# Homomorphisms to oriented cycles.

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March 7, 2003

**Abstract**

We discuss the existence of homomorphisms to oriented cycles and give, for a special class of cycles $C$, a characterization of those digraphs that admit a homomorphism to $C$. Our characterization can be used to prove the multiplicativity of these cycles, as well as the membership of the corresponding decision problem in the class $NP \cap coNP$. We also mention a conjecture on the existence of homomorphisms to general oriented cycles.

## 1 Introduction.

The problem of existence of graph homomorphisms has attracted considerable attention, [1], [2], [8], [4], [7], [9], [6], [5], [20], [22], [16]. ¿From an algorithmic point of view, the problem is known to be NP-complete when the target is a fixed undirected graph $G$ of chromatic number greater than two, and polynomial for all other undirected graphs, [5]. No such clear distinction is known for digraphs, [4], [7], [9],[20], although some conjectures for special cases have been proposed, [6]. One result of note, [19], is a polynomial algorithm for the existence of a homomorphism to an oriented path. However, the existence of homomorphisms to oriented cycles appears to be a harder problem and no general polynomial algorithm is known at this time.

Another recent source of interest in the same problem is Hedetniemi's conjecture [13], [14], [15], [10], [11], which states that the chromatic number of the product of two $n$-chromatic graphs is $n$. (The product here is the conjunction [26], also known as the categorical product [14], in which $(a, b)$ is adjacent to $(c, d)$ just if $a$ is adjacent to $c$ and $b$ to $d$.) This led to the definiton of a *multiplicative* (directed or undirected) graph $W$, [3], (see also [21]), as one for which graphs non-homomorphic to $W$ are closed under taking products. In other words, $W$ is multiplicative just if $G \not\mapsto W$ and $G' \not\mapsto W$ implies that $G \times G' \not\mapsto W$. It is easy to see that Hedetniemi's conjecture asserts that complete graphs are multiplicative. Some multiplicative graphs and digraphs were given in [3], [17], [18]. Again, multiplicative oriented paths have been completely characterized, [17], while the situation for oriented cycles is more difficult. In particular, [17] introduced a special class of cycles, called $\mathcal{C}$-cycles, and showed that among oriented cycles only the $\mathcal{C}$-cycles could be multiplicative. (A simpler proof of this result is given in [18].) However, the problem of whether or not all $\mathcal{C}$-cycles were multiplicative, remained open.

We introduce a more general class of oriented cycles, called $\mathcal{B}$-cycles, and give a characterization of those digraphs which are homomorphic to a fixed $\mathcal{B}$-cycle. This result will allow us to prove (in a subsequent paper [23]) that all $\mathcal{C}$-cycles are indeed multiplicative, thus completing the characterization of multiplicative oriented cycles. It will also follow from our result that the existence problem for homomorphism to a fixed $\mathcal{B}$-cycle is in $NP \cap coNP$. We shall mention corresponding results about homomorphisms to a fixed oriented path, and a possible extension of our main result to any oriented cycle. Proving this extension would verify that the existence problem for homomorphism to any oriented cycle is in $NP \cap coNP$, and possibly even suggest a polynomial algorithm for it.

A *homomorphism* of a digraph $G$ to a digraph $H$ is a mapping of the vertex sets $V(G) \mapsto V(H)$ which preserves the edges, i.e., such that $xy \in E(G)$ implies $f(x)f(y) \in E(H)$. If such a homomorphism exists, we say $G$ is *homomorphic* to $H$ and write $G \mapsto H$. Otherwise we write $G \not\mapsto H$.

An *oriented path* $P$ is a digraph obtained from an undirected path by orienting its edges and assigning to it a positive direction. Thus an oriented path $P$ is a digraph given by its sequence of vertices $< p_0, p_1, \ldots, p_n >$, such that, for each $i \in \{0, 1, \ldots, n-1\}$, either $p_i p_{i+1} \in E(P)$ (a *forward* edge of $P$), or $p_{i+1} p_i \in E(P)$ (a *backward* edge of $P$), and such that $P$ has no other edges. The direction of $P$ is emphasized by saying that $p_0$ the *initial point*, $i(P)$, of $P$, and $p_n$ the *terminal point*, $t(P)$, of $P$, respectively. Expressions such as "$u$ precedes (or follows) $v$ on $P$", or "$z$ is between $x$ and $y$ on $P$", also refer to this order on $P$. Changing the direction of $P$ results in the path $P^T =< p_n, p_{n-1}, \ldots, p_0 >$. Note that $P^T$ is the same digraph as $P$, only traversed in the opposite order. If $P =< p_0, p_1, \ldots, p_n >$ and $P' =< p'_0, p'_1, \ldots, p'_m >$ are oriented paths with disjoint vertex-sets, the *concatenation* of $P$ and $P'$ is the oriented path $P \bullet P' =< p_0, p_1, \ldots, p_n = p'_0, p'_1, \ldots, p'_m >$. We often concatenate given paths with the special oriented path $A =< a, a' >$, consisting of a single forward arc $aa'$.

Let $P =< p_0, p_1, \ldots, p_n >$ be an oriented path and let $u = p_i$ precede $v = p_j$ in $P$ (i.e., let $i < j$). The *interval* of $P$ from $u$ to $v$ is the oriented path $P[u, v] =< p_i, p_i + 1, \ldots, p_j >$. We also let $P[., v] = P[i(P), v]$, and $P[u, .] = P[u, t(P)]$.

Let $P$ be an oriented path. The *length* $l(P)$ of $P$ is the number of forward edges of $P$ minus the number of backward edges of $P$. We say that $P$ is *minimal* if it contains no proper interval of the same length. An interval $I$ of $P$ is called minimal if $I$ is a minimal path. The *distance* from $u$ to $v$ in $P$ is $d_P(u, v) = l(P[u, v])$. The *level* of $u$ in $P$ is $\lambda_P(u) = l(P[., u])$. Note that homomorphisms preserve distance, i.e., if $f : P \mapsto P'$ is a homomorphism and $u, v \in P$ then $d_{P'}(f(u), f(v)) = d_P(u, v)$.

