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HOMOTOPY LIE ALGEBRAS AND FUNDAMENTAL GROUPS VIA DEFORMATION THEORY

by M. MARKL and Ş. PAPADIMA

Introduction and statement of main results.

The aim of our work is a parallel study of the two natural graded Lie algebra objects associated to topological spaces S in traditional homotopy theory : $\mathrm{gr}^*\pi_1 S$ (the graded Lie algebra obtained from the lower central series of the fundamental group of a connected S, with bracket induced by the group commutator) and $\pi_*\Omega S$ (the–connected–graded homotopy Lie algebra of a 1-connected S, with Lie bracket given by the Samelson product) ; in the second case it turns out, as suggested by the analysis of the first case, that more accessible, and still valuable information may be gained on the bigraded homotopy Lie algebra $\mathrm{gr}^*\pi_*\Omega S$, associated to the lower central series of $\pi_*\Omega S$. The main problem one immediately faces here is related to the big difficulties raised by the concrete computation of these invariants, even in rational form. For example, the complete knowledge of the relevant homological information would not be of much help : one knows that $H^*(\mathrm{Sp}(5)/SU(5); \mathbb{Q}) \cong H^*((S^6 \times S^{25})\#(S^{10} \times S^{21}); \mathbb{Q})$ as algebras [30], while the rational homotopy Lie algebras have a quite contrasting behavior, the first one being finite dimensional ([30],[14]), hence nilpotent and the second one being infinite dimensional and not even solvable ([15]) ; similarly the nilmanifold $N_{\mathbb{R}}/N_{\mathbb{R}}$ ($N_{\mathbb{K}}$ being the group

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of upper triangular unipotent 3×3 matrices with entries in $\mathbb{K}$) and the connected sum $(S^1 \times S^2)\#(S^1 \times S^2)$ have the same rational multiplication table in low dimensions (i.e. the same $\mu : H^1 \wedge H^1 \to H^2$) and still their fundamental groups are strongly different, the first one being two-stage nilpotent and the other one being free on two generators (see [13]).

By contrast to these examples we are going to look at some fixed graded-commutative unital $\mathbb{Q}$-algebra A^*, supposed to be connected ($A^0 \cong \mathbb{Q}$) and finite dimensional in each degree, and use the deformation-theoretic methods of rational homotopy theory, which provide various convenient algebraic parametrizations of the spaces S with $H^*(S;\mathbb{Q}) \cong A^*$ as algebras (see e.g. [17], [23], [24], and many others), in order to exhibit conditions on A^* guaranteeing that the (bi)graded Lie algebra invariants of S described above are constant within A^*.

Various such so called intrinsic properties of A^* have been considered in the literature. For example S is called *formal* if its rational homotopy type is entirely determined by its rational cohomology algebra and a basic result of the theory says that any (1-connected) algebra (i.e. $A^1 = 0$) is realized by exactly one formal homotopy type [28]; S is called *spherically generated* if the image of its rational Hurewicz morphism coincides with the primitives of its rational homology coalgebra and a formal space is always spherically generated [17; 8.13]. Accordingly A^* is said to be *intrinsically formal (spherically generated)* if any S with $H^*(S;\mathbb{Q}) \cong A^*$ is formal (spherically generated). Various sufficient conditions for intrinsic formality (spherical generation) are known (see e.g. [10],[32],[24]), via deformation theory.

Let us say that A^* is *graded* (respectively *1-graded*) *intrinsically formal* if the bigraded (resp. graded) Lie algebra $\mathrm{gr}^*\pi_*\Omega S \otimes \mathbb{Q}$ (resp. $\mathrm{gr}^*\pi_1 S \otimes \mathbb{Q}$) is constant within A^*.

Our first goal is to produce examples (both general and concrete classes of them) of natural sufficient conditions for the above mentioned intrinsic properties (and at the same time weaker that those already known for the other two mentioned intrinsic properties, which are generally very restrictive). Secondly we will give explicit computations of homotopy Lie algebras, in the presence of these conditions, and also give bounds for the size of homotopy Lie algebras, some of them quite generally (Propositions 3.3, 3.4 and 5.1). A unitary frame will be provided by what we call "a deformation method for the fundamental group".

Our hypotheses on A^* are related to the Hopf algebra $\mathrm{Ext}_A^{*,*}(\mathbb{Q},\mathbb{Q})$, where the first degree is the resolution degree and the second is the total degree, as usual, more precisely to its indecomposables $\mathcal{Q}\mathrm{Ext}_A^{*,*}(\mathbb{Q},\mathbb{Q})$ and to its primitives $\mathcal{P}\mathrm{Ext}_A^{*,*}(\mathbb{Q},\mathbb{Q})$. The explicit description of rational homotopy Lie algebras involves : if A^* is 1-connected, there is the (minimal) Quillen model of the formal space S_A associated to A^*, to be denoted by $\mathcal{L}_A$, which is a bigraded differential Lie algebra ([27],[31 ; III.3.(1)], see also the next section), and thus $H_*^*\mathcal{L}_A$ becomes a bigraded Lie algebra (for an even more precise computation, see Theorem B(i) below) ; to a connected A^* one may associate the dual of the cup product pairing, $\mu : A^1 \wedge A^1 \to A^2$, to be denoted by $\partial : Y \to X \wedge X$, and then the graded Lie algebra L_A^*, defined as the quotient of the free Lie algebra on X, $\mathbb{L}^*X$, graded by the bracket length, by the ideal generated by ∂Y - under the obvious identification $\mathbb{L}^2X \cong X \wedge X$ (see also Lemma 1.8(i) for a further construction, related to the explicit computation of the rational nilpotent completion of $\pi_1 S_A$).

Before stating our first results, let us make one more definition : the *cup-genus* of A^*, to be denoted by $\mathrm{cg}(A)$, is defined to be the maximal dimension of the graded vector subspaces $N \subset A^+$, having the property that $N \cdot N = 0$ and $N \cap (A^+ \cdot A^+) = 0$; the same definition obviously applies to a vector valued 2-form, $\mu: A^1 \wedge A^1 \to A^2$, where $\mathrm{cg}(\mu)$ equals the maximal dimension of the vector subspaces $N \subset A^1$ for which $\mu(N \wedge N) = 0$. In the classical case of the cohomology of a closed oriented surface, the two definitions coincide, their common value being just the genus of the surface, hence the terminology.

THEOREM A. — *Let A^* be a 1-connected graded algebra.*

(i) *If $\mathcal{Q}\mathrm{Ext}_A^{>1,*}(\mathbb{Q},\mathbb{Q}) = 0$ and if A^* is intrinsically spherically generated then A^* is graded intrinsically formal and the constant value of the bigraded homotopy Lie algebra equals $H_*^*\mathcal{L}_A$.*

(ii) *Suppose A^* is graded intrinsically formal. If $\mathrm{cg}(A) > 1$ then, for any 1-connected S with $H^*(S;\mathbb{Q}) \cong A^*$, the graded rational homotopy Lie algebra $\pi_*\Omega S \otimes \mathbb{Q}$ contains a free graded Lie algebra on two generators.*

THEOREM A'. — *Let A^* be a connected graded algebra, with associated vector-valued 2-form μ.*

(i) *If $\mathcal{P}\mathrm{Ext}_A^{*,1}(\mathbb{Q},\mathbb{Q}) = 0$ then A^* is 1-graded intrinsically formal and the constant value of the rational graded Lie algebra associated to the fundamental group equals L_A^*.*

(ii) *Suppose A^* is 1-graded intrinsically formal. If* $\mathrm{cg}(\mu) > 1$ *then, for any connected S with $H^*(S;\mathbb{Q}) \cong A^*$ and with $H_1(S;\mathbb{Z})$ finitely generated, $\pi_1 S$ contains a free group on two generators.*

We must point out that the condition of intrinsic spherical generation is necessary in Theorem A(i), see 2.3. As for the vanishing of $\mathcal{Q}\mathrm{Ext}_A^{>1,*}(\mathbb{Q},\mathbb{Q})$, this condition, while not really necessary (see again 2.3), seems to be a most natural one, as it follows for example from the proof given in Section 2. On the other hand, it is both a familiar condition, being first considered in [25] in connection with the cohomology of the Steenrod algebra and then intensively studied, see [22] for the connection with the cohomology of local rings, and there are many other interesting examples (see the next theorems, and also [26], as explained in 2.3). Theorem A(ii) should be related to the (yet unsolved) Félix-Avramov conjecture [12], claiming that, for a space S of finite rational Lusternik-Schnirelmann category, $\pi_*\Omega S \otimes \mathbb{Q}$ contains a free graded Lie algebra on two generators as soon as it is infinite dimensional. Similarly, the vanishing of $\mathcal{P}\mathrm{Ext}_A^{*;1}(\mathbb{Q},\mathbb{Q})$ in A'(i) above is not necessary (see 2.3) but again most natural (see the proof), and there are also many examples, as we shall see below. Theorem A'(ii) should be compared to similar results from [6],[7]; the methods are however entirely different and the hypotheses are rather complementary (see the remarks made after the proof of A'(ii) given in 3.2, and Proposition 3.3). Our Theorems A and A' emerged from our belief that the main results of [18], namely that the cohomology algebras $H^*(\vee S^1;\mathbb{Q})$ and $H^*((\vee S^1) \times S^1;\mathbb{Q})$ are 1-graded intrinsically formal - in our terminology (which were proved there "by hand"), ask for a proper generalization in a deformation theoretic framework.

Moving to our more concrete classes of examples of (finitely generated) graded algebras A^*, we shall focus here our attention to two parallel types, namely : *2-skeletal* algebras defined by the requirement that $A^{>2} = 0$ (which are plainly uniquely described by a vector-valued 2-form, $\mu\colon A^1 \wedge A^1 \to A^2$, for example the cohomology algebras of connected 2-complexes), and on the other hand *2-stage* (1-connected) algebras, defined by the condition $(A^+)^3 = 0$ (which correspond, by [11; Corollary 4.10], to formal spaces with rational Lusternik-Schnirelmann category ≤ 2, an intensively studied case). The resemblance is quite clear; to be more precise, there is a one-to-one onto correspondence between 2-skeletal algebras with $\mu =$ onto and 2-stage algebras generated in dimension 3, given by just tripling degrees. Given a 2-skeletal algebra A^*, we first associate to it by duality, as above, the map ∂, then construct a bigraded connected Lie algebra as follows : $E_*^* = \mathbb{L}X/\mathrm{ideal}(\partial Y)$, where the upper degree comes from bracket length and the lower degree is obtained by assigning to X the (lower) degree 2; at the same time we may pick bases $\{x_1, \dots, x_m\}$ for X and $\{y_1, \dots, y_n\}$ for Y and consider the sequence of elements $\partial y_j \in \mathbb{T}^2 X$ ($\mathbb{T}^* X = \bigoplus \mathbb{T}^p X$, $\mathbb{T}^p X = X^{\otimes p}$ is the tensor algebra). Similarly, a 2-stage A^* gives rise to a Quillen model of the form $\mathcal{L}_A = (\mathbb{L}(X_* \oplus Y_*), \partial)$, where $\partial X_* = 0$ and $\partial|_{Y_*}: Y_* \rightarrowtail (\mathbb{L}^2 X)_{*-1}$ (see §1 and §4); we may thus define, exactly as before, a bigraded Lie algebra E_*^* and elements $\partial y_j \in \mathbb{T}^2 X_*$ (note that in the correspondence given by tripling degrees these objects are *the same*).

In the results below we shall make use of Anick's notion of *strongly free* set of elements in a connected graded associative algebra [1; page 127] and of Halperin and Lemaire's natural specialization to *inert sequences* of elements of a connected graded Lie algebra [15; Définition 3.1].

THEOREM B. — *Let A^* be a 2-stage algebra.*

(i) *The following conditions are equivalent :*

– $\mathcal{Q}\mathrm{Ext}_A^{>1,*}(\mathbb{Q}, \mathbb{Q}) = 0$

– $\mathrm{gl\ dim}\ E_* \leq 2$

– $\{\partial y_j\}_{1 \leq j \leq n}$ *is strongly free in* $\mathbb{T}X$

Any of them implies that $H_*^* \mathcal{L}_A \cong E_*^*$ (as bigraded algebras).

(ii) *Assume $\mathcal{Q}\mathrm{Ext}_A^{>1,*}(\mathbb{Q}, \mathbb{Q}) = 0$. If $\mathcal{Q}A^*$ is concentrated in odd degrees or in an interval of degrees of the form $[l, 3l-2]$, then A^* is also intrinsically spherically generated.*

THEOREM B'. — *Let A^* be a 2-skeletal algebra.*

(i) *The following are equivalent :*

– $\mathcal{P}\mathrm{Ext}_A^{*,1}(\mathbb{Q}, \mathbb{Q}) = 0$

– $\mathcal{P}\mathrm{Ext}_A^{*,\geq 1}(\mathbb{Q}, \mathbb{Q}) = 0$

– $\mathrm{gl\ dim}\ E_* \leq 2$ *and μ is onto*

– $\{\partial y_j\}_{1 \leq j \leq n}$ *is strongly free in $\mathbb{T}X$ (automatically μ must be onto).*

(ii) *Any of the above implies the equality of formal power series*

$$\prod_{p=1}^{\infty} (1 - z^p)^{\dim L_A^p} = 1 - mz + nz^2, \quad \text{where} \quad m = \dim X.$$

The main source of examples for B(i) and B'(i) is provided by Anick's combinatorial criterion [1; Theorem 3.2] for strong freeness in tensor algebras (see 4.3, 6.1).

