HOMOTOPY GROUPS OF COMPACT LIE GROUPS E_6, E_7 AND E_8

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§ 1. Introduction

Let G be a simple, connected, compact and simply-connected Lie group. If k is the field with characteristic zero, then the algebra of cohomology $H^*(G;k)$ is the exterior algebra generated by the elements $x_1, \cdots, x_l$ of the odd dimension $n_1, \cdots, n_l$; the integer l is the rank of G and $n = \sum_{i=1}^{l} n_i$ is the dimension of G. Let X be the direct product of spheres of dimension $n_1, \cdots, n_l$, then there exists a continuous map $f: G \longrightarrow X$ which induces isomorphisms of $H^i(X;k)$ to $H^i(G;k)$ for all i (cf. [8]). From this we deduce by Serre's C-theory [8] that $f_*: \pi_i(G) \longrightarrow \pi_i(X)$ are C-isomorphisms for all i, where C is the class of finite abelian groups. Therefore the rank of $\pi_q(G)$ is equal to the number of such i that n_i is equal to q, and particularly if q is even, then $\pi_q(G)$ is finite. It is a classical fact that $\pi_2(G) = 0$ and $\pi_3(G) = Z$.

According to Bott-Samelson [6];

$$\pi_i(E_6) = 0 \quad \text{for } 4 \leqslant i \leqslant 8, \qquad \pi_9(E_6) = Z,$$
$$\pi_i(E_7) = 0 \quad \text{for } 4 \leqslant i \leqslant 10, \qquad \pi_{11}(E_7) = Z,$$
$$\pi_i(E_8) = 0 \quad \text{for } 4 \leqslant i \leqslant 14, \qquad \pi_{15}(E_8) = Z.$$

where E_6, E_7 and E_8 are compact exceptional Lie groups.

In this paper, using the killing method we compute the 2-components of homotopy group $\pi_j(G)$, where $G = E_6, E_7$ and E_8. The resuls are stated as follows;

j	$4 \leqslant j \leqslant 14$	15	16	17	18	19	20	21	22	23
$\pi_j(E_8:2)$	0	Z	Z_2	Z_2	Z_8	0	0	Z_2	0	$Z+Z_2$

j	24	25	26	27	28
$\pi_j(E_8:2)$	Z_2+Z_2	Z_2	0	Z	0

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j	$4 \leqslant j \leqslant 10$	11	12	13	14	15	16	17	18	19
$\pi_j(E_7:2)$	0	Z	Z_2	Z_2	0	Z	Z_2	Z_2	Z_4	$Z+Z_2$

j	20	21	22	23	24	25
$\pi_j(E_7:2)$	Z_2	Z_2	Z_4	$Z_2+Z_2+Z_2$	$Z_2+Z_2+Z_2$	Z_2+Z_2

j	$4 \leqslant j \leqslant 8$	9	10	11	12	13	14	15	16	17
$\pi_j(E_6:2)$	0	Z	0	Z	Z_4	0	0	Z	0	$Z+Z_2$

j	18	19	20	21	22
$\pi_j(E_6:2)$	$Z_{16}+Z_2$	0	Z_8	0	0

All spaces that we concider in this paper are those which have the homotopy groups of finite type. Let G be such a space, then $\pi_i(G)$ is isomorphic to the direct sum of a free part F and the p-components of $\pi_i(G)$ for every prime p. We denote by $\pi_i(G:p)$ the direct sum of a certain subgroup F' of F and the p-component of $\pi_i(G)$, where the index $[F;F']$ is prime to p.

Given an exact sequence for such A, B and C

$$\cdots \longrightarrow \pi_i(A) \longrightarrow \pi_i(B) \longrightarrow \pi_i(C) \longrightarrow \cdots,$$

then we can form the following exact one in our case

$$\cdots \longrightarrow \pi_i(A:p) \longrightarrow \pi_i(B:p) \longrightarrow \pi_i(C:p) \longrightarrow \cdots.$$

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§ 2. The cohomology of the 3-connective fibre spaces of E_6, E_7 and E_8.

H. Cartan and J.P. Serre introduced a method to calculate the homotopy group in [7]. Let $K(\pi, n)$ be an Eilenberg-Mac-Lane space of type (π, n).

THEOREM 2. 1. *Let X be an arcwise connected topological space, then there exists a sequence of $(n-1)$-connected spaces (X, n) $(n=1, 2, \cdots,$ and $(X, 1)=X)$ and continuous maps $f_n:(X, n+1) \longrightarrow (X, n)$ such that:*

(I) *the triple* $((X, n+1), f_n, (X, n))$ *is a fibre space with a fibre* $K(\pi_n(X), n-1)$.
(II) *there exists a fibre space* X'_n *over the base space* $K(\pi_n(X), n)$, *where* X'_n *and* (X, n) *are of the same homotopy type, such that the fibre is* $(X, n+1)$.

Hence $f_1 \circ f_2 \circ \cdots \circ f_{n-1}$ *defines the isomorphism of* $\pi_i(X, n)$ *to* $\pi_i(X)$ *for* $i \geqslant n$.

LEMMA 2. 2. *Let* X *be a* 2*-connected topological space. Assume that* X *satisfies the following conditions,*

(1) $\pi_3(X)$ *is isomorphic to an infinite cyclic group,*

(2) $H^*(X; Z_2) = A_0 \otimes A_1 \otimes \cdots \otimes A_r \otimes B$

where x_3 *is a generator of* $H^3(X; Z_2) \approx Z_2$, $A_0 = Z_2[x_3]/(x_3)^{s_0}$, $A_i = Z_2[Sq^{2^i}Sq^{2^{i-1}} \cdots Sq^2x_3]/(Sq^{2^i}Sq^{2^{i-1}} \cdots Sq^2x_3)^{2^{s_i}}$ $(s_i \geqslant 1)$ $1 \leqslant i \leqslant r$, *and* $Sq^{2^{r+1}}Sq^{2^r} \cdots Sq^2x_3 = 0$, *then*

$$H^*((X, 4); Z_2) = Z_2[w] \otimes \Delta(a_0, a_1, \cdots, a_r) \otimes B'$$

where the $deg.a_i = (2^{i+1}+1)(2^{s_i}-1) + 2^{2^i}$, $deg.w = 2^{2^{r+1}}$, $\Delta(a_0, a_1, \cdots, a_r)$ *indicates a submodule having* $a_0, \cdots, a_r$ *as a simple system of generators and* B' *is isomorphic to* B *by* $(f_1 \circ f_2 \circ f_3)^* : H^*(X; Z_2) \longrightarrow H^*((X, 4); Z_2)$.

Proof. From the above theorem, there exists a fibre space $((X, 4), f_1 \circ f_2 \circ f_3, X)$ with a fibre $K(Z, 2)$. Since $K(Z, 2)$ is the infinite dimensional complex projective space, its mod 2 cohomology structure is $H^*(Z, 2; Z_2) \approx Z_2[u]$, where u is a generator of $H^2(Z, 2; Z_2)$. Let $\{E_r^{**}\}$ be the mod 2 spectral sequence associated to the above fibration $((X, 4), X, K(Z, 2))$, then

$$E_2^{**} = A_0 \otimes A_1 \otimes \cdots \otimes A_r \otimes B \otimes Z_2[u].$$

Clearly we have $d_3(1 \otimes u) = x_3 \otimes 1$. Hence if n is even, $d_3(1 \otimes u^n) = 0$, if n is odd, $d_3(1 \otimes u^n) = x_3 \otimes u^{n-1}$, and $d_3(x_3^{s_0-1} \otimes u^n) = 0$ for all $n > 0$. Thus we obtain

$$E_4^{**} = \Delta(\bar{a}_0) \otimes A_1 \otimes A_2 \otimes \cdots \otimes A_r \otimes B \otimes Z_2[u^n]$$

where $\bar{a}_0 = (x_3)^{s_0-1} \otimes u$.

Let τ be the transgression, $\tau(u^2) = Sq^2x_3$, since the transgression commutes the Steenrod operation. Thus $d_5(1 \otimes u^2) = Sq^2x_3 \otimes 1$. Since d_t is derivative, $d_5(1 \otimes u^{2n}) = 0$ if n is even, $d_5(1 \otimes u^{2n}) = Sq^2x_3 \otimes u^{2(n-1)}$ if n is odd, and $d_5((Sq^2x_3)^{2^{s_i}-1} \otimes u^{2n}) = 0$ for all $n \geqslant 1$. Thus

$$E_6^{**} = \Lambda(\bar{a}_0, \bar{a}_1) \otimes A_2 \otimes A_3 \otimes \cdots \otimes A_r \otimes B \otimes Z_2[u^4]$$

where $\bar{a}_1 = (Sq^2x_3)^{2^{s_1}-1} \otimes u^2$.

Carrying on similarly, we have

$$E_{2^{r+1}+2}^{**} = \Lambda(\bar{a}_0, \bar{a}_1, \cdots, \bar{a}_r) \otimes B \otimes Z_2[u^{2^{r+1}}]$$

where $\bar{a}_i = (Sq^{2^i}Sq^{2^{i-1}} \cdots Sq^2x_3)^{2^{s_i}-1} \otimes u^{2^i}$, $i = 0, 1, \cdots, r$, and $s_i \geqslant 1$. Clearly $d_t = 0$ for all $t \geqslant 2^{r+1}+2$. Thus we obtain

$$E_\infty^{**} = \Lambda(a_0, a_1, \cdots, a_r) \otimes B \otimes Z_2[u^{2^{r+1}}].$$

Since E_∞^{**} is the graded algebra associated to $H^*((X,4)\,;Z_2)$, assume that a_i, w, B' correspond to $\bar{a}_i, u^{2^{r+1}}, B$ respectively. We have the lemma.

Particularly, we can assume that B is mapped isomorphically onto B' by the homomorphism $(f_1 \circ f_2 \circ f_3)^*\,; H^*(X\,;Z_2) \longrightarrow H^*((X,4)\,;Z_2)$. Thus the relation of B are arranged in B'.

The mod 2 cohomology algebra of the exceptional Lie groups have been determined by S. Araki [2] and S. Araki-Y. Shikata [3]. These algebra are as follow.

(2. 1) $H^*(F_4\,;Z_2) = Z_2[x_3]/(x_3^4) \otimes \Lambda(Sq^2x_3, x_{15}, Sq^8x_{15})$,

(2. 2) $H^*(E_6\,;Z_2) = Z_2[x_3]/(x_3^4) \otimes \Lambda(Sq^2x_3, Sq^4Sq^2x_3, x_{15}, Sq^8Sq^4Sq^2x_3, Sq^8x_{15})$,

(2. 3) $H^*(E_7\,;Z_2) = Z_2[x_3, Sq^2x_3, Sq^4Sq^2x_3]/(x_3^4, (Sq^2x_3)^4, (Sq^4Sq^2x_3)^4)$
$\otimes \Lambda(x_{15}, Sq^8Sq^4Sq^2x_3, Sq^8x_{15}, Sq^4Sq^8x_{15})$,

(2. 4) $H^*(E_8\,;Z_2) = Z_2[x_3, Sq^2x_3, Sq^4Sq^2x_3, x_{15}]/(x_3^{16}, (Sq^2x_3)^8, (Sq^4Sq^2x_3)^4, x_{15}^4)$
$\otimes \Lambda(Sq^8Sq^4Sq^2x_3, Sq^8x_{15}, Sq^4Sq^8x_{15}, Sq^2Sq^4Sq^8x_{15})$

where x_i denotes a generator of degree i.

(2. 5) In the inclusion $F_4 \subset E_6 \subset E_7 \subset E_8$, every subgroup is totally non-homologous to zero mod 2 in any bigger group containing it, where each exceptional group denotes simply-connected one. (See, S. Araki and Y. Shikata [3], Theorem 3).

If $Sq^{16}Sq^8Sq^4Sq^2x_3 = 0$ in E_8, then this is a primitive element. By (2. 4), there is no primitive element of degree 33. Thus $Sq^{16}Sq^8Sq^4Sq^2x_3 = 0$ in E_8. Similarly we have $Sq^{16}Sq^8Sq^4Sq^2x_3 = 0$ in E_6, E_7 and $Sq^4Sq^2x_3 = 0$ in F_4.

COROLLARY 2. 3. *Let $\tilde{G}$ be the 3-connective fibre space over G: where $G = F_4, E_6, E_7, E_8$, then*

$$H^*(\tilde{F}_4\,;Z_2)=Z_2[y_8]\otimes\Delta(y_9,y_{11},y_{15},y_{23}),$$
$$H^*(\tilde{E}_6\,;Z_2)=Z_2[y_{32}]\otimes\Delta(y_9,y_{11},y_{15},y_{17},y_{23},y_{33}),$$
$$H^*(\tilde{E}_7\,;Z_2)=Z_2[y_{32}]\otimes\Delta(y_{11},y_{15},y_{19},y_{23},y_{27},y_{33},y_{35}),$$
$$H^*(\tilde{E}_8\,;Z_2)=Z_2[y_{15},y_{32}]/(y_{15}^4)\otimes\Delta(y_{23},y_{27},y_{29},y_{33},y_{35},y_{39},y_{47}),$$

where y_i *denotes a generator of degree* i. *By the naturality of the homomorphism* $p^*=(f_1f_2f_3)^*$, *we have*

$$Sq^8y_{15}=y_{23}\quad\text{in } \tilde{E}_6,\tilde{E}_7,\tilde{E}_8 \text{ and } \tilde{F}_4,$$
$$Sq^4y_{23}=y_{27}\quad\text{in } \tilde{E}_7,\tilde{E}_8,$$
$$Sq^2y_{27}=y_{29}\quad\text{in } \tilde{E}_8.$$

LEMMA 2. 4. *We have the following relations,*

(i) $Sq^1y_8=y_9,\ Sq^2y_9=y_{11}$ in $\tilde{F}_4$,

(ii) $Sq^2y_9=y_{11},\ Sq^8y_9=y_{17}$ in $\tilde{E}_6$,

(iii) $Sq^8y_{11}=y_{19}$ in $\tilde{E}_7$.

