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The primary objective of this research is to extend and develop homotopy algorithms for a variety of computational problems in control. In addition, these problems are being examined in the context of the algebraic and differential geometry on which the homotopy methods are based. This will enable the classification of solutions to a particular problem and consequently will allow the analyst or designer to extract the most desirable solution.

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FINAL REPORT FOR AFOSR GRANT 89-0497

HOMOTOPY METHODS IN CONTROL SYSTEM DESIGN AND ANALYSIS

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Period: 7/1/89 – 2/28/90

Recent technologies have led to stringent control system requirements. This has increased the importance and complexity of the analysis and design of control systems. For example, the design of such control systems often requires the solution of systems of nonlinear equations of high order. Homotopy algorithms which solve systems of nonlinear equations form the basis of the software package known as HOMPACK. However, because of the high dimension and structure of modern control problems, the HOMPACK code cannot be applied directly to many of the problems which arise in control system design and analysis. Thus, the primary objective of the current research is to extend and develop homotopy algorithms for a variety of computational problems in control. In addition, these problems are being examined in the context of the algebraic and differential geometry on which the homotopy methods are based. This will enable the classification of solutions to a particular problem and consequently will allow the analyst or designer to extract the most desirable solution.

Since the funding arrived too late to recruit a student for Fall 1989, work was begun on preconditioned conjugate gradient algorithms for solving large, sparse systems of linear equations. Regardless of what control algorithms are eventually developed, they will depend on sparse matrix technology. Good progress has been made in this regard, as numerous experiments on realistic large problems have been conducted with various PCG algorithms, GMRES, and Orthomin(k). This work has been submitted to the *SIAM Journal on Optimization*, and current analyses of SYMLQ will be presented at the Copper Mountain Conference on Iterative Methods in April, 1989.

In January, 1990, a graduate student, Dragan Žigić, began work on the optimal projection equations in papers of Bernstein, Hyland, and Richter. After some background study in linear systems theory and optimal control, Žigić should be able to make some progress on the control design problems described in Section 2 of the proposal.

Fifteen conference presentations were given in 1989. Those published in proceedings were:

- Two on parallel homotopy algorithms for a hypercube – A. Chakraborty, D. C. S. Allison, C. J. Ribbens, and L. T. Watson, "Parallel orthogonal decompositions of rectangular matrices for curve tracking on a hypercube", in *Proc. Fourth Conf. on Hypercube Concurrent Computers and Applications*, J. Gustafson (ed.), ACM, Monterey, CA, 1989; A. Chakraborty, D. C. S. Allison, C. J. Ribbens, and L. T. Watson, "Parallel unit tangent vector computation for homotopy curve tracing on a hypercube", in *Proc. 1990 ACM Eighteenth Annual Computer Science Conference*, Washington, DC, 1990, 103–108.
- Two optimal control problems – G. Vasudevan, L. T. Watson, and F. H. Lutze, "A homotopy approach for solving constrained optimization problems", in *Proc. Amer. Control Conf.*, Pittsburgh, PA, 1989, 780–785; G. Vasudevan, F. H. Lutze, and L. T. Watson, "A homotopy

method for space flight rendezvous problems", in *Proc. AAS/AIAA Astrodynamics Specialist Conf.*, AAS, Stowe, VT, 1989.

- Two chapters in books – L. T. Watson, "Modern homotopy methods in optimization", in *Impacts of Recent Computer Advances on Operations Research*, R. Sharda, B. L. Golden, E. Wasil, O. Balci, W. Stewart (eds.), North-Holland, New York, 1989, 555–565; L. T. Watson and M. P. Kamat, "Homotopy algorithms for engineering analysis", in *Supercomputing in Engineering Analysis*, H. Adeli (ed.), Marcel Dekker, New York, 1990.

Journal papers completed and submitted were:

- A fluid mechanics application – C. J. Ribbens, C. Y. Wang, L. T. Watson, and K. A. Alexander, "Vorticity induced by a moving elliptic belt", *Comput. & Fluids.*
- Two papers on truss design via homotopy methods – V. Arun, C. F. Reinholtz, and L. T. Watson, "Enumeration and analysis of variable geometry truss manipulators", *ASME J. Mechanisms, Transmissions Automation Design*; V. Arun, C. F. Reinholtz, and L. T. Watson, "New homotopy solution techniques applied to variable geometry trusses", *ASME J. Mechanisms, Transmissions Automation Design.*
- A paper on parallel curve tracking – A. Chakraborty, D. C. S. Allison, C. J. Ribbens, and L. T. Watson, "Unit tangent vector computation for homotopy curve tracking on a hypercube", *Parallel Comput.*
- A preliminary study of linear algebra techniques applicable to large sparse control problems – K. M. Irani, C. J. Ribbens, H. F. Walker, L. T. Watson, and M. P. Kamat, "Preconditioned conjugate gradient algorithms for homotopy curve tracking", *SIAM J. Optim.*
- A solid mechanics application – C. Y. Wang and L. T. Watson, "Rotation of polygonal space structures", *J. Astronaut. Sci.*

Period: 3/1/90 – 2/28/91

During the spring and summer Dragan Žigić worked through most of Kwakernaak and Sivan's optimal control book, read several homotopy papers, and studied the optimal projection papers of Bernstein, Collins, Hyland, and Richter in depth. The collaboration with Bernstein and Collins of Harris Corporation in Melbourne has been extensive and very productive. They have provided us with test data and guidance as to what is important, and we have improved their theoretical results and numerical algorithms. The first fruit of this collaboration is a conference paper entitled "A homotopy method for solving the optimal projection equations for the reduced order model problem" to be presented at the IEEE Southeastcon meeting in Williamsburg in April 1991. This work, also part of Žigić's M.S. thesis, is summarized here:

The optimal projection approach is utilized on various problems arising in optimal control. Hyland and Bernstein [1] give theoretical results for the application of that method to the reduced order model problem, which is to find a reduced order model

$$\dot{x}_m(t) = A_m\, x_m(t) + B_m\, u(t),$$
$$y_m(t) = C_m\, x_m(t),$$

for the system

$$\dot{x}(t) = A\, x(t) + B\, u(t),$$
$$y(t) = C\, x(t),$$

that minimizes the quadratic model reduction error

$$J(A_m, B_m, C_m) = \lim_{t\to\infty} E\left[(y(t) - y_m(t))^t R\,(y(t) - y_m(t))\right].$$

It is assumed that both the system and the model are asymptotically stable, controllable and observable. Necessary conditions for the optimal reduced order model can be expressed in the form:

$$0 = \tau[A\,\hat{Q} + \hat{Q}\,A^t + B\,V\,B^t], \tag{1}$$

$$0 = [A^t\,\hat{P} + \hat{P}\,A + C^t\,R\,C]\,\tau, \tag{2}$$

$$\text{rank}\ (\hat{Q}) = \text{rank}\ (\hat{P}) = \text{rank}\ (\hat{Q}\,\hat{P}) = n_m, \tag{3}$$

where n_m is the degree of the model, $\hat{Q}$ and $\hat{P}$ are *pseudo-Gramians* (analogous to Gramians and have rank deficiency) and the skew projection operator τ is a nonlinear function of $\hat{Q}$ and $\hat{P}$. The optimal model (A_m, B_m, C_m) can be computed as a nonlinear function of (A, B, C) and $\hat{Q}$ and $\hat{P}$.

