. A Closer Look at L^∞

In this section we deal with isomorphisms between lattices of bounded measurable functions. Given a measure space $(X, \mathfrak{X}, \mu)$, we write $L^0(\mu)$ for the set of all measurable functions on X equipped with 'pointwise' operations, and the traditional convention about identifying functions equal almost everywhere.

The subset of essentially bounded functions in $L^0(\mu)$ with the essential supremum norm is a Banach algebra denoted $L^\infty(\mu)$.

By general representation theorems $L^\infty(\mu)$ is isometrically isomorphic to the Banach algebra $C(\mathfrak{M})$, where $\mathfrak{M}$ is some compact space – actually, the set all nonzero multiplicative linear functionals on $L^\infty(\mu)$ equipped with the relative weak* topology of $L^\infty(\mu)^*$. Here, we need only a few elementary facts concerning $\mathfrak{M}$. Mainly, that each measurable $A \subset X$ corresponds to a clopen subset of $\mathfrak{M}$. This is obvious since 1_A is an idempotent in $L^\infty(\mu) = C(\mathfrak{M})$.

Otherwise the topology of $\mathfrak{M}$ is very involved: it is always an extremely disconnected space. Moreover, if μ is not atomic, then $\mathfrak{M}$ has no isolated points and, in fact, no point is G_δ. However, nonempty zero sets may have empty interior (these correspond to noninvertible functions in $L^\infty(\mu)$ vanishing on no set of positive measure).

As a Banach lattice, however, $L^\infty(\mu)$ displays some nice features: the most remarkable is order completeness: if $S \subset L^\infty(\mu)$ is (order) bounded, then $\bigvee S$ exists in $L^\infty(\mu)$. Note that if $S \subset L^\infty(\mu)$ is countable, then $\bigvee S$ can be computed pointwise.

We will exploit this fact thanks to the following.

Lemma 8. *Isomorphisms $L^\infty(\nu) \to L^\infty(\mu)$ preserve almost everywhere convergence.*

Proof. Note that $f_n \to f$ almost everywhere if and only if

$$f = \bigwedge_n \bigvee_{k \geq n} f_k = \bigvee_n \bigwedge_{k \geq n} f_k$$

and that isomorphisms preserve arbitrary joins and meets. □

Before going further, we simplify the treatment of lattice isomorphisms as follows. Let $L : L^\infty(\nu) \to L^\infty(\mu)$ be a lattice isomorphism and let U be the associated isomorphism of algebras. Then $T = L \circ U^{-1}$ is a lattice automorphism of $L^\infty(\mu)$ whose associated algebra automorphism is the identity – we restrict ourselves to this case in the ensuing discussion. By Corolary 2 (b) we have that if $f, g \in L^\infty(\mu)$ agree almost everywhere on a measurable $A \subset X$, then so Tf and Tg do.

We define $t : X \times \mathbb{R} \to \mathbb{R}$ taking $t(x, c) = Tc(x)$ – any version of Tc suffices. Now, if f is simple (that is, it takes only finitely many values), we have $f = \sum_{i=1}^{n} c_i 1_{A_i}$ and so

$$Tf(x) = t(x, f(x))$$

almost everywhere.

One may expect the same holds for every $f \in L^\infty(\mu)$, but simple examples show this is not the case if $t(\cdot, c)$ are arbitrary representatives of Tc. The following result, which is the main step in representing automorphisms of $L^\infty(\mu)$, shows that a judicious choice of the versions works.

Lemma 9. *Let T be an automorphism of $L^\infty(\mu)$ whose associated algebra automorphism is the identity. Then there is a measurable admissible $t : X \times \mathbb{R} \to \mathbb{R}$ such that, for every $f \in L^\infty(\mu)$, one has*

$$Tf(x) = t(x, f(x)) \tag{3}$$

almost everywhere.