However, we should point out that we are not bound by our method to restrict ourselves, neither to 2-skeletal algebras (see e.g. the discussion around the Kohno example in 4.7, 4.8), nor to the rational coefficients.

Our general approach to the computation of the *integral* associated graded Lie algebra $\mathrm{gr}^*\pi_1 S$ (S = connected, of finite type) assumes $H_1 S$ to be a free abelian group. We shall naturally associate to S a $\mathbb{Z}$-Lie algebra with grading, denoted by $L_S^* = \mathbb{L}^*(H_1 S)$ modulo the ideal generated by $\mathrm{Im}\partial$, where ∂ is the $\mathbb{Z}$-dual of the cup product $\mu\colon H^1 S \wedge H^1 S \to H^2 S$ ($\mathbb{Z}$-coefficients throughout, this time!). Plainly L_S^* depends only on the cohomology algebra $H^*(S;\mathbb{Z})$ – in low dimensions – and $L_S^* \otimes \mathbb{Q} \cong L_A^*$, with $A^* = H^*(S;\mathbb{Q})$.

THEOREM C. — *If $H^*(S;\mathbb{Q})$ is 1-graded intrinsically formal and L_S^* is torsion free (as a graded abelian group), then $\mathrm{gr}^*\pi_1 S \cong L_S^*$, as graded Lie algebras.*

In practice, one is led to verify that $\mathcal{P}\mathrm{Ext}^{*,1}_{H^*(S;\mathbb{Q})}(\mathbb{Q},\mathbb{Q}) = 0$ and L_S^* = torsion free, and it often happens to be possible to work these conditions together, by examining $L_S^* \otimes \mathbb{Z}/p\mathbb{Z}$ (for all primes p), see §6.

We end by illustrating with a simple example, namely the case when the cohomology of S, in low dimensions, looks like that of a classical link complement (both $\mathbb{Q}$ and $\mathbb{Z}$ coefficients), in §6. Under certain combinatorial assumptions (formulated in terms of linking numbers, i.e. of $\mu\colon H^1 S \wedge H^1 S \to H^2 S$) we shall use our method to deduce the rigidity of $\mathrm{gr}^*\pi_1 S$ (see 6.2).

For S = classical link complement, we are just reobtaining a result due to Anick ([3], see also [21]). The methods are however entirely different : Anick-Labute first rely on explicit group presentations for $\pi_1 S$ and second (and even more important) on the verification of a certain independence criterion, formulated in [20]. Here is the common point of the two methods : this criterion just expresses an inertia (or, equivalently, strong freeness) condition. The two methods should be regarded as complementary. There are a few examples (of link complements) in [20] where Labute's criterion works, even in a nonrigid situation. There are also many examples (such as closed oriented generic 3-manifolds) where the method in our Theorem C applies, while the approach of Labute fails, see [5].

Finally, we ought to mention that in general given a connected algebra A^* with associated graded Lie algebra L_A^*, the vanishing of $\mathcal{P}\mathrm{Ext}_A^{*,\geq 1}(\mathbb{Q},\mathbb{Q})$ provides one with a very convenient framework for "the deformation theory of the fundamental group". In particular, one has the following (stronger) rigidity result : if moreover $H^{2,\geq 1}(L_A^*;L_A^*)=0$ (where the first degree of the above Lie algebra cohomology is the resolution degree and the other comes from the grading of L_A^* as usual) then the rational nilpotent completion of $\pi_1 S$ is constant within A^* (for instance Kojima's [18] rigidity result for $A^*\cong H^*(\vee S^1;\mathbb{Q})$ immediately follows, since in this case L_A^* is free). Results along these lines, related to the precise description of the variation of $\pi_1 S$"$\otimes$"$\mathbb{Q}$, may be found in [5].

Here is the plan of our paper :

1. Algebraic models and deformation theory
2. Rigidity results (proofs of A(i) and A'(i))
3. Bounds for homotopy Lie algebras (proofs of A(ii) and A'(ii))
4. Rigid examples and inert sequences (proofs of B and B')
5. An integral variant
6. An example (link-algebras and link-groups)

A preliminary version of our results was announced by the second author in a lecture given at the Conference OATE 2, September 1989, Craiova, Romania.

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1. Algebraic models and deformation theory.

In this first section we will purely algebraically reformulate theorems A and A' and prove a deformation-theoretic result which will be the key step in the proofs of A(i) and A'(i) to be given in the next section.

We shall deal with *bigraded Lie algebras* (bglie) L_*^*, $L=\bigoplus L_n^p$, $n\geq 0$ and $p>0$; ignoring upper degrees, we are thus considering just an usual *graded Lie algebra* (glie) L_*, with the standard sign conventions related to the skew-commutativity and the Jacobi identity [31; 0.4.(1)]; the upper degrees are only required to be compatible with the bracket, i.e. $[L^p, L^q] \subset L^{p+q}$; we shall also frequently have the occasion to meet bigraded Lie algebras whose lower degrees are concentrated in dimension zero; by just ignoring them, we shall speak of a *Lie algebra with grading* (grlie) L^* (no extra signs!). We shall also consider the *lower central series* of a (graded) Lie algebra L, the descending chain of (graded) ideals inductively defined by $L^{(1)} = L$ and $L^{(p+1)} = [L, L^{(p)}]$, and the *associated (bi)graded Lie algebra* $\mathrm{gr}^* L = \bigoplus_{p \geq 1} \mathrm{gr}^p L$, $\mathrm{gr}^p L = L^{(p)}/L^{(p+1)}$; the topological examples we have in mind are the *homotopy Lie algebra* $\pi_* \Omega S$ of a 1-connected space S, and its associated graded, $\mathrm{gr}^* \pi_* \Omega S$. Similarly one may consider the *lower central series* of a group π, denoted by $\pi^{(p)}$, $p \geq 1$, and the *associated Lie algebra with grading* $\mathrm{gr}^* \pi = \bigoplus_{p \geq 1} \mathrm{gr}^p \pi$, $\mathrm{gr}^p = \pi^{(p)}/\pi^{(p+1)}$, see e.g. [29; II.2], for example the *homotopy* grlie *algebra* of a connected space S, $\mathrm{gr}^* \pi_1 S$.

We are going to exploit the duality between Lie algebras and (quadratic free) differential graded commutative algebras (dga's)–see [31; Proposition I.1.(5)] (in particular we shall follow the notation of [31] and constantly denote vector space duals by #). We thus recall that there is a (categorial) equivalence between bigraded Lie algebras, which are of finite type with respect to the lower degree, and free dga's of the form $(\wedge Z^*_*, D)$, where $Z = \bigoplus Z^n_p$, $n > 0$ and $p \geq 0$, $\dim Z^n < \infty$ for all n and the differential D is quadratic and bihomogeneous, i.e. $DZ^n_p \subset (\wedge^2 Z)^{n+1}_{p-1}$. The equivalence is described by $L \longmapsto \mathcal{C}^*(L) = (\wedge Z, D)$ [31; I.1], where $Z^n_p = \# L^{p+1}_{n-1}$ and $D \colon Z \to Z \wedge Z$ is dual to the Lie bracket.

Let $(\wedge V^*, D)$ be a free dga, which is of finite type and with quadratic differential, and let L_* be the dual glie. By a *nilpotent filtration* on $(\wedge V, D)$ we shall mean an ascending filtration on V^*, $\{F_p\}_{p \geq 0}$, with $F_0 = 0$ and $DF_p \subset \wedge^2 F_{p-1}$, any $p \geq 1$, which will be called *exhaustive* if $V = \bigcup F_p$. The *canonical filtration* $\{F_p V\}_{p \geq 0}$ is defined by $F_0 V = 0$ and, inductively, $F_p V = (D|_V)^{-1} \wedge^2 F_{p-1} V$, for $p \geq 1$. By construction, it is nilpotent.

1.1. Lemma. — *For any $p \geq 0$, $F_p V$ coincides, by duality, with the space orthogonal to $L^{(p+1)}$, ${L^{(p+1)}}^{\perp}$.*

Proof. — This lemma is both elementary and perhaps well known. For the reader's convenience we are going to sketch a proof.

For $p = 0$, this is obvious. For $v \in V$ and $f, g \in L$ recall the basic duality equation [31; page 26] :

$$\langle v, [f,g]\rangle = -(-1)^{\deg g} \cdot \langle Dv, f \wedge g\rangle. \tag{1.2}$$

This equation immediately gives our claim also for $p = 1$. The induction step will be based on the equation $N^\perp \wedge N^\perp \cong (M \wedge N)^\perp$, where M is a (graded) vector space and $N \subset M$ a (graded) subspace, which follows from elementary multilinear algebra.

Suppose then $F_{p-1}V \cong {L^{(p)}}^\perp$. By definition, $v \in F_pV$ if and only if $Dv \in \wedge^2 F_{p-1}V$. Invoking the above equation (with $M = L$ and $N = L^{(p)}$), we have $\wedge^2 F_{p-1}V \cong \wedge^2 {L^{(p)}}^\perp \cong (L \wedge L^{(p)})^\perp$. But $Dv \in (L \wedge L^{(p)})^\perp$ means, by (1.2), exactly $v \in [L, L^{(p)}]^\perp \cong {L^{(p+1)}}^\perp$, which completes the proof. $\square$

We are now going to briefly review the algebraic models of rational homotopy Lie algebras. Any connected space S has a so-called (Sullivan) *minimal model* $\mathcal{M}_S$: it is a free dga $(\wedge Z^*, d)$, which is both *nilpotent*, i.e. Z is increasingly filtered by subspaces $\{F_p\}_{p \geq 0}$, with $F_0 = 0$ and $dF_p \subset \wedge F_{p-1}$ for $p > 0$, and *minimal*, that is $d|_Z = d_2 + d_3 + \ldots$, where each d_i takes values in $\wedge^i Z$, where $i =$ monomial length; when S is 1-connected with finite Betti numbers, Z^* is of finite type. In this latter case there is also the *Quillen model* of S, $\mathcal{L}_S$; this is a free dgLie $(\mathbb{L}W_*, \partial)$, which is also *minimal*, i.e. $\partial W \subset [\mathbb{L}W, \mathbb{L}W]$. The first basic result reads

THEOREM ([27],[28],[31]). — *$\pi_*\Omega S \otimes \mathbb{Q} \cong \mathcal{C}^{*^{-1}}(\wedge Z^*, d_2) \cong H_*\mathcal{L}_S$ as graded Lie algebras.*

For a general connected S one may also consider the *1-minimal model*, namely the sub dga $\mathcal{M}_1 \subset \mathcal{M}$ given by $\mathcal{M}_1 = (\wedge V, d)$ (where we put $Z^1 = V$); this will be a *1-minimal* algebra, i.e. a free nilpotent dga generated in degree one. For such algebras, one may still define the canonical filtration exactly as above : this will be an exhaustive nilpotent filtration on $(\wedge V, d)$, $\mathcal{F} = \{F_pV\}_{p \geq 0}$. If moreover the first Betti number of S is finite, then it easily follows that $\dim F_pV < \infty$, for any p, and we may safely dualize. We thus define the associated Lie algebra with grading of a 1-minimal algebra $(\wedge V, d)$, to be denoted by $\mathrm{gr}^*(\wedge V, d)$, by

$$\mathrm{gr}^*(\wedge V, d) = \varprojlim_p \mathrm{gr}^* \mathcal{C}^{*^{-1}}(\wedge F_pV, d). \tag{1.3}$$

THEOREM ([28],[9]). — *$\mathrm{gr}^*\pi_1 S \otimes \mathbb{Q} \cong \mathrm{gr}^*(\wedge V, d)$ as Lie algebras with grading.*

Given a 1-minimal $(\wedge V, d)$ and assuming $\dim H^1(\wedge V, d) < \infty$ in order to smoothly dualize, we may also start with an arbitrary exhaustive nilpotent filtration $\mathcal{F}' = \{F'_p V\}_{p \geq 0}$. It readily follows by induction that $F'_p \subset F_p$, for any p, hence we have a natural grlie map

$$\mathrm{gr}^*_{\mathcal{F}}(\wedge V, d) \cong \varprojlim_p \mathrm{gr}^* \mathcal{C}^{*^{-1}}(\wedge F_p, d) \to \varprojlim_p \mathrm{gr}^* \mathcal{C}^{*^{-1}}(\wedge F'_p, d) \cong \mathrm{gr}^*_{\mathcal{F}'}(\wedge V, d).$$