Proof. (i) From Theorem 2. 1, there exists a fibration $(\bar{F}_4,K(Z,3),\tilde{F}_4)$, where $\bar{F}_4$ denotes the space which has same homotopy type as F_4. We consider the spectral sequence $\{E_r^{**}\}$ over Z_2 associated with the above fibration. Then

$$E_2^{**}=H^*(Z,3\,;Z_2)\otimes H^*(\tilde{F}_4\,;Z_2).$$

It is known that

$$H^*(Z,3\,;Z_2)=Z_2[v,Sq^2v,Sq^4Sq^2v,\cdots]$$

where v is a fundamental class of $H^3(Z,3:Z_2)$. From the mod 2 cohomology algebra of $F_4, Sq^4v\otimes 1,(Sq^2v)^2\otimes 1$ and $v^4\otimes 1$ must be d_r-images for some r. If $p\neq 0$ and $0<q<8$, or $q\neq 0$ and $0<p<3$, then $E_r^{p,q}=0$ for all r. Since $E_r^{0,8}$ has only one element $1\otimes y_8$ for $r\leqslant 9$, $Sq^4Sq^2v\otimes 1$ is not a d_r-image for $r\leqslant 8$. Thus τ be the transgression, we have $\tau(y_8)=Sq^4Sq^2v$. Since $E_r^{0,9}$ has only one generator $1\otimes y_9$ and $(Sq^2v)^2\otimes 1$ is not a d_r-image for $r\leqslant 10$, we have that $\tau(y_9)=(Sq^2v)^2$. Consider

$$d_r: E_r^{p,q}\longrightarrow E_r^{12,0}\qquad\text{for } p+q=11 \text{ and } r=q+1.$$

From Corollary 2. 3, we have $E_r^{p,q}=0$ for $q\neq 8,9$. But $E_r^{2,9}=0$. $E_r^{3,8}$ has one generator $v\otimes y_8$ and $d_9(v\otimes y_8)=vSq^4Sq^2v\otimes 1\neq 0$, for $d_9(1\otimes y_8)$

$= Sq^4Sq^2v \otimes 1$. Thus $E_{12}^{0,11}$ has only one generator $1 \otimes y_{11}$ and $v^4 \otimes 1$ is not a d_r-image for $r \leqslant 11$. Therefore we have that $\tau(y_{11}) = v^4$. Using Adem's relation, from $Sq^1Sq^4Sq^2v = Sq^5Sq^2v = (Sq^2v)^2$, $Sq^2(Sq^2v)^2 = Sq^2Sq^5Sq^2v = Sq^6Sq^3v = v^4$, we obtain $Sq^1y_8 = y_9$, and $Sq^2y_9 = y_{11}$.

(ii) From Theorem 2. 1, there exists a fibration $(\bar{E}_6, K(Z,3), \tilde{E}_6)$ where $\bar{E}_6$ denotes the space which has the same homotopy type as E_6. Let τ be the transgression associated with this fibration. Let $\{E_r^{p,q}\}$ be the mod 2 spectral sequence associated with this fibration. Then

$$E_2^{**} = H^*(Z,3\,;Z_2) \otimes H^*(\tilde{E}_6\,;Z_2).$$

By the same argument as in $\tilde{F}_4$, we have that $\tau(y_9) = (Sq^2v)^2$ and $\tau(y_{11}) = v^4$. Concider

$$d_r\,;E_r^{p,q} \longrightarrow E_r^{18,0} \text{ for } p+q=17 \text{ and } r=q+1.$$

From Corollary 2. 3, we have $E_r^{p,q} = 0$ for $q \neq 9, 11, 15$ and 17 $(q \leqslant 22)$. But $E_r^{2,15} = 0$. $E_{10}^{8,9}$ has one generator $(vSq^2v) \otimes y_9$ and $d_{10}((vSq^2v) \otimes y_9) = v(Sq^2v)^3 \otimes 1 \neq 0$, for $d_{10}(1 \otimes y_9) = (Sq^2v)^2 \otimes 1$. $E_{12}^{6,11}$ has one generator $v^2 \otimes y_{11}$ and $d_{12}(v^2 \otimes y_{11}) = v^6 \otimes 1 \neq 0$ for $d_{12}(1 \otimes y_{11}) = v^4 \otimes 1$. Thus, since $E_{17}^{0,17}$ has one generator y_{17} and $(Sq^4Sq^2v)^2 \otimes 1$ is not a d_r-image for $r \leqslant 16$, $d_{18}(1 \otimes y_{17}) = (Sq^4Sq^2v)^2 \otimes 1$, i.e. $\tau(y_{17}) = (Sq^4Sq^2v)^2$. Using Adem's relation, $Sq^2(Sq^2v)^2 = Sq^2Sq^5Sq^2v = Sq^6Sq^3v = v^4$ and $Sq^8(Sq^2v)^2 = Sq^8Sq^5Sq^2v = Sq^9Sq^4Sq^2v = (Sq^4Sq^2v)^2$. From the commutativity of the Steenrod operation and the transgression, we obtain $Sq^2y_9 = y_{11}$ and $Sq^8y_9 = y_{17}$.

(iii) Consider the fibration $(\bar{E}_7, K(Z,3), \tilde{E}_7)$ of theorem 2. 1 (II), where $\bar{E}_7$ has the same homotopy type as E_7. Let $\{E_r^{p,q}\}$ be the mod 2 spectral sequence associated with this fibration. Then

$$E_r^{**} = H^*(Z,3\,;Z_2) \otimes H^*(\tilde{E}_7\,;Z_2).$$

From the mod 2 cohomology algebra of E_7, $v^4 \otimes 1$ and $(Sq^2v)^4 \otimes 1$ must be the d_r-images for some r. Since $H^*(\bar{E}_7\,;Z_2) = 0$ for degree $\leqslant 10$, we have $E_r^{p,q} = 0$ for $p \neq 0$ and $0 < q < 10$. Thus we have that $\tau(y_{11}) = v^4$, where τ is the transgression. Consider

$$d_r\,;E_r^{p,q} \longrightarrow E_r^{20,0} \quad \text{for } p+q=19 \text{ and } r=q+1.$$

From $H^i(\tilde{E}_7\,;Z_2) = 0$ for $i \neq 11, 15$ and $i < 19$, it follow that $E_r^{p,q} = 0$ for $(p,q) \neq (4,11)$ and $(2,15)$. On the other hand $H^i(Z,3\,;Z_2) = 0$ for $i = 2, 4$

and $i \leqslant 4$. Thus $E_r^{p,q}=0$ for $(p,q)=(4,11)$ and $(2,15)$. From this we obtain $\tau(y_{19})=(Sq^2v)^4$. By Adem's relation $Sq^8v^4=Sq^8Sq^6Sq^3v=Sq^{10}Sq^4Sq^3v+Sq^{11}Sq^3Sq^3v=Sq^{10}Sq^5Sq^2+Sq^{11}Sq^5Sq^1v=(Sq^2v)^4$. Thus we obtain $Sq^8y_{11}=y_{19}$.

LEMMA 2.5. *Let a topological space X be 2-connected and the homology of finite type. Assume that $H^*(X;Z_2)$ has the additive basis $a_1,\cdots,a_s$ for dim. $<N$. Then there exist a finite cell complex $K=_*\cup e_1\cup e_2\cup\cdots\cup e_s$, where dim. e_i = degree $a_i=n_i$ and a continuous map $f;K\longrightarrow X$ such that f induces isomorphism of $H^*(X;Z_2)$ onto $H^*(K;Z_2)$ for dim. $<N$.*

Particularly if $\pi_{n_i-1}(K^{n_i-1})$ is finite, then we can assume that the class of attaching map of e_i belong to the 2-components. Here $_*$ denotes a vertex and K^n the n-skelton of K.

Proof. We prove this by induction on dimension N. Suppose that there exist a finite cell complex $K_0=K^{N-1}$ and a continuous map $f_0;K_0\longrightarrow X$ satisfying lemma 2.5 for dim. $<N$. Here we may assume that $f_0;K_0\longrightarrow X$ is the injection by the mapping-cylinder argument. Suppose that $H^N(X;Z_2)$ has generator $a_{s+1},\cdots,a_r$.

From the cohomology exact sequence for pair (X,K_0) and the assumption of the induction, we have

$$H^i(X,K_0;Z_2)=0 \qquad \text{for } i<N,$$
$$H^N(X,K_0;Z_2)\approx H^N(X;Z_2).$$

By the duality, we obtain

$$H_i(X,K_0;Z_2)=0 \qquad \text{for } i<N$$

and

$$H_N(X,K_0;Z_2) \text{ has the generators } \bar{a}_{s+1},\cdots,\bar{a}_r.$$

By Serre's C-theory [8], we have that $\pi_N(X,K_0)\otimes Z_2\longrightarrow H_N(X,K_0)\otimes Z_2$ is an isomorphism. Let $f_i:(E^N,S^{N-1})\longrightarrow(X,K_0)$ $(i=1,2,\cdots,r-s)$ be the generators of $\pi_N(X,K_0)$ such that they correspond to $\bar{a}_{s+i}$ by the above isomorphism and construct a cell complex K which is obtained from the disjoint union of $C(S_1^{N-1}\vee\cdots\vee S_{r-s}^{N-1})$ and K_0 by identifying $S_1^{N-1}\vee\cdots\vee S_{r-s}^{N-1}$ with its image under a map $(f_1|S_1^{N-1})\vee\cdots\vee(f_{r-s}|S_{r-s}^{N-1})$; $S_1^{N-1}\vee\cdots\vee S_{r-s}^{N-1}\longrightarrow K_0$, where CY is a cone over the space Y and S_i^{N-1} is a $(N-1)$-sphere. Using the map f_i the inclusion map $f_0;K_0\longrightarrow X$ has an extension over K and we denote this extension by $g : K \longrightarrow X$. Then $g : K \longrightarrow X$ induce an isomorphism $H_N(K, K_0 ; Z_2)$ onto $H_N(X, K_0 ; Z_2)$ and from the duality between homology and cohomology, it follows that $g^* : H^N(X, K_0 ; Z_2) \longrightarrow H^N(K, K_0 ; Z_2)$ is an isomorphism onto.

Applying the five lemma to the diagram

$$\begin{array}{ccccccc} H^{N-1}(K_0 ; Z_2) & \longrightarrow & H^N(X, K_0 ; Z_2) & \longrightarrow & H^N(X ; Z_2) & \longrightarrow & H^N(K_0 ; Z_2) = 0 \\ \downarrow \approx & & \downarrow g^* & & \downarrow g^{**} & & \downarrow \approx \\ H^{N-1}(K_0 ; Z_2) & \longrightarrow & H^N(K, K_0 ; Z_2) & \longrightarrow & H^N(K ; Z_2) & \longrightarrow & H^N(K_0 ; Z_2) = 0, \end{array}$$

we obtain that

$$g^* : H^N(X ; Z_2) \longrightarrow H^N(K ; Z_2)$$

is an isomorphism.

Particularly if $\pi_{N-1}(K_0)$ is finite, then there exists an odd integer q such that $q\{f_i | S^{N-1}\}$ belongs to the 2-component of $\pi_{N-1}(K_0)$. Displacing f_i by qf_i, it is sufficient for the last statement that we construct a cell complex K from K_0. Consequently the lemma is proved.

Let α be an element of $\pi_{n+i-1}(S^n)$ and consider a cell complex $K_\alpha = S^n \cup_\alpha e^{n+i}$ which is uniquely determined by α up to homotopy type.

THEOREM 2. 6. *Let $n > i$ and $i = 2$ (4 or 8 respectively), then $Sq^i : H^n(K_\alpha ; Z_2) \longrightarrow H^{n+i}(K_\alpha ; Z_2)$ is an isomorphism onto if and only if $\alpha \equiv \eta_n$, (ν_n or σ_n respectively) mod $2\pi_{n+i-1}(S^n)$. (For the proof see H. Tada;* [11] *Proposition* 8. 1)

From Lemma 2. 5 and Corollary 2. 3, there exist a cell complex $M = S^8 \cup e^9 \cup e^{11} \cup e^{15}$ and a continuous map $f : M \longrightarrow \tilde{F}_4$ such that f induces an C_2-isomorphisms $\pi_i(M)$ onto $\pi_i(\tilde{F}_4)$ for $i \leqslant 14$, where C_2 is the classes of finite abelian group whose 2-primary components are zero. Since $Sq^1 y_8 = y_9$ in $\tilde{F}_4$, we may assume that e^9 is attached to S^8 by a map of degree two. Then we have

$$(2.\,6) \qquad \pi_{13}(S^8 \cup_2 e^9 : 2) = 0,$$

$$\pi_{14}(S^8 \cup_2 e^9 : 2) \approx \pi_{14}(S^8 : 2) = Z_2 \qquad \text{generated by } \nu_8^2,$$

we denote by ν_8^2 a generator of $\pi_{14}(S^8 \cup_2 e^9 : 2)$ identifying with that of $\pi_{14}(S^8 : 2)$ by the inclusion $S^8 \subset S^8 \cup_2 e^9$.

