

Homotopies in Classical and Paraconsistent Modal Logics

Can BAŞKENT

The Graduate Center of the City University of New York
cbaskent@gc.cuny.edu www.canbaskent.net

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Outlook of the Presentation

- ▶ Introduction
- ▶ Homotopies and Modal Logic
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Introduction I

Paraconsistent logic is an umbrella term for the formal systems where *ex contradictione quodlibet* ($\varphi, \neg\varphi \vdash \psi$ for all φ, ψ) fails. Apart from the standard proof-theoretical approach, paraconsistent logics have been analyzed from intuitionistic logical (Priest, 2009), category theoretical (Lawvere, 1969) and algebraic (Lawvere, 1991; Awodey, 2006) perspectives.

In this work, we discuss both classical and paraconsistent modal logics with topological semantics. As well known, topological semantics for classical modal logics was introduced in 1940s by McKinsey and Tarski (McKinsey & Tarski, 1946; McKinsey & Tarski, 1944). Similarly, topological semantics for paraconsistent systems was suggested by Goodman in late 1970s (Goodman, 1981).



Introduction II

Our contribution is to introduce (homoeomorphisms and) homotopies in the context of aforementioned logics in a validity preserving way.

How does homotopies can be defined in a topological system which gives meaning to paraconsistency?



Modal Semantics I

In classical modal logic, we associate the extension (i.e. the points that satisfy a given formula) of a modal formula with its interior (or dually, with its closure). Therefore, the extension $|\Box\varphi|$ of $\Box\varphi$ in the model M , is the interior of the extension of φ , i.e. $|\Box\varphi|^M = \text{Int}(|\varphi|^M)$. In this respect, there is a well-known connection between S4 Kripke models and modal topological models.

A topology σ on a (non-empty) set S is a collection of subsets of S , such that:

- ▶ $\emptyset, S \in \sigma$.
- ▶ σ is closed under arbitrary union.
- ▶ σ is closed under finite intersection.



Modal Semantics II

Formally, we define a topological model as $M = (S, \sigma, V)$ where S is a non-empty set, σ is a topology on S , and V is a valuation function taking propositional variables and returning subsets of S . In topological semantics, extensions of modal formulas are open or closed sets (though not necessarily *only* of modal formulas).



Paraconsistent and Intuitionistic Semantics

We stipulate that the extension of *any* formula be open (or dually, closed). In this case, we obtain the intuitionistic logic with the topology of opens (or dually, the paraconsistent logic with the topology of closed sets).

The problem here is the negation: because the complement of an open (closed) is not an open (closed) in general. For this reason, the intuitionistic negation in topological models is defined as the “interior of the complement” (Mints, 2000). Similarly, the paraconsistent negation is defined as the “closure of the complement” (Goodman, 1981; Mortensen, 2000). Under these assumptions, observe that in the topology of opens (closed sets), any theory that includes the theory of the propositions that are true at the boundary is incomplete (inconsistent) (Goodman, 1981; Mortensen, 2000).



Basics I

Two topological spaces are called *homeomorphic* if there is a continuous bijection with a continuous inverse from one space to the other.

Moreover, two continuous functions are called *homotopic* if there is a continuous deformation between the two.

Figure



Basics II

An immediate observation yields that since extensions of all formulae in paraconsistent models are closed (respectively, open in paracomplete models), the topologies obtained in both cases are discrete (Başkent, 2011). For a given model M , let $|M|$ denote the size of M 's carrier set.



Basics III

Theorem

Let M_1 and M_2 be paraconsistent and paracomplete topological models respectively. If $|M_1| = |M_2|$, then there is a homeomorphism from a paraconsistent topological model to the paracomplete one, and vice versa.

Theorem

Let $M = (S, \sigma, V)$ and $M' = (S', \sigma', V')$ be two paraconsistent topological models with a homeomorphism f from S to S' .

Assume $V'(p) = f(V(p))$. Then, $M \models \varphi$ iff $M' \models \varphi$ for all φ .

Here, note that one direction of the biconditional can be satisfied by the continuity whereas the other direction is satisfied by the openness of f .



Homotopies and Paraconsistency I

We can now introduce homotopies to paraconsistent topological models.

A *homotopy* between f and f' is a family of continuous functions $H_t : S \rightarrow S'$ such that for $t \in [0, 1]$ we have $H_0 = f$ and $H_1 = g$ and the map $h : t \rightarrow H_t$ is continuous from $[0, 1]$ to the space of all continuous functions from S to S' .

Given a model $M = (S, \sigma, V)$, we call the family of models $\{M_t = (S_t \subseteq S, \sigma_t, V_t)\}_{t \in [0,1]}$ generated by homotopic functions and M as homotopic models. In the generation, we put $V_t = f_t(V)$.