1.4. Proposition. — *For any exhaustive nilpotent filtration $\mathcal{F}'$ the above map $\mathrm{gr}^*_{\mathcal{F}}(\wedge V, d) \to \mathrm{gr}^*_{\mathcal{F}'}(\wedge V, d)$ is an isomorphism.*

Proof. — For a fixed n and an arbitrary exhaustive nilpotent filtration $\{\mathcal{F}'_p\}$, we have to evaluate the vector space $\varprojlim_p \mathrm{gr}^n \mathcal{C}^{*^{-1}}(\wedge F'_p, d)$. By Lemma 1.1 this is naturally isomorphic to $\varprojlim_p \#(F_n F'_p / F_{n-1} F'_p) \cong \#(\varinjlim_p F_n F'_p / \varinjlim_p F_{n-1} F'_p)$. It plainly suffices to show that the natural map $\varinjlim_p F_n F'_p \to F_n V$ is isomorphic (to be more precise epic). For $n = 1$ this is obvious (recall that $V = \varinjlim_p F'_p$) and the induction goes well on, by the very definition of the canonical filtration. □

We move to the algebraic models of formal spaces. Given a connected algebra of finite type A^*, it is constructed in [17; pages 242-243], the *bigraded model*, $\mathcal{B}_A = (\wedge Z^*_*, D)$; it is a minimal dga, which also carries a second (lower) graduation, with respect to which D is homogeneous of (lower) degree -1 (it is a minimal bdga). It is uniquely characterized by the existence of a bdga map $\mathcal{B}_A \to (A^*, 0)$ (where A^* is concentrated in lower degree zero, and is endowed with trivial differential) inducing a cohomology isomorphism; forgetting the lower degrees, it represents the minimal model of the formal space S_A. If A^* is 1-connected, there is also a formal Quillen model (see [31; III.4.(5)]) $\mathcal{L}_A = (\mathbb{L}W_*, \partial_2)$, which corresponds to A^* by duality : $W = \#s^{-1}\overline{A}$, and the quadratic Lie differential ∂_2 is essentially dual to the multiplication; this dg Lie model also carries a second (upper) grading, given by bracket length, and ∂_2 is homogeneous of upper degree $+1$. We have a well-known [31] equality of bglie

$$\mathcal{C}^{*^{-1}}(\wedge Z^*_*, D_2) \cong H^*_* \mathcal{L}_A. \tag{1.5}$$

Given a bglie L^*_* (of finite type with respect to the lower degree) its dual quadratic bdga, $(\wedge Z^*_*, D)$ also carries a third grading, coming from monomial length; the induced grading on cohomology will be denoted by $H^*_*(\wedge Z, D) = \bigoplus_{p\geq 1} {}^p H^*_*(\wedge Z, D)$.

1.6. LEMMA. — *Consider $\mathcal{C}^*(L^*_*) = (\wedge Z^*_*, D)$. The following are equivalent :*

(i) $L^{(p)} \cong L^{\geq p}$, *any* p

(ii) $F_p Z \cong Z_{<p}$, *any* p

(iii) L^* *is generated by* L^1

(iv) ${}^1H^*_+(\wedge Z, D) = 0$

(v) $\mathrm{gr}^* L_* \cong L^*_*$

(vi) $L^* \cong \varprojlim_p \mathrm{gr}^*(L/L^{(p)})$.

Proof. — By Lemma 1.1, $F_p Z \cong Z_{<p}$ is equivalent to $L^{(p+1)} \cong L^{\geq p+1}$. Since L^* is strictly positively graded, we have in general an inclusion $L^{(p)} \subset L^{\geq p}$, for any p; an easy homogeneity argument shows that the equality is in fact equivalent to $L^1 \to L/[L, L]$ being onto, hence, by duality, to the fact that the canonical projection $\mathrm{Ker}(D|_{Z_*}) \to Z_0$ is monic; this last condition precisely says that ${}^1H_+(\wedge Z_*, D) = 0$. In general, $\mathrm{gr}^* L$ is always generated by $\mathrm{gr}^1 L$; conversely, assuming L^* is generated by L^1, it easily follows from (i) that $\mathrm{gr}^* L \cong L^*$. Finally, by an obvious stability argument, one always knows that $\varprojlim_p \mathrm{gr}^*(L/L^{(p)}) \cong \mathrm{gr}^* L$. □

A convenient set-up for the description of formal 1-minimal models is provided by considering Lie algebras with grading, L^*, which are required to be generated by L^1, $\dim L^1 < \infty$. Given such L^*, consider the inverse system $\ldots \twoheadrightarrow L/L^{(p+1)} \twoheadrightarrow L/L^{(p)} \twoheadrightarrow \cdots$ of central extensions of finite-dimensional grlie algebras (considered as bglie concentrated in lower degree zero). Set

$$(\wedge V_*, d) = \varinjlim_p \mathcal{C}^*(L^*/L^{(p)*}). \tag{1.7}$$

It is a 1-minimal dga, by the well-known duality between central extensions of Lie algebras and elementary extensions of dga's [13], which is also a bdga. It carries a natural (nilpotent exhaustive) filtration $\mathcal{F}$ given by $F_p = V_{<p}$, for which one has by construction and the preceding lemma $\mathrm{gr}^*_{\mathcal{F}}(\wedge V, d) \cong L^*$. Starting with an algebra A^* (connected and of finite type as usual), first construct a gr Lie algebra L^*_A as in the introduction, namely

$L_A^* = \mathbb{L}_X^*/\mathrm{ideal}(\partial Y)$, where $\partial\colon Y \to X \wedge X \cong \#(\mu\colon A^1 \wedge A^1 \to A^2)$ (also noticing that L_A^* depends only on $\mu\colon A^1 \wedge A^1 \to \mathrm{Im}\mu \subset A^2$), and then associate to L_A^* the 1-bigraded model $(\wedge V_*, d)$ as in (1.7). The next lemma seems to be folklore, but we chose to include a proof, being unable to find a reference (not to speak of the fact that the construction (1.7) will be again useful later on, see the proof of Proposition 6.3).

1.8. LEMMA. — *Let A^*, L_A^* and $(\wedge V_*, d)$ be as above. Then :*

(i) *$(\wedge V_*, d)$ is the 1-bigraded model of the formal space S_A*

(ii) $\mathrm{gr}^*\pi_1 S_A \otimes \mathbb{Q} \cong L_A^*$.

Proof. — Given the general theory, Proposition 1.4 and the above remarks, (ii) will follow at once from (i).

As far as (i) is concerned, we start by constructing a bdga map $f\colon (\wedge V_*, d) \to (A^*, 0)$. We set $f|_{V_+} = 0$, notice that $V_0 \cong \#L_A^1 \cong \#X \cong A^1$, and put $f|_{V_0} = \mathrm{id}$; due to the homogeneity property of d with respect to lower degrees, checking that f commutes with the differentials is reduced to showing that $fdV_1 = 0$, i.e. the composition $V_1 \xrightarrow{d} V_0 \wedge V_0 \xrightarrow{f\wedge f} A^1 \wedge A^1 \xrightarrow{\mu} A^2$ equals zero. Taking duals, this amounts to seeing that $L_A^2 \xleftarrow{[\,,\,]} L_A^1 \wedge L_A^1 = X \wedge X \xleftarrow{\partial} Y$ equals zero, which is obvious by the construction of L_A^*. By the uniqueness of 1-minimal models we must only verify that $H^1 f$ is an isomorphism and $H^2 f$ is monic [28]. But we know that $H^1(\wedge V_*, d) \cong \varinjlim_p {}^1H(C^*(L_A^*/L_A^{(p)*}))$ and, by Lemma 1.6, ${}^1H(C^*(L_A^*/L_A^{(p)*})) \cong {}^1H_0(C^*(L_A^*/L_A^{(p)*})) \cong V_0$, which takes care of the condition on $H^1 f$. On the other hand $H^2(\wedge V_*, d) \cong H_0^2(\wedge V_*, d) \oplus H_+^2(\wedge V_*, d)$, and $\mathrm{Im} H_0^2 f \cong \mathrm{Im}\mu$, while $\mathrm{Im} H_+^2 f = 0$, by the construction of f. We may thus use a dimension argument : $H^2 f$ is monic is equivalent to $\dim H^2(\wedge V, d) = \dim \mathrm{Im}\,\mu$. We may notice again that $H^k(\wedge V, d) \cong {}^kH(\wedge V, d)$, and this in turn equals $H^k(L_A; \mathbb{Q})$ - classical Lie algebra cohomology with trivial coefficients via the Koszul resolution, see e.g. [31] - for any k. We are thus led to compute $\dim H_2(L_A; \mathbb{Q})$, and we may use for this purpose the description of the second homology group of a Lie algebra of the form $\mathbf{f}/\mathbf{r}$, where $\mathbf{f}$ is a free Lie algebra and $\mathbf{r}$ an ideal, given in [16; page 238, Exercise 3.2] : $H_2(\mathbf{f}/\mathbf{r}; \mathbb{Q}) \cong [\mathbf{f}, \mathbf{f}] \cap \mathbf{r}/[\mathbf{f}, \mathbf{r}]$. We infer that $H_2(L_A; \mathbb{Q}) \cong I/[\mathbb{L}X, I]$, where I is the ideal generated by ∂Y. The dimension of the last object plainly equals $\dim \mathrm{Im}\,\partial$, and finally $\dim \mathrm{Im}\,\partial = \dim \mathrm{Im}\,\mu$, by duality. □

Next we are going to conveniently rephrase the conditions on the Hopf algebra $\mathrm{Ext}_A^{*,*}(\mathbb{Q},\mathbb{Q})$ stated in the introduction.

1.9. LEMMA. — *Let A^* be a connected algebra, with bigraded model $\mathcal{B}_A = (\wedge Z_*^*, D)$ and Quillen model $\mathcal{L}_A$ (in the 1-connected case).*

(i) *$Q\mathrm{Ext}_A^{>1,*}(\mathbb{Q},\mathbb{Q}) = 0$ if and only if $H^*\mathcal{L}_A$ is generated (as a Lie algebra) by $H^1\mathcal{L}_A$.*

(ii) *$\mathcal{P}\mathrm{Ext}_A^{*,1}(\mathbb{Q},\mathbb{Q}) = 0$ if and only if $Z_*^2 = 0$.*

Proof. — (i) The condition $Q\mathrm{Ext}_A^{>1,*}(\mathbb{Q},\mathbb{Q}) = 0$ simply means that the Yoneda Ext-algebra of A is generated (as an algebra) by $\mathrm{Ext}_A^1(\mathbb{Q},\mathbb{Q})$. Consider then the formal space S_A and its formal Quillen dgrlie model $\mathcal{L}_A^*$, graded by the bracket length. In [2; Theorem 2], is established a graded isomorphism $\bigoplus_{i\geq 0}\mathrm{Ext}_A^{i,*} \cong \bigoplus_{i\geq 0} H_*^i\mathcal{U}\mathcal{L}_A$ (where $\mathcal{U}$ = universal enveloping algebra functor), which is also compatible with the algebra structures, up to sign. Since $H^*\mathcal{U}\mathcal{L}_A \cong \mathcal{U}H^*\mathcal{L}_A$ as algebras and $Q\mathcal{U}H^*\mathcal{L}_A \cong QH^*\mathcal{L}_A \cong H^*\mathcal{L}_A/[H^*\mathcal{L}_A, H^*\mathcal{L}_A]$ as graded vector spaces [27], our assertion follows.

(ii) It is proven in [17; Corollary 7.17], that one has $Z_*^n \cong \# \mathcal{P}\mathrm{Ext}_A^{*,n-1}(\mathbb{Q},\mathbb{Q})$ for any n. □

We describe now the algebraic parametrization of rational homotopy types with fixed cohomology algebra A^* in terms of deformation theory, following [17] and [23] (see also [24]). In the dga setting of [17], one starts with the bigraded model $\mathcal{B}_A = (\wedge Z_*^*, D)$; it is convenient to set $D = D^1$. Then any space S with $H^*(S;\mathbb{Q}) \cong A^*$ has a free dga model of the form $(\wedge Z_*^*, D^1 + p)$, where the algebraic parameter, the perturbation p, may be written $p = p^2 + p^3 + \ldots$, each p^i being homogeneous of lower degree $-i$; the trouble comes from the fact that this (nilpotent) model may fail to be minimal (this is geometrically related to the collapsing of the Eilenberg-Moore spectral sequence of S, see the next section). If A^* is 1-connected, there is the alternative dglie setting of [23], where one starts with the Quillen formal model $\mathcal{L}_A = (\mathbb{L}^*W_*, \partial)$; here we set $\partial = \partial^1$. One may represent any S within A^* by a (minimal) dglie model of the form $(\mathbb{L}W_*, \partial^1 + p)$, where, again, $p = p^2 + p^3 + \ldots$, each p^i being homogeneous of upper degree i. Finally, here comes our basic deformation-theoretic result.