Consider the following exact sequence

$$\pi_i(S^8:2) \longrightarrow \pi_i(S^8:2) \longrightarrow \pi_i(S^8 \cup e^9:2) \longrightarrow \pi_i(S^9:2) \longrightarrow \pi_i(S^9:2)$$

for $i \leqq 15$. From $\pi_{12}(S^8)=\pi_{13}(S^9)=\pi_{14}(S^9)=0$ and $\pi_{14}(S^8)=\{\nu_8^2\}=Z_2$, (2. 6) is obtained.

Consider the exact sequence

$$\pi_{14}(S^{10}:2) \longrightarrow \pi_{14}(S^8 \underset{2}{\cup} e^9:2) \xrightarrow{i^*} \pi_{14}(S^8 \underset{2}{\cup} e^9 \cup e^{11}:2) \xrightarrow{j_*} \pi_{14}(S^{11}:2)$$
$$\longrightarrow \pi_{14}(S^9 \underset{2}{\cup} e^{10}:2)$$

where i is the inclusion $S^8 \underset{2}{\cup} e^9 \subset S^8 \underset{2}{\cup} e^9 \cup e^{11}$, and $j: S^8 \underset{2}{\cup} e^9 \cup e^{11} \longrightarrow S^{11}$ is the projection. From (2. 6), we have the following exact sequence

(2. 7) $$0 \longrightarrow \pi_{14}(S^8 \underset{2}{\cup} e^9:2) \xrightarrow{i^*} \pi_{14}(S^8 \underset{2}{\cup} e^9 \cup e^{11}:2) \xrightarrow{j_*} \pi_{14}(S^{11}:2) \longrightarrow 0\,.$$

Then there exists a coextension (in the sense of [11]) $\tilde{\nu}_{10}$ of ν_{10} and $j_*\tilde{\nu}_{10}=\nu_{11}$. Assume that $8\tilde{\nu}_{10}=0$, then $-i_*\nu_8^2=i_*\nu_8^2=8\tilde{\nu}_{10}$. Let $f: S^{14} \vee S^{11} \longrightarrow S^8 \underset{2}{\cup} e^9 \cup e^{11}$ be a map such that $f|S^{14}$ and $f|S^{11}$ representative of $8\iota_{14} \oplus \nu_{11}$, then $f \circ g: S^{14} \longrightarrow S^8 \underset{2}{\cup} e^9 \cup e^{11}$ is homotopic to zero. Consider a mapping cone C_f of f, then there exists a coextension $G: S^{15} \longrightarrow C_f$ of g. Let K be a mapping cone of G, then we have a complex

$$K=S^8 \cup e_6 \cup e^{11} \cup e^{12} \cup e^{15} \cup e^{16}$$

and $Sq^4u_8=u_{12}$, $Sq^4u_{12}=u_{16}$, where u_8, u_{12} and u_{16} are cohomology classes mod 2 which are represented by S^8, e^{12} and e^{16} respectively. Thus it is verified that $Sq^4Sq^4u_8 \neq 0$ in K. By use of Adem's relation

$$Sq^4Sq^4u_8=Sq^6Sq^2u_8+Sq^2Sq^6u_8.$$

Since there is no cell of dimension 10 or 14 in K, the right side of the above equation vanishes in K, but this is a contradiction. Thus we have proved that $8\tilde{\nu}_{10}=0$. Therefore, from the exact sequence (2. 7), we obtain

$$\pi_{14}(S^8 \cup e^9 \cup e^{11}:2)=\{i_*\nu_8^2\}+\{\tilde{\nu}_{10}\} \approx Z_2+Z_8\,.$$

In the complex $M=S^8 \underset{2}{\cup} e^9 \cup e^{11} \cup e^{15}$, let e^{15} be attached to $S^8 \underset{2}{\cup} e^9 \cup e^{11}$ by a map $h: S^{14} \longrightarrow S^8 \underset{2}{\cup} e^9 \cup e^{11}$, then we have the sequence

$$\pi_{14}(S^{14}:2) \xrightarrow{h_*} \pi_{14}(S^8 \underset{2}{\cup} e^9 \cup e^{11}:2) \longrightarrow \pi_{14}(M:2) \longrightarrow \pi_{14}(S^{15}:2)=0$$

is exact. By Lemma 5. 5 of [10], $\pi_{14}(F_4)=Z_2$. Thus $\pi_{14}(M:2)\approx Z_2$ and

$$h_*\iota_{14}=b\bar{\nu}_{10}+a(i_*\nu_8^2) \quad \text{where } a=0 \text{ or } 1,$$

for an odd integer b. Thus

$$j_*h_*\iota_{14}=\nu_{11} \mod 2\pi_{14}(S^{11}).$$

By theorem 2. 6, we have the following important lemma.

LEMMA 2. 7. $Sq^4y_{11}=y_{15}$ in $\tilde{F}_4$.

Considering the natural inclusions $\tilde{F}_4\subset\tilde{E}_6\subset\tilde{E}_7$, we have

COROLLARY 2. 8. *$Sq^4y_{11}=y_{15}$ in $\tilde{E}_6$ and $\tilde{E}_7$.*

§ 3. Homotopy group of some cell complexes.

Let X be an m-connected CW-complex and let α be an element of $\pi_{n-1}(X)$ $(n>m)$. Consider a CW-complex $K_\alpha=X\underset{\alpha}{\cup}e^n$.

LEMMA 3. 1. *Let i be an injection $X\longrightarrow K_\alpha$ and let $p:K_\alpha\longrightarrow S^n$ be a mapping which shrinks X to a point. Then the following sequence is exact for $j\leqslant m+n-1$*

$$(3.1)\qquad \cdots\longrightarrow\pi_j(S^{n-1})\xrightarrow{\alpha_*}\pi_j(X)\xrightarrow{i_*}\pi_j(K_\alpha)\longrightarrow\pi_{j-1}(S^{n-1})\xrightarrow{\alpha_*}\pi_{j-1}(X)\longrightarrow\cdots.$$

Here ∂ is a composition $E^{-1}\circ p_*:\pi_j(K_\alpha)\longrightarrow\pi_{j-1}(S^{n-1})$, and $E:\pi_{j-1}(S^{n-1})\longrightarrow\pi_j(S^n)$ is the suspension homomorphism. If α is of order a power of 2, then the above sequence is exact for the 2-primary components.

Proof. See Blakers-Massey [4].

We introduce necessary results on the homotopy group of spheres. According to [11], the results are listed in the following table;

(i) $n>k+1$

(3. 2)

$k=$	0	1	2	3	4	5	6	7	8
$\pi_{n+k}(S^n:2)$	Z	Z_2	Z_2	Z_8	0	0	Z_2	Z_{16}	Z_2+Z_2
Generator	ι_n	η_n	η_n^2	ν_n			ν_n^2	σ_n	$\bar{\nu}_n, \varepsilon_n$

$k=$	9	10	11	12	13
$\pi_{n+k}(S^n:2)$	$Z_2+Z_2+Z_2$	Z_2	Z_8	0	0
Generator	$\nu_n^3, \eta_n\varepsilon_{n+1}, \mu_n$	$\eta_n\mu_{n+1}$	ζ_n		

(ii) $n \leqslant k+1$ $\quad n=9,\ 10,\ 11,\ 13,\ 14.$

(3. 3)

$k=$	8	9	10	11
$\pi_{k+9}(S^9:2)$	$Z_2+Z_2+Z_2$	$Z_2+Z_2+Z_2+Z_2$	Z_8+Z_2	Z_8+Z_2
Generator	$\sigma_9\eta_{16}, \bar{\nu}_9, \varepsilon_9$	$\sigma_9\eta_{16}^2, \nu_9^3, \mu_9, \eta_9\varepsilon_{10}$	$\sigma_9\nu_{16}, \eta_9\mu_{10}$	$\zeta_9, \bar{\nu}_9\nu_{17}$
$\pi_{k+10}(S^{10}:2)$		$Z+Z_2+Z_2+Z_2$	Z_4+Z_2	Z_8
Generator		$\Delta(\iota_{21}), \nu_{10}^3, \mu_{10}, \eta_{10}\varepsilon_{11}$	$\sigma_{10}\nu_{17}, \eta_{10}\mu_{11}$	ζ_{10}
$\pi_{k+11}(S^{11}:2)$			Z_2+Z_2	Z_8
Generator			$\sigma_{11}\nu_{18}, \eta_{11}\mu_{12}$	ζ_{11}
$\pi_{k+13}(S^{13}:2)$				
Generator				
$\pi_{k+14}(S^{14}:2)$				
Generator				

$k=$	12	13	14
$\pi_{k+9}(S^9:2)$	0	Z_2	$Z_{16}+Z_4$
Generator		$\sigma_9\nu_{16}^2$	σ_9^2, κ_9
$\pi_{k+10}(S^{10}:2)$	Z_4	Z_2	$Z_{16}+Z_2$
Generator	$\Delta(\nu_{21})$	$\sigma_{10}\nu_{17}^2$	$\sigma_{10}^2, \kappa_{10}$
$\pi_{k+11}(S^{11}:2)$	Z_2	Z_2+Z_2	$Z_{16}+Z_2$
Generator	θ'	$\theta'\eta_{23}, \sigma_{11}\nu_{18}^2$	$\sigma_{11}^2, \kappa_{11}$
$\pi_{k+13}(S^{13}:2)$	Z_2	Z_2	$Z_{16}+Z_2$
Generator	$E\theta$	$E\theta\eta_{25}$	$\sigma_{13}^2, \kappa_{13}$
$\pi_{k+14}(S^{14}:2)$		Z	Z_8+Z_2
Generator		$\Delta(\iota_{29})$	$\sigma_{14}^2, \kappa_{14}$

We shall use the following relations;

(3. 4) $\qquad \sigma_n\circ\mu_{n+7}=\eta_n\circ\sigma_{n+1}=\bar{\nu}_n+\varepsilon_n \qquad$ for $n\geqslant 10$
by Lemma 6. 4 of [11],

(3. 5) $\sigma_n \circ \eta^2_{n+7} = \eta^2_n \circ \sigma_{n+2} = \nu^3_n + \eta_n \circ \varepsilon_{n+1}$ for $n \geqslant 10$
by Lemma 6. 3 of [11],

(3. 6) $\sigma_n \circ \nu_{n+7} = 0$ for $n \geqslant 12$
$\nu_n \circ \sigma_{n+3} = 0$ for $n \geqslant 11$,
$2\sigma_{10} \circ \nu_{17} = \nu_{10} \circ \sigma_{13}$ by (7. 20) of [11],
$\varepsilon_n \circ \eta^2_{n+8} = \eta^2_n \circ \varepsilon_{n+2} = 0$ for $n \geqslant 9$ by (7. 10) and (7. 20) of]11],

(3. 7) $\sigma_n \circ \bar{\nu}_{n+7} = 0$ for $n \geqslant 11$ by (10. 8) of [11],
$\sigma_n \circ \varepsilon_{n+7} = 0$ for $n \geqslant 6$ by Lemma 10. 7 of [11],

(3. 8) $\nu_n \circ \varepsilon_{n+3} = \nu_n \circ \nu_{n+3} = 0$ for $n \geqslant 7$ by (7. 17) of [11],
$\nu_n \circ \eta_{n+3} = \eta_n \circ \nu_{n+1} = 0$ for $n \geqslant 6$ by (5. 9) of [11],

(3. 9) $\nu_n \circ \mu_{n+3} = 0$ for $n \geqslant 7$ by Theorem 7. 6 of [11],

(3. 10) $\Delta(\iota_{21}) \circ \eta_{19} = 2\sigma_{10} \circ \nu_{17}$ by (7. 21) of [11].

Consider a generator σ_n of $\pi_{n+7}(S^n : 2) \approx Z_{16}$ for $n \geqslant 9$ and a cell complex $K_{\sigma_n} = S^n \cup_{\sigma_n} e^{n+8}$. Let $i : S^n \longrightarrow K_{\sigma_n}$ be the injection.