Proof. For each rational r, let us fix a bounded version of Tr, so that we consider $Tr(x)$ defined for every $x \in X$. Note that if $r < s$, then $Tr(x) < Ts(x)$ almost everywhere. Clearly, the set $N = \{x \in X : Tr(x) \geq Ts(x) \text{ for some } r < s\}$ has measure zero. We first define $t^* : X \times \mathbb{Q} \to \mathbb{R}$ taking

$$t^*(x, c) = \begin{cases} Tr(x) & \text{if } x \notin N \\ r & \text{if } x \in N \end{cases}$$

Now, for $c \in \mathbb{R}$, put

$$t(x, c) = \sup\{t^*(x, r) : r \in \mathbb{Q}, r < c\}.$$

It follows from Lemma 8 that $t(\cdot, c)$ is a bounded representative of Tc for each $c \in \mathbb{R}$.

Notice that all functions $t(x, \cdot)$ are strictly increasing: choose $c, d \in \mathbb{R}$ such that $c < d$ and pick any $x \in X$. If p and q are rational numbers such that $c < p < q < d$, we have

$$t(x, c) \leq t^*(x, p) < t^*(x, q) \leq t(x, d),$$

as desired.

Next we verify that $t(x, \cdot)$ is left continuous: it suffices to see that for each $c \in \mathbb{R}, x \in X$ there is a sequence (c_n) with $c_n < c$ and $t(x, c_n) \to t(x, c)$. Let r_n be a strictly increasing sequence of rational numbers converging to c such that $t^*(x, r_n) \to t(x, c)$. We then have

$$\lim_n t^*(x, r_n) = \lim_n t(x, r_{n+1}) = t(x, c).$$

We now derive the formula (3) from that we know so far, namely that $t(\cdot, c)$ is a version of Tc and the left continuity of every $t(x, \cdot)$. Fix $f \in L^\infty(\mu)$. Let (s_n) be an increasing sequence of simple functions converging pointwise to f. By the remark before the Lemma we have $Ts_n(x) = t(x, s_n(x))$ almost everywhere. But, by left continuity one also has

$$t(x, s_n(x)) \to t(x, f(x)) \qquad (x \in X).$$

On the other hand Ts_n converges almost everywhere to Tf and so $Tf(x) = t(x, f(x))$ almost everywhere.

Let us complete the proof that t is admissible. As T^{-1} is an automorphism, there is $s : X \times \mathbb{R} \to \mathbb{R}$, with each $s(x, \cdot)$ strictly increasing, each $s(x, \cdot)$ a version of $T^{-1}c$, and such that, for every $f \in L^\infty(\mu)$, one has $T^{-1}f(x) = s(x, f(x))$ almost everywhere. In particular we have that, for each $r \in \mathbb{Q}$, $s(x, t(x, r)) = r$ holds almost everywhere and thus the set

$$M = \{x \in X : s(x, t(x, r)) \neq r \text{ for some } r \in \mathbb{Q}\}$$

is null. It follows from Lemma 3 that $t(x, \cdot)$ is an automorphism of $\mathbb{R}$ for x outside M. Now, if we redefine t taking $t(x, c) = c$ for $x \in M$ and $c \in \mathbb{R}$, we get an admissible t for which (3) still holds.

We conclude the proof by showing that t is measurable. For each n, define $t_n : X \times \mathbb{R} \to \mathbb{R}$ taking $t_n(x, c) = t(x, k/2^n)$, where k is the unique integer such that $k/2^n \leq c < (k+1)/2^n$. Notice that $t_n \leq t_{n+1}$. By construction, each t_n is measurable and since t_n converges pointwise to t, so is t. □

Finally, we get a transparent description of the isomorphisms between L^∞ lattices when the underlying measures are 'standard'. Recall that $(X, \mathfrak{X}, \mu)$ is standard if $\mathfrak{X}$ is the algebra of Borel sets of a Polish space (a separable complete metrizable space) X.

Also, given measure spaces $(X, \mathfrak{X}, \mu)$ and $(Y, \mathfrak{Y}, \nu)$, a measurable isomorphism is a bijection $\tau : X \to Y$, measurable in both directions and such that $A \in \mathfrak{X}$ has measure zero if and only if $\nu(\tau(A)) = 0$.