1.10. PROPOSITION. — *Consider a bigraded Lie algebra L_*^* (of finite type with respect to lower degrees) and its quadratic* bdga *dual, $(\wedge Z_*^*, D)$. Suppose that L^* is generated by L^1. Then for any quadratic* dga *of the form*

$(\wedge Z^*, d)$, where $d = D^1 + p^2 + p^3 + \ldots$, $D^1 = D$ and each p^i is homogeneous of degree $-i$ with respect to lower degrees, we have an isomorphism of bigraded Lie algebras :

$$\mathrm{gr}^* \mathcal{C}^{*^{-1}}(\wedge Z^*, d) \cong \mathrm{gr}^* \mathcal{C}^{*^{-1}}(\wedge Z^*, D).$$

Proof. — Set $\mathcal{C}^{*^{-1}}(\wedge Z^*, d) = (L_*, [\ ,\ ]_p)$. By just dualizing the decomposition of the perturbed differential, $d = D + p$, one infers that the perturbed Lie bracket $[\ ,\ ]_p$ has the following property : for any $x \in L^m$, $y \in L^n$, $[x, y]_p = [x, y]$ modulo $L^{>m+n}$ (and of course $[x, y] \in L^{m+n}$). By Lemma 1.6 one may precisely describe the lower central series of the original graded Lie algebra $(L_*, [\ ,\ ])$, in terms of the upper graduation. Our assertion will immediately follow, as soon as we prove that the lower central series of the perturbed Lie algebra is the same, or equivalently (by Lemma 1.1) the canonical filtration of $(\wedge Z, d)$, denoted by ${}^p\mathcal{F} = \{{}^pF_n Z\}_{n \geq 0}$ coincides with the canonical filtration of $(\wedge Z, D)$, which is, again by 1.6, $\mathcal{F} = \{F_n Z = Z_{<n}\}_{n \geq 0}$. By (lower) degree inspection, $dZ_n \subset \wedge^2 Z_{<n}$, for any n, and this inductively implies that $F_n Z \subset {}^pF_n Z$, for any n. The remaining inclusion will also be proven by induction, trivially starting with $n = 0$. Assume then $z \in {}^pF_n Z$ and write $z = z_0 + \ldots + z_m$, where $z_i \in Z_i$ and $z_m \neq 0$. By the definition of the canonical filtration and by the induction hypothesis, we know that $dz \in \wedge^2 Z_{<n-1}$; writing $d = D + p$ and examining the top component of dz with respect to lower degree, we infer that $Dz_m \in \wedge^2 Z_{<n-1}$, hence $z_m \in F_n Z = Z_{<n}$, therefore $m < n$ and $z \in Z_{<n} = F_n Z$, as desired. □

2. Rigidity results.

This section contains the (almost simultaneous) proofs of Theorems A(i) and A'(i), and the first examples.

2.1. *Proof of Theorem A(i).* — Represent any S with $H^*(S; \mathbb{Q}) \cong A^*$ by the free dga model $(\wedge Z^*, D + p)$, as explained before. We claim that it will be enough to show that the Eilenberg-Moore spectral sequence of S collapses at the E_2 term. Indeed, we know from [17; Theorem 7.20], that this is equivalent to the minimality of $(\wedge Z, D + p)$, and also equivalent to $\dim \pi_k S \otimes \mathbb{Q} = \dim \pi_k S_A \otimes \mathbb{Q}$, for any k. Once we may assume this, we know that $\mathcal{C}^* \pi_* \Omega S \otimes \mathbb{Q} \cong (\wedge Z^*, D_2 + p_2)$ where the subscript 2 indicates that we have taken the quadratic parts. We may now use the previous proposition, by taking $L^*_* = H^*_*\mathcal{L}_A$, with dual $(\wedge Z^*_*, D_2)$, see (1.5); our hypothesis on $\mathcal{Q}\mathrm{Ext}_A$ guarantees that L^* is generated by L^1 (Lemma 1.9(i)). We deduce a bglie isomorphism $\mathrm{gr}^*\pi_*\Omega S\otimes\mathbb{Q} \cong \mathrm{gr}^* L_*$, the second bglie being isomorphic to L^*_*, again due to the above-mentioned hypothesis (see Lemma 1.6(v)).

In order to establish the EMss collapse property, we are going to use the numerical criterion in terms of ranks of homotopy groups and the dglie approach. Represent then S by a Quillen minimal model of the form $(\mathbb{L}W_*, \partial + p)$, as described before. The hypothesis that A^* is also intrinsically spherically generated comes now into play and allows us to suppose moreover (see [24; Proposition 1.8]) that $p|_{\mathcal{P}} = 0$, where the primitive subspace $\mathcal{P}$ equals $\mathrm{Ker}(\partial|_W)$. Filtering $\mathbb{L}W$ by bracket length, we obtain a well-known [27] spectral sequence of graded Lie algebras, converging to $H_*(\mathbb{L}W, \partial+p) \cong \pi_*\Omega S\otimes\mathbb{Q}$ and starting with $E^1_* \cong (\mathbb{L}W_*, \partial)$ and $E^2_* \cong H_*\mathcal{L}_A \cong \pi_*\Omega S_A \otimes \mathbb{Q}$. On the other hand we invoke again our assumption on $\mathcal{Q}\mathrm{Ext}_A$, recalling that $H^*\mathcal{L}_A$ is generated as a Lie algebra by $H^1\mathcal{L}_A \cong \mathcal{P}$, which consists only of permanent cycles, by the spherical generation property, hence $E^2 \cong E^\infty$ and $\dim \pi_k\Omega S\otimes\mathbb{Q} = \dim \pi_k\Omega S_A\otimes\mathbb{Q}$, as claimed. Our proof is complete. □

2.2. *Proof of Theorem A'(i).* — This is similar but simpler. Use again the perturbed free dga model $(\wedge Z^*, D+p)$ of S and set $Z^1_* = V_*$. Since $Z^2 = 0$ (Lemma 1.9(ii)) $(\wedge V, D+p)$ is a subgda of $(\wedge Z, D+p)$, for trivial degree reasons; it is equally trivial to see that the above dga inclusion induces an isomorphism at the H^1 level and is monic at the H^2 level, hence [28] $(\wedge V, D+p)$ represents the 1-minimal model of S (the nilpotence condition is easily checked along the lower degree filtration of V_*). Set then $F_n = V_{<n}$. By Proposition 1.4, $\mathrm{gr}^*\pi_1 S\otimes\mathbb{Q} \cong \varprojlim_n \mathrm{gr}^*\mathcal{C}^{*^{-1}}(\wedge V_{<n}, D+p)$.

We may apply Proposition 1.10 to the finitely generated quadratic bdga $(\wedge V_{<n}, D)$. The requirement that the dual Lie algebra L^* be generated by L^1 is now automatically satisfied. Indeed we may check the equivalent condition given by Lemma 1.6(iv) by noticing that obviously

$$^1H_+(\wedge V^1_{<n}, D) \cong H^1_+(\wedge V^1_{<n}, D) \subset H^1_+(\wedge Z^1_*, D) \cong H^1_+(\wedge Z^*_*, D)$$

and that the cohomology of the bigraded model $(\wedge Z^*_*, D)$ is concentrated by definition in lower degree zero. We infer that $\mathrm{gr}^*\mathcal{C}^{*^{-1}}(\wedge V_{<n}, D+p) \cong \mathrm{gr}^*\mathcal{C}^{*^{-1}}(\wedge V_{<n}, D)$, which is independent of p, for any n. Finally, for the formal space S_A corresponding to $p = 0$, Lemma 1.8(ii) tells us that $\mathrm{gr}^*\pi_1 S_A\otimes\mathbb{Q} \cong L^*_A$, which was the last assertion to be proved. □

2.3. *Remarks and examples.* — First we ought to notice that intrinsic spherical generation is a necessary condition for graded intrinsic formality (in the 1-connected case), indeed, if $H^*(S;\mathbb{Q}) \cong A^*$, the isomorphism $\mathrm{gr}^*\pi_*\Omega S \otimes \mathbb{Q} \cong \mathrm{gr}^*\pi_*\Omega S_A \otimes \mathbb{Q}$ evidently implies that $\dim \pi_k S \otimes \mathbb{Q} = \dim \pi_k S_A \otimes \mathbb{Q}$, for any k, hence $E_2 \cong E_\infty$ in the EMss, and this in turn forces S to be spherically generated, as shown in [17; 8.13]. On the other hand the assumption on the Yoneda Ext-algebra of A^* made in A(i), albeit very natural, is not strictly necessary, as shows the following very simple example, namely $A^* = H^*\mathbb{P}^2\mathbb{C}$. This is an intrinsically formal (hence graded intrinsically formal) example – this is very easy, see e.g. [28]. A short direct computation gives that $H^*_*\mathcal{L}_A$ is a 2-dimensional abelian Lie algebra with basis $a \in H^1_1$ and $b \in H^2_4$, therefore (Lemma 1.9(i)) $\mathcal{Q}\mathrm{Ext}_A^{>1,*}(\mathbb{Q};\mathbb{Q}) \neq 0$.

As a first natural series of examples where $\mathcal{Q}\mathrm{Ext}_A^{>1,*}(\mathbb{Q};\mathbb{Q}) = 0$ we may quote $A^* = H^*MG$, where MG is the universal Thom space associated to an arbitrary orthogonal representation of the compact connected Lie group G, see [26]. In the other direction, any homogeneously generated algebra A^* is intrinsically spherically generated ([24; 2.4, see also 4.4]).

We also have to notice that the hypotheses of A(i) are independent. We have just seen that $H^*\mathbb{P}^2\mathbb{C}$ is intrinsically spherically generated and still the condition on $\mathcal{Q}\mathrm{Ext}_A$ is violated. Let us now define an algebra A^* by describing its Quillen formal model, as in [31] : $\mathcal{L}_A = (\mathbb{L}(x_1, x_2, x_3, x, y), \partial)$, where $\deg x_1 = \deg x_2 = \deg x_3 = 2$, $\deg x = 7$ and $\deg y = 5$, and the only nontrivial action of ∂ is on y, namely $\partial y = [x_1, x_2]$. Anticipating a little (see 4.1 and 4.3), we know that $\mathcal{Q}\mathrm{Ext}_A^{>1,*}(\mathbb{Q},\mathbb{Q}) = 0$. Defining a perturbation p by the requirement that the only nontrivial action be $px = [x_1, [x_2, x_3]]$, we get a space S with $H^*(S;\mathbb{Q}) \cong A^*$, whose minimal Quillen model is $(\mathbb{L}, \partial + p)$ [23]. Finally, due to the fact that, in $\mathbb{L}(x_1, x_2, x_3)$, $px \notin \mathrm{ideal}(\partial Y)$, a simple computation with the algebraic rational Hurewicz homomorphism as in [31; III.3.(5)] shows that the primitive element of $H_*(S;\mathbb{Q})$ corresponding to x is not spherical, hence A^* is not intrinsically spherically generated.

The first nontrivial examples of 1-graded intrinsically formal algebras are those of [18], namely $A^*_m = H^*((\vee_{m-1} S^1) \times S^1)$, for $m > 2$; these fit into our theory and satisfy the condition $\mathcal{P}\mathrm{Ext}_A^{*,1}(\mathbb{Q},\mathbb{Q}) = 0$ (see 4.3).

Finally consider A^* = the cohomology algebra of the 2-skeleton of the m-torus, $m > 2$. Concerning the bigraded model $\mathcal{B}_A$ of [17], it may be easily seen that the 1-bigraded model is $(\wedge Z^1_0, D = 0)$, with $\dim Z^1_0 = m$,

$Z_0^2 = 0$ and $\dim Z_1^2 = m(m-1)(m-2)/6$. Using deformation theory exactly as in the proof of A'(i), it immediately follows that even π_1"$\otimes$"$\mathbb{Q}$ is constant within A^*, hence A^* is 1-graded intrinsically formal, though $\mathcal{P}\mathrm{Ext}_A^{*,1}(\mathbb{Q},\mathbb{Q}) \neq 0$. However it seems that this is the right kind of condition for having a reasonable "deformation theory" for the fundamental group.

3. Bounds for homotopy Lie algebras.

Here we give the proofs of A(ii) and A'(ii) and also exhibit two quite general types of bounds for homotopy Lie algebras directly related to the main ideas of the paper.