PROPOSITION 3. 2. *We have the following tables of the homotopy groups* $\pi_j(K_{\sigma_n} : 2)$ *for* $n = 9, 10, 11, 14$ *and* 15, *and generator of their* 2-*primary components.*

(3. 11)

j	$j \leq 8$	9	10	11	12	13	14	15	16
$\pi_j(K_{\sigma_9} : 2)$	0	Z	Z_2	Z_2	Z_8	0	0	Z_2	0
Generator		$i_*\iota_9$	$i_*\eta_9$	$i_*\eta^2_9$	$i_*\nu_9$			$i_*\nu^2_9$	

j	17	18	19	20	21	22
$\pi_k(K_{\sigma_9} : 2)$	$Z + Z_2 + Z_2$	$Z_2 + Z_2 + Z_2$	Z_2	$Z_8 + Z_2$	0	0
Generator	$\widetilde{16\iota_{16}}$, $i_*\varepsilon_9$, $i_*\bar{\nu}_9$	$i_*\eta_9\varepsilon_{10}$, $i_*\nu^3_9$, $i_*\mu_9$	$i_*\eta_9\mu_{10}$	$i_*\zeta_9$, $i_*\bar{\nu}_9\nu_{17}$		

(3. 12)

j	$j \leq 9$	10	11	12	13	14	15	16	17
$\pi_j(K_{\sigma_{10}} : 2)$	0	Z	Z_2	Z_2	Z_8	0	0	Z_2	0
Generator		$i_*\iota_{10}$	$i_*\eta_{10}$	$i_*\eta^2_{10}$	$i_*\nu_{10}$			$i_*\nu^2_{10}$	

j	18	19	20	21	22	23
$\pi_j(K_{\sigma_{10}}:2)$	$Z+Z_2$	$Z+Z_2+Z_2$	Z_2	Z_{16}	Z_4	0
Generator	$\widetilde{16\iota_{17}}, i_*\varepsilon_{10}$	$i_*\Delta(\iota_{21}), i_*\eta_{10}\varepsilon_{11}, i_*\mu_{10}$	$i_*\eta_{10}\mu_{11}$	$\widetilde{4\nu_{17}}$	$i_*\Delta(\nu_{21})$	

(3. 13)

j	$j\leq 9$	11	12	13	14	15	16	17	18
$\pi_j(K_{\sigma_{11}}:2)$	0	Z	Z_2	Z_2	Z_8	0	0	Z_2	0
Generator		$i_*\iota_{11}$	$i_*\eta_{11}$	$i_*\eta_{11}^2$	$i_*\nu_{11}$			$i_*\nu_{11}^2$	

j	19	20	21	22	23	24	25
$\pi_j(K_{\sigma_{11}}:2)$	Z_2+Z	Z_2+Z_2	Z_2	Z_{32}	Z_2	Z_2	Z_2
Generator	$i_*\varepsilon_{14}, \widetilde{16\iota_{18}}$	$i_*\mu_{11}, i_*\eta_{11}\varepsilon_{12}$	$i_*\eta_{11}\mu_{12}$	$\widetilde{2\nu_{18}}$	$i_*\theta'$	$i_*\theta'\eta_{23}$	$i_*\kappa_{11}$

(3. 14)

j	$j\leq 13$	14	15	16	17	18	19	20	21
$\pi_j(K_{\sigma_{14}}:2)$	0	Z	Z_2	Z_2	Z_8	0	0	Z_2	0
Generator		$i_*\iota_{14}$	$i_*\eta_{14}$	$i_*\eta_{14}^2$	$i_*\nu_{14}$			$i_*\nu_{14}^2$	

j	22	23	24	25	26	27
$\pi_j(K_{\sigma_{14}}:2)$	$Z+Z_2$	Z_2+Z_2	Z_2	Z_{64}	0	Z
Generator	$\widetilde{16\iota_{21}}, i_*\varepsilon_{14}$	$i_*\mu_{14}, i_*\eta_{14}\varepsilon_{15}$	$i_*\eta_{14}\mu_{15}$	$\widetilde{\nu_{21}}$		$i_*\Delta(\iota_{29})$

(3. 15)

j		15	16	17	18	19	20	21	22
$\pi_j(K_{\sigma_{15}}:2)$	0	Z	Z_2	Z_2	Z_8	0	0	Z_2	0
Generator		$i_*\iota_{15}$	$i_*\eta_{15}$	$i_*\eta_{15}^2$	$i_*\nu_{15}$			$i_*\nu_{15}^2$	

j	23	24	25	26	27	28
$\pi_j(K_{\sigma_{15}}:2)$	$Z+Z_2$	Z_2+Z_2	Z_2	Z_{64}	0	0
Generator	$\widetilde{16\iota_{22}}, i_*\varepsilon_{15}$	$i_*\mu_{15}, i_*\eta_{15}\varepsilon_{16}$	$i_*\eta_{15}\mu_{16}$	$\widetilde{\nu_{22}}$		

Here we denote by $\tilde{\beta}$ *an element of* $\pi_i(K_{\sigma_n}:2)$ *such that* $\partial\tilde{\beta}=\beta\in\pi_{i-1}(S^{n+7}:2)$ *i.e. we may consider that* $\tilde{\beta}$ *is a coextension of* β.

Proof. Consider the exact sequence

$$\cdots\longrightarrow\pi_j(S^{n+7}:2)\xrightarrow{\sigma_{n*}}\pi_j(S^n:2)\xrightarrow{i_*}\pi_j(K_{\sigma_n}:2)\xrightarrow{\partial}\pi_{j-1}(S^{n+7}:2)$$
$$\xrightarrow{\sigma_{n*}}\pi_{j-1}(S^n:2)\longrightarrow\cdots$$

of (3. 1) for $j\leqslant 2n+5$. From $\pi_j(S^{n+7}:2)=0$ for $j\leqslant n+6$ and from the exactness of the above sequence, it follows that

$$i_*:\pi_j(S^n:2)\longrightarrow\pi_j(K_{\sigma_n}:2)$$

are isomorphisms onto for $j\leqslant n+6$, and $n=9,10,11,14,15$.

It follows from (3. 1) that the sequence

$$\pi_{n+7}(S^{n+7}:2)\xrightarrow{\sigma_{n*}}\pi_{n+7}(S^n:2)\xrightarrow{i_*}\pi_{n+7}(K_{\sigma_n}:2)\xrightarrow{\partial}\pi_{n+6}(S^{n+7}:2)=0$$

is exact for $n\geqslant 9$. From $\pi_{n+7}(S^n:2)\approx\{\sigma_n\}\approx Z_{16}$, we have that

$$\sigma_{n*}:\pi_{n+7}(S^{n+7}:2)\longrightarrow\pi_{n+7}(S^n:2) \tag{3. 16}$$

is an epimorphism. Thus we obtain $\pi_{n+7}(K_{\sigma_n}:2)=0$ for $n=9,10,11,14$ and 15.

Consider the exact sequence

$$\pi_{n+8}(S^{n+7}:2)\xrightarrow{\sigma_n}\pi_{n+8}(S^n:2)\xrightarrow{i_*}\pi_{n+8}(K_{\sigma_n}:2)\xrightarrow{\partial}Z=\{16\iota_{n+7}\}\longrightarrow 0$$

of (3. 1) for $n\geqslant 9$. From (3. 2), (3. 3) and (3. 4) we have that

$$\sigma_{n*}:\pi_{n+8}(S^{n+7}:2)\longrightarrow\pi_{n+8}(S^n:2) \tag{3. 17}$$

are monomorphisms for $n\geqslant 9$. Thus it follows from the exactness of the above sequence that the table is true for $\pi_{n+8}(K_{\sigma_n}:2)$, $n=9,10,11,14,15$.

From (3. 17) and the exact sequence (3. 1), it follows that the sequence

$$\pi_{n+9}(S^{n+7}:2)\xrightarrow{\sigma_{n*}}\pi_{n+9}(S^n:2)\xrightarrow{i_*}\pi_{n+9}(K_{\sigma_n}:2)\longrightarrow 0$$

is exact for $n\geqslant 9$. From (3. 5), (3. 2) and (3. 3), we have that

$$\sigma_{n*}:\pi_{n+9}(S^{n+7}:2)\longrightarrow\pi_{n+9}(S^n:2) \tag{3. 18}$$

is monomorphisms for $n\geqslant 9$. Thus we obtain that

$$\pi_{n+9}(K_{\sigma_n}:2) \approx \pi_{n+9}(S^n:2)/\{\sigma_n \circ \eta_{n+7}\}.$$

From (3. 18) and the exact sequence (3. 1), it follows that the sequence

$$\pi_{n+10}(S^{n+7}:2) \xrightarrow{\sigma_{n*}} \pi_{n+10}(S^n:2) \xrightarrow{i_*} \pi_{n+10}(K\sigma_n:2) \longrightarrow 0$$

is exact for $n \geqslant 9$. From (3. 2), (3. 3) and (3. 6), it follows that

(3. 19) $\sigma_{9*}:\pi_{19}(S^{16}:2) \longrightarrow \pi_{19}(S^9:2)$ is a monomorphism,
$\sigma_{n*}:\pi_{n+10}(S^{n+7}:2) \longrightarrow \pi_{n+10}(S^n:2)$ is trivial for $n=14, 15$,
the kernel of $\sigma_{10*}:\pi_{20}(S^{17}:2) \longrightarrow \pi_{20}(S^{10}:2)$ is

generated by $\{4\nu_{17}\}$, and

the kernel of $\sigma_{11*}:\pi_{21}(S^{18}:2) \longrightarrow \pi_{21}(S^{11}:2)$ is

generated by $\{2\nu_{18}\}$.
Thus it follows that the table is true for $\pi_{n+10}(K_{\sigma_n}:2)$ $n=9, 10, 11, 14$ and 15.

In the stable rangs, we have the exact sequence

$$0 \longrightarrow \pi_{n+11}(S^n:2) \xrightarrow{i_*} \pi_{n+11}(K_{\sigma_n}:2) \xrightarrow{\partial} \pi_{n+10}(S^{n+7}:2) \longrightarrow 0$$

of (3. 1) for $n \geqslant 13$. Moreover we have the following relation in the stable secondary compositions

$$\zeta \in \langle \sigma, 4\nu, 2\iota \rangle \mod 2G_{11} \qquad \text{from Lemma 9. 1 of [11],}$$
$$\supset \langle \sigma, \nu, 8\iota \rangle \qquad \text{from Proposition 1. 2 of [11],}$$

and $\langle \sigma, \nu, 8\iota \rangle$ is a coset of the subgroup $\sigma \circ G_4 + 8G_{11} = 8G_{11}$. Thus

$$\zeta \equiv \langle \sigma, \nu, 8\iota \rangle \mod 2\ G_{11}$$

where G_n is the n-th stable homotopy group of the sphere and ζ is a generator of the 2-components of G_{11}.

From Proposition 1. 8 of [11], we obtain

$$i_*\xi = i_* \langle \sigma, \nu, 8\iota \rangle \mod 2\ i_*G_{11}$$
$$= -8\tilde{\nu}$$

where $\tilde{\alpha} \in \pi_i(K_{\sigma_n}:2)$ is a coextension of $\alpha \in \pi_{i-1}(S^{n+7}:2)$. Thus, from this and from the exactness of the above sequence it follows that

(3. 20) $$\pi_{n+11}(K_{\sigma_n}:2) = \{\tilde{\nu}\} = Z_{64}$$

for $n \geqslant 13$

From (3. 1), (3. 19) and from $\pi_{n+11}(S^{n+7}:2)=0$ for $n\geqslant 0$, it follows the next four exact sequences and the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_{20}(S^9:2) & \xrightarrow{i_*} & \pi_{20}(K_{\sigma_9}:2) & \longrightarrow & 0 & & \\
 & & \downarrow E & & \downarrow E & & & & \\
0 & \longrightarrow & \pi_{21}(S^{10}:2) & \xrightarrow{i_*} & \pi_{21}(K_{\sigma_{10}}:2) & \xrightarrow{\partial} & \{4\nu_{17}\} & \longrightarrow & 0 \\
 & & \downarrow E & & \downarrow E & & \downarrow E & & \\
0 & \longrightarrow & \pi_{22}(S^{11}:2) & \xrightarrow{i_*} & \pi_{22}(K_{\sigma_{11}}:2) & \xrightarrow{\partial} & \{2\nu_{18}\} & \longrightarrow & 0 \\
 & & \downarrow E^{n-11} & & \downarrow E^{n-11} & & \downarrow E^{n-11} & & \\
0 & \longrightarrow & \pi_{n+11}(S^n:2) & \xrightarrow{i_*} & \pi_{n+11}(K_{\sigma_n}:2) & \xrightarrow{\partial} & \{\nu_{n+7}\} & \longrightarrow & 0
\end{array}$$

for $n\geqslant 13$, where $E:\pi_{21}(S^{10}:2)\longrightarrow\pi_{22}(S^{11}:2)$ and $E^{n-11}:\pi_{22}(S^{11}:2)\longrightarrow\pi_{n+11}(S^n:2)$ are isomorphisms. From (3. 20) and the above diagram, we obtain that

$$\pi_{20}(K_{\sigma_9}:2)=\{i_*\zeta_9\}+\{i_*\bar{\nu}_9\circ\nu_{17}\}\approx Z_8+Z_2,$$
$$\pi_{21}(K_{\sigma_{11}}:2)=\{\widetilde{4\nu}_{17}\}\approx Z_{16},$$
$$\pi_{22}(K_{\sigma_{11}}:2)=\{\widetilde{2\nu}_{18}\}\approx Z_{32},$$
$$\pi_{n+11}(K_{\sigma_n}:2)=\{\tilde{\nu}_{n+7}\}\approx Z_{64}\qquad\text{for } n\geqslant 13.$$

It is easily seen the results of $\pi_{n+12}(K_{\sigma_n}:2)$ and $\pi_{n+13}(K_{\sigma_n}:2)$ from the exact sequence of (3. 1), the table (3. 2), (3. 3) and the relation (3. 6).