Theorem 3. *Let $(X, \mathfrak{X}, \mu)$ and $(Y, \mathfrak{Y}, \nu)$ be standard measure spaces, $\tau : X \to Y$ a measurable isomorphism and $t : X \times \mathbb{R} \to \mathbb{R}$ a measurable admissible function – with respect to the Borel structures. Then the map $T : L^\infty(\nu) \to L^\infty(\mu)$ given by*

$$Tf(x) = t(x, f(\tau(x))) \tag{4}$$

is an isomorphism. All isomorphims arise in this way.

Proof. The hypotheses on the measures guarantee that every algebra isomorphism $L^\infty(\nu) \to L^\infty(\mu)$ is given by composition with a measurable isomorphism $\tau : X \to Y$. Thus, the second part follows from Lemma 9.

To prove the first part, we may assume $\mu = \nu$ and $\tau = \mathbf{1}_X$ and everything is obvious but that T is onto. Consider the 'admissible inverse' of t given by $s(x, d) = c$ if and only if $t(x, c) = d$. This is a measurable function since its graph

$$G(s) = \{(x, d, c) \in X \times \mathbb{R} \times \mathbb{R} : c = s(x, d)\} = \{(x, d, c) \in X \times \mathbb{R} \times \mathbb{R} : d = t(x, c)\}$$

is Borel (t is measurable) and a function on a Polish space (as $X \times \mathbb{R}$) is Borel measurable if and only if its graph is a Borel set (see [7, Proposition 8.3.4]). Now, the map S defined on $L^\infty(\mu)$ by

$$Sf(x) = s(x, f(x))$$

is easily seen to be the inverse of T. □

Example 2. *A compact space $\mathfrak{M}$ such that for every $\mathfrak{m} \in \mathfrak{M}$ there is an automorphism T of $C(\mathfrak{M})$ such that $T^*\delta_{\mathfrak{m}}$ is discontinuous.*

Proof. Let μ be Lebesgue measure on the Borel sets of the unit interval. Our compact space will be $\mathfrak{M}$, the maximal ideal space of the Banach algebra L^∞.

Fix $\mathfrak{m} \in \mathfrak{M}$. We can regard $\mathfrak{m}$ as a unital homomorphism of algebras $\mathfrak{m} : L^\infty \to \mathbb{R}$. It is clear that there is a unique $y \in [0,1]$ such that $f(\mathfrak{m}) = \mathfrak{m}(f) = f(y)$ for every $f \in C[0,1]$, that is, $\mathfrak{m}$ is an extension of δ_y. Let (A_n) be a decreasing base of neighborhoods of y in $[0,1]$, with $A_1 = [0,1]$ and put $n(x) = \max\{n : x \in A_n\}$, with $n(y) = 1$.

Now, define $t : [0,1] \times \mathbb{R} \to \mathbb{R}$ by $t(x,c) = \ell(n(x), c)$, where ℓ is the function defined in the introduction and put

$$Tf(x) = t(x, f(x)) \qquad (f \in L^\infty).$$

Clearly, T is an automorphism, and it is not hard to see that $T^*\mathfrak{m}$ is discontinuous at $f = 1$. □

It is well-known that there is no topology on $L^\infty(\mu)$ inducing convergence almost everywhere unless μ is purely atomic. However [7], convergence almost everywhere implies convergence in measure and since each sequence converging in measure has subsequences converging almost everywhere to the same limit we infer from Lemma 8 that isomorphisms $T : L^\infty(\nu) \to L^\infty(\mu)$ are homeomorphisms for the topology of convergence in measure. If μ is finite, that topology is given on $L^0(\mu)$ by the F-norm

$$\|f\|_0 = \int_X \frac{|f(x)|}{1 + |f(x)|} d\mu \qquad (f \in L^0(\mu)).$$

Perhaps the most important topology on $L^\infty(\mu)$ is the weak* topology. Here, we treat $L^\infty(\mu)$ as the dual of the Banach space $L^1(\mu)$. Theorem 3 could suggest that automorphisms of $L^\infty(\mu)$ must be weak* continuous. We have, however, the following result, based on a remark of [5].