3.1. *Proof of Theorem A(ii).* — Assuming $H^*(S;\mathbb{Q}) \cong A^*$, we know that $\mathrm{gr}^*\pi_*\Omega S \otimes \mathbb{Q} \cong \mathrm{gr}^* H_*\mathcal{L}_A$, by the graded intrinsic formality of A^*. We claim that from $\mathrm{cg}(A) \geq n$ it follows that there exists a bglie onto map $f\colon \mathrm{gr}^* H_*\mathcal{L}_A \to \mathbb{L}_n^*$, where $\mathbb{L}_n^*$ is a free graded Lie algebra on n homogeneous generators of strictly positive degrees, which is bigraded by using the bracket length as upper degree. Postponing for the moment the proof of the claim, we finish by observing that a bglie onto map $f\colon \mathrm{gr}^*\pi_*\Omega S \otimes \mathbb{Q} \to \mathbb{L}^*(x,y)$ gives rise to a bglie monic section $s\colon \mathbb{L}^*(x,y) \to \mathrm{gr}^*\pi_*\Omega S \otimes \mathbb{Q}$, which, by freeness of $\mathbb{L}(x,y)$ and by lifting in upper degree one, finally provides a glie map $h\colon \mathbb{L}(x,y) \to \pi_*\Omega S \otimes \mathbb{Q}$. Since the free Lie algebra is generated in upper degree one, we know that $\mathrm{gr}^*\mathbb{L} \cong \mathbb{L}^*$ and thus $\mathrm{gr}^* h = s$, therefore h is also monic.

Coming back to our claim, we recall that we have an n-dimensional graded vector subspace $N \subset A^+$, with $N \cdot N = 0$ and $N \subset A^+ \twoheadrightarrow QA$ monic. We therefore have a graded algebra map $\mathbb{Q} \cdot 1 \oplus N \overset{j}{\hookrightarrow} A^*$, where the multiplication in the first algebra is defined by $(q \oplus n) \cdot (q' \oplus n') = qq' \oplus (qn' + q'n)$. This gives rise by duality to a bdglie map $g\colon \mathcal{L}_A \to \mathcal{L}_{\mathbb{Q}\cdot 1 \oplus N} = (\mathbb{L}^* V_*, \partial = 0)$, where $V_* = \# s^{-1} N^*$ [31]. We may take then $f = \mathrm{gr}^* H_* g$, and it will be plainly enough to show that $H_* g$ is onto; since g is bihomogeneous and $\mathbb{L}$ is generated by $\mathbb{L}^1$, this is equivalent to $H^1 g$ being onto or, by duality, to $Q(j)$ being monic, which is precisely the injectivity condition for $N \subset A^+ \twoheadrightarrow QA$. $\square$

3.2. *Proof of Theorem A'(ii).* — Here we know that $\mathrm{gr}^*\pi_1 S \otimes \mathbb{Q} \cong \mathrm{gr}^*\pi_1 S_A \otimes \mathbb{Q} \cong L_A^*$ (see 1.8.(ii)). We now claim that $\mathrm{cg}(\mu)$ = maximal n for which there is a grlie onto map $f\colon L_A^* \to \mathbb{L}_n^*$. Temporarily taking this for granted, we are going to complete the proof, in a way similar with the preceding one, by first taking a grlie section of f, $s\colon \mathbb{L}^*(x,y) \to L_A^* \cong \mathrm{gr}^*\pi_1 S \otimes \mathbb{Q}$. Due to the finite generation property of $H_1(S;\mathbb{Z})$ we may also suppose (possibly after replacing x and y by suitable nonzero multiples) that sx and sy lift to $\mathrm{gr}^1\pi_1 S$, hence to $\pi_1 S$. We have thus obtained a group homomorphism from the free group on two generators, $h\colon \mathbb{F}_2 = \mathbb{F}(x,y) \to \pi_1 S$, with the property that $\mathrm{gr}^* h \otimes \mathbb{Q} = s$ ($\mathrm{gr}^*\mathbb{F}(x,y) \cong \mathbb{L}_{\mathbb{Z}}^*(x,y)$, [29; IV.6, Theorem 1]). Consequently $\mathrm{gr}^* h$ is monic. A rather standard argument shows then h to be monic, by working with the nilpotent quotients : for any n consider the induced map $h_n\colon \mathbb{F}_2/\mathbb{F}_2^{(n)} \to \pi_1 S/\pi_1 S^{(n)}$. An easy inductive argument based on the commutative diagram

$$\begin{array}{ccccccccc}
\{0\} & \longrightarrow & \mathrm{gr}^n\mathbb{F}_2 & \longrightarrow & \mathbb{F}_2/\mathbb{F}_2^{(n+1)} & \longrightarrow & \mathbb{F}_2/\mathbb{F}_2^{(n)} & \longrightarrow & \{1\} \\
 & & \downarrow{\scriptstyle \mathrm{gr}^n h} & & \downarrow{\scriptstyle h_{n+1}} & & \downarrow{\scriptstyle h_n} & & \\
\{0\} & \longrightarrow & \mathrm{gr}^n\pi_1 S & \longrightarrow & \pi_1 S/\pi_1 S^{(n+1)} & \longrightarrow & \pi_1 S/\pi_1 S^{(n)} & \longrightarrow & \{1\}
\end{array}$$

with exact rows shows that h_n is monic for each n, therefore $\mathrm{Ker}\, h \subset \bigcap_n \mathbb{F}_2^{(n)} = \{1\}$.

The truth of the claim may be easily seen, by observing first that the graded Lie maps $f\colon \mathbb{L}^*X/\mathrm{ideal}(\partial Y) \to \mathbb{L}^*V$ are in a bijective correspondence, by duality, with the linear maps $g\colon N \to A^1$ with the property that $\mu \circ (g \wedge g) = 0$, and next that f is onto if and only if g is monic. $\square$

Remarks. — Chen's method of iterated integrals ([6],[7]) allows one to obtain results which are of a similar nature with the above A'(ii), but it requires the presence of conditions imposed at the level of differential forms, and not just at the level of the de Rham cohomology; for example our condition $\mathrm{cg}(\mu) > 1$ is replaced by : there exist closed 1-forms ω_1 and ω_2 on the manifold M, representing independent cohomology classes, and such that $\omega_1 \wedge \omega_2 = 0$ *as a form.* From this point of view the two approaches are to be considered as complementary : the manifolds $N_{\mathbb{R}}/N_{\mathbb{Z}}$ and $(S^1 \times S^2)\#(S^1 \times S^2)$ mentioned in the introduction have the same cohomology algebra and $\mathrm{cg}(\mu) = 2$, but in the case of the first (nil)manifold it is impossible to find representatives with $\omega_1 \wedge \omega_2 = 0$ (this would imply by [7] that its fundamental - nilpotent - group contains an $\mathbb{F}_2$!), while in the other case this is easily done geometrically (by taking the two closed 1-forms dual to two disjointly embedded 2-spheres, one in each term of the connected sum), but A'(ii) cannot be used since $Z_0^2 \neq 0$.

The next result is also complementary to Chen's [6], but this time concerning its conclusion. For an integral version see Proposition 5.1.

3.3. PROPOSITION. — *Let A^* be given, with vector-valued 2-form μ and associated grlie algebra L_A^*. If S is any connected space whose cohomology algebra has μ as associated vector-valued two-form then there exists a grlie epimorphism $L_A^* \twoheadrightarrow \mathrm{gr}^*\pi_1 S \otimes \mathbb{Q}$.*

Proof. — Take the 1-minimal model of S, $(\wedge V, d)$, consider the canonical filtration $\{F_n V\}$ and set $L_n \cong \mathcal{C}^{*^{-1}}(\wedge F_n V, d)$; we know that $\mathrm{gr}^*\pi_1 S \otimes \mathbb{Q} \cong \varprojlim_n \mathrm{gr}^* L_n$. Fixing the 1-minimal dga $(\wedge F_n V, d)$, we obviously have, by naturality, $F_m(F_n V) \subset F_m V$, for any m; a straightforward induction, which only uses the definition of the canonical filtration, shows that we have in fact $F_m(F_n V) \cong F_m V$, for $m \leq n$; the general equality $\#\mathrm{gr}^p \cong F_p/F_{p-1}$ following from Lemma 1.1 and the preceding remark show that the inverse limit $\varprojlim_n \mathrm{gr}^p L_n$ stabilizes for $n \geq p$, for any p. For $p = 1$, L_1 is just the abelian Lie algebra on $\#F_1 V \cong \#H^1(\wedge V, d) \cong \#H^1 S \cong \#A^1 \cong X$. By stability we have a tower of grlie maps $g_n : \mathbb{L}^* X \to \mathrm{gr}^* L_n$, given by $g_n|_X = \mathrm{id}$ (whence they are all onto). In order to check that they all factor through L_A^* giving thus rise to a tower $f_n : \mathbb{L}_A^* \to \mathrm{gr}^* L_n$ (consisting of epic grlie maps), hence to a grlie epimorphism $f : L_A^* \to \varprojlim_n \mathrm{gr}^* L_n \cong \mathrm{gr}^*\pi_1 S \otimes \mathbb{Q}$ as desired, it would suffice to check that $g_2 \partial Y = 0$ in $\mathrm{gr}^2 L_2$, again by stability, i.e. that the composition $Y \xrightarrow{\partial} X \wedge X \cong \mathrm{gr}^1 L_2 \wedge \mathrm{gr}^1 L_2 \xrightarrow{[\ ,\ ]} \mathrm{gr}^2 L_2$ equals zero. By duality, this is equivalent to seeing that $F_2/F_1 \xrightarrow{d} F_1 \wedge F_1 \cong A^1 \wedge A^1 \xrightarrow{\mu} A^2$ equals zero. Denoting by $\mathcal{D}$ the decomposable elements of a graded algebra, plainly $dF_2 = 0$ in $\mathcal{D}H^2(\wedge V, d) \cong \mathcal{D}H^2 S \cong \mathcal{D}A^2$, the last equality coming from our assumption on the vector-valued 2-form associated to $H^* S$. The proof is now complete. □

The following interesting numerical test for 1-graded intrinsic formality may be immediately deduced.

COROLLARY. — *A^* is 1-graded intrinsically formal if and only if $\mathrm{rk}\,\mathrm{gr}^p \pi_1 S$ is constant within A^* (and equal to $\dim L_A^p$), for any p.*

Our last result in this direction is somewhat surprising, since the other known qualitative numerical results indicate (see [10; Chapitre 7]) that the numerical invariants of the formal space S_A would represent an upper bound for the corresponding numerical invariants of S if $H^*(S; \mathbb{Q}) \cong A^*$.

3.4. Proposition. — *Let S be a 1-connected space of finite type, with rational cohomology algebra A^*. If the Eilenberg-Moore spectral sequence of S collapses at the E_2 term, then we have inequalities* $\mathrm{rk}(\pi_*\Omega S)_n^{(m)} \geq \dim(H_*\mathcal{L}_A)_n^{(m)}$, *for any n, m.*

Proof. — Represent S by a (minimal !) model of the form $(\wedge Z^*, D+p)$, as in 2.1, and note that $\pi_*\Omega S \otimes \mathbb{Q} \cong \mathcal{C}^{*^{-1}}(\wedge Z^*, D_2 + p_2)$, and $H_*^*\mathcal{L}_A \cong \mathcal{C}^{*^{-1}}(\wedge Z_*^*, D_2)$. We thus have a bigraded vector space $L_*^* = \# Z_{*-1}^{*+1}$ and two graded Lie brackets, $[\ ,\ ]_p$ and $[\ ,\ ]$; the second is actually bihomogeneous and the conditions on the perturbation p translate, as in the proof of 1.10, to the fact that, for any $x \in L^m$, $y \in L^n$, $[x,y]_p = [x,y]$ modulo $L^{>m+n}$. We may also suppose that $\dim L < \infty$. This can be seen as follows : for any fixed lower degree n, as in our statement, the vector space $L_n^{(q)}$ remains unchanged (for any q) after taking the quotient of L by the graded Lie ideal $L_{>n}$.

Having established this framework, let us denote by $\{F_p^m\}_{m\geq 0}$ respectively by $\{F^m\}_{m\geq 0}$, the lower central series corresponding to $[\ ,\ ]_p$, respectively to $[\ ,\ ]$. We have to show that $\dim\left(F_p^m\right)_n \geq \dim F_n^m$, for any m, n. Consider then the (decreasing and finite) filtration on F_p^m defined by $G^k F_p^m$ = vector subspace spanned by $[x_1,[\ldots,[x_{m-1},x_m]_p\ldots]_p]_p$, where $x_i \in L_{s_i}^{r_i}$ and $\sum r_i \geq k$ (and similarly for $G^k F^m$). In the bihomogeneous case we evidently have $G^k F^m \cong F^m \cap L^{\geq k}$.

For fixed m and k, define $f\colon G^k F_p^m / G^{k+1} F_p^m \to G^k F^m / G^{k+1} F^m$ by $f(\Sigma_p) = \Sigma$, where Σ_p is a sum of terms of the form $[x_1,[\ldots,[x_{m-1},x_m]_p\ldots]_p]_p$ in $G^k F_p^m$ (modulo $G^{k+1}F_p^m$) and Σ is the sum (in $G^k F^m / G^{k+1} F^m$) which is obtained by replacing the above monomials by $[x_1,[\ldots,[x_{m-1},x_m]\ldots]]$. The map f obviously being onto (if well-defined !) and compatible with lower degrees, the desired inequalities will follow (both filtrations being finite).