Consider the exact sequence

$$\pi_{25}(S^{18}:2)\xrightarrow{\sigma_{11*}}\pi_{25}(S^{11}:2)\xrightarrow{i^*}\pi_{25}(K_{\sigma_{11}}:2)\xrightarrow{\partial}\pi_{24}(S^{18}:2)\xrightarrow{\sigma_{n*}}\pi_{24}(S^{11}:2)$$

of (3. 1). From (3. 2), (3. 3) it follows that

(3. 21) $$\sigma_{11*}:\pi_j(S^{18}:2)\longrightarrow\pi_j(S^{11}:2)\qquad\text{for } j=24,\ 25$$

are monomorphisms. Thus from the exactness of the above sequence we have

$$\pi_{25}(K_{\sigma_{11}}:2)\approx\pi_{25}(S^{11}:2)/\{\sigma_{11}^2\}=\{\kappa_{11}\}\approx Z_2$$

From (3. 1) and (3. 2), we have the exact sequence

$$\pi_{26}(S^{18}:2)\xrightarrow{\sigma_{11*}}\pi_{26}(S^{11}:2)\xrightarrow{i_*}\pi_{26}(K_{\sigma_{11}}:2)\longrightarrow 0.$$

From (3. 7) and (3. 2), we have that

}. 22) $$i_* : \pi_{26}(S^{11} : 2) \longrightarrow \pi_{26}(K_{\sigma_{11}} : 2)$$

an isomorphism onto.

Next consider a generator ν_{10} of $\pi_{13}(S^{10} : 2)$ of order 8 and an element $= \Delta(\iota_{21}) + \gamma$ of $\pi_{19}(S^{10} : 2)$ of order infinite order, where γ is an element $\eta_{20} \circ \varepsilon_{11} + b\nu_{10}^3$ of $\pi_{19}(S^{10} : 2)$ with the order at most 2 $(a, b = 0$ or $1)$. Let a ell complex $K = S^{11} \cup C(S^{13} \vee S^{19})$ be obtained by attaching $C(S^{13} \vee S^{19})$ to ^{10} by $\nu_{10} \vee \beta : S^{13} \vee S^{19} \longrightarrow S^{10}$. Then we have the following lemma.

LEMMA 3. 3. *We have the following table of homotopy group* $\pi_j(K : 2)$ *for* $\leqslant 21$;

j	$j \leqslant 9$	10	11	12	13	14	15	16
$\pi_j(K : 2)$	0	Z	Z_2	Z_2	0	Z	Z_2	Z_2
Generator		$i_*\iota_{10}$	$i_*\eta_{10}$	$i_*\eta_{10}^2$		$\widetilde{8\iota_{13}}$	$\widetilde{\eta_{13}}$	$\widetilde{\eta_{13}^2}$

j	17	18	19	20	21
$\pi_j(K : 2)$	$Z_{16} + Z_4$	$Z_2 + Z_2$	$Z_2 + Z_2$	$Z_2 + Z_2$	Z_{128}
Generator	$i_*\sigma_{10}, \widetilde{2\nu_{13}}$	$i_*\nu_{10}, i_*\varepsilon_{10}$	$i_*\eta_{10}\varepsilon_{11}, i_*\mu_{10}$	$i_*\sigma_{10}\nu_{17}, i_*\eta_{10}\mu_{11}$	$\widetilde{\sigma_{13} \oplus \eta_{19}}$

Iere $i : S^{10} \longrightarrow K$ *is an injection and we denote by* $\tilde{\alpha}$ *an element of* $\pi_j(K : 2)$ *such hat* $\tilde{\alpha}$ *is a coextension of* $\alpha \in \pi_{j-1}(S^{13} \vee S^{19} : 2)$.

Proof. By (3. 1), we have an exact sequence

}. 23) $$\cdots \longrightarrow \pi_j(S^{13} \vee S^{19} : 2) \xrightarrow{(\nu_{10} \vee \beta)_*} \pi_j(S^{1}{}_{0} : 2) \xrightarrow{i_*} \pi_j(K : 2) \xrightarrow{\partial} \pi_{j-1}(S^{13} \vee S^{19} : 2) \xrightarrow{(\nu_{10} \vee \beta)_*} \pi_{j-1}(S^{10} : 2) \longrightarrow \cdots$$

or $j \leqslant 21$. We can identify $\pi_j(S^{13} \vee S^{19} : 2)$ $((\nu_{10} \vee \beta)_*$ respectively) with $_j(S^{13} : 2) \oplus \pi_j(S^{19} : 2)$ $(\nu_{10*} + \beta_*$ respectively) for $j \leqslant 21$ and we shall use the otation $\alpha = \nu_{10*} + \beta_*$.

From the tables (3. 2), (3. 3), the relations (3. 6), (3. 8) and the exact equence (3. 23), it is easy to see the results of $\pi_j(K : 2)$ for $j \neq 17$, 21.

Consider the exact sequence

$$\pi_{17}(S^{13} : 2) \oplus \pi_{17}(S^{19} : 2) \xrightarrow{\alpha} \pi_{17}(S^{10} : 2) \xrightarrow{i_*} \pi_{17}(K : 2) \xrightarrow{\partial} \pi_{16}(S^{13} : 2) \oplus \pi_{16}(S^{19} : 2) \xrightarrow{\alpha} \pi_{16}(S^{10} : 2)$$

of (3. 23), where $\pi_{16}(S^{13}:2)\oplus\pi_{16}(S^{19}:2)=\pi_{16}(S^{13}:2)=\{\nu_{13}\}\approx Z_8$ and $\pi_{17}(S^{13}:2)\oplus\pi_{17}(S^{19}:2)=0$ by (3. 2). We have that the homomorphism $\alpha:\pi_{16}(S^{13}:2)\oplus\pi_{16}(S^{19}:2)\longrightarrow\pi_{16}(S^{10}:2)$ is an epimorphism and its kernal is generated by $\{2\nu_{13}\}$. Thus we obtain the following sequence

(3. 24) $$0\longrightarrow\{\sigma_{10}\}\xrightarrow{i_*}\pi_{17}(K:2)\xrightarrow{\partial}\{2\nu_{13}\}\longrightarrow 0.$$

By Adams [1],

$$\{\nu_{10},2\nu_{13},4\iota_{16}\}\equiv 0 \quad \bmod 4\pi_{17}(S^{10}:2)$$

and we have, by Proposition 1. 8 of [11], $4\,\widetilde{2\nu_{13}}=-i_*\{\nu_{10},2\nu_{13},4\iota_{16}\}\in 4\,i_*\pi_{17}(S^{10}:2)$. Thus $4(\widetilde{2\nu_{13}}+i_*\alpha)=0$ for some $\alpha\in\pi_{17}(S^{10}:2)$. We may replace $\widetilde{2\nu_{13}}+i_*\alpha$ by $\widetilde{2\nu_{13}}$. Thus, from (3. 24), follows that

$$\pi_{17}(K:2)=\{i_*\sigma_{10}\}+\{\widetilde{2\nu_{13}}\}\approx Z_{16}+Z_4.$$

From (3. 23), we have the exact sequence

$$\pi_{21}(S^{13}:2)\oplus\pi_{21}(S^{19}:2)\xrightarrow{\alpha}\pi_{21}(S^{10}:2)\xrightarrow{i_*}\pi_{21}(K:2)$$
$$\xrightarrow{\partial}\pi_{20}(S^{13}:2)\oplus\pi_{20}(S^{19}:2)\xrightarrow{\alpha}\pi_{20}(S^{10}:2).$$

By (3. 6), (3. 10) and the diagram (3. 2), (3. 3), we have

$$\alpha\{\sigma_{13}\}=\nu_{10}\circ\sigma_{13}=2\sigma_{10}\circ\nu_{17}=\Delta(\iota_{21})\circ\eta_{19}=\alpha\{\eta_{19}\}.$$

Thus we obtain that

(3. 25) the kernel of $\alpha:\pi_{20}(S^{13}:2)\oplus\pi_{20}(S^{19}:2)\longrightarrow\pi_{20}(S^{10}:2)$

is generated by $\{\sigma_{13}\oplus\eta_{19}\}\approx Z_{16}$.

By (3. 8), (3. 10) and the diagram (3. 2),

(3. 26)
$$\begin{aligned}\alpha\{\bar\nu_{13}\}&=\nu_{10}\circ\bar\nu_{13}=0,\\ \alpha\{\varepsilon_{13}\}&=\nu_{10}\circ\varepsilon_{13}=0,\\ \alpha\{\eta_{19}^2\}&=\beta\{\eta_{19}^2\}=\Delta(\iota_{21})\circ\eta_{19}^2+a\eta_{10}\circ\varepsilon_{11}\circ\eta_{19}^2+b\nu_{10}^3\circ\eta_{19}^2\\ &=2\sigma_{10}\circ\nu_{18}\circ\eta_{19}^2+4a\nu_{10}\circ\varepsilon_{13}\\ &=0.\end{aligned}$$

Thus, from (3. 25), (3. 26) and the from above sequence, it follows that the sequence

$$0 \longrightarrow \{\zeta_{10}\} \xrightarrow{i_*} \pi_{21}(K:2) \xrightarrow{\partial} \{\sigma_{13} \oplus \eta_{19}\} \longrightarrow 0$$

is exact. By (9. 3) of [11],

$$\zeta_{10} \in \{\nu_{10}, 2\sigma_{13}, 8\iota_{20}\} \qquad \text{mod } 8\pi_{21}(S^{10}:2)$$

and by Proposition 1. 3 of [11]

$$\begin{aligned} i_*\zeta_{10} &\in i_*\{\nu_{10}, 2\sigma_{13}, 8\iota_{20}\} \\ &= -8\ \widetilde{2\sigma_{13}} \\ &= -16\ \widetilde{\sigma_{13} \oplus \eta_{19}}. \end{aligned}$$

Thus we obtain that

$$\pi_{21}(K:2) = \{\widetilde{\sigma_{13} \oplus \eta_{19}}\} \approx Z_{128}.$$

§ 4. Homotopy groups of exceptional Lie groups E_6, E_7 and E_8.

(I) HOMOTOPY GROUPS $\pi_j(E_8:2)$ for $j \leqslant 28$.

From Corollary 2. 3, Lemma 2. 5, there exist a cell complex $K_{\tilde{E}_8} = S^{15} \cup_{\sigma_{15}} e^{23} \cup e^{27} \cup e^{29}$ and a continuous map $f: K_{\tilde{E}_8} \longrightarrow \tilde{E}_8$, from which the following isomorphism f_*, induced by a map f, is obtained;

(4. 1) $$f_*: \pi_j(S^{15} \cup_{\sigma_{15}} e^{23} \cup e^{27} \cup e^{29}:2) \approx \pi_j(\tilde{E}_8:2) \qquad \text{for } j \leqslant 28.$$

Let e^{27} be attached to $K_{\sigma_{15}} = S^{15} \cup_{\sigma_{15}} e^{23}$ by a map $g: S^{26} \longrightarrow K_{\sigma_{15}}$ and e^{29} be attached to $S^{15} \cup_{\sigma_{15}} e^{23} \cup e^{27}$ by a map $h: S^{28} \longrightarrow S^{15} \cup_{\sigma_{15}} e^{23} \cup e^{27}$, then, from Corollary 2. 3 and Theorem 2. 6, it follows that the next diagrams are commutative

(4. 2) (i) $$\begin{array}{ccc} S^{26} & \xrightarrow{g} & K_{\sigma_{15}} \\ & \searrow^{\nu_{23}} & \downarrow p \\ & & S^{23} \end{array}$$ (ii) $$\begin{array}{ccc} S^{28} & \xrightarrow{h} & S^{15} \cup_{\sigma_{15}} e^{23} \cup e^{27} \\ & \searrow^{\eta_{27}} & \downarrow p' \\ & & S^{27} \end{array}$$

where p, p' are the maps which shrink $S^{15}, S^{15} \cup_{\sigma_{15}} e^{23}$ are respectively to a point. From (4. 1),

$$\pi_j(\tilde{E}_8:2) \approx \pi_j(S^{15} \cup_{\sigma_{15}} e^{23} \cup e^{27}:2) \qquad \text{for } j \leqslant 27.$$

Consider the exact sequence

$$\pi_{26}(S^{26}:2)\xrightarrow{g_*}\pi_{26}(K_{\sigma_{15}}:2)\xrightarrow{i'_*}\pi_{26}(S^{15}\underset{\sigma_{15}}{\cup}e^{23}\cup e^{27}:2)\xrightarrow{\partial}\pi_{25}(S^{26}:2)$$

of (3. 1), where $i':K_{\sigma_{15}}\longrightarrow S^{15}\underset{\sigma_{15}}{\cup}e^{23}\cup e^{27}$ is the inclusion map. From (i) of (4. 2) and the table (3. 15), we have that

$$g_*:\pi_{26}(S^{26}:2)\longrightarrow\pi_{26}(K_{\sigma_{15}}:2) \tag{4. 3}$$

is an epimorphism. Thus, from the exactness of the above sequence, we obtain

$$\pi_{26}(S^{15}\underset{\sigma_{15}}{\cup}e^{23}\cup e^{27}:2)=0. \tag{4. 4}$$