Equations (1) and (2), called *modified Lyapunov equations*, resemble standard matrix Lyapunov equations, but are highly nonlinear since they contain τ. The algorithm proposed in this paper utilizes probability-one homotopy theory as the main tool for solving the system (1)–(3). There is a family of systems (a homotopy) which make a continuous transformation from some initial system to the final system (1)–(3). Each system along the homotopy path is itself solved by a homotopy algorithm–a homotopy within a homotopy, so to speak. The central theorem of the paper shows the validity of the whole process, i.e., determines the class of initial systems which certainly lead to the final system along a homotopy path. Another, significantly simpler, homotopy is used to solve the initial problem. Finally, it is shown how the optimal solution to the reduced order model problem can be computed in an easy way from a solution to the system (1)–(3).

[1] D. C. Hyland and D. S. Bernstein, *The Optimal Projection Equations for Model Reduction and the Relationships Among the Methods of Wilson, Skelton and Moore*, IEEE Transactions on Automatic Control, Vol. AC-30, No. 12, December 1985, pp. 1201–1211.

Nine conference presentations were given in 1990. Those published in proceedings were:

- Two on parallel homotopy algorithms for a hypercube—A. Chakraborty, D. C. S. Allison, C. J. Ribbens, and L. T. Watson, "Low dimensional homotopy curve tracking on a hypercube", in *Proc. 1990 Internat. Conf. on Parallel Processing*, Vol. III, P.-C. Yew (ed.), St. Charles, IL, 1990, 44–51; A. Chakraborty, D. C. S. Allison, C. J. Ribbens, and L. T. Watson, "Parallel homotopy curve tracking on a hypercube", in *Parallel Processing for Scientific Computing*, J. Dongarra, P. Messina, D. C. Sorensen, and R. G. Voigt (eds.), SIAM, Philadelphia, PA, 1990, 149–153.
- Two mechanisms problems—V. Arun, C. F. Reinholtz, and L. T. Watson, "Application of new homotopy continuation techniques to variable geometry trusses", in *Cams, Gears, Robot and Mechanism Design*, DE-Vol. 26, A. Pisano, M. McCarthy, S. Derby (eds.), ASME, New York, 1990, 87–92; V. Arun, C. F. Reinholtz, and L. T. Watson, "Enumeration and analysis

of variable geometry truss manipulators", in *Cams, Gears, Robot and Mechanism Design*, DE-Vol. 26, A. Pisano, M. McCarthy, S. Derby (eds.), ASME, New York, 1990, 93–98.

- A NATO ASI invited lecture—L. T. Watson, "Numerical analysis of nonlinear equations in computer vision and robotics", in *Numerical Linear Algebra, Digital Signal Processing and Parallel Algorithms*, NATO ASI Series F, G. Golub and P. Van Dooren (eds.), Springer-Verlag, Berlin, 1990, 695–704.

Journal papers completed and submitted were:

- Four solid mechanics applications—L. T. Watson and C. Y. Wang, "Large deformations of rotating polygonal space structures", *Comput. Math. Appl.*; C. Y. Wang and L. T. Watson, "Large deformations of a whirling elastic cable", *Acta Mech.*; J. Rakowska, R. T. Haftka, and L. T. Watson, "An active set algorithm for tracing parametrized optima", *Struct. Optim.*; J. Rakowska, R. T. Haftka, and L. T. Watson, "Multi-objective control-structure optimization via homotopy methods", *SIAM J. Optim.*
- A fluid mechanics problem in biology—Zs. Nagy-Ungvarai, J. J. Tyson, S. C. Müller, L. T. Watson, and B. Hess, "Experimental study of spiral waves in the Ce-catalyzed Belousov-Zhabotinskii reaction", *J. Phys. Chem.*
- Three survey papers on optimization—L. T. Watson and A. P. Morgan, "Serial and parallel global optimization of polynomial programs via homotopy algorithms", *SIAM J. Optim.*; L. T. Watson, "Globally convergent homotopy algorithms for nonlinear systems of equations", *Nonlinear Dynamics*; L. T. Watson, "A survey of probability-one homotopy methods for engineering optimization", *Arabian J. Sci. Engrg.*
- A paper on sparse matrix technology—C. deSa, K. M. Irani, C. J. Ribbens, L. T. Watson, and H. F. Walker, "Preconditioned iterative methods for homotopy curve tracking", *SIAM J. Sci. Stat. Comput.*
- An analysis of parallel homotopy algorithms—A. Chakraborty, D. C. S. Allison, C. J. Ribbens, and L. T. Watson, "Analysis of function component complexity for hypercube homotopy algorithms", *IEEE Trans. Parallel Distrib. Sys.*
- A theoretical numerical analysis paper—G. Ulrich and L. T. Watson, "Positivity conditions for quartic polynomials", *SIAM J. Sci. Stat. Comput.*
- Another survey paper representing a major application of homotopy methods to circuit simulation—R. C. Melville, Lj. Trajković, S.-C. Fang, and L. T. Watson, "Globally convergent homotopy methods for the DC operating point problem", *SIAM J. Optim.*

HOMPACK has become the basis for a major circuit simulation code development effort within AT&T Bell Labs. Over 200 requests for HOMPACK have been received, indicating that the work is having an impact, thanks to AFOSR support.

Period: 3/1/91 – 4/30/92

The "homotopy within a homotopy" scheme described in the previous section, similar to the homotopy algorithms of Collins and Richter, turned out to have theoretical and computational flaws. Žigić devised a smooth homotopy based on the Drazin inverse $(\hat{Q}\hat{P})^{\sharp}$ of $\hat{Q}\hat{P}$, and successfully solved a number of problems with this homotopy. This Drazin inverse based homotopy is described in the Williamsburg IEEE Southeastcon proceedings (April, 1991), and in a chapter in Žigić's MS thesis.

In late spring of 1991 Žigić discovered a new formulation of (1)–(3), based on the contragredient transformation, that does not involve the explicit calculation of $\hat{Q}$ and $\hat{P}$ or any generalized inverse. A homotopy based on this contragredient formulation successfully solved every problem we could find in the literature, as well as a realistic space structure control problem from Marshall Space Flight Center in Huntsville. This algorithm became the basis for three journal papers (listed below) and Žigić's MS thesis.

Žigić spent the summer of 1991 working on technical details for choosing a good starting point, and submitted this work for the First Conference on Control Applications to be held in Dayton in September 1992. This paper was accepted, and ranked among the top 10% of all papers submitted. Žigić graduated and returned to Yugoslavia in August 1991.

Yuzhen Ge, a Ph.D. in mathematics, took over from Žigić in fall 1991. She spent six months reading Žigić's thesis, homotopy papers, and reports by Bernstein, Collins, Hyland, and Richter. While the Žigić algorithm is accurate and robust, it involves too many unknowns to be practical for large scale problems. Ge has spent all of 1992 implementing and testing an input normal form homotopy suggested by Collins, which has a very small number of unknowns and might be practical for large scale problems. That work is reported in the attached document "An input normal form homotopy for the L^2 optimal model order reduction problem". Unfortunately the input normal form equations are inherently unstable, so a practical, robust homotopy for large scale problems remains an open question.

Since the beginning of this grant, two MS students and one Ph.D. student have been supported. The MS theses are

- Kashmira M. Irani, "Preconditioned sequential and parallel conjugate gradient algorithms for homotopy curve tracking," M.S. thesis, Dept. of Computer Sci., Virginia Polytechnic Institute and State Univ., Blacksburg, VA, May 1990.
- Dragan Žigić, "Homotopy methods for solving the optimal projection equations for the reduced order model problem," M.S. thesis, Dept. of Computer Sci., Virginia Polytechnic Institute and State Univ., Blacksburg, VA, June 1991.