It remains to be shown that $\Sigma_p = 0$ in $G^k F_p^m$ implies $\Sigma = 0$ in $G^k F^m / G^{k+1} F^m$. By expanding the brackets in Σ_p and replacing them by unperturbed brackets we find out that $\Sigma_p = \Sigma - z$, with $z \in L^{>k}$; if $\Sigma_p = 0$, then $\Sigma = z \in F^m \cap L^{>k} \cong G^{k+1} F^m$ and we are done. □

4. Rigid examples and inert sequences.

We prove Theorems B and B' and we indicate a source of examples, based on Anick's [1 ; page 133], notion of combinatorial freeness.

4.1. *Proof of Theorem B(i).* — By duality (see [31; III.4.(5) and I.1.(7)]) the condition $(A^+)^3 = 0$ may be rephrased as follows : $\mathcal{L}_A = (\mathbb{L}(X_* \oplus Y_*), \partial)$, where $X_* \oplus Y_* = \# s^{-1}\overline{A}^*$, $\partial X_* = 0$ and $\partial|_{Y_*}$ is monic and takes values in $(\mathbb{L}^2 X)_{*-1}$. The fact that the ideal generated by ∂Y in $\mathbb{L}X$ is *inert* (in the sense of [15; Définition 3.1]) is equivalent to the fact that the sequence $\partial y_1, \ldots, \partial y_n$ is inert in $\mathbb{L}X$ (which means by definition that the sequence is *strongly free* - in the sense of [1; page 127] - when viewed in $\mathbb{T}X$), and this is also equivalent to $H^*\mathcal{L}_A$ being generated as a Lie algebra by $H^1\mathcal{L}_A$ (i.e. $\mathcal{Q}\mathrm{Ext}_A^{>1,*}(\mathbb{Q},\mathbb{Q}) = 0$, see Lemma 1.9(i)), and further equivalent to the fact that the natural projection $\mathbb{L}X_* \to E_* = \mathbb{L}X_*/\mathrm{ideal}(\partial Y_*)$ induces a monomorphism on $\mathrm{Tor}_2^{\mathcal{U}(.)}(\mathbb{Q},\mathbb{Q})$ and an isomorphism on $\mathrm{Tor}_{\geq 3}^{\mathcal{U}(.)}(\mathbb{Q},\mathbb{Q})$; all these are to be found in [15; Proposition 3.2], Théorème 1.1. It is also proven there that if they are fulfilled then $H_*^*\mathcal{L}_A \cong E_*^*$. In our case $\mathrm{Tor}_{\geq 2}^{\mathcal{U}\mathbb{L}X}(\mathbb{Q},\mathbb{Q}) = 0$, due to the freeness of $\mathbb{L}X$, and thus the above conditions on Tor_2 and $\mathrm{Tor}_{\geq 3}$ simply reduce to $\mathrm{gl}\ \dim E_* \leq 2$. This remark completes our proof. □

4.2. *Proof of Theorem B'(i).* — Consider the bigraded model of A^*, $\mathcal{B}_A = (\wedge Z_*^*, D)$. As remarked in the proof of Lemma 1.9(ii) $\mathcal{P}\mathrm{Ext}_A^{*,1}(\mathbb{Q},\mathbb{Q}) = 0$ if and only if $Z^2 = 0$ and $\mathcal{P}\mathrm{Ext}_A^{*,\geq 1} = 0$ if and only if $Z^{\geq 2} = 0$. By the general uniqueness results for k-stage minimal models (i.e. minimal algebras generated in degree $\leq k$ together with modelling dga maps inducing a cohomology isomorphism up to degree k and a cohomology monomorphism in degree $k+1$) $Z^2 = 0$ is equivalent to the fact that the (b)dga 1-modelling canonical map of [17], $f\colon (\wedge Z_*^1, D) \to (A^*, 0)$ is isomorphic in cohomology in degrees 1 and 2 and monomorphic in degree 3; similarly $Z^{\geq 2} = 0$ is equivalent to H^*f being an isomorphism.

Since f is a 1-modelling map, $H^1 f$ is isomorphic and $H^2 f$ is a monomorphism onto the decomposables $\mathcal{D}A^2$; on the other hand $A^3 = 0$, the algebra A^* being 2-skeletal. These remarks show that $Z^2 = 0$ is equivalent to the surjectivity of μ plus $H^3(\wedge Z_*^1, D) = 0$. As we have already noticed in the proof of Lemma 1.8, $H^k(\wedge Z_*^1, D) \cong H^k(L_A^*; \mathbb{Q})$, for any k. Now use an innocuous but very useful trick (which will enable us to use freely the results obtained in [1] and [15] for *connected graded* algebras) : we replace the grlie algebra L_A^* by the connected glie algebra E_* constructed in the introduction. We have changed nothing except doubling upper degrees and then transforming them into lower degrees; consequently $H^k(\wedge Z^1, D) \cong \#\mathrm{Tor}_k^{\mathcal{U}E_*}(\mathbb{Q},\mathbb{Q})$, any k; since the $\mathrm{Tor}_k^{\mathcal{U}(.)}$ test of [15] is enough to be checked only for $k = 2$ and 3 (Proposition 3.2 of [15]), we conclude that $\mathcal{P}\mathrm{Ext}_A^{*,1}(\mathbb{Q},\mathbb{Q}) = 0$ is equivalent to μ being onto and gl dim$E_* \leq 2$ (which implies $H^{\geq 3}(\wedge Z^1, D) = 0$, therefore H^*f is an isomorphism and $\mathcal{P}\mathrm{Ext}_A^{*,\geq 1}(\mathbb{Q},\mathbb{Q}) = 0$) and also equivalent to ∂ being monic and the sequence $\partial y_1, \dots, \partial y_n$ being inert in $\mathbb{L}X$ (or strongly free in $\mathbb{T}X$), as before. Noting that the strong freeness of a sequence implies the linear independence of its elements, the surjectivity of μ follows, and our proof is complete. □

We point out that we have a characterization of the vanishing property for $\mathcal{P}\mathrm{Ext}_A^{*,1}(\mathbb{Q},\mathbb{Q})$ valid for general algebras A^* (as usual, connected and of finite type), similar to the one given in Theorem B'(i). We choose not to give it here, because we are not going to use it here.

4.3. *Combinatorial conditions for strong freeness.* — We shall describe now, following Anick [1], a very useful combinatorial test for the strong freeness of sequences of elements in graded tensor algebras over an arbitrary field. Let then $\mathbb{T}X$ be the (connected graded) tensor algebra on a positively graded vector space X, $\dim X = m$. Pick an *ordered* homogeneous basis of X, say $\{x_1, \dots, x_m\}$, and then extend this order to a total order on the monomials $u = x_{i_1} \otimes \dots \otimes x_{i_r}$ of $\mathbb{T}X$, having the properties : $\deg u < \deg v \Longrightarrow u < v$ and $u < v \Longrightarrow zut < zvt$, for any z and t (we shall explicitely use, on the monomials of the same degree, the *lexicographic order from the right*). Given any nonzero element $y \in \mathbb{T}X$, write $y = c_1u_1 + \dots + c_ru_r$, where c_i are constants and u_i are monomials, and define *the highest term* of y, to be denoted by $\overline{y}$, by $\overline{y} = u_i$, where $u_i =$ the largest u_j for which $c_j \neq 0$. For a given monomial $u = x_{i_1} \otimes \dots \otimes x_{i_r}$, define its *origin* by $\mathbf{o}(u) = i_1$ and its *end* by $\mathbf{e}(u) = i_r$. To simplify matters (having in mind our applications via B(i) and B'(i)) we shall only consider sequences $y_1, \dots, y_n$ of tensor degree two, i.e. $y_i \in \mathbb{T}^2X$, for any i. Anick's result reads then :

THEOREM (see [1 ; theorems 3.2 and 3.1]). — *The sequence $y_1, \dots, y_n$ is strongly free in $\mathbb{T}X$ if the monomial sequence of its highest terms $\bar{y}_1, \dots, \bar{y}_n$, is* combinatorially free, *i.e.*

($\star$) *the monomials $\bar{y}_1, \dots, \bar{y}_n$ are distinct, and*

($\star\star$) *the sets of indices $\{\mathbf{o}(\bar{y}_1), \dots, \mathbf{o}(\bar{y}_n)\}$ and $\{\mathbf{e}(\bar{y}_1), \dots, \mathbf{e}(\bar{y}_n)\}$ are disjoint.*

As a first example, both simple and instructive, we shall again follow Anick and take $\{x'_1, \dots, x'_r\} \cup \{x''_1, \dots, x''_s\}$ as basis for X (concentrated in lower degree 2) – in this order – and consider the sequence of Lie elements

$\{y_{ij} = [x'_i, x''_j] \in \mathbb{L}^2 X \subset \mathbb{T}^2 X\}$, $i = 1, \dots, r$ and $j = 1, \dots, s$. Then any subsequence is strongly free, being combinatorially free (since $\bar{y}_{ij} = x'_i \otimes x''_j$, and the combinatorial conditions, $(\star)$ and $(\star\star)$, are obviously satisfied).

For $A^* = H^*((\bigvee_{m-1} S^1) \times S^1)$, $\dim Y = m-1$, $\partial y_i = [x_i, x_m]$, $i = 1, \dots, m-1$, and the 1-graded intrinsic formality of A follows, via Theorem B'(i).

4.4. *Proof of Theorem B(ii).* — A general sufficient condition for the intrinsic spherical generation of a 1-connected algebra A^*, with Quillen model $\mathcal{L}_A = (\mathbb{L}^* W_*, \partial)$, may be found in [24; 1.9. and 1.10]; it only requires the vanishing of $\mathrm{Hom}_{-1}(H^1_* \mathcal{L}_A, H^{\geq 2}_* \mathcal{L}_A)$, where $\mathrm{Hom}_{-1}(\ ,\ )$ denotes linear maps which are homogeneous of lower degree -1. Recall next that we know, by assumption, that $H^* \mathcal{L}_A$ is generated by $H^1 \mathcal{L}_A$ (cf. Lemma 1.9(i)). On the other hand $H^1_* \mathcal{L}_A \cong \mathrm{Ker}(\partial|_{W_*}) \cong \# Q A^{*+1}$. A simple degree argument (based on our hypotheses on the degrees of QA^*) shows that $\mathrm{Hom}_{-1}(H^1_* \mathcal{L}_A, H^{\geq 2}_* \mathcal{L}_A) = 0$ and finishes the proof. □

Note that the same argument gives the intrinsic spherical generation property, for a general 1-connected algebra A^*, subject only to the condition that the degrees of QA^* be concentrated in an interval of the form $[l, 3l-2]$; on the other hand, the condition $\deg(QA^*) =$ odd does not imply alone, in general, the intrinsic spherical generation, even for 2-stage algebras A^*.

Here is a slightly more general version of B'(ii) :

4.5. LEMMA. — *Let A^* be any algebra (connected and of finite type), with associated grlie L^*_A. If $\mathcal{P}\mathrm{Ext}^{*,\geq 1}_A(\mathbb{Q}, \mathbb{Q}) = 0$ then we have an equality of formal power series*

$$\prod_{p=1}^{\infty} (1 - z^p)^{\dim L^p_A} = A^*(-z),$$

where $A^*(z)$ is the Hilbert series of A^*, $\sum_{n \geq 0} \dim A^n \cdot z^n$.

Proof. — As indicated in the proof of B'(i), the assumption $\mathcal{P}\mathrm{Ext}^{*,\geq 1}_A$ $(\mathbb{Q}, \mathbb{Q}) = 0$ simply means that the bigraded model $\mathcal{B}_A$ coincides with the 1-bigraded model, $(\wedge Z^1_*, D)$. By a general formula of [17; Proposition 3.10], we know that $A^*(-z) = \prod_{n=0}^{\infty} (1 - z^{n+1})^{\dim Z^1_n}$. On the other hand, Lemma 1.8 tells us that $Z^1_n \cong \# L^{n+1}_A$ (see (1.7)), which completes the proof. □

Remarks. — The above formula is in fact quite effective : one may uniquely express $\dim L_A^p$, for any p, with the aid of the Möbius function and of certain universal polynomials in the coefficients of $A^*(z)$, if $\dim A^* < \infty$, see [4].

4.6. PROPOSITION. — *Given an arbitrary algebra A^* (connected and of finite type), the condition $\mathcal{P}\mathrm{Ext}_A^{*,\geq 1}(\mathbb{Q},\mathbb{Q}) = 0$ is equivalent to the fact that A^* is generated (as a graded algebra) by A^1, plus the equality of Hilbert series :* $\mathrm{Tor}_*^{\mathcal{U}L_A}(\mathbb{Q},\mathbb{Q})(z) = A^*(z)$.