A *directed path (interval)* is an oriented path (interval) with all the edges in the same direction. If they are forward edges it is called a *forward directed path (interval)*, otherwise it is called a *backward directed path (interval)*.

An *oriented cycle* $C$ is a digraph obtained from an undirected cycle by orienting its edges and assigning to it a positive direction. Thus an oriented cycle

$C$ is a digraph given by its circular sequence of vertices $< c_0, c_1, \ldots, c_n, c_0 >$, such that, for each $i \in \{0, 1, \ldots, n\}$, either $c_i c_{i+1} \in E(C)$ (a *forward* edge of $C$), or $c_{i+1} c_i \in E(C)$ (a *backward* edge of $C$), and such that $C$ has no other edges. (Subscript addition modulo $n$.) Since we do not distinguish an initial vertex of an oriented cycle, $< c_0, c_1, \ldots, c_n, c_0 >=< c_1, c_2, \ldots, c_n, c_0, c_1 >$, and we usually choose a most convenient vertex to start listing $C$. Note that we can view an oriented cycle as an oriented path in which the initial and terminal vertices have been identified, and in this spirit we shall use some of the definitions given for oriented paths also for oriented cycles. In particular, the length of the oriented cycle $C$ is the difference between the number of forward edges and the number of backward edges of $C$; an interval of $C$ is an interval of $< c_0, c_1, \ldots, c_n >$, where $< c_0, c_1, \ldots, c_n, c_0 >$ is any of the diferent ways of listing $C$.

Let $P =< p_0, p_1, \ldots, p_m >$ be an oriented path and $C =< c_0, c_1, \ldots, c_n >$ an oriented cycle. Consider a homomorphism $f : P \mapsto C$ such that $f(p_i) = f(p_{i+2})$ for some $i$. Define $P' =< p_0, p_1, \ldots, p_i, p_{i+3}, \ldots, p_m >$, and define $f'(p_j) = f(p_j)$ for $j = 0, 1, \ldots, i, i+3, \ldots, m$. Note that $f' : P' \mapsto C$ is a homomorphism; we shall say that it is obtained from $f : P \mapsto C$ by a *simplification step*. If $f'' : P'' \mapsto C$ is obtained from $f : P \mapsto C$ by a sequence of simplification steps, we shall say that $f : P \mapsto C$ *simplifies* to $f'' : P'' \mapsto C$. We shall say that the homomorphism $f : P \mapsto C$ *wraps* $P$ around $C$ if $f$ simplifies to $f'' : P'' \mapsto C$ where $P'' =< p''_0, p''_1, \ldots, p''_{n+1} >$ and $f''(p''_j) = c_j$, for $j = 0, 1, \ldots, n$,and $f''(p''n + 1) = c_0$. We shall say that the homomorphism $f : P \mapsto C$ *winds* $P$ around $C$ if some restriction of $f$ to an interval of $P$ wraps the interval around $C$. Finally, we shall say that the oriented path $P$ *can (can not) be wound around* $C$ if there is (isn't) a homomorphism $f : P \mapsto C$ which winds $P$ around $C$.

**DEFINITION 1** *An oriented cycle $C =< c_0, c_1, \ldots, c_n, c_{n+1}, \ldots, c_m, c_0 >$, with $n \geq 2$, is a $\mathcal{B}$-cycle (with parameter $n$), if $< c_0, c_1, \ldots, c_n >$ is a forward directed path (of length $n$), and $< c_0, c_m, c_{m-1}, \ldots, c_{n+1}, c_n >$ an oriented path of length $n-1$ which does not contain an interval of length $n$.*

Note that $C$ has length 1. Note further that $I =< c_0, c_1, \ldots, c_n >$ is the only minimal interval of $C$ of length $n$, i.e., every interval of $C$ is of length at most $n$, and every interval of length $n$ must contain $I$. Therefore, if $P$ is any minimal oriented path of length $n$ and if $h : P \mapsto C$ is a homomorphism, then $h(i(P)) = c_0$ and $h(t(P)) = c_n$. Furthermore, if $P$ is an oriented path which contains an interval of length greater than $n$ and if $h : P \mapsto C$ is a homomorphism, then $h$ must wind $P$ around $C$.

We now give our main result, a characterization of the class of digraphs which are homomorphic to a fixed $\mathcal{B}$-cycle.

**THEOREM 2** *Let $C$ be a $\mathcal{B}$-cycle and $G$ any digraph. Then $G \not\mapsto C$ if and only if there exists an oriented path $P$ such that $P \mapsto G$ but $P \not\mapsto C$.*

If $G \mapsto C$ and $P$ is an oriented path such that $P \mapsto G$, then of course $P \mapsto C$ by composition. Thus the sufficiency of the condition is obvious. The remainder of this paper consists of proving the necessity. Thus we shall prove that $G$ is homomorphic to $C$ provided all paths homomorphic to $G$ are also homomorphic to $C$.

Note that an image of a path $P =< p_0, p_1, \ldots, p_m >$ under homomorphism $f$ to $G$ may be viewed as a walk in $G$, simply by identifying it with the sequence of vertices $f(p_0), f(p_1), \ldots, f(p_m)$. We could also call a *walk pattern of* $G$ any path $P$ which is homomorphic to $G$. In this terminology, our main theorem would assert that $G$ is homomorphic to $C$ if and only if each walk pattern of $G$ is homomorphic to $C$. Since this terminology is somewhat unusual, we shall avoid it in the sequel. However, it may help the reader to bear this point of view in mind when reading the proofs. In particular, we frequently define paths $P =< p_0, p_1, \ldots, p_m >$ and homomorphisms $f : P \mapsto G$, having first in mind the walk $f(p_0), f(p_1), \ldots, f(p_m)$ in $G$.