It follows from (3. 1), (3. 15) and (4. 3) that the sequence

$$0=\pi_{27}(K_{\sigma_{15}}:2)\xrightarrow{i'_*}\pi_{27}(S^{15}\underset{\sigma_{15}}{\cup}e^{23}\cup e^{27}:2)\xrightarrow{\partial}\pi_{26}(S^{26}:2)$$
$$\xrightarrow{g_*}\pi_{26}(K_{\sigma_{15}}:2)\longrightarrow 0$$

is exact. Thus we obtain

$$\pi_{27}(S^{15}\underset{\sigma_{15}}{\cup}e^{23}\cup e^{27}:2)=Z. \tag{4. 5}$$

Next consider the diagram;

$$\begin{array}{ccccccc}
 & & \pi_{28}(S^{15}\underset{\sigma_{15}}{\cup}e^{23}:2)=0 & & & & \\
 & & \downarrow i'_* & & & & \\
\pi_{28}(S^{28}:2) & \xrightarrow{h_*} & \pi_{28}(S^{15}\cup e^{23}\cup e^{27}:2) & \xrightarrow{i''_*} & \pi_{28}(S^{15}\cup e^{23}\cup e^{27}\cup e^{29}:2) & \xrightarrow{\partial} & \pi_{27}(S^{28}:2)=0 \\
\downarrow \eta_{27*} & \swarrow p'_* & \downarrow \partial & & & & \\
\pi_{28}(S^{27}:2) & \xleftarrow[E]{} & \pi_{27}(S^{26}:2) & & & & \\
 & & \downarrow g_* & & & & \\
 & & \pi_{27}(S^{15}\underset{\sigma_{15}}{\cup}e^{23}:2)=0 & & & &
\end{array}$$

where i'' is a inclusion map. From (3. 1) the row and column sequences are exact, and from (ii) of (4. 2) and from the definition of ∂, it follows that the diagram is commutative. By (3. 15), $\partial:\pi_{28}(S^{15}\cup e^{23}\cup e^{27}:2)\longrightarrow\pi_{27}(S^{26}:2)$ is an isomorphism, and $E:\pi_{27}(S^{26}:2)\longrightarrow\pi_{28}(S^{27}:2)$ is an isomorphism. Thus, from the commutativity of the above diagram, it follows that

$$h_*:\pi_{28}(S^{28}:2)\longrightarrow\pi_{28}(S^{15}\underset{\sigma_{15}}{\cup}e^{23}\cup e^{27}:2)$$

is epimorphic. Thus, from the exactness of the column sequence, we obtain

$$\pi_{28}(S^{15} \underset{\sigma_{16}}{\cup} e^{23} \cup e^{27} \cup e^{29} : 2) = 0. \tag{4. 6}$$

From (4. 1), (3. 15) and (4. 4) (4. 9), it follows the next table of the homotopy groups of exceptional Lie group E_8.

PROPOSITION 4. 1.

j	1, 2	3	$4 \leqslant j \leqslant 14$	15	16	17	18	19	20
$\pi_j(E_8 : 2)$	0	Z	0	Z	Z_2	Z_2	Z_8	0	0

j	21	22	23	24	25	26	27	28
$\pi_j(E_8 : 2)$	Z_2	0	$Z+Z_2$	Z_2+Z_2	Z_2	0	Z	0

(II) HOMOTOPY GROUPS $\pi_j(E_7 : 2)$ for $j \leqslant 25$.

From Lemma 2. 5, there exist a cell complex $K_{\tilde{E}_7} = S^{11} \cup e^{15} \cup e^{19} \cup e^{23} \cup e^{26} \cup e^{27}$ and a continuous map $k : K_{\tilde{E}_7} \longrightarrow \tilde{E}_7$ such that $k_* : \pi_j(K_{\tilde{E}_7}) \longrightarrow \pi_j(\tilde{E}_7)$ are C_2-isomorphism onto for $j \leqslant 28$. By Corollary 2. 8 and Lemma 2. 4, e^{15} is attached to S^{11} by a representative of $\nu_{11} \in \pi_{14}(S^{11} : 2)$.

Consider the diagram

$$\begin{array}{ccccc}
S^{11} \underset{\nu_{11}}{\cup} e^{15} \cup e^{19} & \longrightarrow & S^{15} \underset{\beta}{\cup} e^{19} & & \\
\downarrow k & & \downarrow \bar{k} & \swarrow f & S^{15} \\
\tilde{E}_7 & \subset & \tilde{E}_8 & &
\end{array}$$

where p is a map which shrinks S^{11} to a point and $\tilde{E}_7 \subset \tilde{E}_8$ is the natural inclusion. Since $\pi_i(\tilde{E}_8) = 0$ for $i \leqslant 14$, $k|S^{11} \simeq 0$ in E_8. Thus there exists a map $\bar{k} : S^{15} \cup e^{19} \longrightarrow \tilde{E}_8$ such that the above diagram is homotopy commutative. A generator $x_{15} \in H^{15}(\tilde{E}_7 : Z_2)$ corresponds to a generator $x_{15} \in H^{15}(\tilde{E}_8 ; Z_2)$ by the natural inclusion $\tilde{E}_7 \subset \tilde{E}_8$. Thus, from the commutativity of the above diagram, $x_{15} \in H^{15}(\tilde{E}_8 ; Z_2)$ corresponds to a generator of $H^{15}(S^{15} \cup e^{19} ; Z_2)$ by $\bar{k}^*$. Let $f : S^{15} \longrightarrow \tilde{E}_8$ be a representative of a generator $\{f\}$ of $\pi_{15}(\tilde{E}_8) = Z$, then $\bar{k}|S^{15}$ is homotopic to $x\{f\}$ for some odd integer x. Let e^{19} be attached to S^{15} by $\beta : S^{18} \longrightarrow S^{15}$ for a cell complex $S^{15} \cup e^{19}$ of the above diagram.

Since $\bar{k}$ is extended over e^{19}, we have

$$0 = (\bar{k}|S^{15})_* \beta = x(f_* \beta) \quad \text{in 2-component.}$$

By (4. 1), $f_* : \pi_j(S^{15}) \longrightarrow \pi_j(\tilde{E}_8)$ are C_2-isomorphism onto for $j \leqslant 21$. Thus it follows $\beta = 0$. From this we have that $S^{11} \cup e^{19}$ is a subcomplex of $K_{\tilde{E}_7}$, and e^{19} is attached to S^{11} by σ_{11}.

LEMMA 4. 2. *We may regard the inclusion* $j : K_{\sigma_{11}} = S^{11} \underset{\sigma_{11}}{\cup} e^{19} \subset K_{\tilde{E}_7}$ *as the fibre map. Let* F *be the fibre, then* $H^*(F\,;Z_2)$ *has additive basis* $\{1, a_{14}, a_{22}, a_{26}\}$ *for degree* < 29, *where* a_i *denote a generator of degree* i.

Proof. From lemma 2. 5, $H^*(K_{\tilde{E}_7}\,;Z_2) = \Delta(x_{11}, x_{15}, x_{19}, x_{23}, x_{27})$ for degree < 30 and $Sq^4x_{11} = x_{15}$, $Sq^8x_{15} = x_{23}$, $Sq^4x_{23} = x_{27}$, $Sq^8x_{11} = x_{19}$. Let $\{E_r^{**}\}$ be the mod 2 spectral sequence associated with the above fibering, then we have

$$E_2^{**} = H^*(K_{\tilde{E}_7}\,;Z_2) \otimes H^*(F\,;Z_2)$$

and

$$E_\infty^{**} = \Delta(x_{11}, x_{19}) \text{ for degree} < 30.$$

Clearly $K_{\tilde{E}_7}$ and F are 10-and 13-connected respectively. We have the following cohomology exact sequence $\cdots \longrightarrow H^*(K_{\tilde{E}_7}\,;Z_2) \xrightarrow{j^*} H^*(K_{\sigma_{11}}\,;Z_2) \longrightarrow H^*(F\,;Z_2) \xrightarrow{\tau} H^*(K_{\tilde{E}_7}\,;Z_2) \longrightarrow \cdots$ for degree $\leqslant 24$. It follows that $H^*(F\,;Z_2) = \{1, a_{14}, a_{22}\}$ for degree < 24 where $\tau(a_{14}) = x_{15}$ and $\tau(a_{22}) = x_{23}$, i.e, $d_{15}(1 \otimes a_{14}) = x_{15} \otimes 1$ and $d_{23}(1 \otimes a_{22}) = x_{23} \otimes 1$. For $24 \leqslant q \leqslant 29$, any non-zero element of $E_2^{0,q}$ must be cancelled by d_r with some element of $E_r^{r,q-r+1}$. By the dimensional reason, the only posibilities of such q are $q = 24, 25, 26$ corresponding to $x_{11} \otimes a_{14}$, $x_{11}x_{15} \otimes 1$ and $x_{27} \otimes 1$ respectively. Thus $H^q(F\,;Z_2) = 0$ for $q = 27, 28, 29$. Since $d_{15}(x_{11} \otimes a_{14}) = x_{11}x_{15} \otimes 1 \neq 0$, $x_{11} \otimes a_{14}$ is not a d_{15}-image, hence $H^{24}(F\,;Z_2) = 0$. We have also $H^{25}(F\,;Z_2) = 0$ since $x_{11}x_{15} \otimes 1 = 0$ in $E_{26}^{26,0}$. By the dimensional reason, we see that $x_{27} \otimes 1 \neq 0$ in $E_{27}^{27,0}$, hence there exists an element a_{26} such that $d_{28}(1 \otimes a_{26}) = x_{27} \otimes 1$ and a_{26} generates $H^{26}(F\,;Z_2) \approx Z_2$.

From the proof of this lemma, we have that a_{14}, a_{22}, a_{26} are transgressive elements. Since $Sq^8x_{15} = x_{23}$, $Sq^4x_{23} = x_{27}$, it follows, from the commutativity of the Steenrod operation and the transgression, that

$$Sq^8a_{14} = a_{22}, \quad Sq^4a_{22} = a_{26}. \tag{4. 7}$$

By Lemma 2. 5 and Theorem 2. 6, there exists a cell complex $K_F = S^{14} \cup e^{22} \cup e^{26}$ and a continuous map from K_F to F which induces isomorphisms from $\pi_j(K_F:2)$ onto $\pi_j(F:2)$ for $j \leqslant 26$. Let $f: K_F \longrightarrow K_{\sigma_{11}} = S^{11} \underset{\sigma_{11}}{\cup} e^{19}$ be the mapping from a fibre to the total space identifying F with K_F for dimension $\leqslant 26$. Then $f|S^{14}$ is a representative of ν_{11}.

Consider the exact sequence

$$(4.8) \qquad \cdots \longrightarrow \pi_j(K_F:2) \xrightarrow{f_*} \pi_j(K_{\sigma_{11}}:2) \xrightarrow{j_*} \pi_j(K_{\tilde{E}_7}:2) \xrightarrow{\partial} \pi_{j-1}(K_F:2) \xrightarrow{f_*} \pi_{j-1}(K_{\sigma_{11}}:2) \longrightarrow \cdots$$

associated with the above fibering for $j \leqslant 26$ and the following homotopy commutative diagram

$$(4.9) \qquad \begin{array}{ccc} S^{14} & \xrightarrow{i} & K_F \\ \downarrow{\scriptstyle \nu_{11}} & & \downarrow{\scriptstyle f} \\ S^{11} & \xrightarrow{i} & K_{\sigma_{11}}. \end{array}$$

From (3. 1), (3. 14) and from the fact that e^{26} is attached to $K_{\sigma_{14}}$ by a coextension of ν_{22}, we have the next table ;

(4. 10)

j	$j \leqslant 13$	14	15	16	17	18	19	20
$\pi_j(K_F:2)$	0	Z	Z_2	Z_2	Z_8	0	0	Z_2
Generator		$i_*\iota_{14}$	$i_*\eta_{14}$	$i_*\eta_{14}^2$	$i_*\nu_{14}$			$i_*\nu_{14}^2$

j	21	22	23	24	25	26
$\pi_j(K_F:2)$	0	$Z+Z_2$	Z_2+Z_2	Z_2	0	Z
Generator		$\widetilde{16\iota_{21}}, i_*\varepsilon_{14}$	$i_*\mu_{14}, i_*\eta_{14}\varepsilon_{15}$	$i_*\eta_{14}\mu_{15}$		$\widetilde{64\iota_{25}}$

LEMMA 4. 3. *For the homomorphism* $f_*: \pi_j(K_F:2) \longrightarrow \pi_j(K_{\sigma_{11}}:2)$, *we have the following table;*

(4. 11)

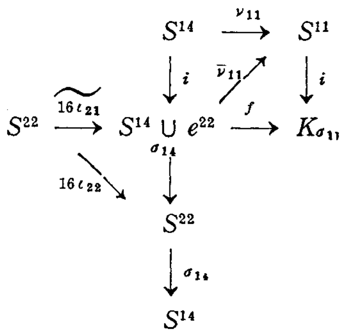
$\alpha =$	$i_*\iota_{14}$	$i_*\eta_{14}$	$i_*\eta_{14}^2$	$i_*\nu_{14}$	$i_*\nu_{14}^2$	$i_*\widetilde{16\iota_{21}}$	$i_*\varepsilon_{14}$	$i_*\mu_{14}$	$i_*\eta_{14}\varepsilon_{15}$	$i_*\eta_{14}\mu_{15}$
$f_*\alpha =$	$i_*\nu_{14}$	0	0	$i_*\nu_{14}^2$	$i_*\eta_{11}\varepsilon_{12}$	$4\,\widetilde{2\nu_{18}}$	0	0	0	0

Proof. From (4. 9), (3. 8), (3. 9), it follows that the table is true excepting for $\alpha = i_*\widetilde{16\iota_{21}}$, $i_*\nu_{14}^2$.