Proof. — Recall the 1-modelling (b)dga map $f\colon (\wedge Z_*^1, D) \to (A^*, 0)$, where $(\wedge Z_*^1, D) \to (A^*, 0)$ is constructed out of L_A^* as explained in (1.7), see Lemma 1.8. The vanishing condition on $\mathcal{P}\mathrm{Ext}_A^{*,\geq 1}(\mathbb{Q},\mathbb{Q})$ is then equivalent to the fact that H^*f is an isomorphism. But we know that $H^*(\wedge Z_*^1, D) \cong H_0^*(\wedge Z_*^1, D) \oplus H_+^*(\wedge Z_*^1, D)$, where $H_0^* \cong \wedge^* Z_0^1/\mathrm{ideal}(DZ_1^1)$, and $H^*f(H_+^*) = 0$, by construction. Recalling from the proof of 1.8 that ($V_* \cong Z_*^1$!) $Z_0^1 \cong A^1$ and $DZ_1^1 \cong \mathrm{Ker}\ \mu$, H_0^*f being the canonical map, it follows, if H^*f is an isomorphism, that then A^* is generated by A^1, and we have an equality between the Hilbert series of A^* and of $H^*(\wedge Z^1, D)$. On the other hand we already have remarked (again in the proof of Lemma 1.8) that this last Hilbert series equals $H^*(L_A;\mathbb{Q})(z)$, hence also $\mathrm{Tor}_*^{\mathcal{U}L_A}(\mathbb{Q},\mathbb{Q})(z)$, which completes half of our proof, the other implication being immediate, with a dimension argument. $\square$

4.7. *Example* [19]. — Denote by P_n the n-th pure braid group and consider $A_n^* = H^*(P_n;\mathbb{Q})$. Then $\mathcal{P}\mathrm{Ext}_{A_n}^{*,\geq 1}(\mathbb{Q},\mathbb{Q}) = 0$, for any n. This may be seen as follows : it is known that A_n^* is generated by A_n^1, for any n; the Hilbert series $A_n^*(z)$ equals $(1+z)(1+2z)\ldots(1+nz)$ (hence A_n^* is *not* 2-skeletal for $n > 2$). The main result of [19] also gives the equality $\mathrm{Tor}_*^{\mathcal{U}L_{A_n}}(\mathbb{Q},\mathbb{Q})(z) = A_n^*(z)$, for any n.

4.8. *Remarks.* — Our method of the computation of $\mathrm{gr}^*\pi_1$ is based on the rigidity assumption $\mathcal{P}\mathrm{Ext}_A^{*,1}(\mathbb{Q},\mathbb{Q}) = 0$ (Theorem A'(i)). If A^* is 2-skeletal, this is equivalent (B'(i)) to the inertia of ∂Y in $\mathbb{L}X$ (or the strong freeness of ∂Y in $\mathbb{T}X$). In general, this inertia is a much stronger hypothesis than the vanishing of $\mathcal{P}\mathrm{Ext}^{*,1}$. This may be seen as follows : if ∂Y is inert in $\mathbb{L}X$ then the (b)dga 1-modelling map $f\colon (\wedge Z_*^1, D) \to (A^*, 0)$, constructed out of L_A^* in Lemma 1.8(i), is actually a 2-model (since $H^3(\wedge Z^1, D) = 0$, by inertia), therefore $\mathcal{P}\mathrm{Ext}_A^{*,1}(\mathbb{Q},\mathbb{Q}) = 0$ (see 4.2). On the other hand, consider $A_n^* = H^*(P_n;\mathbb{Q})$, as in the previous example; here we know that $\mathcal{P}\mathrm{Ext}_{A_n}^{*,\geq 1}(\mathbb{Q},\mathbb{Q}) = 0$, but in general ∂Y is *not* inert in $\mathbb{L}X$, since

$\mathrm{Tor}_3^{\mathcal{U}L_{A_n}}(\mathbb{Q},\mathbb{Q}) \neq 0$, for $n > 2$. Far more interesting 1-graded intrinsically formal examples (which may be handled by our method but fall out of the range of inertia required by the method of Labute-Anick [20],[3]) see [5].

5. An integral variant.

This section is devoted to the proof of Theorem C, which will follow from the following elementary result (the integral version of Proposition 3.3) :

5.1. PROPOSITION. — *For any connected space S of finite type, with $H_1(S;\mathbb{Z})$ free, there is a natural* grlie *map map $L_S^* \twoheadrightarrow \mathrm{gr}^*\pi_1 S$ (where L_S^* is naturally associated to S, as explained in the introduction).*

Proof. — By the Hurewicz theorem, there is a natural grlie epimorphism $\mathbb{L}^*(H_1 S) \twoheadrightarrow \mathrm{gr}^*\pi_1 S$, given by the identification $H_1 S \cong \mathrm{gr}^1\pi_1 S$. We only have to show that it factors through the ideal generated by the defining relations of L_S^*. We are thus going to prove this (by using naturality and systematically avoiding group presentations).

To see that the composition $H_2 S \xrightarrow{\partial} H_1 S \wedge H_1 S \cong \mathrm{gr}^1\pi \wedge \mathrm{gr}^1\pi \xrightarrow{[\ ,\ ]} \mathrm{gr}^2\pi$ equals zero (where $\pi = \pi_1 S$), we may first replace S by π, and then, again by naturality, we are left with checking it for $\pi/\pi^{(3)}$. We may thus suppose that π is two-step nilpotent and (recalling that $H_1 S$ was supposed to be free) it fits into a central extension of the form

$$\text{(E)} \qquad 0 \to A \to \pi \xrightarrow{p} \mathbb{Z}^k \to 1$$

inducing a diagram

$$\begin{array}{ccc}
H_2\pi & \xrightarrow{H_2 p} & H_2\mathbb{Z}^k \\
\partial \big\downarrow & & \cong \big\downarrow \partial \\
H_1\pi \wedge H_1\pi & \underset{\cong}{\xrightarrow{\wedge^2 H_1 p}} & H_1\mathbb{Z}^k \wedge H_1\mathbb{Z}^k \\
[\ ,\ ] \big\downarrow & & \big\downarrow t\circ\partial^{-1} \\
\mathrm{gr}^2\pi & = & H_1 A
\end{array}$$

where t is the homology transgression in the integral Serre spectral sequence of (E). Since $t \circ H_2 p = 0$, as well-known, everything will follow from the commutativity of the bottom square. In order to compute $t \circ \partial^{-1}$ we may once more involve the naturality, thus supposing $k = 2$, that is $K(\pi, 1)$ is the total space of an orientable $K(A,1)$-fibration over the 2-torus T. We claim that the transgression of $[T]$, the fundamental class of T, is represented by the commutator $xyx^{-1}y^{-1}$, where $x, y \in \pi$ are elements projected by p onto the standard generators of $\pi_1 T$ – and this will plainly finish the proof. Here is a simple geometric argument for establishing the claim : represent $[T] \in H_2(T, \mathrm{Point})$ as $k_*[S]$, where $S = T\backslash\{\text{open 2-disk}\}$ and k collapses ∂S to a point ; lift k to $l\colon (S, \partial S) \to (K(\pi,1), K(A,1))$ and set $x = l_* u$, $y = l_* v$, where u and v are the free generators of $\pi_1 S$ corresponding, via k_*, to the standard generators of $\pi_1 T$. Hence $t[T] = \partial l_*[S] = l_*[\partial S]$ is represented by $xyx^{-1}y^{-1}$, as claimed. □

5.2. *Proof of Theorem C.* — Given that L_S^* is torsion free, in particular $H_1(S;\mathbb{Z})$ is free, the preceding proposition applies and gives a grlie epimorphism $f\colon L_S^* \twoheadrightarrow \mathrm{gr}^*\pi_1 S \otimes \mathbb{Q}$, where $A^* = H^*(S;\mathbb{Q})$. According to the Corollary of Proposition 3.3, the 1-graded intrinsic formality of A^* may be used, via a dimension argument, to infer that $f \otimes \mathbb{Q}$ is also monic. The torsion-freeness of L_S^* is now invoked to see the injectivity of the natural map $L_S^* \to L_S^* \otimes \mathbb{Q} = L_A^*$, and this finally gives that f is monic, hence a grlie isomorphism. □

6. An example.

Consider an $m \times m$ symmetric matrix with zero on the diagonal, with entries in R ($R = \mathbb{Q}$, $\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z}$, p a prime), $\ell = (l_{ij})_{(i,j)\in I\times I}$, where $\mathrm{card}(I) = m$. We shall associate to ℓ a so-called *link-algebra* (with coefficients in R). It is a 2-skeletal connected R-algebra given by : $A^1 =$ free R-module generated by e_i, $i \in I$; $A^2 = A^1 \wedge A^1$ modulo the relations $e_i \wedge e_j + e_j \wedge e_k = e_i \wedge e_k$, $i, j, k \in I$. The multiplication table, $\mu\colon A^1 \wedge A^1 \to A^2$ is given by $\mu(e_i \wedge e_j) =$ class of $l_{ij} e_i \wedge e_j$ in A^2, for any $i, j \in I$. Of course, here we have in mind the cohomology algebra with R-coefficients of the complement of a classical linking of m circles $\mathcal{C}_i$, $i \in I$, where $l_{ij} = \mathrm{lk}(\mathcal{C}_i, \mathcal{C}_j)$ are the linking numbers.

We further associate to a given link-algebra over R a grlie algebra over R, denoted by L_R^* and defined by : $L_R^* = \mathbb{L}^*(x_i | i \in I)$ modulo the relations $r_i = \sum_{j\in I} l_{ij}[x_i, x_j]$, $i \in I$. It is clear that, for $R = \mathbb{Q}$ or $\mathbb{Z}$, if the above μ coincides with the multiplication table in low dimensions of some space S, then L^*_R is nothing else but $L^*_S \otimes R$ (note however that the relations are not independent, since plainly $\sum_{i \in I} r_i = 0$).

Finally, construct out of ℓ a combinatorial object, namely a finite unoriented graph Γ, with vertices v_i, $i \in I$, and arrows $\{v_i, v_j\}$ given by the condition $l_{ij} \neq 0$.

We shall next recall the following basic combinatorial result :

6.1. Proposition [3]. — *For field coefficients ($R = \mathbb{Q}$, $\mathbb{Z}/p\mathbb{Z}$) the connectedness of Γ implies the existence of a total ordering, $x_1 < \ldots < x_m$, with respect to which the sequence of highest terms $\bar{r}_1, \ldots, \bar{r}_{m-1}$ is combinatorially free in $\mathbb{T}(x_1, \ldots, x_m)$, hence $r_1, \ldots, r_{m-1}$ is strongly free (as recalled in 4.3).*

For integral coefficients ($R = \mathbb{Z}$), the connectedness of all (obvious) mod p reductions Γ_p (p = prime) implies the torsion freeness of the associated integral grlie $L^*_{\mathbb{Z}}$.

Remarks. — Anick derives the integral result from the statement for R = field, by taking mod p reductions, inferring that $r_1, \ldots, r_{m-1}$ (mod p) is strongly free in $\mathbb{T}_{\mathbb{Z}/p\mathbb{Z}}(x_1, \ldots, x_m)$, hence concluding that the Hilbert series of $L^*_{\mathbb{Z}} \otimes \mathbb{Z}/p\mathbb{Z}$ depends only on m (as a consequence of a general result on Hilbert series of quotients of connected graded algebras modulo strongly free sequences of relations, asserting that the Hilbert series depends only on the sequence of degrees of the relations, see [1]), therefore $L^*_{\mathbb{Z}}$ must be torsion free.

The same connectedness hypothesis for all mod p reductions Γ_p of Γ simultaneously gives (via the obvious resulting connectedness of $\Gamma_{\mathbb{Q}}$) the strong freeness of $r_1, \ldots, r_{m-1}$ in $\mathbb{T}_{\mathbb{Q}}(x_1, \ldots, x_m)$, hence the 1-graded intrinsic formality of $A^* = H^*(S; \mathbb{Q})$, as soon as $\mu_A = \mu_{\mathbb{Q}}$, as explained in the remarks made in 4.8.

6.2. Corollary. — *If $\mu\colon H^1(S;\mathbb{Q}) \wedge H^1(S;\mathbb{Q}) \to H^2(S;\mathbb{Q})$ equals some $\mathbb{Q}$-link-algebra multiplication $\mu_{\mathbb{Q}}$ and $\Gamma_{\mathbb{Q}}$ is connected then* $\mathrm{gr}^*\pi_1 S \otimes \mathbb{Q} \cong L^*_{\mathbb{Q}}$, *as Lie algebras with grading.*

If $\mu\colon H^1(S;\mathbb{Z}) \wedge H^1(S;\mathbb{Z}) \to H^2(S;\mathbb{Z})$ equals some $\mathbb{Z}$-link-algebra multiplication $\mu_{\mathbb{Z}}$, the abelian group $H_1(S;\mathbb{Z})$ is free and all mod p reductions Γ_p are connected, then the grlie $\mathrm{gr}^*\pi_1 S$ *is torsion free (as a graded abelian group) and equals the integral grlie $L^*_{\mathbb{Z}}$.*

Proof. — For the rational statement, the above proposition guarantees the strong freeness of ∂Y in $\mathbb{T}(X)$ (in our standard general notation, cf. the introduction). This in turn implies that $A^* = H^*(S;\mathbb{Q})$ is 1-graded intrinsically formal (see 4.8), therefore $\mathrm{gr}^*\pi_1 S \otimes \mathbb{Q} \cong L_A^*$ (cf. e.g. Lemma 1.8(ii)). On the other hand $L_A^* \cong L_{\mathbb{Q}}^*$, as we have already noticed.