# 2 The mapping $\psi$

¿From now on we assume that $C =< c_0, c_1, \ldots, c_n, c_{n+1}, \ldots, c_m, c_0 >$ is a fixed $\mathcal{B}$-cycle with parameter $n$, and that $G$ is a fixed digraph such that every path $P$ homomorphic to $G$ is also homomorphic to $C$. We proceed to construct a homomorphism $G \mapsto C$.

First we associate with $C$ a path $R =< r_0, r_1, \ldots, r_n, r_{n+1}, \ldots, r_m, r_{m+n}, r_{m+n-1}, \ldots, r_{m+1} >$, such that $r_i r_j$ is a forward (respectively backward) arc just if $c_i c_j$ is a forward (respectively backward) arc of $C$, where we let $c_{m+n-i} = c_i, i = 0, 1, \ldots, n-1$. Note that there is a natural homomorphism $R \mapsto C$, taking $r_i$ to $c_i$. For $v \in R$, $Ind(v)$ denotes the index of $v$, i.e., $Ind(v) = i$ just if $v = r_i$. We write $u \leq v$ (or $u < v$) if $Ind(u) \leq Ind(v)$ (respectively $Ind(u) < Ind(v)$) on $R$; thus $r_i \leq r_j$ just if $i \leq j$. When we speak of the maximum of a set of vertices of $R$, we are referring to this order. (Note that this is not the ordering imposed by $R$, in which $r_m$ is followed by $r_{m+n}$, then $r_{m+n-1}$, etc.)

It follows from the definition of $R$ that $D =< r_0, r_1, \ldots, r_n >$ is a directed interval of $R$ of length $n$. It also follows that for every vertex $x$ of $R$ we have $0 \leq \lambda_R(x) \leq n$, and $\lambda_R(x) = 0$ if and only if $x = r_0$. Thus $D$ is the only minimal interval of $R$ of length $n$. Hence if $P$ is any minimal oriented path of length $n$ and if $h : P \mapsto R$ is any homomorphism, then $h(i(P)) = r_0$ and $h(t(P)) = r_n$.

We shall denote by $\mathcal{P}$ the set of all paths $P$ homomorphic to $G$ such that $0 < \lambda_P(x) \leq n$ holds for all vertices of $P$ except for $i(P)$. Each interval of any $P \in \mathcal{P}$ has length at most $n$. It is well known, cf. [21], [3], that this implies that $P$ is homomorphic to $D$, and hence also homomorphic to $R$.

**DEFINITION 3** *Define $\phi : \mathcal{P} \mapsto V(R)$ as follows: For $P \in \mathcal{P}$,*

$$\phi(P) = max\{h(t(P)) : h : P \mapsto R\}.$$

*Define $\psi : V(G) \mapsto V(R)$ by*

$$\psi(x) = min\{\phi(P) : P \in \mathcal{P}, \text{ and for some } h : P \mapsto G, h(t(P)) = x\}.$$

Since there is a natural homomorphism from $R$ to $C$, $\psi$ induces a mapping of $G$ to $C$. In this section we show that this induced mapping has some nice properties. However it is not, in general, a homomorpism. In the next section, we will use $\psi$ to construct a true homomorphism of $G$ to $C$.

**DEFINITION 4** *Put*

$$K = \{x \in V(G) : \psi(x) = r_{m+n}\}$$

$$K_1 = \{x \in V(G) : r_1 \leq \psi(x) \leq r_m\} \textit{ and}$$

$$K_2 = \{x \in V(G) : r_{m+1} \leq \psi(x) \leq r_{m+n-1}\}.$$

*Put* $L = K_1 \cup K_2$.

It follows from the definition of $P \in \mathcal{P}$ (and from the fact that homomorphisms preserve distances), that $\phi(P) \neq c_0$; whence each $\psi(x) \neq c_0$ and $V(G) = K \cup L$.

**LEMMA 5** *For each vertex $x \in K_2$, there exists a path $P \in \mathcal{P}$ and homomorphisms $g : P \mapsto G$, $h : P \mapsto R$ such that $g(t(P)) = x, h(i(P)) = r_{m+n}$, and $h(t(P)) = \psi(x)$. In particular, $P$ contains no interval of length $n$.*

**Proof**. The definiton of $\psi(x)$ implies that there is a path $P'$ and homomorphisms $g' : P' \mapsto G$, $h' : P' \mapsto R$ such that $g'(t(P')) = x$ and $h'(t(P')) = \psi(x) = \phi(P')$. Since $< r_{m+n}, r_{m+n-1}, \ldots, r_m + 1 >$ is a directed path, it follows from the definition of $\phi(P')$ that $h'$ maps some vertex of $P'$ to $r_{m+n}$. Let $v$ be the last vertex of $P'$ such that $h'(v) = r_{m+n}$. Let $P = P'[v, .]$, and let $g, h$ be the corresponding restrictions of $g', h'$. It is now easy to see that the conclusions hold.

**Remark**. The situation is different for vertices $x \in K_1$. In fact, any $P$ with homomorphisms $g : P \mapsto G$, $h : P \mapsto R$ such that $g(t(P)) = x$ and $h(t(P)) = \phi(P) = \psi(x)$ does contain an interval of length $n$. Indeed, if $\lambda_P(v) < n$ for all $v$ then $P$ maps to $< r_{m+n}, r_{m+n-1}, \ldots, r_{m+1} >$ contradicting the fact that $\phi(P) \leq r_m$. Let $v$ be the first vertex on $P$ of level $n$, and let $f$ be any homomorphism of $P$ to $R$ (respectively to $C$). Since $P[., v]$ is a minimal interval of length $n$, we have $f(i(P)) = r_0$ (respectively $f(i(P)) = c_0$) and $f(v) = r_n$ (respectively $f(v) = c_n$). This implies in particular that the length of $P$ is determined by $x$, namely $l(P) = \lambda_R(\psi(x))$.