The relation $i_*\eta_{11}\circ\varepsilon_{12}=i_*\nu_{11}^3$ in $\pi_{20}(K\sigma_{11}:2)$ imply the formula

$$f_*(i_*\nu_{14}^2)=i_*\eta_{11}\circ\varepsilon_{12}.$$

Consider the following commutative diagram



where $\widetilde{16\iota_{21}}$ is a coextension of $16\iota_{21}$ and $\bar{\nu}_{11}$ is an extension of ν_{11}. We have

$$\begin{aligned} f_*\widetilde{16\iota_{21}} &= i_*\bar{\nu}_{11}\circ\widetilde{16\iota_{21}} \\ &= -\,i_*\{\nu_{11},\sigma_{14},16\iota_{21}\} \quad \text{by Proposition 1. 8 of [11],} \\ &= -\,i_*\zeta_{11} \quad \text{by (9. 3) of [11],} \\ &= -4\ \widetilde{2\nu_{21}}. \end{aligned}$$

PROPOSITION 4. 4. *The homotopy groups $\pi_j(E_7:2)$ for $j\leqslant 25$ are listed in the following table;*

j	1,2	3	$4\leqslant j\leqslant 10$	11	12	13	14	15
$\pi_j(E_7:2)$	0	Z	0	Z	Z_2	Z_2	0	Z

j	16	17	18	19	20	21	22	23
$\pi_j(E_7:2)$	Z_2	Z_2	Z_4	$Z+Z_2$	Z_2	Z_2	Z_4	$Z+Z_2+Z_2$

j	24	25
$\pi_j(E_7:2)$	$Z_2+Z_2+Z_2$	Z_2+Z_2

Proof. The results of $\pi_j(E_7:2)$ for $j\leqslant 22$ follow immediately from the tables (4. 10), (3. 13), (4. 11) and from the exactness of the sequence of (4. 8).

$$\{\nu_{11}, \varepsilon_{14}, 2\iota_{22}\} \supset E^4\{\nu_7, \varepsilon_{10}, 2\iota_{18}\} \subset E^4\pi_{19}(S^7) = 0.$$

Thus we have

(4. 12) $$\{\nu_{11}, \varepsilon_{14}, 2\iota_{22}\} \equiv 0 \mod 2\pi_{23}(S^{11}).$$

Similarly we have

(4. 13) $$\{\nu_{11}, \mu_{14}, 2\iota_{23}\} \equiv 0 \mod 2\pi_{24}(S^{11}).$$

$\{\nu_{11}, \eta_{14}\circ\varepsilon_{15}, 2\iota_{23}\} \supset \{\nu_{11}\circ\eta_{14}, \varepsilon_{15}, 2\iota_{23}\} = \{0, \varepsilon_{15}, 2\iota_{23}\} \equiv 0$ by Proposition 1. 2 of [11]. Thus we have

(4. 14) $$\{\nu_{11}, \eta_{14}\circ\varepsilon_{15}, 2\iota_{23}\} \equiv 0 \mod 2\pi_{24}(S^{11}:2).$$

Similarly,

(4. 15) $$\{\nu_{11}, \eta_{14}\circ\mu_{15}, 2\iota_{24}\} \equiv 0 \mod 2\pi_{25}(S^{11}:2).$$

Consider the commutative diagram

(4. 16)

$$\begin{array}{ccccccccc}
\pi_j(K_F:2) & \xrightarrow{f_*} & \pi_j(K\sigma_{11}:2) & \xrightarrow{j_*} & \pi_j(K_{\tilde{E}_7}:2) & \xrightarrow{\partial} & \pi_{j-1}(K_F:2) & \xrightarrow{f_*} & \pi_{j-1}(K\sigma_{11}:2) \\
\uparrow i_* & & \uparrow i_* & & \uparrow i_* & & \uparrow i_* & & \uparrow i_* \\
\pi_j(S^{14}:2) & \xrightarrow{\nu_{11*}} & \pi_j(S^{11}:2) & \xrightarrow{i_*} & \pi_j(S^{11}\cup e^{15}:2) & \xrightarrow{\partial} & \pi_{j-1}(S^{14}:2) & \xrightarrow{\nu_{11*}} & \pi_{j-1}(S^{11}:2)
\end{array}$$

where i, j are inclusions.

From Proposition 1. 8 of [11] and the above secondary composition, coextension $\tilde{\varepsilon}_{14}, \tilde{\mu}_{14}, \widetilde{\eta_{14}\circ\varepsilon_{15}}$ and $\widetilde{\eta_{14}\circ\mu_{15}}$ of $\varepsilon_{14}, \mu_{14}, \eta_{14}\circ\varepsilon_{15}$ and $\eta_{14}\circ\mu_{15}$ respectively are elements of order 2. Thus from the commutativity and the exactness of the above diagram. (4. 16), the results of $\pi_j(K_{E_7}:2)$ for $j = 23, 24, 25$, are obtained.

(III) HOMOTOPY GROUPS $\pi_j(E_6:2)$ for $j \leqslant 22$.

By Corollary 2. 3,

$$H^*(\tilde{E}_6; Z_2) = Z_2[y_{32}] \otimes \Delta(y_9, y_{11}, y_{15}, y_{17}, y_{23}, y_{33})$$

and

$$Sq^2y_9 = y_{11},\ Sq^8y_9 = y_{17},\ Sq^4y_{11} = y_{15},\ Sq^8y_{15} = y_{23}.$$

From Lemma 2. 5, there exists a cell complex $K_{\tilde{E}_6}$ and a continuous map $l : K_{\tilde{E}_6} \longrightarrow \tilde{E}_6$ such that $l_* : \pi_j(K_{\tilde{E}_6}) \longrightarrow \pi_j(\tilde{E}_6)$ are C_2-isomorphism onto for $j \leqslant 24$, i.e, $K_{\tilde{E}_6} = S^9 \cup e^{11} \cup e^{15} \cup e^{17} \cup e^{20} \cup e^{23} \cup e^{24}$.

By Corollary 2. 8, e^{11} is attached to S^9 by η_9.

LEMMA 4. 5. *$K_{\sigma_9} = S^9 \underset{\sigma_9}{\cup} e^{17}$ is a subcomplex of $K_{\tilde{E}_6}$. Exchanging an inclusion map $K_{\sigma_9} \longrightarrow K_{\tilde{E}_6}$ by a fibre map, we denote by F the fibre of this fibering. Then $H^*(F ; Z_2)$ has the additive basis $\{1, a_{10}, a_{14}, a_{20}, a_{22}\}$ for degree $\leqslant 25$ such that $Sq^4 a_{10} = a_{14}$, $Sq^8 a_{14} = a_{22}$, where a_i denotes agenerator of degree i.*

Proof. From Lemma 2. 5, $H^*(K_{\tilde{E}_6} ; Z_2) = \Delta(x_9, x_{11}, x_{15}, x_{17}, x_{23})$ for degree < 32 and $Sq^2 x_9 = x_{11}$, $Sq^4 x_{11} = x_{15}$, $Sq^8 x_{15} = x_{23}$, $Sq^8 x_9 = x_{17}$.

By use of Adem's relation we have relations

$$Sq^6 x_{11} = Sq^6 Sq^2 x_9 = Sq^4 Sq^4 x_9 + Sq^7 Sq^1 x_9,$$
$$Sq^2 x_{15} = Sq^2 Sq^4 x_{11} = Sq^5 Sq^1 x_{11} + Sq^6 x_{11}.$$

Since there is no cell of dimension 10 and 13, $Sq^6 x_{11} = 0$ in $K_{\tilde{E}_6}$. Since there is no cell of dimension 12 and $Sq^6 x_{11} = 0$, $Sq^2 x_{15} = 0$ in $K_{\tilde{E}_6}$. Then e^{17} is inessential to e^{15}, that is, up to homotopy type $S^9 \cup e^{11} \cup e^{17}$ is a subcomplex. Since $\pi_{16}(S^9 \cup e^{11}, S^9) \approx \pi_{16}(S^{11}) = 0$, we have that $S^9 \cup e^{17}$ is a subcomplex. Then, by Theorem 2. 6, we may consider that $S^9 \cup e^{17} = K_{\sigma_9}$ is a subcomplex of $K_{\tilde{E}_6}$.

Let $\{E_r^{**}\}$ be the mod 2 spectral sequence associated with a fibering $\{K_{\sigma_9}, i, K_{\tilde{E}_6}\}$ with the fibre F, then

$$E_2^{**} = H^*(K_{\tilde{E}_6} ; Z_2) \otimes H^*(F ; Z_2)$$

and

$$E_\infty^{**} = \wedge (x_9, x_{17}) \qquad \text{for degree} \leqslant 25.$$

By concerning the cohomology exact sequence associated with this fibering, we have $H^*(F ; Z_2) = \{1, a_{10}, a_{14}\}$ for degree < 18 with generator a_{10}, a_{14} such that $d_{11}(1 \otimes a_{10}) = x_{11} \otimes 1$ and $d_{15}(1 \otimes a_{14}) = x_{15} \otimes 1$. For the total degree < 27, E_2^{**} is the sum of $'E_2^{**} = H^*(K_{E_6} ; Z_2) \otimes \{1, a_{10}, a_{14}\}$ and $\sum_{q \geqslant 18} 1 \otimes H^q(F ; Z_2)$. From $'E_2^{**}$ we compute $'E_r^{**}$ giving d_r trivially except $d_r(b \otimes a_{10}) = bx_{11} \otimes 1$ and $d_r(b \otimes a_{14}) = bx_{15} \otimes 1$, $b \in H^*(K_{E_6} ; Z_2)$. Then we have for the total degree < 30, $'E_\infty^{**} = \Delta(x_9, x_{17}, x_{23}) \otimes 1 + \{x_{11} \otimes a_{10}, x_{15} \otimes a_{14}\}$, where we use the fact $x_{11}^2 = x_{15}^2 = 0$. Compare this with E_∞^{**}, we conclude that $x_{23} \otimes 1$, $x_{11} \otimes a_{10}$ must be cancelled by some elements a_{22}, a_{20}, i.e,

$d_{23}(1\otimes a_{22}) = x_{23}\otimes 1$ and $d_{11}(1\otimes a_{20}) = x_{11}\otimes a_{10}$. Moreover, no other non-zero elements exists in $H^*(F; Z_2)$ for degree $\leqslant 25$. Thus $H_*(F; Z_2) = \{1, a_{10}; a_{14}, a_{20}, a_{22}\}$ for degree $\leqslant 25$.

From the above proof, a_{10}, a_{14} and a_{22} are transgressive element. Since $Sq^4x_{11} = x_{15}$ and $Sq^8x_{15} = x_{23}$, using the commutativity of Steenrod operation and transgression we have $Sq^4a_{10} = a_{14}$ and $Sq^8a_{14} = a_{22}$.

By Lemma 2. 5, there exists a cell complex $K_F = S^{10}\cup e^{14}\cup e^{20}\cup e^{22}$ and a continuous map which induce C_2-isomorphisms from $\pi_j(K_F)$ to $\pi_j(F)$ for $j\leqslant 24$. We identify the fiber to the total space, then we have a commutative diagram

(4. 17)
$$\begin{array}{ccc} S^{10} & \xrightarrow{i} & K_F \\ \downarrow{\scriptstyle \eta_9} & & \downarrow{\scriptstyle f} \\ S^9 & \xrightarrow{i} & K_{\sigma_9} \end{array}$$

where i is inclusion map, and the exact sequence

(4. 18)
$$\cdots \longrightarrow \pi_j(K_F:2) \longrightarrow \pi_j(K_{\sigma_9}:2) \longrightarrow \pi_j(K_{\tilde{E}_6}:2) \longrightarrow \pi_{j-1}(K_F:2) \longrightarrow \pi_{j-1}(K_{\sigma_9}:2) \longrightarrow \cdots.$$

Consider the cell complex $K_F = S^{10}\cup e^{14}\cup e^{20}\cup e^{22}$. Since $Sq^4a_{10} = a_{14}$, e^{14} is attached to S^{10} by a representative of ν_{10}.

From $\pi_{19}(S^{10}\cup e^{14}, S^{10}) \approx \pi_{18}(S^{13}) = 0$, we may assume that $K_F = S^{10}\cup C(S^{13}\vee S^{19})\cup e^{22}$.