Fourteen conference presentations were given during the current period (3/1/91–4/30/92):

- Fifth SIAM Conference on Parallel Processing for Scientific Computing, Houston, TX, March, 1991 (2 papers).
- IEEE Southeastcon, Williamsburg, VA, April, 1991.
- Sixth Distributed Memory Computing Conference, Portland, OR, April, 1991.
- Second International Conference on Industrial and Applied Mathematics, Washington, DC, July, 1991 (4 papers).
- Computational Structures Technology, Edinburgh, Scotland, August, 1991 (keynote lecture).
- 1991 International Conference on Parallel Processing, St. Charles, IL, August, 1991.
- Fourth SIAM Conference on Applied Linear Algebra, Minneapolis, MN, Sept., 1991.
- Sixth IIMAS–UNAM Workshop on Numerical Analysis and Optimization, Oaxaca, Mexico, January, 1992.
- Copper Mountain Conference on Iterative Methods, Copper Mountain, CO, April, 1992.
- Scalable High Performance Computing Conference, Williamsburg, VA, April, 1992.

Those conference presentations published in refereed proceedings were:

- L. T. Watson, "Numerical analysis of nonlinear equations in computer vision and robotics", in *Numerical Linear Algebra, Digital Signal Processing and Parallel Algorithms*, NATO ASI Series F, Computer and Systems Sciences, Vol. 70, G. Golub and P. Van Dooren (eds.), Springer-Verlag, Berlin, 1991, 695–704.
- D. Žigić, E. G. Collins, S. Richter, and L. T. Watson, "A homotopy method for solving optimal projection equations for the reduced order model problem", in *Proc. IEEE Southeastcon '91*, Vol. 2, IEEE, New York, 1991, 1193–1197.
- L. T. Watson, R. T. Haftka, F. H. Lutze, R. H. Plaut, and P. Y. Shin, "The application of globally convergent homotopy methods to nonlinear optimization", in *Advances in Numerical Partial Differential Equations and Optimization*, S. Gómez, J. P. Hennart, and R. A. Tapia (eds.), SIAM, Philadelphia, PA, 1991, 284–298.
- J. R. Weimar, L. T. Watson, and J. J. Tyson, "Cellular automaton models for reaction diffusion equations", in *Proc. Sixth Distributed Memory Computing Conf.*, Q. Stout and M. Wolfe (eds.), IEEE Computer Soc., Los Alamitos, CA, 1991, 431–434.
- D. C. S. Allison, K. M. Irani, C. J. Ribbens, and L. T. Watson, "Shared memory parallel algorithms for homotopy curve tracking", in *Proc. 1991 Internat. Conf. on Parallel Processing*, Vol. III, K. So (ed.), CRC Press, Boca Raton, FL, 1991, 17–20.
- L. S. Auvil, C. J. Ribbens, and L. T. Watson, "Problem specific environments for parallel computing", in *Proc. Scalable High Performance Computing Conference*, R. Voigt (ed.), IEEE Computer Soc. Press, Los Alamitos, CA, 1992, 149–152.

Journal papers completed and submitted were:

- Parallel sparse matrix homotopy algorithm—D. C. S. Allison, K. M. Irani, C. J. Ribbens, and L. T. Watson, "High dimensional homotopy curve tracking on a shared memory multiprocessor", *J. Supercomputing*, 5 (1991) 347–366.
- Two mathematical biology papers—J. R. Weimar, J. J. Tyson, and L. T. Watson, "Diffusion and wave propagation in cellular automaton models of excitable media", *Phys. D*, 55 (1992) 309–327; J. R. Weimar, J. J. Tyson, and L. T. Watson, "Third generation cellular automaton for modeling excitable media", *Phys. D*, 55 (1992) 328–339.
- Three control theory papers—D. Žigić, L. T. Watson, E. G. Collins, Jr., and D. S. Bernstein, "Homotopy methods for solving the optimal projection equations for the H_2 reduced order model problem", *Internat. J. Control*; D. Žigić, L. T. Watson, E. G. Collins, Jr., and D. S. Bernstein, "Homotopy approaches to the H_2 reduced order model problem", *J. Math. Systems, Estimation, Control*; D. Žigić, L. T. Watson, and C. A. Beattie, "Contragredient transformations applied to the optimal projection equations", *Linear Algebra Appl.*
- Two engineering applications—J. Rakowska, R. T. Haftka, and L. T. Watson, "Tracing the efficient curve for multi-objective control-structure optimization", *Comput. Systems Engrg.*; C. J. Ribbens, L. T. Watson, and C. Y. Wang, "Steady viscous flow in a triangular cavity", *Comput. & Fluids.*
- Parallel mathematical software—C. J. Ribbens, L. T. Watson, and C. deSa, "Toward parallel mathematical software for elliptic partial differential equations", *ACM Trans. Math. Software.*
- Application of robust statistics to image processing—Y. Mainguy, J. B. Birch, and L. T. Watson, "A robust variable order facet model for image data", *Comput. Vision, Graphics, Image Processing: Image Understanding.*
- Sparse matrix technology—W. D. McQuain, C. J. Ribbens, L. T. Watson, and R. C. Melville, "Preconditioned iterative methods for sparse linear algebra problems arising in circuit simulation", *SIAM J. Sci. Stat. Comput.*

ATTACHMENT — AFOSR 89-0497

AN INPUT NORMAL FORM HOMOTOPY FOR THE L^2 OPTIMAL MODEL ORDER REDUCTION PROBLEM

Yuzhen Ge[†], Emmanuel G. Collins, Jr.[*], Layne T. Watson[†]

1. Introduction.

The L^2 optimal model reduction problem, i.e., the problem of approximating a higher order dynamical system by a lower order one so that a model reduction criterion is minimized, is of significant importance and is under intense study. Several earlier attempts to apply homotopy methods to the L^2 optimal model order reduction problem were not entirely satisfactory. Richter [1], [2] devised a homotopy approach which only estimated certain crucial partial derivatives and employed relatively crude curve tracking techniques. Žigić, Bernstein, Collins, Richter, and Watson [3], [4], [5], [6] formulated the problem so that numerical linear algebra techniques could be used to explicitly calculate partial derivatives, and employed sophisticated homotopy curve tracking algorithms, but the number of variables made large problems intractable. We propose here several ways to reduce the dimension of the homotopy map so that large problems are computationally feasible.

The problem can be formulated as: given the asymptotically stable, controllable, and observable time invariant continuous time system

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t), \\ y(t) &= C\,x(t), \end{aligned} \tag{1}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{l\times n}$, the goal is to find a reduced order model

$$\begin{aligned} \dot{x}_m(t) &= A_m\,x_m(t) + B_m\,u(t), \\ y_m(t) &= C_m\,x_m(t), \end{aligned} \tag{2}$$

where $A_m \in \mathbf{R}^{n_m\times n_m}$, $B_m \in \mathbf{R}^{n_m\times m}$, $C_m \in \mathbf{R}^{l\times n_m}$, $n_m < n$ which minimizes the cost function

$$J(A_m, B_m, C_m) \equiv \lim_{t\to\infty} E\left[(y - y_m)^t\, R\,(y - y_m)\right], \tag{3}$$

where the input $u(t)$ is white noise with symmetric and positive definite intensity V and R is a symmetric and positive definite weighting matrix.

The optimal projection equations of Hyland and Bernstein [10], [11], described in Section 5, are basis independent and correspond to the maximum number of degrees of freedom one could plausibly use. Richter and Collins [3] use this maximum number, and Žigić [4] reduced it somewhat. At the other extreme, the minimal number of unknowns corresponds to the input normal form described in Section 2, and developed into a homotopy algorithm in Sections 3 and 4. Subtle differences between the optimal projection equations and input normal form formulations are explored in Section 5. Section 6 gives numerical results for the input normal form homotopy on the test set of Žigić [4].

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2. Input normal form formulations.

The following theorem is needed to present the homotopy method for the input normal form.