For $x \in L$, let $\mathcal{P}_x$ denote the set of all paths $P \in \mathcal{P}$ which admit homomorphisms $g : P \mapsto G$ and $h : P \mapsto R$ such that $g(t(P)) = x$, $h(t(P)) = \phi(P) = \psi(x)$, and $h(i(P)) = r_{m+n}$ if $x \in K_2$ or $h(i(P)) = r_0$ if $x \in K_1$. (According to the last remark, $h(i(P)) = r_0$ is automatic for $x \in K_1$.) The above remark also implies that the length of all $P \in \mathcal{P}_x$ is the same for $x \in K_1$; a similar argument shows the same for $x \in K_2$.

**LEMMA 6** *Let $x \in L$ and let $P \in \mathcal{P}_x$. Then $P$ can not be wound around $C$.*

**Proof**. Suppose $P \in \mathcal{P}_x$ and $f : P \mapsto C$ is a homomorphism which winds $P$ around $C$.

Assume first that $x \in K_2$: Note that $f(i(P)) \neq c_0$ since $P$ does not contain an interval of length $n$. Thus $f(v) \neq c_0$ for all $v \in P$, since the distance from $f(i(P))$ to $c_0$ along any direction of $C$ is non-positive and the distance from $i(P)$ to any other point of $P$ is positive. Therefore $f$ does not wind $P$ around $C$.

Assume now that $x \in K_1$: Then $f(i(P)) = c_0$, according to the remark. Since all $\lambda_P(v) \geq 0$, $f$ must map $P$ around $C$ in the positive direction. Since $f$ winds $P$ around $C$, some vertex $v \neq i(P)$ of $P$ has $f(v) = c_0$. Then $l_P(v) = 1$ and so $P' = P[v, .]$ contains no interval of length $n$. Furthermore $P'$ contains no vertex of negative level. Therefore there is a homomorphism $h : P' \mapsto R[r_{m+n}, r_{m+1}]$ such that $h(v) = r_{m+n}$. We may view $f$ restricted to $P[., v]$ as a homomorphism to $R$, with $f(v) = c_{m+n}$. This restriction of $f$, together with the homomorphism $h$ then yield a homomorphism $g : P \mapsto R$, such that $g(t(P)) \geq r_{m+1}$. This contradicts the assumption that $\phi(P) = \psi(x) \leq r_m$.

**LEMMA 7** *Let $x, y \in L$ and $xy \in E(G)$. Then any $P_x \in \mathcal{P}_x$ has length less than $n$, and any $P_y \in \mathcal{P}_y$ length more than 1.*

**Proof**. Suppose $P_x \in \mathcal{P}_x$ has length $n$, and let $P' = P_x \bullet A$. (Recall that $A$ is the path consisting of a single forward arc $aa'$.) Then $P'$ has length $n+1$ and is homomorphic to $G$ (extend any homomorphism $P_x \mapsto G$ to $P'$ by mapping $a'$ to $y$). According to our assumption, it is also homomorphic to $C$. Any homomorphism $P' \mapsto C$ must wind $P'$ around $C$, in order to achieve the length $n+1$. In fact, even its restriction to $P_x$ must wind $P_x$ around $C$, for the length of $C$ is 1. Since this contradicts Lemma 6, we have $l(P_x) < n$.

Suppose $P_y \in \mathcal{P}_y$ has length 1, and let $P'' = P_x \bullet A \bullet P_y^T$. Then the length of $P''[a, .]$ is zero, and hence for each $u \in P_y^T \backslash i(P_y)$ in $P''$ the distance to $a$, being the same as the distance to $i(P_y)$, is positive. We first show that there is no $u \in P_y^T \backslash i(P_y)$ for which $\lambda_{P_y}(u) = n$. Suppose there is; then we have $\lambda_{P''}(u) \geq n+1$. Thus any homomorphism $P'' \mapsto C$ must wind $P''$ around $C$. On the other hand, there exists such a homomorphism $f : P'' \mapsto C$, because $P''$ is obviously homomorphic to $G$ (take $a$ to $x$ and $a'$ to $y$). Since $i(P'')$ is the initial vertex of a minimal interval of $P''$ of length $n$, we have $f(i(P'')) = c_0$. Let $v \in P''$ be the first vertex of $P''$ after $i(P'')$ such that $f(v) = c_0$. Then $v \in P_y$, for otherwise $f$ would wind $P_x$ around $C$ contrary to Lemma 6. Also $v$ must precede $u$ in $P''$ because $\lambda_{P''}(u) \geq n+1$. Therefore $v \neq i(P_y)$. This is a contradiction because it implies that $d_C(f(a), f(v)) \leq 0$ while $d_{P''}(a, v) > 0$.

Therefore all $\lambda_{P_y}(u) \leq n-1$, and hence $y \in K_2$. Then $l(P_y) = 1$ implies that $\psi(y) = \phi(P_y) = r_{m+n-1}$. Let, as above, $P' = P_x \bullet A$. Then $P' \in \mathcal{P}$ because $l(P_x) \leq n-1$. There is a homomorphism of $P'$ to $G$ which takes $a'$ to $y$. Thus we must have $\phi(P') \geq \psi(y) \geq r_{m+n-1}$. Let $h : P' \mapsto R$ be a homomorphism such that $h(t(P')) \geq r_{m+n-1}$. Now $h(t(P')) = h(a') \neq r_{m+n}$, because $h(a')$ is the end of the edge starting in $h(a)$, while $r_{m+n}$ has indegree zero. Also $h(t(P')) = h(a') \neq r_{m+n-1}$, otherwise $h(a) = r_{m+n}$ which contradicts the assumption that $\phi(P_x) = \psi(x) \leq r_{m+n-1}$. This final contradiction proves the lemma.