Corollary 5. *Let ν and μ be finite standard measures without atoms. A lattice isomorphism $T : L^\infty(\nu) \to L^\infty(\mu)$ is weak* continuous if and only if it is an affine homeomorphism in the norm topology.*

Proof. We can assume throughout the proof that $\nu = \mu$ is Lebesgue measure on the unit interval and also that $T0 = 0$ – hence T is affine if and only if it is linear. The 'if part' is clear: if T is linear, then $Tf(x) = \omega(x) f(\tau(x))$, where $\omega = T1_Y$.

These operators are always weak* continuous since both composition operators and multiplication operators are.

To prove the converse, after composing with τ^{-1} we can assume T has the form $f \mapsto t(\cdot, f(\cdot))$. This is equivalent to say that two functions agree on a given Borel subset of the unit interval if and only if so do their images under T.

Let (r_n) denote the sequence of Rademacher's functions, that is, $r_n(x)$ is the signum of $\cos(2^n \pi x)$ for $0 \leq x \leq 1$. In is well-known that (r_n) converges to zero in the weak* topology of L^∞. Set $U_n^\pm = \{x \in [0,1] : r_n(x) = \pm 1\}$. Now, since $1_{U_n^\pm} = (1 \pm r_n)/2$ we have $1_{U_n^\pm} \to \frac{1}{2}$ weakly* and since multiplication on L^∞ is separately weak* continuous we see that, for each $f, g \in L^\infty$, one has

$$\lim_{n\to\infty} \left(1_{U_n^+} f + 1_{U_n^-} g\right) = \frac{f+g}{2} \qquad \text{(weak*)}.$$

Notice that $1_{U_n^+} f + 1_{U_n^-} g$ agrees with f on U_n^+ and agrees with g on U_n^-, so

$$T(1_{U_n^+} f + 1_{U_n^-} g) = 1_{U_n^+} Tf + 1_{U_n^-} Tg \quad \text{and}$$
$$T\left(\frac{f+g}{2}\right) = \lim_{n\to\infty} \left(1_{U_n^+} Tf + 1_{U_n^-} Tg\right) = \frac{Tf + Tg}{2},$$

with the limit taken in the weak* sense. Hence T preserves midpoints and zero (in both directions). It follows that it is additive (in both directions) and since T is order preserving linearity and norm continuity easily follows. □

6. Lattices of Measurable Functions

We now extend the results of the preceding Section to other function lattices. As before $(X, \mathfrak{X}, \mu)$ will denote a standard measure. Let us say that L is a function lattice on μ if $L^\infty(\mu) \subset L \subset L^0(\mu)$, as lattices. If μ is finite this includes most lattices one encounters in analysis including the spaces $L^p(\mu)$, for $0 \leq p \leq \infty$. If μ is infinite then, e. g., $L^2(\mu)$ does not contain $L^\infty(\mu)$, but note that if μ is σ-finite, then there is a finite measure υ with the same null sets (hence $L^0(\upsilon) = L^0(\mu)$ and $L^\infty(\upsilon) = L^\infty(\mu)$) and such that $L^p(\mu)$ is isomorphic to a function lattice on υ for all p.

Theorem 4. *Let $T : M \to L$ be an isomorphism, where M and L are function lattices on the standard measures ν and μ, respectively. Then there exist a measurable isomorphism $\tau : X \to Y$ and a measurable function $t : X \times \mathbb{R} \to \mathbb{R}$ such that, for every $f \in M$,*

$$Tf(x) = t(x, f(\tau(x))) \tag{5}$$

holds almost everywhere. Moreover t can be chosen so that, for each fixed x, the map $t(x, \cdot)$ is an automorphism of $\mathbb{R}$.