THEOREM 1 [7]. *Suppose $\bar{A}_m$ is asymptotically stable. Then for every minimal $(\bar{A}_m, \bar{B}_m, \bar{C}_m)$, i.e., $(\bar{A}_m, \bar{B}_m)$ is controllable and $(\bar{A}_m, \bar{C}_m)$ is observable, there exist a similarity transformation U and a positive definite matrix $\Omega = \mathrm{diag}(\omega_1, \cdots, \omega_{n_m})$ such that $A_m = U^{-1}\bar{A}_m U$, $B_m = U^{-1}\bar{B}_m$, and $C_m = \bar{C}_m U$ satisfy*

$$\begin{aligned} 0 &= A_m + A_m^T + B_m V B_m^T, \\ 0 &= A_m^T \Omega + \Omega A_m + C_m^T R C_m. \end{aligned} \tag{4}$$

In addition,

$$\begin{aligned} (A_m)_{ii} &= -\frac{1}{2}(B_m V B_m^T)_{ii}, \\ \omega_i &= \frac{(C_m^T R C_m)_{ii}}{(B_m V B_m^T)_{ii}}, \\ (A_m)_{ij} &= \frac{(C_m^T R C_m)_{ij} - \omega_j (B_m V B_m^T)_{ij}}{\omega_j - \omega_i}, \qquad \text{if } \omega_i \neq \omega_j. \end{aligned} \tag{5}$$

DEFINITION 1. The triple (A_m, B_m, C_m) satisfying (4) or (5) is said to be in *input normal form.*

Under the assumption that a solution (A_m, B_m, C_m) in input normal form is sought, the only independent variables are B_m and C_m, and in this case the domain is

$$\{(A_m, B_m, C_m) : \ A_m \text{ is stable, } (A_m, B_m, C_m) \text{ is minimal and in input normal form}\}.$$

The cost function (3) can be written as

$$J(A_m, B_m, C_m) = \mathrm{tr}\ (\tilde{Q}\tilde{R}) \tag{6}$$

where $\tilde{Q}$ is a symmetric and positive definite matrix satisfying

$$\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{V} = 0, \tag{7}$$

and

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & A_m \end{pmatrix}, \qquad \tilde{R} = \begin{pmatrix} C^T R C & -C^T R C_m \\ -C_m^T R C & C_m^T R C_m \end{pmatrix}, \qquad \tilde{V} = \begin{pmatrix} B V B^T & B V B_m^T \\ B_m V B^T & B_m V B_m^T \end{pmatrix}.$$

$\tilde{Q}$ can be written as

$$\tilde{Q} = \begin{pmatrix} \tilde{Q}_1 & \tilde{Q}_{12} \\ \tilde{Q}_{12}^T & \tilde{Q}_2 \end{pmatrix},$$

where $\tilde{Q}_1 \in \mathbf{R}^{n \times n}$, $\tilde{Q}_{12} \in \mathbf{R}^{n \times n_m}$, and $\tilde{Q}_2 \in \mathbf{R}^{n_m \times n_m}$.

The goal of minimizing (6) under the constraints (4) and (7) leads to the Lagrangian

$$\begin{aligned} L(A_m, B_m, C_m, \Omega, \tilde{Q}) = \mathrm{tr}[&\tilde{Q}\tilde{R} + (A_m + A_m^T + B_m V B_m^T) M_c \\ &+ (A_m^T \Omega + \Omega A_m + C_m^T R C_m) M_o + (\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{V})\tilde{P}], \end{aligned} \tag{8}$$

where the symmetric matrices M_o, M_c, and $\tilde{P}$ are Lagrange multipliers.

Setting $\partial L/\partial \tilde{Q} = 0$ gives

$$\tilde{A}^T\tilde{P} + \tilde{P}\tilde{A} + \tilde{R} = 0, \tag{9}$$

where $\tilde{P}$ is symmetric positive definite and can be partitioned as

$$\tilde{P} = \begin{pmatrix} \tilde{P}_1 & \tilde{P}_{12} \\ \tilde{P}_{12}^T & \tilde{P}_2 \end{pmatrix}.$$

$\partial L/\partial \Omega = 0$ and $\partial L/\partial A_m = 0$ yield

$$0 = 2M_c + 2\Omega M_o + 2(\tilde{P}_{12}^T\tilde{Q}_{12} + \tilde{P}_2\tilde{Q}_2), \qquad 0 = (A_m M_o)_{ii}, \quad 1 \le i \le n_m. \tag{10}$$

A straightforward calculation shows

$$\begin{aligned} \frac{\partial L}{\partial B_m} &= 2(\tilde{P}_{12}^T B + \tilde{P}_2 B_m)V + 2M_c B_m V, \\ \frac{\partial L}{\partial C_m} &= 2R(C_m\tilde{Q}_2 - C\tilde{Q}_{12}) + 2RC_m M_o. \end{aligned} \tag{11}$$

THEOREM 2 [8]. *The matrices M_c and M_o in (11) satisfy*

$$\begin{aligned} M_c &= -(\frac{1}{2}S + \Omega M_o), \\ (M_o)_{ii} &= -\frac{1}{(A_m)_{ii}} \sum_{\substack{j=1 \\ j\neq i}}^{n_m} (A_m)_{ij}(M_o)_{ji}, \\ (M_o)_{ij} &= \frac{(S)_{ij} - (S)_{ji}}{2(\omega_j - \omega_i)}, \quad \text{if } \omega_j \neq \omega_i, \end{aligned} \tag{12}$$

where

$$S = 2(\tilde{P}_{12}^T\tilde{Q}_{12} + \tilde{P}_2\tilde{Q}_2). \tag{13}$$

3. A homotopy approach.

A homotopy approach following [8] is now described. Let A_f, B_f, C_f , R_f, and V_f denote A, B, C, R, and V in the above and define

$$\begin{aligned} A(\lambda) &= A_0 + \lambda(A_f - A_0), \\ B(\lambda) &= B_0 + \lambda(B_f - B_0), \\ C(\lambda) &= C_0 + \lambda(C_f - C_0), \end{aligned} \qquad \begin{aligned} R(\lambda) &= R_0 + \lambda(R_f - R_0), \\ V(\lambda) &= V_0 + \lambda(V_f - V_0). \end{aligned}$$

For brevity, $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $V(\lambda)$, and $R(\lambda)$ will be denoted by A, B, C, V, and R respectively in the following. Let

$$\begin{aligned} H_{B_m}(\theta,\lambda) &= \frac{\partial L}{\partial B_m} = 2(\tilde{P}_{12}^T B + \tilde{P}_2 B_m)V + 2M_c B_m V, \\ H_{C_m}(\theta,\lambda) &= \frac{\partial L}{\partial C_m} = 2R(C_m\tilde{Q}_2 - C\tilde{Q}_{12}) + 2RC_m M_o, \end{aligned} \tag{14}$$

where

$$\theta \equiv \begin{pmatrix} \text{Vec }(B_m) \\ \text{Vec }(C_m) \end{pmatrix}$$

denotes the independent variables B_m and C_m, M_o and M_c satisfy (12), and $\tilde{Q}$ and $\tilde{P}$ satisfy respectively (7) and (9). Vec(P) for a matrix $P \in \mathbf{R}^{p\times q}$ is the concatenation of its columns:

$$\text{Vec}(P) \equiv \begin{pmatrix} P_{\cdot 1} \\ P_{\cdot 2} \\ \vdots \\ P_{\cdot q} \end{pmatrix} \in \mathbf{R}^{pq}.$$

The homotopy map is defined as

$$\rho(\theta,\lambda) = \begin{pmatrix} \text{Vec }\left[H_{B_m}(\theta,\lambda)\right] \\ \text{Vec }\left[H_{C_m}(\theta,\lambda)\right] \end{pmatrix}, \tag{15}$$

and its Jacobian matrix is

$$D\rho(\theta,\lambda) = \big(D_\theta\rho(\theta,\lambda), D_\lambda\rho(\theta,\lambda)\big). \tag{16}$$