**COROLLARY 8** *Assume $x, y \in L, xy \in E(G)$. If $P_x \in \mathcal{P}_x, P_y \in \mathcal{P}_y$ then $P_x \bullet A \in \mathcal{P}$, $P_y \bullet A^T \in \mathcal{P}$.*

**LEMMA 9** *Assume $x, y \in K_1$. If $xy \in E(G)$, then $\psi(x)\psi(y) \in E(R)$.*

**Proof**. Suppose first that $\psi(x) < \psi(y)$. Let $P_x \in \mathcal{P}_x$ and let $P' = P_x \bullet A$. There is a homomorphism $g : P' \mapsto G$ with $g(a) = x$ and $g(a') = y$. Since $P' \in \mathcal{P}$ (by the above Corollary) and $g(t(P')) = y$, there is a homomorphism $h' : P' \mapsto R$ such that $h'(a') \geq \psi(y)$. On the other hand, $h'(a) \leq \phi(P_x) = \psi(x) < \psi(y)$, since $h'$ restricted to $P_x$ is a homomorphism. Now $h'(a)h'(a') \in E(R)$, and so $Ind(h'(a')) \leq Ind(h'(a)) + 1$. (Since $\psi(x) < \psi(y), \psi(x)$ can not be $r_m$). Then $Ind(h'(a)) \leq Ind(\psi(x)) < Ind(\psi(y)) \leq Ind(h'(a')) \leq Ind(h'(a)) + 1$ implies that $h'(a) = \psi(x), h'(a') = \psi(y)$ and therefore $\psi(x)\psi(y) \in E(R)$.

A similar argument applies in the case $\psi(x) > \psi(y)$. One only needs to use $P_y \in \mathcal{P}_y$ and $P'' = P_y \bullet A^T$, and a homomorphism $h'' : P'' \mapsto R$ such that $h''(a') \geq \psi(x)$.

It remains to consider the case $\psi(x) = \psi(y)$. Let $P', h', P''$ and $h''$ be defined as above. Let $\psi(x) = \psi(y) = r_i$. Then as above, $h'(a) \leq \psi(x) = r_i = \psi(y) \leq h'(a')$. Now $h'(a)h'(a') \in E(R)$ implies that $Ind(h'(a)) \geq Ind(h'(a')) - 1 \geq i - 1$. Therefore either $h'(a) = r_i$, or $h'(a) = r_{i-1}$. But $h'(a)$ cannot be $r_{i-1}$, because the homomorphism inherent in the definition of $\phi(P_x)$ maps $P_x$ to a path that starts at $r_0$ and ends at $r_i$, so (as homomorphisms preserve distances) $h'$ cannot map $P_x$ to a path that starts at $r_0$ and ends at $r_{i-1}$. (By the remark, $h'(i(P_x)) = r_0$). Therefore $h'(a) = \psi(x) = r_i$ and $h'(a') = r_{i+1}$ (or $r_{m+n}$ if $i = m$). So $r_i r_{i+1}$ (or $r_m r_{m+n}$) $\in E(R)$. The same argument applied to $P''$ and $h''$ will show that $r_{i+1} r_i$ (or $r_{m+n} r_m$) $\in E(R)$. This is a contradiction because $R$ has no pair of opposite edges. Therefore this case can not happen and the lemma is proved.

**LEMMA 10** *Assume $x, y \in K_2$. If $xy \in E(G)$, then $\psi(x)\psi(y) \in E(R)$.*

**Proof**. Again, we take $P_x \in \mathcal{P}_x, P_y \in \mathcal{P}_y$, $P' = P_x \bullet A$, and $P'' = P_y \bullet A^T$; note that $P' \mapsto G$ and $P'' \mapsto G$ with $a$ going to $x$ and $a'$ to $y$. Since $P' \in \mathcal{P}$, there exists a homomorphism $h' : P' \mapsto R$ with $h'(a') \geq \psi(y)$. Clearly, $h'(a) \leq \psi(x)$. As there is an edge from $h'(a)$ to $h'(a')$, $Ind(h'(a')) = Ind(h'(a)) - 1$; hence $Ind(\psi(y)) \leq Ind(\psi(x)) - 1$. A similar argument applied to $P''$ shows that $Ind(\psi(y)) \geq Ind(\psi(x)) - 1$. Thus $Ind(\psi(y)) = Ind(\psi(x)) - 1$ and $\psi(x)\psi(y) \in E(R)$.

**LEMMA 11** *Assume $x \in K, y \in K_2$. If $xy \in E(G)$, then $\psi(x)\psi(y) \in E(R)$.*

**Proof**. Note that we cannot use our Corollary, as $x \notin L$. However, proceeding by contradiction, since we have $\psi(x) = r_{m+n}$, we may assume that $\psi(y) \leq r_{m+n-2}$. In this case we still may take any $P_y \in \mathcal{P}_y$ and be assured of $l(P_y) \geq 2$, or else $\phi(P_y) \geq r_{m+n-1}$. Thus letting $P'' = P_y \bullet A^T$, we have $P'' \in \mathcal{P}$. There is a homomorphism $P'' \mapsto G$ taking $t(P'') = a$ to $x$. Therefore there is a homomorphism $h'' : P'' \mapsto R$such that $h''(a) = r_{m+n}$. Now $h''(i(P'')) = r_0$ because $r_0$ is the only point in $C$ which has positive distance to $r_{m+n}$. However this is a contradiction because $P_y$ contains no subpath of length $n$.