Proof. We may and do assume $T0 = 0$. Consider the composition

$$S : L^\infty(\nu) \longrightarrow M \xrightarrow{T} L \longrightarrow L^0(\mu) \xrightarrow{\rho_*} L^\infty(\mu)$$

where the unlabelled arrows are plain inclusions and $\rho_*(f) = \rho \circ f$, with

$$\rho(t) = \frac{t}{1+|t|} \qquad (t \in \mathbb{R}).$$

As S is injective we know from Corollary 2 that if $f, g \in L^\infty(\nu)$ agree on some set A of positive measure then Sf and Sg (and so Tf and Tg) agree on some set of positive measure, say B, depending only on A. Suppose now that $f, g \in M$ agree on A. Then there are sequences (f_n) and (g_n) in $L^\infty(\nu)$ with f_n and g_n equal on A, $f_n \to f$ and $g_n \to g$ almost everywhere. But T preserves almost everywhere convergence (same proof as Lemma 8) and thus Tf and Tg agree almost everywhere on B. Applying the same reasoning to T^{-1}, we have in particular that $f, g \in M$ agree on some set of positive measure if and only if so Tf and Tg do.

Consequently $T1_Y > 0$ almost everywhere and so $\eta(x) = 1/T1_Y(x)$ makes sense as a function of $L^0(\mu)$. Now, consider the composition

$$T' : L^\infty(\nu) \longrightarrow M \xrightarrow{\ T\ } L \longrightarrow L^0(\mu) \xrightarrow{\ \cdot\eta\ } L^0(\mu) \xrightarrow{\ \rho_*\ } L^\infty(\mu),$$

where $\cdot\eta$ denotes multiplication by η, which is a linear automorphism of $L^0(\mu)$. We can regard T' as a homomorphism from $C(\mathfrak{N})$ into $C(\mathfrak{M})$, where $L^\infty(\mu) = C(\mathfrak{M})$ and $L^\infty(\nu) = C(\mathfrak{N})$. Clearly, $T'0 = 0$ and $T'1 = 1/2$. Hence T' induces a continuous map $\phi : \mathfrak{M} \to \mathfrak{N}$ and also a homomorphism of unital algebras $\phi^* : C(\mathfrak{N}) \to C(\mathfrak{M})$. As ϕ^* sends idempotents to idempotents we can define a mapping $\Phi : \mathfrak{Y} \to \mathfrak{X}$ by the rule

$$1_{\Phi(A)} = \phi^*(1_A).$$

Our immediate aim is to show that Φ is an isomorphism. Notice that the same argument as before proves the following.

Claim. Let $f, g \in M$ agree almost everywhere on A. Then Tf and Tg agree almost everywhere on $\Phi(A)$.

As T^{-1} has similar properties, we get a mapping $\Psi : \mathfrak{X} \to \mathfrak{Y}$ and we have that if Tf and Tg agree on B, then f and g agree almost everywhere on $\Psi(B)$.

Let us show that Ψ is an inverse for Φ. Pick $A \in \mathfrak{Y}$ and consider the functions 1 and 1_A. We have that $T1$ and $T1_A$ agree almost everywhere on $\Phi(A)$ and so $1 = T^{-1}T1$ and $1_A = T^{-1}T1_A$ agree on $\Psi\Phi(A)$. Hence $\Psi\Phi(A) \subset A$ for each measurable A and also $(\Psi\Phi(A))^c = \Psi\Phi(A^c) \subset A^c$, that is, $\Psi\Phi(A) = A$.

Hence Φ is an isomorphism and the hypotheses about the measures guarantee the existence of a bimeasurable $\tau : X \to Y$ such that $\Phi(A) = \tau^{-1}(A)$.

The rest of the proof follows the lines of Section 5. We already know that $f < g$ almost everywhere if and only if $Tf < Tg$ almost everywhere. In particular we can choose versions of Tr for $r \in \mathbb{Q}$ such that $Tr(x) < Ts(x)$ for all $r < s$ and all $x \in X$. Now, define $t : X \times \mathbb{R} \to \mathbb{R}$ by letting $t(x, 0) = 0$ and

$$t(x,c) = \begin{cases} \inf\{Tr(x) : r > c, r \in \mathbb{Q}\} & \text{for } c < 0 \\ \sup\{Tr(x) : r < c, r \in \mathbb{Q}\} & \text{for } c > 0 \end{cases}$$

It is clear that for every real c the function $t(\cdot, c)$ is a version of Tc and also that for every fixed x the real function $t(x, \cdot)$ is left continuous on $(0, \infty)$ and right continuous on $(-\infty, 0)$.