Define

$$\begin{aligned} \hat{H}_{B_m}(\tilde{P}^{(j)}, M_c^{(j)}) &= 2(\tilde{P}_{12}^{T\,(j)}B + \tilde{P}_2^{(j)}B_m)V + 2M_c^{(j)}B_mV, \\ \hat{H}_{C_m}(\tilde{Q}^{(j)}, M_o^{(j)}) &= 2R(C_m\tilde{Q}_2^{(j)} - C\tilde{Q}_{12}^{(j)}) + 2RC_mM_o^{(j)}, \end{aligned} \tag{17}$$

where the superscript (j) means $\partial/\partial\theta_j$: $Y^{(j)} \equiv \frac{\partial Y}{\partial\theta_j}$. Using the above definitions, we have for $\theta_j = (B_m)_{kl}$,

$$\begin{aligned} \frac{\partial H_{B_m}}{\partial(B_m)_{kl}} &= \hat{H}_{B_m}(\tilde{P}^{(j)}, M_c^{(j)}) + 2(\tilde{P}_2 + M_c)E^{(k,l)}V, \\ \frac{\partial H_{C_m}}{\partial(B_m)_{kl}} &= \hat{H}_{C_m}(\tilde{Q}^{(j)}, M_o^{(j)}), \end{aligned} \tag{18}$$

and for $\theta_j = (C_m)_{kl}$,

$$\begin{aligned} \frac{\partial H_{B_m}}{\partial(C_m)_{kl}} &= \hat{H}_{B_m}(\tilde{P}^{(j)}, M_c^{(j)}), \\ \frac{\partial H_{C_m}}{\partial(C_m)_{kl}} &= \hat{H}_{C_m}(\tilde{Q}^{(j)}, M_o^{(j)}) + 2RE^{(k,l)}(\tilde{Q}_2 + M_o), \end{aligned} \tag{19}$$

where $E^{(k,l)}$ is a matrix of the appropriate dimension whose only nonzero element is $e_{kl} = 1$. $\tilde{P}^{(j)}$ and $\tilde{Q}^{(j)}$ can be obtained by solving the Lyapunov equations

$$\begin{aligned} 0 &= \tilde{A}^{(j)}\tilde{Q} + \tilde{A}\tilde{Q}^{(j)} + \tilde{Q}^{(j)}\tilde{A}^T + \tilde{Q}\tilde{A}^{T\,(j)} + \tilde{V}^{(j)}, \\ 0 &= \tilde{A}^{T\,(j)}\tilde{P} + \tilde{A}^T\tilde{P}^{(j)} + \tilde{P}^{(j)}\tilde{A} + \tilde{P}\tilde{A}^{(j)} + \tilde{R}^{(j)}. \end{aligned} \tag{20}$$

Similarly for λ, using a dot to denote $\partial/\partial\lambda$,

$$\begin{aligned} \frac{\partial H_{B_m}}{\partial\lambda} &= \hat{H}_{B_m}(\dot{\tilde{P}}, \dot{M}_c) + 2\tilde{P}_{12}^T(\dot{B}V + B\dot{V}) + 2(\tilde{P}_2 + M_c)B_m\dot{V}, \\ \frac{\partial H_{C_m}}{\partial\lambda} &= \hat{H}_{C_m}(\dot{\tilde{Q}}, \dot{M}_o) + 2\dot{R}C_m(\tilde{Q}_2 + M_o) - 2(\dot{R}C + R\dot{C})\tilde{Q}_{12}, \end{aligned} \tag{21}$$

where $\dot{\tilde{P}}$ and $\dot{\tilde{Q}}$ are obtained by solving the Lyapunov equations

$$0 = \dot{\tilde{A}}\tilde{Q} + \tilde{A}\dot{\tilde{Q}} + \dot{\tilde{Q}}\tilde{A}^T + \tilde{Q}\dot{\tilde{A}}^T + \dot{\tilde{V}},$$
$$0 = \dot{\tilde{A}}^T\tilde{P} + \tilde{A}^T\dot{\tilde{P}} + \dot{\tilde{P}}\tilde{A} + \tilde{P}\dot{\tilde{A}} + \dot{\tilde{R}}.$$

4. Numerical algorithm for input normal form homotopy.

The initial point $(\theta, \lambda) = (\theta_0, 0) = \big((B_m)_0, (C_m)_0, 0\big)$ is chosen so that the triple $\big((A_m)_0, (B_m)_0, (C_m)_0\big)$ is in input normal form and satisfies $\rho(\theta_0, 0) = 0$.

THEOREM 3 [9]. *Suppose $\bar{A}$ is asymptotically stable. Then for every minimal $(\bar{A}, \bar{B}, \bar{C})$, i.e., $(\bar{A}, \bar{B})$ is controllable and $(\bar{A}, \bar{C})$ is observable, there exist a similarity transformation T and a positive definite matrix $\Lambda = \mathrm{diag}\,(d_1, d_2, \cdots, d_n)$ such that $A = T^{-1}\bar{A}T$, $B = T^{-1}\bar{B}$, and $C = \bar{C}T$ satisfy*

$$0 = A\Lambda + \Lambda A^T + BVB^T,$$
$$0 = A^T\Lambda + \Lambda A + C^TRC.$$

DEFINITION 2. The triple (A, B, C) in the above theorem is *balanced.*

According to Moore [9], under certain conditions, the leading principal $n_m \times n_m$ block of A, the leading principal $n_m \times m$ block of B, and the leading principal $l \times n_m$ block of C in balanced form are good approximations to the reduced order model. This suggests that the initial point $(\theta_0, 0)$ be chosen as follows:

1) Transform the given triple (A_f, B_f, C_f) to balanced form (A_b, B_b, C_b).
2) Partition (A_b, B_b, C_b) as

$$A_b = \begin{array}{cc} & \overbrace{\phantom{A_{11}}}^{n_m} \\ n_m\{ & \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \end{array}, \quad B_b = n_m\{ \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \qquad C_b = \begin{array}{c} \overbrace{}^{n_m} \\ \begin{pmatrix} C_1 & C_2 \end{pmatrix} \end{array}.$$

3) (A_0, B_0, C_0) is chosen as

$$A_0 = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \qquad B_0 = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \qquad C_0 = \begin{pmatrix} C_1 & 0 \end{pmatrix}.$$

4) The initial point for the reduced order model is chosen as

$$\bar{\theta}_0 = \begin{pmatrix} \mathrm{Vec}\ (\bar{B}_m)_0 \\ \mathrm{Vec}\ (\bar{C}_m)_0 \end{pmatrix} = \begin{pmatrix} \mathrm{Vec}\ B_1 \\ \mathrm{Vec}\ C_1 \end{pmatrix},$$

and $(\bar{A}_m)_0 = A_{11}$ by construction.

5) Transform the initial point $\big((\bar{A}_m)_0, (\bar{B}_m)_0, (\bar{C}_m)_0\big)$ to input normal form so that the initial reduced order model is

$$\big((A_m)_0, (B_m)_0, (C_m)_0\big) = \big(T^{-1}\,(\bar{A}_m)_0\,T, \quad T^{-1}\,(\bar{B}_m)_0, \quad (\bar{C}_m)_0\,T\big).$$

The initial point for the homotopy map is then $(\theta_0, 0)$, where

$$\theta_0 = \begin{pmatrix} \mathrm{Vec}\ (B_m)_0 \\ \mathrm{Vec}\ (C_m)_0 \end{pmatrix}.$$

Once the initial point is chosen, the rest of the computation is as follows:

1) Set $\lambda := 0$, $\theta := \theta_0$.
2) Calculate A_m from (5), $\tilde{R}$, $\tilde{V}$, and compute $\tilde{Q}$ and $\tilde{P}$ according to (7) and (9).
3) Evaluate S from (13) and M_o and M_c according to (12).
4) Evaluate the homotopy map $\rho(\theta,\lambda)$ in (15) and $D\rho(\theta,\lambda)$ in (16).
5) Predict the next point $Z^{(0)} = (\theta^{(0)}, \lambda^{(0)})$ on the curve γ.
6) For $k := 0, 1, 2, \cdots$ until convergence do

$$Z^{(k+1)} = \left[D\rho(Z^{(k)})\right]^{\dagger}\rho(Z^{(k)}),$$

where $\left[D\rho(Z)\right]^{\dagger}$ is the Moore-Penrose inverse of $D\rho(Z)$. Let $(\theta_1, \lambda_1) = \lim_{k\to\infty} Z^{(k)}$.