**LEMMA 12** *Assume $x \in K_1, y \in K_2$. Then $xy \notin E(G)$, and if $yx \in E(G)$, then $l(P_x) = 2$ and $l(P_y) = 1$ for any $P_x \in \mathcal{P}_x, P_y \in \mathcal{P}_y$.*

**Proof**. Suppose $xy \in E(G)$ and let $P' = P_x \bullet A$, for $P_x \in \mathcal{P}_x$. Thus $P' \in \mathcal{P}$ by the Corollary. There is a homomorphism $P' \mapsto G$ taking $a$ to $x$ and $a'$ to $y$. Hence there is a homomorphism $h' : P' \mapsto R$ such that $h'(a') \geq \psi(y) \geq r_{m+1}$. Also, we have $h'(a) \leq \phi(P_x) = \psi(x) \leq r_m$. Hence $h'(a) = r_m$ and $h'(a') = r_{m+n}$. This is impossible, as $r_m r_{m+n} \notin E(R)$.

Now assume that $yx \in E(G)$. An argument identical to the above (with $A =< a, a' >$ consisting of the single *backward* arc $a'a$) shows that $h'(a) = r_m$ and $h'(a') = r_{m+n}$. Since $d_R(r_0, r_m) = 2$, we have $l(P_x) = d_{P_x}(i(P_x), a) = d_R(h'(i(P_x)), h'(a)) = d_R(r_0, r_m) = 2$, because $h'(i(P_x)) = r_0$ and $h'(a) = r_m$.

Let $P_y \in \mathcal{P}_y$; we prove that $l(P_y) = 1$. Let $P'' = P_x \bullet A \bullet (P_y)^T$, and let $f : P'' \mapsto C$ be a homomorphism. Since $i(P'')$ is the initial point of some minimal interval of $P''$ of length $n$, we have $f(i(P'')) = c_0$, and $f$ begins by mapping $P''$ to $C$ in the positive direction. If $l(P_y) = q \geq 2$, then $l(P) = 1 - q < 0$, and $f$ must eventually wind $P''$ around $C$ in the negative direction. But this is impossible, since $P_y$ can not wind around $C$. Therefore $l(P_y) = 1$.

## 3 The homomorphism $h_\psi$

In the previous section we constructed a mapping $\psi$ from $G$ to $R$, and so, by composition with the natural homomorphism $R \mapsto C$, a mapping from $G$ to $C$. The above lemmas suggest that $\psi$ is very close to being a homomorphism; however it is not a homomorphism in general. In this section we will modify this mapping to construct a true homomorphism from $G$ to $C$. Roughly speaking, we shall make a correction for those vertices $x$ that are forced into $K_1$ by a path in $\mathcal{P}_x$ which would allow mapping $x$ further along $R$ if a length zero portion of it were cut out. Specifically:

**DEFINITION 13** *Let $M$ denote the set of all vertices $x \in K_1$ for which there exists a $P_x \in \mathcal{P}_x$, a homomorphism $g : P_x \mapsto G$ with $g(t(P_x)) = x$, and a pair of vertices $u < v \in P_x$ such that $g(u) = g(v), l(P_x[u, v]) = 0$, and $P_x[., u] \bullet P_x[v, .]$ contains no interval of length $n$. If $x \in M$ then any $P_x$ as above has the same length, and we denote it by $i(x)$.*

*Define the mapping $h_\psi : G \mapsto V(C)$ as follows:*

$h_\psi(x) = c_0$ *if* $x \in K$.
$h_\psi(x) = c_i$ *where* $i = Ind(\psi(x))$ *if* $x \in K_1 \backslash M$.
$h_\psi(x) = c_j$ *if* $x \in K_2$ *and* $\psi(x) = c_{m+n-j}$.
$h_\psi(x) = c_i$ *if* $x \in M$ *and* $i(x) = i$.

**THEOREM 14** *The mapping $h_\psi$ is a homomorphism of $G$ to $C$.*

**Proof**. Suppose $x, y \in V(G)$ and $xy \in E(G)$. We shall show that $h_\psi(x)h_\psi(y) \in E(C)$. By considering the natural homomorphism $R \mapsto C$ and the definition of $h_\psi$, the above lemmas imply the following:

If both $x$ and $y$ are in $K_1 \backslash M$ then $h_\psi(x)h_\psi(y) \in E(C)$.
If both $x$ and $y$ are in $K_2$ then $h_\psi(x)h_\psi(y) \in E(C)$.
If $x \in K$ and $y \in K_2$ then $h_\psi(x)h_\psi(y) \in E(C)$.

We have also proved that it is never the case that $x \in K_1$ and $y \in K_2$. Note that it is also never the case that $y \in K$, since $r_{m+n}$ has indegree zero. Therefore we complete the proof of the theorem by showing the following assertions:

If $x \in K$ and $y \in K_1$ then $h_\psi(x)h_\psi(y) \in E(C)$.
If both $x \in M$ and $y \in M$ then $h_\psi(x)h_\psi(y) \in E(C)$.
If $x \in M, y \in K_1 \backslash M$ or $y \in M, X \in K_1 \backslash M$ then $h_\psi(x)h_\psi(y) \in E(C)$.
If $x \in K_2$ and $y \in K_1$ then $h_\psi(x)h_\psi(y) \in E(C)$.

We proceed to prove these four assertions in a sequence of four lemmas.

**LEMMA 15** *Assume $x \in K, y \in K_1$. If $xy \in E(G)$ then $h_\psi(x)h_\psi(y) \in E(C)$.*

**Proof**. Let $P_y \in \mathcal{P}_y$ and $P = P_y \bullet A^T$. Then all vertices $v \in P$ have $0 \leq \lambda_P(v) \leq n$.