The integral statement follows from Theorem C. Note first that under our assumptions $L_S^* \cong L_{\mathbb{Z}}^*$, which is known to be torsion free, by the above proposition. As we have just remarked, the same connectedness hypotheses guarantee the fact that $H^*(S;\mathbb{Q})$ is 1-graded intrinsically formal. The method of Theorem C gives then all the desired conclusions. □

As pointed out in 2.3, the vanishing of $\mathcal{P}\mathrm{Ext}_A^{*,1}(\mathbb{Q},\mathbb{Q})$ is not a necessary condition for the 1-graded intrinsic formality of A^*, even in the 2-skeletal case. It is interesting to see, thus concluding our example, that this condition is really necessary, for link-algebras over $\mathbb{Q}$. We shall derive this from the following general implication of the 1-graded intrinsic formality property.

6.3. Proposition. — *Let A^* be a 2-skeletal algebra, with associated 2-form $\mu: A^1 \wedge A^1 \to A^2$ and Lie algebra with grading L_A^*. If A^* is 1-graded intrinsically formal and μ is* not *onto, then $L_A^{\geq 3} = 0$.*

Corollary. — *For a $\mathbb{Q}$-link-algebra A^*, the following are equivalent :*

(i) *A^* is 1-graded intrinsically formal.*

(ii) $\mathcal{P}\mathrm{Ext}_A^{*,1}(\mathbb{Q},\mathbb{Q}) = 0$.

(iii) *The associated graph Γ is connected.*

Proof of the Corollary. — Let us note first that the condition $L_A^{\geq 3} = 0$ in Proposition 6.3 is, in general, a very restrictive one; it implies, for instance, that the 1-bigraded model of A^* (see Lemma 1.8(i)) is of the form $(\wedge Z_{\leq 1}^1, D)$, therefore (by using deformation theory as in the proof of A'(i)) π_1"$\otimes$"$\mathbb{Q}$ is rigid, not just its associated graded. We shall see just how restrictive it is, in the case of link-algebras.

We have already noticed the implications (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i), while proving the first half of Corollary 6.2. Assuming A^* to be 1-graded intrinsically formal, we know from the previous proposition that either μ is onto (and we are done, since it is elementary to see that this condition is equivalent to the connectedness of Γ, compare with [21]) or $L_A^3 = 0$, i.e. the linear map $f: X \otimes Y \to \mathbb{L}^3 X$ given by $f(x \otimes y) = [x, \partial y]$ is onto. Counting dimensions, this gives the inequalities $(m^3 - m)/3 \leq m \cdot \dim(\partial Y) = m \cdot \dim(\mathrm{Im}\mu) \leq m(m-1)$, hence $m = 1$ and $A^2 = 0$, a contradiction.

□

Proof of the Proposition. — Assuming that μ is not onto, one has a decomposition $Y = Y' \oplus C$, where $\partial|_{Y'}$ is monic, $\partial C = 0$ and $C \neq 0$, say C has $\{z_1, \ldots, z_c\}$ as basis. We must see that $L_A^3 = 0$. Pick then an arbitrary element $p \in \mathbb{L}^3 X$; we have to prove that $p \in I =$ the ideal generated by $\partial Y = \partial Y'$. Suppose on the contrary that $p \notin I$ and consider then the (larger) perturbed ideal $I_p =$ ideal generated by ∂Y and p and the natural grlie surjection f, $f\colon L_A^* = \mathbb{L}^* X/I \to \mathbb{L}^* X/I_p =: L_p^*$. Perform then on L_p^* the construction described in (1.7) to obtain a 1-minimal (b)dga $(\wedge V_*, d)$, with the property that $\mathrm{gr}^*(\wedge V, d) = L_p^*$ (see also Proposition 1.4). If we are able to exhibit a dga $\mathcal{M}$ with the property that $H^*\mathcal{M} \cong A^*$ and having $(\wedge V, d)$ as 1-minimal model, then the property of A^* of being 1-graded intrinsically formal may be eventually exploited, giving a grlie isomorphism between L_p^* and L_A^*; by a dimension argument f must be an isomorphism and consequently $p \in I$, a contradiction.

Construct $\mathcal{M}$ by starting with $(\wedge V, d) \otimes (\wedge(z_2, \ldots, z_c), 0)$, where $\deg z_i = 2$ and the second differential is trivial. Consider then $H^{>2}((\wedge V, d) \otimes (\wedge(z_2, \ldots, z_c), 0)$ and add new generators, $U_1^* = \bigoplus_{k \geq 2} U_1^k$, so as to kill $H^{\geq 3}$; look next at $H^{>2}(\wedge V \otimes \wedge(z) \otimes \wedge U_1)$ and kill it by adding new generators U_2^*, iterate and obtain $\mathcal{M}$ as the inductive limit of this process. By construction $H^*\mathcal{M}$ will be a 2-skeletal algebra. It is equally easy to see that when killing $H^{\geq 3}\mathcal{M}$ as above one does not change $H^{\leq 2}\mathcal{M}$, hence $(\wedge V, d)$ is indeed the 1-minimal model of $\mathcal{M}$ and $H^*\mathcal{M} \cong H^{\leq 2}((\wedge V, d) \otimes (\wedge(z_2, \ldots, z_c), 0) \cong \mathbb{Q} \cdot 1 \oplus H^1(\wedge V, d) \oplus H^2(\wedge V, d) \oplus \mathrm{span}_{\mathbb{Q}}\{z_2, \ldots, z_c\}$. Recalling the construction (1.7), Lemma 1.6(iv) tells that $H^1(\wedge V_*, d) \cong H_0^1(\wedge V_*, d) \cong V_0 \cong \# L_p^1 \cong \# X \cong A^1$. Likewise $H^2(\wedge V_*, d) \cong H_0^2(\wedge V_*, d) \oplus H_+^2(\wedge V_*, d)$, where $H_0^2(\wedge V_*, d) \cong V_0 \wedge V_0 / dV_1$ and the multiplication $H^1(\wedge V, d) \wedge H^1(\wedge V, d) \twoheadrightarrow H_0^2(\wedge V, d)$ is by construction the dual of the inclusion $K \rightarrowtail L_p^1 \wedge L_p^1$, where $K = \mathrm{Ker}(L_p^1 \wedge L_p^1 \xrightarrow{[\ ,\ ]} L_p^2)$. Because f is an isomorphism in degrees 1 and 2, we may safely replace $L_p^{\leq 2}$ by $L_A^{\leq 2}$ and thus identify $\#(K \rightarrowtail L_p^1 \wedge L_p^1)$ with $\#(\partial\colon Y' \rightarrowtail X \wedge X) \cong \mu\colon A^1 \wedge A^1 \twoheadrightarrow \mathrm{Im}\,\mu$. In order to show that $H^*\mathcal{M} \cong A^*$ and thus finish our proof, we only have to see that $\dim H_+^2(\wedge V_*, d) = 1$, or equivalently that $\dim H^2(\wedge V, d) = 1 + \dim Y'$.

This will be accomplished by remarking that in general one has $H^k(\wedge V, d) = H^k(L_p; \mathbb{Q}) = \# H_k(L_p; \mathbb{Q})$, for any k, and the same reasoning as in 1.8 indicates that $\dim H_2(L_p;\mathbb{Q}) = \dim I_p/[\mathbb{L}X, I_p]$, and our assumption $p \notin I$ helps to conclude that this last dimension equals $\dim Y' + 1$, as claimed. $\square$

BIBLIOGRAPHY

[1] D. ANICK, Non-commutative graded algebras and their Hilbert series, J. of Algebra, (1)78 (1982), 120-140.

[2] D. ANICK, Connections between Yoneda and Pontrjagin algebras, Lect. Notes in Math. 1051, Springer-Verlag, 1984, pp. 331-350.

[3] D. ANICK, Inert sets and the Lie algebra associated to a group, Journ. of Algebra, 111-1 (1987), 154-165.

[4] I.K. BABENKO, On analytic properties of the Poincaré series of loop spaces, Matem. Zametki, 27 (1980), 751-765, in Russian; English transl. in Math. Notes 27 (1980).

[5] B. BERCEANU, Şt. PAPADIMA, Cohomologically generic 2-complexes and 3-dimensional Poincaré complexes, preprint.

[6] K.-T. CHEN, Iterated integrals of differential forms and loop space homology, Ann. of Mathematics, (2)97 (1973), 217-246.

[7] K.-T. CHEN, Differential forms and homotopy groups, J. of Differential Geometry, 6 (1971), 231-246.

[8] K.-T. CHEN, Commutator calculus and link invariants, Proc. Amer. Math. Soc., 3 (1952), 44-55.

[9] B. CENKL, R. PORTER, Malcev's completion of a group and differential forms, J. of Differential Geometry, 15 (1980), 531-542.

[10] Y. FÉLIX, Dénombrement des types de **k**-homotopie. Théorie de la déformation, Bull. Soc. Math. France, (3)108 (1980).

[11] Y. FÉLIX, S. HALPERIN, Rational LS category and its applications, Trans. Amer. Math. Society, 273 (1982), 1-38.

[12] Y. FÉLIX, J.-C. THOMAS, Sur la structure des espaces de LS catégorie deux, Illinois J. of Math., (4)30 (1986), 574-593.

[13] P.A. GRIFFITHS, J.W. MORGAN, Rational homotopy theory and differential forms, Progress in Math. 16, Birkhäuser, 1981.

[14] W. GREUB, S. HALPERIN, R. VANSTONE, Connections, curvature and cohomology, vol. III, Academic Press, 1976.

[15] S. HALPERIN, J.-M. LEMAIRE, Suites inertes dans les algèbres de Lie graduées, Mat. Scand., 61 (1987), 39-67.

[16] P.J. HILTON, U. STAMBACH, A course in homological algebra, Graduate texts in Mathematics 4, Springer-Verlag, 1971.

[17] S. HALPERIN, J.D. STASHEFF, Obstructions to homotopy equivalences, Advances in Math., 32 (1979), 233-279.

[18] S. KOJIMA, Nilpotent completions and Lie rings associated to link groups, Comment. Math. Helv., 58 (1983) 115-134.

[19] T. KOHNO, Série de Poincaré-Koszul associée aux groupes de tresses pures, Invent. Math., 82 (1985), 57-75.

[20] J.P. LABUTE, The determination of the Lie algebra associated to the lower central series of a group, Trans. Amer. Math. Society, (1) 288 (1985), 51-57.

[21] J.P. LABUTE, The Lie algebra associated to the lower central series of a link group and Murasugi's conjecture, Proc. Amer. Math. Soc., 109,4 (1990), 951-956.

[22] C. LÖFWALL, On the subalgebra generated by the one-dimensional elements in the Yoneda Ext-algebra, Algebra, algebraic topology and their interactions, Proc. Stockholm 1983, Lect. Notes in Math. 1183, Springer-Verlag, 1986, pp. 291-338.

[23] J.-M. LEMAIRE, F. SIGRIST, Dénombrement des types d'homotopie rationnelle, C. R. Acad. Sci. Paris, 287 (1978), 109-112.

[24] M. MARKL, Şt. PAPADIMA, Geometric decompositions, algebraic models and rigidity theorems, Journ. of Pure and Appl. Algebra, 71 (1991), 53-73.

[25] S.B. PRIDDY, Koszul resolutions, Trans. Amer. Math. Society, 152 (1970), 39-60.

[26] ŞT. PAPADIMA, The rational homotopy of Thom spaces and the smoothing of homology classes, Comment. Math. Helv., 60 (1985), 601-614.

[27] D. QUILLEN, Rational homotopy theory, Ann. of Math., 90 (1969), 205-295.

[28] D. SULLIVAN, Infinitesimal computations in topology, Publ. Math. IHES, 47 (1977), 269-331.

[29] J.-P. SERRE, Lie algebras and Lie groups, Benjamin, 1965.

[30] J.D. STASHEFF, Rational Poincaré duality spaces, Illinois J. of Math., 27 (1983), 104-109.

[31] D. TANRÉ, Homotopie rationnelle. Modèles de Chen, Quillen, Sullivan, Lect. Notes in Mathem. 1025, Springer-Verlag, 1983.

[32] D. TANRÉ, Cohomologie de Harrison et type d'homotopie rationnelle, Algebra, algebraic topology and their interactions, Proc. Stockholm 1983, Lect. Notes in Math. 1183, Springer-Verlag, 1986, pp. 361-370.

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