7) If $\lambda_1 < 1$, then set $\theta := \theta_1$, $\lambda := \lambda_1$, and go to step 2).
8) If $\lambda_1 \geq 1$, compute the solution $\bar{\theta}$ at $\lambda = 1$. A_m is then obtained from (5).

5. Comparison with optimal projection equations approach.

THEOREM 4 [10] [11]. *Suppose (A_m, B_m, C_m) solves the problem (1)–(3). Then there exist* pseudogramians $\hat{Q}$, $\hat{P}$ *that are a solution to* modified Lyapunov equations

$$\begin{aligned} 0 &= \tau[A\,\hat{Q} + \hat{Q}\,A^t + B\,V\,B^t], \\ 0 &= [A^t\,\hat{P} + \hat{P}\,A + C^t\,R\,C]\,\tau, \end{aligned} \tag{22}$$

and satisfy rank conditions

$$rank\,(\hat{Q}) = rank\,(\hat{P}) = rank\,(\hat{Q}\,\hat{P}) = n_m,$$

such that the optimal model is given by

$$\begin{aligned} A_m &= \Gamma\,A\,G^t, \\ B_m &= \Gamma\,B, \\ C_m &= C\,G^t, \end{aligned} \tag{23}$$

where G and Γ come from a (G, M, Γ)-factorization *of $\hat{Q}\hat{P}$:*

$$\begin{aligned} \hat{Q}\,\hat{P} &= G^t\,M\,\Gamma, \\ \Gamma\,G^t &= I_{n_m}, \end{aligned} \tag{24}$$

$G, \Gamma \in \mathbf{R}^{n_m \times n}$, $M \in \mathbf{R}^{n_m \times n_m}$ is positive semisimple and $\tau \equiv G^t\,\Gamma$.

Equations (22) are called the optimal projection equations, which after a lot of algebra described in [5], can be written in a form suitable for computation as

$$\begin{aligned} U_1\,A\,W_1\Sigma\,W_1^t + \Sigma\,W_1^t\,A^t + U_1\,B\,V\,B^t &= 0, &\quad (n_m\;n) \\ A^t\,U_1^t\,\Sigma + U_1^t\,\Sigma\,U_1\,A\,W_1 + C^t\,R\,C\,W_1 &= 0, &\quad (n\;n_m) \\ U_1\,W_1 - I &= 0. &\quad (n_m^2) \end{aligned} \tag{25}$$

The unknowns are $W_1 \in \mathbf{R}^{n \times n_m}$, $U_1 \in \mathbf{R}^{n_m \times n}$ and symmetric $\Sigma \in \mathbf{R}^{n_m \times n_m}$.

Hyland and Bernstein [11] stated that the optimal projection equations can have at most $\begin{pmatrix} n \\ n_m \end{pmatrix}$ solutions. It is shown by the following 2-dimensional example that this is not true in general.

The system [12] is given by

$$A = \begin{pmatrix} -0.05 & -0.99 \\ -0.99 & -5000.0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 100 \end{pmatrix}, \quad C = (1 \quad 100). \tag{26}$$

PROPOSITION: *For the system* (1) *defined by* (26), *the solution set of the optimal projection equations contains three isolated solutions and a one-dimensional manifold parametrized by one element of either W_1 or U_1.*

Proof. The three isolated solutions are

$$A_m = (-0.005004234), \quad B_m = (1.0002127), \quad C_m = (1.0002127),$$
$$A_m = (-4998.0786133), \quad B_m = (100.0001907), \quad C_m = (100.0001907),$$
$$A_m = (-0.4659163), \quad B_m = (-1.9404824), \quad C_m = (-1.9404824),$$

which were obtained by both POLSYS from HOMPACK [17] and by a homotopy approach [4]–[6]. The one-dimensional manifold of solutions corresponds to

$$A_m = (-0.4851515), \quad B_m = (0.0), \quad C_m = (0.0),$$

which can be derived directly from the optimal projection equations as follows.

Let $W_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $U_1 = (x_3, x_4)$, and $\Sigma = x_5$. The optimal projection equations (25) for this problem can be written as

$$\begin{aligned}
0 &= a_{11}x_1^2x_3x_5 + a_{12}x_1x_2x_3x_5 + a_{21}x_1^2x_4x_5 + a_{22}x_1x_2x_4x_5 \\
&\quad + a_{11}x_1x_5 + a_{12}x_2x_5 + (BVB^t)_{11}x_3 + (BVB^t)_{21}x_4, \\
0 &= a_{11}x_1x_2x_3x_5 + a_{12}x_2^2x_3x_5 + a_{21}x_1x_2x_4x_5 + a_{22}x_2^2x_4x_5 \\
&\quad + a_{21}x_1x_5 + a_{22}x_2x_5 + (BVB^t)_{12}x_3 + (BVB^t)_{22}x_4, \\
0 &= a_{11}x_1x_3^2x_5 + a_{12}x_2x_3^2x_5 + a_{21}x_1x_3x_4x_5 + a_{22}x_2x_3x_4x_5 \\
&\quad + a_{11}x_3x_5 + a_{21}x_4x_5 + (C^tRC)_{11}x_1 + (C^tRC)_{12}x_2, \\
0 &= a_{11}x_1x_3x_4x_5 + a_{12}x_2x_3x_4x_5 + a_{21}x_1x_4^2x_5 + a_{22}x_2x_4^2x_5 \\
&\quad + a_{12}x_3x_5 + a_{22}x_4x_5 + (C^tRC)_{21}x_1 + (C^tRC)_{22}x_2, \\
0 &= x_1x_3 + x_2x_4 - 1.
\end{aligned} \tag{27}$$

The triple (A_m, B_m, C_m) is given by

$$\begin{aligned}
A_m &= \Gamma A G^t = (x_3 \ x_4) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= x_1(a_{11}x_3 + a_{21}x_4) + x_2(a_{12}x_3 + a_{22}x_4), \\
B_m &= \Gamma B = (x_3 \ x_4) \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = b_{11}x_3 + b_{21}x_4, \\
C_m &= CG^T = (c_{11} \ c_{12}) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_{11}x_1 + c_{12}x_2,
\end{aligned} \tag{28}$$

where $\Gamma = U_1$ and $G = W_1^t$. Substituting (26) into (27) and (28), setting $B_m = x_3 + 100x_4 = 0$ and $C_m = x_1 + 100x_2 = 0$ gives $x_1 = -100x_2$, $x_3 = -100x_4$, and $A_m = -4852x_2x_4$. Equations (27) become

$$0 = 485200x_2^2x_4x_5 - 0.49x_2x_5, \tag{29}$$

$$0 = 485200x_2x_4^2x_5 - 0.49x_4x_5, \tag{30}$$

$$0 = 4852x_2^2x_4x_5 + 4901x_2x_5, \tag{31}$$

$$0 = 4852x_2x_4^2x_5 + 4901x_4x_5, \tag{32}$$

$$0 = 10001x_2x_4 - 1. \tag{33}$$

If $x_2 = 0$ or $x_4 = 0$, equation (33) will not be satisfied. Only the situation that $x_2 \neq 0$ and $x_4 \neq 0$ is possible. Then equations (29)–(33) can be reduced to

$$\begin{aligned} 0 &= 485200x_2x_4x_5 - 0.49x_5, \\ 0 &= 4852x_2x_4x_5 + 4901x_5, \\ 0 &= 10001x_2x_4 - 1. \end{aligned} \tag{34}$$

If $x_5 \neq 0$ then (34) becomes

$$\begin{aligned} 0 &= 485200x_2x_4 - 0.49, \\ 0 &= 4852x_2x_4 + 4901, \\ 0 &= 10001x_2x_4 - 1, \end{aligned} \tag{35}$$

which does not have a solution.