Suppose first that $l(P) > 0$: Then $P \in \mathcal{P}$ and $P \mapsto G$ so that $t(P) = a$ is taken to $x$. Thus there is a homomorphism $h : P \mapsto R$ such that $h(a) = r_{m+n}$, which implies $h(a') = r_m$ because $h(a') \leq \psi(y) \leq r_m$. Hence $\psi(y) = r_m$. If $y \notin M$, then $h_\psi(y) = c_m$, (because $\psi(y) = r_m$) which implies $h_\psi(x)h_\psi(y) \in E(C)$. Hence we assume that $y \in M$. We may also assume that $P_y$ fulfills the requirements of the definition of $M$, i.e., that there exists a homomorphism $g : P_y \mapsto G$ with $g(t(P_y) = x$, and a pair of vertices $u < v \in P_y$ such that $g(u) = g(v), l(P_y[u, v]) = 0$, and $P' = P_y[., u] \bullet P_y[v, .]$ contains no interval of length $n$. Let $P'' = P' \bullet A^T$. It is clear that $P'' \mapsto G$ taking $a$ to $x$ (since $g(u) = g(v)$ we can use the restriction of $g$), and that $P'' \in \mathcal{P}$ (because $l(P'') = l(P)$). Since $x \in K$, there is a homomorphism $h'' : P'' \mapsto R$ such that $h''(a) = r_{m+n}$. Now the length of $P''$ is positive, and the only vertex of $R$ with a positive distance to $r_{m+n}$ is $r_0$. Hence $h''(i(P'')) = r_0$, contrary to $P'$ not containing an interval of length $n$.

Hence $l(P) = 0$. Then for each vertex $u \in P, d_P(i(P), u) = d_P(t(P), u)$; in particular, $l(P_y) = 1$. If $\psi(y) = r_1$, then $h_\psi(y) = c_1$ (whether or not

$y \in M$); hence $h_\psi(x)h_\psi(y) \in E(C)$. Thus suppose that $\psi(y) > r_1$. Since $y \in K_1$, there exists in $P_y$ a vertex of level $n$. Let $u$ be the last such vertex on $P_y$. Let P* be a path isomorphic to, but disjoint from, P; let $v*$ be the vertex of $P*$ corresponding to the vertex $v$ of $P$. (For notational reasons, we denote the terminal point by $(a')* = a''$.) Let $P' = A \bullet P_y[u,.]^T \bullet P_y^*[u,.]$. Then $P' \mapsto G$ so that the image starts with the edge $xy$ and returns to $y$. It is easy to verify that $P' \in \mathcal{P}$. Now $P'$ and $P_y$ are two paths in $P$ which admit homomorphisms to $G$ with the terminal points $a', a''$ taken to $y$. We claim that $\phi(P') = \phi(P_y)$, i.e., that for any homomorphism $h : P' \mapsto R$ there exists a homomorphism $h' : P_y \mapsto R$ such that $h'(a') = h(a'') = y$. Thus let $h : P' \mapsto R$ be a homomorphism. Since $P'[.,u]$ is a minimal interval of length $n$, we have $h(u) = r_n$. Define $h' : P_y \mapsto R$ as follows:

for $v \in P_y[.,u]$ let $h'(v) = r_i$ where $i = \lambda_{P_y}(v)$
for $v \in P_y[u,.]$ let $h'(v) = h(v*)$.

Then $h'$ is a homomorphism from $P_y$ to $R$ and $h'(a') = h(a")$. Therefore $\phi(P') = \phi(P_y)$ which implies $P' \in \mathcal{P}_y$. Then the path $P'$ shows that $y \in M$. Since $l(P') = 1$ we have $h_\psi(y) = c_1$ and $h_\psi(x)h_\psi(y) \in E(C)$.

**LEMMA 16** *Assume both $x \in M$ and $y \in M$. If $xy \in E(G)$ then $h_{\psi(x)}h_\psi(y) \in E(C)$.*

**Proof**. By an earlier lemma, $\psi(x)\psi(y) \in E(R)$. Let $P_x \in \mathcal{P}_x, P_y \in \mathcal{P}_y$, let $h : P_x \mapsto R$ be a homomorphism such that $h(t(P_x)) = \psi(x)$ and $h' : P_y \mapsto R$ a homomorphism such that $h'(t(P_y)) = \psi(y)$. Since $h(i(P_x)) = h'(i(P_y)) = r_0$, $l(P_x) = \lambda_R(\psi(x)) = \lambda_R(\psi(y) - 1 = l(P_y) - 1$. Suppose $l(P_x) = i$ and $l(P_y) = j$: since $i = j - 1$, $h_\psi(x)h_\psi(y) = c_{j-1}c_j \in E(C)$.

**LEMMA 17** *Assume $x \in M, y \in K_1 \backslash M$, or $y \in M, x \in K_1 \backslash M$. If $xy \in E(G)$ then $h_\psi(x)h_\psi(y) \in E(C)$.*

**Proof**. Take paths $P_x \in \mathcal{P}_x, P_y \in \mathcal{P}_y$, and let $P' = P_x \bullet A \bullet P_y^T$. As in lemma 16, we find that $l(P_x) = l(P_y) - 1$, i.e., that $l(P') = 0$. Assume first that $x \in M, y \in K_1 \backslash M$. Then we may assume that $P_x$ contains vertices $u < v$ such that some homomorphism of $P_x$ to $G$ which takes $t(P_x)$ to $x$ maps $u$ and $v$ to the same vertex of $G$, and such that $l(P_x[u,v]) = 0$ and $P_x[.,u] \bullet P_x[v,.]$ contains no interval of length $n$. Let also $z$ be the last vertex of $P_y$ of level $n$. Let $P'' = P_x[.,u] \bullet P_x[v,.] \bullet A \bullet (P_y[z,.])^T \bullet P_y[z,.]$. Obviously $z$ is the only vertex of $P''$ with level $n$. By the same argument as used in the proof of lemma 15, we can show that $P'' \in \mathcal{P}_y$. This would mean that $y \in M$ unless $y = u$. Thus we must have $y = u$ and $\psi(y) = \phi(P'') = r_n$. Therefore $l(P_x) = n-1$ and $h_\psi(x)h_\psi(y) = c_{n-1}c_n \in E(C)$. If $y \in M, x \in K_1 \backslash M$, then one finds analogous vertices $u, v \in P_y, z \in P_x$, and a corresponding argument applied to $P'' = P_y[.,u] \bullet P_y[v,.] \bullet A^T \bullet (P_x[z,.])^T \bullet P_x[z,.]$ shows that $x \in M$, as $l(P_x) \neq n$ by lemma 7. Thus this case can not happen, and the lemma is proved.