It follows from the Claim that if s is simple, then

$$Ts(x) = t(x, s(\tau(x)))$$

holds almost everywhere. For an arbitrary $f \in M$, write $f = f^+ - f^-$, where f^+ and f^- are non-negative functions with disjoint supports. Clearly, there are increasing sequences $(s_n^\pm)$ of simple functions with $s_n^\pm \to f^\pm$ almost everywhere, with $\operatorname{supp} s_n^\pm \subset \operatorname{supp} f^\pm$. Writing $s_n = s_n^+ - s_n^-$ we have $s_n \to f$ almost everywhere. Moreover, for every x one has $t(x, s_n(\tau(x))) \to t(x, f(\tau(x)))$, so

$$Tf(x) = t(x, f(\tau(x)))$$

almost everywhere.

The remainder of the proof, namely, that we can modify t in such a way that each $t(x, \cdot)$ is an increasing homeomorphism of $\mathbb{R}$ and that resulting t is jointly measurable goes exactly as in the proof of Lemma 9. □

Of course, the precise conditions on t and τ under which (5) defines an isomorphism will depend on M and L. A trivial condition is that $t(\cdot, f(\cdot))$ belongs to L if and only if $f \in M$. For the 'maximal' lattice $L^0(\mu)$ we have the following.

Corollary 6. *Let μ be a finite standard measure. Let τ be measurable automorphism of X and $t : X \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that for every fixed x, the funcion $t(x, \cdot)$ is an automorphism of $\mathbb{R}$. Then the formula*

$$Tf(x) = t(x, f(\tau(x))) \qquad (x \in X)$$

defines an automorphism of $L^0(\mu)$. All automorphisms of $L^0(\mu)$ arise in this way and they are continuous when $L^0(\mu)$ is furnished with the topology of convergence in measure. □

Another trivial consequence of Theorem 4 is that for $0 \leq p, q \leq \infty$ the lattices L^p and L^q are isomorphic if and only if $0 < p, q < \infty$. However this follows easily by inspection: indeed, if $0 < p < \infty$, then the map sending $f \in L^p$ to $\sigma(f)|f|^p$ gives an isomorphism between L^1 and L^p –here σ denotes 'signum'. Conversely, the statement: 'there is a sequence (f_n) such that for every f there is n such that $f_n \geq f$' holds in L^p if and only if $p = \infty$, while the statement: 'if (f_n) and f are such that $f \leq f_n$ for all n and $f_n \wedge f_m = f$ whenever $n \neq m$, then $\bigvee_n f_n$ exists' holds in L^p if and only if $p = 0$.

7. Concluding Remarks

Very recently J. Marovt has published three papers on preservers [18, 20, 19]. His results can be summarized as follows. Notice that a bijection between lattices is an isomorphism if and only if it preserves the order in both directions.

Let T be a bijection on $C(X, \mathbb{I})$, where X is a metrizable compact space and $\mathbb{I} = [0, 1]$.

(a) If T preserves the order in both directions then it has the form $Tf(x) = t(x, f(\tau(x)))$ where τ is a homeomorphism of X and $t : X \times \mathbb{I} \to \mathbb{I}$ is given by $t(x, c) = Tc(x)$.
(b) If T is multiplicative then $Tf(x) = (f(\tau(x)))^{\kappa(x)}$, where τ is a homeomorphism and $\kappa : X \to (0, \infty)$ is continuous.
(c) If T is affine then it has the form $Tf(x) = 1/2 + u(x)(1/2 - f(\tau(x)))$, where τ is a homeomorphism and $u : X \to \pm 1$ is continuous.