Homotopies and Paraconsistency II

Theorem

Given two topological paraconsistent models $M = \langle S, \sigma, V \rangle$ and $M' = \langle S', \sigma', V' \rangle$ with two continuous functions $f, f' : S \rightarrow S'$ both of which respect the valuation: $V' = f(V) = f'(V)$. If there is a homotopy H between f and f' , then homotopic models satisfy the same formulae.



Homotopies and Paraconsistency III

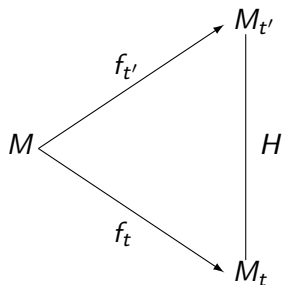


Figure: Homotopic Models



Theorem

Given two classical topological modal models $N = \langle T, \eta, V \rangle$ and $N' = \langle T', \eta', V' \rangle$ with two continuous functions $f, f' : T \rightarrow T'$ both of which respect the valuation: $V' = f(V) = f(V')$. If there is a homotopy H between f and f' , then homotopic models satisfy the same formula.



Comparing Bisimilar Models I

Consider the given two Kripke models M and M' . Assume that w, w' and u, u', y' and v, v', x' satisfy the same propositional letters. Then, it is easy to see that w and w' are bisimilar, and therefore satisfy the same formulae.

We pose a conceptual question about the relation between M and M' . Even if these two models are bisimilar, they are essentially different models, yet they are indistinguishable from modal logic perspective. However, it is topologically possible to *contract* M' to M in a validity preserving fashion. The problem now is the following: Given a model M , how can we *measure* the level of change from M to M' ? We use homotopies as a measure for transformation to solve this problem. We define homeo-topo-bisimulations, and show that they preserve validity.



Comparing Bisimilar Models II

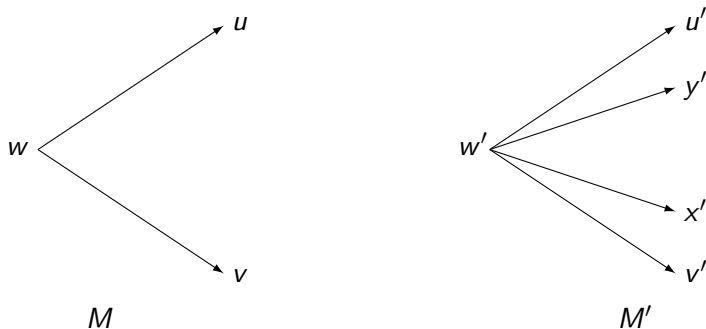


Figure: Two Bisimilar Models M and M' .



Homeo-topo-bisimulation I

Definition

Let $M = \langle S, \sigma, \nu \rangle$ and $M' = \langle S', \sigma', \nu' \rangle$ be two topological models. A homeo-topo-bisimulation is a nonempty relation $\rightleftharpoons_f \subseteq S \times S'$ based on a homeomorphism f from S into S' such that if $s \rightleftharpoons_f s'$, then we have the following:

1. $s \in \nu(p)$ if and only if $s \in \nu'(p)$ for any propositional variable p .
2. $s \in U \in \sigma$ implies that there exists $f(U) \in \sigma'$ such that $s' \in f(U)$ and for all $t' \in f(U)$ there exists $t \in U$ with $t \rightleftharpoons_f t'$
3. $s' \in f(U) \in \sigma'$ implies that there exists $U \in \sigma$ such that $s \in U$ and for all $t \in U$ there exists $t' \in f(U)$ with $t \rightleftharpoons_f t'$



Homeo-topo-bisimulation II

Theorem

Homeo-topo-bisimulations preserve the validity.



Parametrizing Difference Between Bisimilar Models I

This is how we use homotopies to measure the level of change. Let M be a given topological model. Construct M_f and M_g as the homeomorphic image of M respecting the valuation where f and g are homeomorphism (one can take f as identity for simplicity). For simplicity, assume that $M \rightleftharpoons_f M_f$ and $M \rightleftharpoons_g M_g$. Now, if f and g are homotopic, then we have functions h_x for x continuous on $[0, 1]$ with $h_0 = f$ and $h_1 = g$. Therefore, given $x \in [0, 1]$ the model M_x will be obtained by applying h_x to M respecting the valuation. Hence, $M_0 = M_f$ and $M_1 = M_g$.



Parametrizing Difference Between Bisimilar Models II

Therefore, given M , the distance of any homeo-topo-bisimilar model M_x to M will be x , and it will be the measure of non-modal change in the model. In other words, even if $M \Leftrightarrow_{h(x)} M_x$, we will say M and M_x are x -different than each other. This establishes a measurable and observable relation between bisimilar models, and can be used in various modal logical applications.



What was the Motivation?

Now, we can compare and discuss the connection between two *updated epistemic models* within the framework of Dynamic Epistemic Logic.

Homotopies establish a relation between continuous updates!



Thanks!

Thanks for your attention!

Talk slides and the papers are available at:

www.CanBaskent.net



Thanks!

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