Let $\alpha: S^{21}\longrightarrow S^{10}\cup C(S^{13}\vee S^{19})$ be the attaching map of e^{22} and e^{20} be attached to S^{10} by $\beta: S^{19}\longrightarrow S^{10}$. Consider the exact sequence

$$\pi_{21}(S^{10}:2) \longrightarrow \pi_{21}(S^{10}\cup e^{14}\cup e^{20}:2) \xrightarrow{\partial} \pi_{20}(S^{13}\vee S^{19}:2) \xrightarrow{(\nu_{10}\vee\beta)_*} \pi_{20}(S^{10}:2).$$

From the definition of ∂, we have the commutative diagram

$$\begin{array}{ccc} \pi_{21}(S^{10}\cup e^{14}\cup e^{20}:2) & \xrightarrow{\partial} & \pi_{20}(S^{13}\vee S^{19}:2) \\ & {\scriptstyle p_*}\searrow \quad \swarrow{\scriptstyle E} & \\ & \pi_{21}(S^{14}\vee S^{20}:2) = \pi_{21}(S^{14}:2) + \pi_{21}(S^{20}:2) & \end{array}$$

where p is a map which shrinks S^{10} to a point. Since $Sq^8a_{14} = a_{22}$, $p_*\alpha = \sigma_{14} + x\eta_{20}$ for $x = 1$ or 0. From the exactness of the above sequence, $0 = (\nu_{10}\vee\beta)_*\circ\partial\alpha = \nu_{10}\circ\sigma_{13} + x(\beta\circ\eta_{19})$. Thus we have $x(\beta\circ\eta_{19}) = \nu_{10}\circ\sigma_{13} \neq 0$ and $x = 1$.

Put $\beta = a(\varDelta(\iota_{21})) + b\eta_{10}\circ\varepsilon_{11} + c\nu_{11}^3 + d\mu_{10}$ for some integers a, b, c, d, then we have

$$\begin{aligned}\nu_{10}\circ\sigma_{13} &= \beta\circ\eta_{19}\\ &= a(\varDelta(\iota_{21}))\circ\eta_{19} + b\eta_{10}^2\circ\varepsilon_{12} + c\nu_{10}^3\circ\eta_{19} + d\eta_{10}\circ\mu_{11}\\ &= a\nu_{10}\circ\sigma_{13} + 0 + 0 + d\eta_{10}\circ\mu_{11} \qquad \text{by (3. 6) and (3. 10).}\end{aligned}$$

Thus by (3. 3) $a = 1$ and $d = 0$. Therefore

(4. 19) $$\beta = \varDelta(\iota_{21}) + b\eta_{10}\circ\varepsilon_{11} + c\nu_{10}^3 \text{ where } b, c = 0 \text{ or } 1.$$
$$\partial\alpha = \sigma_{13} + \eta_{19}$$

From (4. 19), Lemma 3. 3 and from the exact sequence

$$\cdots \longrightarrow \pi_j(S^{21}:2) \xrightarrow{\alpha_*} \pi_j(S^{10}\cup e^{14}\cup e^{20}:2) \longrightarrow \pi_j(K_F:2) \longrightarrow \pi_{j-1}(S^{21}:2) \xrightarrow{\alpha_*} \cdots$$

of (3. 1), we have the next table;

(4. 20)

j	$j\leqslant 9$	10	11	12	13	14	15	16	17
$\pi_j(K_F:2)$	0	Z	Z_2	Z_2	0	Z	Z_2	Z_2	$Z_{16}+Z_4$
Generator		$i_*\iota_{10}$	$i_*\eta_{10}$	$i_*\eta_{10}^2$		$\widetilde{8\iota_{13}}$	$\widetilde{\eta_{13}}$	$\widetilde{\eta_{13}^2}$	$i_*\sigma_{10}$, $\widetilde{2\nu_{13}}$

j	18	19	20	21
$\pi_j(K_F:2)$	Z_2+Z_2	Z_2+Z_2	Z_2+Z_2	0
Generator	$i_*\overline{\nu_{10}}$, $i_*\varepsilon_{10}$	$i_*\eta_{10}\varepsilon_{11}$, $i_*\mu_{10}$	$i_*\sigma_{10}\nu_{17}$, $i_*\eta_{10}\mu_{11}$	

LEMMA 4. 6. *For the homomorphism* $f_*:\pi_j(K_F:2)\longrightarrow\pi_F(K_{\sigma_9}:2)$, *we have the following table*;

(4. 21)

$\alpha =$	$i_*\iota_{10}$	$i_*\eta_{10}$	$i_*\eta_{10}^2$	η_{13}	$i_*\sigma_{10}$	$2\nu_{13}$	$i_*\nu_{10}$	$i_*\varepsilon_{10}$
$f_*\alpha =$	$i_*\eta_9$	$i_*\eta_9^2$	$4i_*\nu_9$	$i_*\nu_9^2$	$i_*\varepsilon_9 + i_*\overline{\nu_9}$	$i_*\varepsilon_9$	$i_*\nu_9^3$	$i_*\eta_9\varepsilon_{10}$

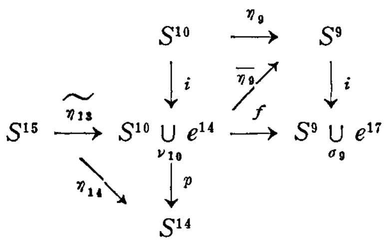
$\alpha =$	$i_*\eta_{10}\circ\varepsilon_{11}$	$i_*\mu_{10}$	$i_*\sigma_{10}\nu_{17}$	$i_*\eta_{10}\mu_{11}$
$f_*\alpha =$	0	$i_*\eta_9\mu_{10}$	$i_*\nu_9\nu_{17}$	$4i_*\zeta_9$

Proof. We shall use the next relations

$\eta_n^3 = 4\nu_n$ for $n \geqslant 5$ by (5. 5) of [11],
$\eta_n \circ \bar{\nu}_{n+1} = \nu_n^3$ for $n \geqslant 6$ by Lemma 6. 3 of [11],
$\eta_9 \circ \sigma_{10} = \bar{\nu}_9 + \varepsilon_9$ by Lemma 6. 4 of [11],
$\eta_n^2 \circ \varepsilon_{n+2} = 0$ for $n \geqslant 9$ by (7. 10), (7. 20) of [11],
$4\zeta_n = \eta_n^2 \circ \mu_{n+2}$ for $n \geqslant 5$ by Lemma 6. 7 of [11].

From (4. 17), (4. 22), it follows that the table is true except for $\alpha = \widetilde{\eta_{13}}$ and $\widetilde{2\nu_{13}}$.

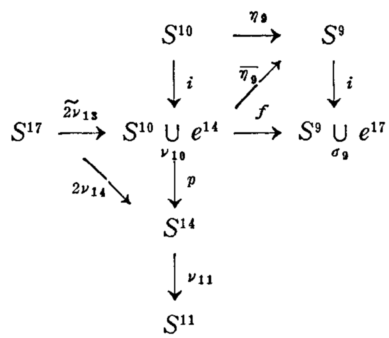
From the definition of $\widetilde{\eta_{13}}$ and (4. 17), we have the commutative diagram



where p is the mapping which shrinks S^{10} to a point and $\bar{\eta}_9$ is a extension of η_9. Thus we have

$$f_*\tilde{\eta}_{13} = i_*\bar{\eta}_9 \circ \tilde{\eta}_{13} = i_*\{\eta_9, \nu_{10}, \eta_{13}\} \ni i_*\nu_9^2 \quad \text{by Lemma 5. 5 of [11].}$$

Consider the commutative diagram



then we have

$$f_*\widetilde{2\nu_{13}} = i_*\bar{\eta}_9 \circ \widetilde{2\nu_{13}} \in i_*\{\eta_9, \nu_{10}, 2\nu_{13}\} \quad \text{by Proposition 1. 7 of [11],}$$
$$\in i_*\varepsilon_9 \quad \text{by (6. 1) of [11].}$$

PROPOSITION 4. 7. *The homotopy groups* $\pi_j(E_6:2)$ *for* $j \leqslant 22$ *are listed in the following table*;

j	1,2	3	$4 \leqslant j \leqslant 8$	9	10	11	12	13	14
$\pi_j(E_6:2)$	0	Z	0	Z	0	Z	Z_4	0	0

j	15	16	17	18	19	20	21	22
$\pi_j(E_6:2)$	Z	0	$Z+Z_2$	$Z_{16}+Z_2$	0	Z_8	0	0

Proof. The results of $\pi_j(E_6:2)$ for $j \neq 18, 20$, follow immediately from the table the (3. 11), (4. 20), (4. 21) and from the exact sequence (4. 18).

By (3. 9) and Proposition 1. 2 of [11], $\mu \in <\eta, 8\iota, 2\sigma> \equiv <\eta, 2\sigma, 8\iota> + <2\sigma, \eta, 8\iota>$ and $<2\sigma, \eta, 8\iota> \equiv <\sigma, 2\eta, 8\iota> \equiv 0$. Then, by concerning the suspension homomorphism, we obtain

$$\{\eta_9, 2\sigma_{10}, 8\iota_{17}\} \ni \mu_9.$$

By Lemma 9. 1 of [11], we have

$$\{\eta_9, \eta_{10} \circ \varepsilon_{11}, 2\iota_{19}\} \ni \zeta_9.$$

Consider the commutative diagram

$$\begin{array}{ccccccccc}
\pi_{18}(K_F:2) & \xrightarrow{f_*} & \pi_{18}(K_{\sigma_9}:2) & \xrightarrow{j_*} & \pi_{18}(K_{E_6}:2) & \xrightarrow{\partial} & \pi_{17}(K_F:2) & \xrightarrow{f_*} & \pi_{17}(K_{\sigma_9}:2) \\
\uparrow i_* & & \uparrow i_* & & \uparrow i_* & & \uparrow i_* & & \uparrow i_* \\
\pi_{18}(S^{10}:2) & \xrightarrow{\eta_{9*}} & \pi_{18}(S^9:2) & \xrightarrow{j_*} & \pi_{18}(S^9 \cup e^{11}:2) & \xrightarrow{\partial} & \pi_{17}(S^{10}:2) & \xrightarrow{\eta_{9*}} & \pi_{17}(S^9:2)
\end{array}$$

where j is a inclusion map $S^9 \longrightarrow S^9 \cup_{\eta_9} e^{11}$.
By Proposition 1. 8 of [11], we have

$$j_*\mu_9 \in j_*\{\eta_9, 2\sigma_{10}, 8\iota_{17}\} = -8\ \widetilde{2\sigma_{10}}.$$

From the above commutative diagram and from the tables (3. 11), (4. 20), (4. 21), we obtain

$$\pi_{18}(K_{\tilde{E}_6}:2) \approx Z_{16} + Z_2.$$

We have the following commutative diagram

$$\begin{array}{ccccccccc}
\pi_{20}(K_F:2) & \xrightarrow{f_*} & \pi_{20}(K\sigma_9:2) & \longrightarrow & \pi_{20}(K_{E_6}:2) & \longrightarrow & \pi_{19}(K_F:2) & \xrightarrow{f_*} & \pi_{19}(K\sigma_9:2) \\
\uparrow i_* & & \uparrow i_* & & \uparrow i_* & & \uparrow i_* & & \uparrow i_* \\
\pi_{20}(S^{10}:2) & \xrightarrow{\eta_9*} & \pi_{20}(S^9:2) & \xrightarrow{j_*} & \pi_{20}(S^9 \cup e^{11}:2) & \xrightarrow{\partial} & \pi_{19}(S^{10}:2) & \xrightarrow{\eta_9*} & \pi_{19}(S^9:2)
\end{array}$$

and from Proposition 1. 7 of [11]

$$j_*\zeta_9 \in j_*\{\eta_9, \eta_{10}\circ\varepsilon_{11}, 2\iota_{19}\} = -2\widetilde{\eta_{10}\circ\varepsilon_{11}}.$$

From the exact sequence (4. 18) and from the table (3. 8), (4. 10), (4. 21), we obtain

$$\pi_{20}(K\widetilde{E}_6:2) \approx Z_8.$$

BIBLIOGRAPHY

[1] J. F. Adams: On the group $J(X)$ IV, Topology, **5-1** (1966), 21–71.
[2] S. Araki: Cohomology modulo 2 of the compact exceptional groups E_6 and E_7, J. of Math. Osaka C.V., Vol. **12** (1961), 43–65.
[3] S. Araki and Y. Shikata: Cohomology mod 2 of the compact exceptional group E_8, Proc. Japan Acad., **37** (1961), 619–622.
[4] A.L. Blakers and W.S. Massey: The homotopy groups of a triad II, Ann. of Math., **55** (1952), 192–201.
[5] R. Bott: The stable homotopy of the classical groups, Ann. of Math., **70** (1959), 313–337.
[6] R. Bott and H. Samelson: Application of the theory of Morse to symmetric spaces, Amer. J. Math., **80** (1958), 964–1029.
[7] H. Cartan and J.P. Serre: Espaces fibrés et groupes d'homotopie I, II, C.R. Acad. Sci. Paris., **234** (1952), 288–290, 393–395.
[8] J.P. Serre: Groupes d'homotopie et classes de groupes abélian, Ann. of Math., **58** (1953), 258–294.
[9] J.P. Serre: Cohomologie modulo 2 des complexes d'Eilenberg Mac-Lane, Comm. Math. Helv., **27** (1953), 198–231.
[10] M. Mimura: The homotopy group of Lie groups of low rank, J. Math. Kyoto Univ., **6-2** (1967), 131–176.
[11] H. Toda: Composition methods in homotopy groups of spheres, Ann. of Math. Studies., (1962).

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