**LEMMA 18** *Assume* $x \in K_2, y \in K_1$. *If* $xy \in E(G)$ *then* $h_\psi(x)h_\psi(y) \in E(C)$.

**Proof**. Let $P_x \in \mathcal{P}_x, P_y \in \mathcal{P}_y$. Let $u$ be the last point of $P_y$ with $\lambda_{P_y}(u) = n$. Let $P = P_x \bullet A \bullet (P_y[u,.])^T \bullet P_y[u,.]$. By lemma 12, $l(P_x) = 1, l(P_y) = 2$. Using an argument from the proof of lemma 15 we can show that $P \in \mathcal{P}_y$. If $u = t(P_y)$ then $n = 2$, and $\psi(y) = \phi(P) = r_2$. Hence $h_\psi(x)h_\psi(y) = c_1c_2 \in E(C)$. If $u \neq t(P_y)$, then $P$ shows that $y \in M$. Again $h_\psi(y) = c_2$ (since $l(P) = 2$) and $h_\psi(x)h_\psi(y) \in E(C)$.

This completes the proof of both our theorems.

# 4 Conclusions

For any $\mathcal{B}$-cycle $C$, our result identifies the obstructions to a possible homomorphism $G \mapsto C$, as oriented paths homomorphic to $G$ but not to $C$. There are similar obstruction theorems for other graphs and digraphs, [24], [21], [3]. For instance, it is well known, [3], [21], that if $C$ is a directed cycle, then $G \mapsto C$ if and only if $G$ contains only cycles of length divisible by the length of $C$. Perhaps the following may hold:

**Conjecture 19** *Let $C$ be any oriented cycle. Then $G \mapsto C$ if and only if*

*each oriented path homomorphic to $G$ is also homomorphic to $C$,*
*and*
*each oriented cycle of $G$ has length divisible by the length of $C$.*

Our main theorem verifies the conjecture for $\mathcal{B}$-cycles, which have length 1 and thus automatically satisfy the divisibility condition. The above example verifies the conjecture for directed cycles, which admit a homomorphism from any oriented path and thus automatically satisfy the first condition. We have also verified the conjecture in a few additional cases.

There are corresponding results for oriented paths, [25]:

**THEOREM 20** *Let $P$ be an oriented path. Then $G \mapsto P$ if and only if each oriented path homomorphic to $G$ is also homomorphic to $P$.*

As mentioned in the introduction, our main motivation in this paper was to prove the following result, [23]:

**THEOREM 21** *Let $C$ be an oriented cycle. Then $C$ is multiplicative if and only if $C$ is a $\mathcal{C}$-cycle.*

**Proof-sketch**. The necessity of the condition was proved in [17] (cf. also [18]). Thus assume that $C$ is a $\mathcal{C}$-cycle, and $G \not\mapsto C, G' \not\mapsto C$. Each $\mathcal{C}$-cycle is a $\mathcal{B}$-cycle, and therefore, according to our main result, there exist paths $P, P'$, homomorphic to $G, G'$ respectively, such that $P \not\mapsto C$ and $P' \not\mapsto C$. We prove in [23] that this implies that there exists a path $P*$ homomorphic to $P \times P'$ (and hence also homomorphic to $G \times G'$) such that $P* \not\mapsto C$. Thus we have $G \times G' \not\mapsto C$ and hence $C$ is multiplicative.

As another application of our main result, we shall prove that, for each $\mathcal{B}$-cycle $C$, the following decision problem is in $NP \cap co-NP$:

**Instance:** A digraph $G$.

**Question:** Is $G$ homomorphic to $C$ ?

It is easy to see that the problem is in $NP$. The fact that it also belongs to $co-NP$ is an easy consequence of the two lemmas below (with $H = C$), from [25]. It should be observed that at this time there is no known polynomial algorithm for this problem.

**DEFINITION 22** *Let $P =< p_0, p_1, \ldots p_m >$ be an oriented path and $H$ any digraph. The* cannonical labeling *of $P$ by $H$ is the unique mapping $l$ of $P$ to the subsets of $V(H)$ for which*

$$
\begin{aligned}
& l(p_0) = V(H) \\
& l(p_{i+1}) = \{v \in V(H) : \textit{ for some } u \in l(p_i), uv \in E(H) \textit{ if } p_i p_{i+1} \in \\
& \quad E(P), \textit{ mboxor } vu \in E(H) \textit{ if } p_{i+1} p_i \in E(P)\}.
\end{aligned}
$$

**LEMMA 23** *Let $P =< p_0, p_1, \ldots p_m >$ be an oriented path and $H$ any digraph. Then $P \mapsto H$ if and only if $l(p_m) = \emptyset$ in the cannonical labeling of $P$ by $H$.*

**LEMMA 24** *Let $H$ be a digraph with $k$ vertices, and $G$ a digraph with $n$ vertices. If there exist oriented paths $P$ is homomorphic to $G$ but not to $H$, then there exists such a path $P$ of length at most $2^k \cdot n$.*

Lemma 24 implies that any digraph $H$ which admits an obstruction characterization in terms of paths (such as our main theorem, or 20) has a certificate for $G \not\mapsto H$, which is a path of length polynomial in the size of $G$. (The digraph $H$ is fixed, thus $2^k$ is a constant.) Then the previous lemma verifies the certificate in polynomial time. We noted this for $H = P$, an oriented path, in [25]. These observations extend to any cycle $C$ for which the above conjecture holds, thus conjecture implies that the existence problem for homomorphism to any oriented cycle is in $NP \cap co-NP$.

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