Thus $x_5 = 0$, and equation (34) reduces to

$$10001x_2x_4 - 1 = 0,$$

which gives $A_m = -4852/10001 = -0.4851515$ corresponding to a one-dimensional manifold parametrized by x_2 or x_4. Hence the solution $A_m = -0.4851515$, $B_m = 0$ and $C_m = 0$ (which is not controllable or observable) corresponds to a one-dimensional manifold of solutions of the optimal projection equations. Q. E. D.

The set of solutions of the input normal form equations contains the same set of isolated solutions as the optimal projection equations, and also a fourth isolated solution given by $A_m = B_m = C_m = 0$. Therefore the solution sets of the two formulations are different.

The input normal form equations can be rewritten as

$$\begin{aligned} 0 &= 2(\tilde{P}_{12}^T B + \tilde{P}_2 B_m)V + 2M_c B_m V, \\ 0 &= 2R(C_m \tilde{Q}_2 - C\tilde{Q}_{12}) + 2RC_m M_o. \end{aligned} \tag{36}$$

Setting $B_m = C_m = 0$, the equations become

$$\begin{aligned} 0 &= \tilde{P}_{12}^T BV, \\ 0 &= RC\tilde{Q}_{12}, \end{aligned} \tag{37}$$

where $\tilde{P}_{12}$ and $\tilde{Q}_{12}$ satisfy respectively

$$0 = A^T \tilde{P}_{12} + \tilde{P}_{12} A_m,$$
$$0 = A\tilde{Q}_{12} + \tilde{Q}_{12} A_m,$$

which has a solution $\tilde{P}_{12} = \tilde{Q}_{12} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. A_m satisfies

$$A_m + A_m^t + B_m V B_m^T = A_m + A_m^t = 0$$

which gives $A_m = 0$.

6. Numerical results and comparisons.

In this section numerical results for the input normal form formulations are given for nine systems. These systems have been studied and solved in [4], [5], [6] using the optimal projection equations approach. Comparisons are made between the input normal form formulations and the optimal projection equations.

The cost J is computed for each model as tr $(\tilde{Q}\tilde{R})$, according to (38). For all examples $V = R = I$. All the answers are given in input normal form. The solutions obtained by the input normal form formulation are the same as those obtained by the optimal projection equation method, unless indicated otherwise.

EXAMPLE 1 [12]. The system is given by

$$A = \begin{pmatrix} -0.05 & -0.99 \\ -0.99 & -5000.0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 100 \end{pmatrix}, \quad C = (1 \quad 100).$$

The homotopy algorithm converges to a solution corresponding to the model of order $n_m = 1$ given by

$$A_m = (-0.00500423), \quad B_m = (-0.100042), \quad C_m = (-10.000021),$$

which is not in the solution set of [4], [5], [6] by the optimal projection equation approach. This model yields the (maximum) cost $J = 10000$.

In the first step of choosing an initial point, (A_f, B_f, C_f) is transformed to (A_b, B_b, C_b), where orthogonal decompositions of two matrices are needed. If the eigenvalues of one of the matrices are rearranged in ascending order, then a different solution is obtained, namely

$$A_m = (-4998.078625), \quad B_m = (-99.980784), \quad C_m = (-100.019608).$$

This model yields the (minimum) cost $J = 96.078058$.

EXAMPLE 2 [13]. The system is given by

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -10 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 70 & 1 \end{pmatrix}, \quad C = (1 \quad -0.2).$$

A model of order $n_m = 1$ is

$$A_m = (-11.979443), \quad B_m = (-4.859135 \quad 0.589656), \quad C_m = (2.760762).$$

This model yields the cost $J = 0.598377$.

EXAMPLE 3 [12]. The system is given by

$$A = \begin{pmatrix} -0.25 & -0.4 \\ -0.4 & -0.72 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1.2 \end{pmatrix}, \quad C = (1 \quad 1.2).$$

A model of order $n_m = 1$ is

$$A_m = (-0.838521), \quad B_m = (-1.295006), \quad C_m = (1.825580).$$

This model yields the cost $J = 0.107256$.

EXAMPLE 4 [14]. The system is given by

$$A = \begin{pmatrix} -1 & 3 & 0 \\ -1 & -1 & 1 \\ 4 & -5 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}, \quad C = (1 \quad 0 \quad 0).$$

A model of order $n_m = 1$ is

$$A_m = (-0.286334), \quad B_m = (-0.756748) \quad C_m = (0.878161).$$

This model yields the cost $J = 1.228834$ and this solution is different from that obtained by the optimal projection equation method [4], [5], [6]. A model of order $n_m = 2$ is

$$A_m = \begin{pmatrix} -0.215037 & 0.753968 \\ -2.513846 & -3.600739 \end{pmatrix}, \quad B_m = \begin{pmatrix} 0.655800 \\ 2.683557 \end{pmatrix}, \quad C_m^t = \begin{pmatrix} 0.888877 \\ -1.090926 \end{pmatrix}.$$

This model yields the cost $J = 0.0197781$.

EXAMPLE 5 [12]. The system is given by

$$A = \begin{pmatrix} -10 & 1 & 0 \\ -5 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad C = (1 \quad 0 \quad 0).$$

A model of order $n_m = 1$ is

$$A_m = (-0.157898), \quad B_m = (0.561956), \quad C_m = (0.318537).$$

This model yields the cost $J = 0.0107792$. A model of order $n_m = 2$ is

$$A_m = \begin{pmatrix} -0.139652 & 0.100607 \\ -0.600971 & -0.448192 \end{pmatrix}, \quad B_m = \begin{pmatrix} 0.528492 \\ 0.946775 \end{pmatrix}, \quad C_m^t = \begin{pmatrix} 0.320438 \\ -0.0961019 \end{pmatrix}.$$

This model yields the cost $J = 0.000329024$.

EXAMPLE 6. The system is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & -0.02 & 1 & 0.01 \\ 0 & 0 & 0 & 1 \\ 0.1 & 0.001 & -0.1 & -0.001 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = (0 \quad 1 \quad 0 \quad 0).$$

For this system, the initial ω's are approximately the same, which leads to a significant numerical error in computing M_o and the numerical failure of the homotopy algorithm. Therefore this technique for choosing initial points fails, and some modification to the algorithm is needed to avoid this kind of ill conditioning. However, it is not at all clear how to systematically avoid nearly equal ω's, and this remains an open question.

