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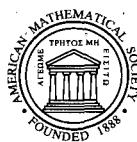
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Number 902

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## Homotopical Algebraic Geometry II: Geometric Stacks and Applications

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May 2008 • Volume 193 • Number 902 (second of 5 numbers) • ISSN 0065-9266

American Mathematical Society  
Providence, Rhode Island

# Contents

Abstract	ix
Introduction	1
Reminders on abstract algebraic geometry	1
The setting	2
Linear and commutative algebra in a symmetric monoidal model category	2
Geometric stacks	3
Infinitesimal theory	4
Higher Artin stacks (after C. Simpson)	4
Derived algebraic geometry: $D^-$ -stacks	4
Complicial algebraic geometry: $D$ -stacks	6
Brave new algebraic geometry: $S$ -stacks	6
Relations with other works	7
Acknowledgments	8
Notations and conventions	9
<b>Part 1. General theory of geometric stacks</b>	11
Introduction to Part 1	13
Chapter 1.1. Homotopical algebraic context	15
Chapter 1.2. Preliminaries on linear and commutative algebra in an HA context	25
1.2.1. Derivations and the cotangent complex	25
1.2.2. Hochschild homology	29
1.2.3. Finiteness conditions	30
1.2.4. Some properties of modules	35
1.2.5. Formal coverings	36
1.2.6. Some properties of morphisms	37
1.2.7. Smoothness	41
1.2.8. Infinitesimal lifting properties	41
1.2.9. Standard localizations and Zariski open immersions	43
1.2.10. Zariski open immersions and perfect modules	47
1.2.11. Stable modules	49
1.2.12. Descent for modules and stable modules	54
1.2.13. Comparison with the usual notions	57
Chapter 1.3. Geometric stacks: Basic theory	61
1.3.1. Reminders on model topoi	61
1.3.2. Homotopical algebraic geometry context	65
1.3.3. Main definitions and standard properties	76
1.3.4. Quotient stacks	80
1.3.5. Quotient stacks and torsors	83

1.3.6. Properties of morphisms	86
1.3.7. Quasi-coherent modules, perfect modules and vector bundles	88
<b>Chapter 1.4. Geometric stacks: Infinitesimal theory</b>	97
1.4.1. Tangent stacks and cotangent complexes	97
1.4.2. Obstruction theory	105
1.4.3. Artin conditions	109
<b>Part 2. Applications</b>	121
Introduction to Part 2	123
<b>Chapter 2.1. Geometric <math>n</math>-stacks in algebraic geometry (after C. Simpson)</b>	129
2.1.1. The general theory	129
2.1.2. Comparison with Artin's algebraic stacks	132
<b>Chapter 2.2. Derived algebraic geometry</b>	135
2.2.1. The HA context	135
2.2.2. Flat, smooth, étale and Zariski open morphisms	139
2.2.3. The HAG context: Geometric $D^-$ -stacks	151
2.2.4. Truncations	154
2.2.5. Infinitesimal criteria for smooth and étale morphisms	159
2.2.6. Some examples of geometric $D^-$ -stacks	164
2.2.6.1. Local systems	164
2.2.6.2. Algebras over an operad	170
2.2.6.3. Mapping $D^-$ -stacks	173
<b>Chapter 2.3. Complicial algebraic geometry</b>	177
2.3.1. Two HA contexts	177
2.3.2. Weakly geometric $D$ -stacks	181
2.3.3. Examples of weakly geometric $D$ -stacks	182
2.3.3.1. Perfect modules	182
2.3.3.2. The $D$ -stacks of dg-algebras and dg-categories	183
2.3.4. Geometric $D$ -stacks	187
2.3.5. Examples of geometric $D$ -stacks	188
2.3.5.1. $D^-$ -stacks and $D$ -stacks	188
2.3.5.2. CW-perfect modules	188
2.3.5.3. CW-dg-algebras	192
2.3.5.4. The $D$ -stack of negative CW-dg-categories	193
<b>Chapter 2.4. Brave new algebraic geometry</b>	199
2.4.1. Two HAG contexts	199
2.4.2. Elliptic cohomology as a Deligne-Mumford $S$ -stack	203
<b>Appendix A. Classifying spaces of model categories</b>	207
<b>Appendix B. Strictification</b>	211
<b>Appendix C. Representability criterion (after J. Lurie)</b>	215
<b>Bibliography</b>	219
<b>Index</b>	223