Marovt conjectures that the same holds without metrizability. This is the case in (c), but neither in (a) or (b). Indeed, the counterexample for (b) in [9] is easily seen to be a counterexample for (a) as well. Actually, I do not know if every multiplicative bijection on $C(X, \mathbb{I})$ must preserve order (in both directions). It is clear that any multiplicative bijection on $C(X, (0, 1))$ does. We refer the reader to the papers [22, 24, 14, 8] for related results on the semigroup $C(X)$.

The affine case is as follows (a slightly different argument appears in [10]): as $\mathbb{I}$ is affine with $[-1, 1]$ we can consider affine bijections on $C(X, [-1, 1])$. Note that $C(X, [-1, 1])$ is just the unit ball of the Banach space $C(X)$. On the other hand, if B is the unit ball of a Banach space then the origin is the only point f in B satisfying the following: given $g \in B$ there is $h \in B$ such that $f = (g + h)/2$ (for if $f \neq 0$ then f is not a midpoint with $g = -f/\|f\|$). It follows that $T0 = 0$ and so $Tcf = cTf$ for all $f \in C(X, [-1, 1])$ and $c \in \mathbb{I}$. Hence we can extend T to a linear bijection thus:

$$f \longmapsto \|f\| T(f/\|f\|).$$

But a linear bijection mapping the unit ball onto itself is also an isometry and so the Banach-Stone theorem gives that T has the form $Tf = u \cdot f \circ \tau$, where τ is a homeomorphism of X and $u : X \to \pm 1$ is continuous. Translating this to $C(X, \mathbb{I})$ one obtains (c) with an X arbitrary compact Hausdorff space (or any metrizable space, compact or not). It is perhaps a little ironic that the Studia paper [20] contains more or less Banach's part of the Banach-Stone theorem (see [13] for an historical account).

The results of this paper remain true *mutatis mutandis* if we replace $\mathbb{R}$ by $\mathbb{I}$. We will refrain from entering into further details here, as this would add nothing to the main ideas.

We close with some questions arising from the results of the paper.

First, it would be interesting to characterize those maps $t : X \times \mathbb{R} \to \mathbb{R}$ such that $t(x, c) = Tc(x)$ for some automorphism T of $C(X)$, with X compact. This would complete Theorem 1.

With the sole exception of Section 6 our results apply only to bounded functions. It would be interesting to obtain a description of the automorphisms of $C(X)$ when X is a metric space. Same question for $U(X)$, with X complete metric.

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Note added in proof. After receiving the proofs, I found out two papers closely related to the subject of these notes, namely

[NS] I. Namioka, S. Saeki, On lattice isomorphisms of $C(X)^+$. Tokyo J. Math. 1 (1978), no 2, 345–368.

[LS] R. Lochan, D. Strauss, Lattice homomorphisms of spaces of continuous functions. J. London Math. Soc. (2) 25 (1982), no 2, 379–384.

I hasten to remark that some results presented here were already proved in these papers. Indeed, Corollary 3 and the remark that closes Sect. 3 are due to Namioka and Saeki. Let me mention the notable result, proved in [NS], that $C(X)$ admits a discontinuous automorphism if and only if $X = \beta S$, for some open F_σ proper subset $S \subset X$ (Theorem 1.5). This rounds off the earlier results by Itô quoted in the Introduction. It seems that Namioka and Saeki were unaware of [15] as well as of Cater's [6].

Also, Lemmas 1 and 2 correspond to Theorem 1 in [LS], while Corollary 2 corresponds to Propositions 1 and 2 in [LS]. Moreover, the first and second parts of Corollary 1 should be credited to Lochan and Strauss as they correspond to their Propositions 3 and 4. Our approach is simpler. By the way, Theorem 4 in [LS] was obtained earlier in [6]. Finally, it is worth noticing that [LS] contains the striking results that it is compatible with ZF (the usual setting of set theory without the axiom of choice) to assume that every surjective homomorphism $C(Y) \to C(X)$ is continuous (Theorem 3).

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