EXAMPLE 7 [9], [15]. The system is given by

$$A = \begin{pmatrix} 0 & 0 & 0 & -150 \\ 1 & 0 & 0 & -245 \\ 0 & 1 & 0 & -1113 \\ 0 & 0 & 1 & -19 \end{pmatrix}, \quad B = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad C = (\,0 \quad 0 \quad 0 \quad 1\,).$$

A model of order $n_m = 1$ is

$$A_m = (\,-0.495187\,), \quad B_m = (\,0.995175\,), \quad C_m = (\,0.0148426\,).$$

This model yields the cost $J = 4.90749 \cdot 10^{-5}$. A model of order $n_m = 2$ is

$$A_m = \begin{pmatrix} -0.437964 & -0.482612 \\ 2.840074 & -3.172419 \end{pmatrix}, \quad B_m = \begin{pmatrix} 0.935911 \\ -2.518896 \end{pmatrix}, \quad C_m^t = (\,0.0149143 \quad 0.00682097\,).$$

This model yields the cost $J = 4.159 \cdot 10^{-7}$. A model of order $n_m = 3$ is

$$A_m = \begin{pmatrix} -0.437810 & -0.483078 & -0.0370108 \\ 2.826317 & -3.135361 & -0.612598 \\ -4.651841 & 13.160394 & -12.554152 \end{pmatrix},$$

$$B_m = \begin{pmatrix} 0.935746 \\ -2.504141 \\ 5.010819 \end{pmatrix}, \quad C_m = (\,0.0149143 \quad 0.00682180 \quad 0.000635413\,).$$

This model yields the cost $J = 4.59 \cdot 10^{-10}$.

EXAMPLE 8 [9]. The system is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -50 & -79 & -33 & -5 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad C = (\,50 \quad 15 \quad 1 \quad 0\,).$$

A model of order $n_m = 1$ is

$$A_m = (\,-0.576205\,), \quad B_m = (\,1.073504\,), \quad C_m = (\,0.588692\,).$$

This model yields the cost $J = 0.104740$. A model of order $n_m = 2$ is

$$A_m = \begin{pmatrix} -0.532330 & -0.598751 \\ 3.800771 & -4.815122 \end{pmatrix}, \quad B_m = \begin{pmatrix} 1.031824 \\ -3.103263 \end{pmatrix}, \quad C_m^t = \begin{pmatrix} 0.588704 \\ 0.278923 \end{pmatrix}.$$

This model yields the cost $J = 0.0269278$. A model of order $n_m = 3$ is

$$A_m = \begin{pmatrix} -0.520312 & -0.731867 & -0.162146 \\ 2.888921 & -2.235622 & -3.721286 \\ -1.084500 & 6.305395 & -0.746729 \end{pmatrix},$$

$$B_m^t = (\,1.020109 \quad -2.114532 \quad 1.222072\,), \quad C_m = (\,0.586461 \quad 0.307967 \quad 0.105043\,).$$

This model yields the cost $J = 0.00148438$.

EXAMPLE 9 [16]. The system is given by

$$A = \begin{pmatrix} -6.2036 & 15.054 & -9.8726 & -376.58 & 251.32 & -162.24 & 66.827 \\ 0.53 & -2.0176 & 1.4363 & 0 & 0 & 0 & 0 \\ 16.846 & 25.079 & -43.555 & 0 & 0 & 0 & 0 \\ 377.4 & -89.449 & -162.83 & 57.998 & -65.514 & 68.579 & 157.57 \\ 0 & 0 & 0 & 107.25 & -118.05 & 0 & 0 \\ 0.36992 & -0.14445 & -0.26303 & -0.64719 & 0.49947 & -0.21133 & 0 \\ 0 & 0 & 0 & 0 & 0 & 376.99 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 89.353 & 0 \\ 376.99 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0.21133 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

A model of order $n_m = 1$ is

$$A_m = (\,-0.199272\,), \quad B_m = (\,0.631300 \quad -0.00187918\,), \quad C_m = (\,-0.187347 \quad -354.430393\,).$$

This model yields the cost $J = 27632.2$. A model of order $n_m = 2$ is

$$A_m = \begin{pmatrix} -0.199608 & -0.0763006 \\ 3.331193 & -13.275827 \end{pmatrix},$$

$$B_m = \begin{pmatrix} 0.631832 & -0.00191612 \\ -5.151821 & -0.101952 \end{pmatrix}, \quad C_m = \begin{pmatrix} -0.201050 & 0.800899 \\ -354.414137 & -66.187284 \end{pmatrix}.$$

This model yields the cost $J = 23262.3$. A model of order $n_m = 3$ is

$$A_m = \begin{pmatrix} -0.198769 & 0.235666 & -0.02481363 \\ -1.087392 & -0.912444 & 9.201811 \\ -0.115288 & -9.502428 & -0.0261157 \end{pmatrix},$$

$$B_m = \begin{pmatrix} -0.630503 & 0.00216112 \\ -1.350879 & -0.00377142 \\ -0.222387 & -0.0526803 \end{pmatrix}, \quad C_m = \begin{pmatrix} 0.291338 & -0.0265117 & -4.035696 \\ 354.222032 & -164.479031 & 26.635498 \end{pmatrix}.$$

This model yields the cost $J = 0.673079$. A model of order $n_m = 4$ is

$$A_m = \begin{pmatrix} -0.198769 & 0.235667 & -0.0248136 & 0.000915746 \\ -1.087390 & -0.912440 & 9.201811 & -0.00904508 \\ -0.115288 & -9.502427 & -0.0261155 & 0.00159031 \\ -5.465132 & -11.698410 & -1.929974 & -37.554401 \end{pmatrix},$$

$$B_m = \begin{pmatrix} -0.630503 & 0.00216112 \\ -1.350876 & -0.00377141 \\ -0.222386 & -0.0526803 \\ -8.666510 & -0.0203036 \end{pmatrix}, \quad C_m^t = \begin{pmatrix} 0.291340 & 354.222032 \\ -0.0265302 & -164.479038 \\ -4.035692 & 26.635453 \\ 0.0861885 & -0.815898 \end{pmatrix}.$$

This model yields the cost $J = 3.22 \cdot 10^{-7}$.

For this example with $n_m = 3$, 4, the columns of the initial Jacobian matrices are so badly scaled that the numerical linear algebra in HOMPACK fails. Modifying HOMPACK to use the LINPACK subroutine DQRDC for the QR factorization of the initial Jacobian matrices enables HOMPACK to successfully overcome the ill conditioning and find a solution.

Table 1 gives the CPU times in seconds and the number of steps needed to obtain the results for each example. The CPU times are for a DEC station 5000/200, using double precision, IEEE arithmetic, and the MIPS RISC f77 compiler. Table 2 gives the comparison of the optimal projection equations approach and the input normal form formulation for Examples 8 and 9. The asterisks in Table 2 indicate cases that required special numerical linear algebra techniques to deal with severe scaling errors.

TABLE 1. ALGORITHM MEASURES FOR INPUT NORMAL FORM HOMOTOPY.

example	n_m	steps	time (sec)
1	1	5	0.06
2	1	21	0.13
3	1	19	0.10
4	1	12	0.14
4	2	7	0.20
5	1	10	0.12
5	2	10	0.22
7	1	11	0.22
7	2	8	0.30
7	3	6	0.46
8	1	10	0.20
8	2	18	0.50
8	3	10	0.65
9	1	11	0.71
9	2	123	8.0
9	3	6	1.3
9	4	6	1.9

TABLE 2. COMPARISON OF METHODS

<table>
<tr><td></td><td colspan="4">Example 8</td></tr>
<tr><td></td><td colspan="2">Optimal projection</td><td colspan="2">Input normal form</td></tr>
<tr><td>n_m</td><td># steps</td><td>time (sec)</td><td># steps</td><td>time (sec)</td></tr>
<tr><td>1</td><td>31</td><td>0.6</td><td>10</td><td>0.20</td></tr>
<tr><td>2</td><td>59</td><td>2.7</td><td>18</td><td>0.50</td></tr>
<tr><td>3</td><td>89</td><td>14</td><td>10</td><td>0.65</td></tr>
<tr><td></td><td colspan="4">Example 9</td></tr>
<tr><td>2</td><td>575</td><td>88</td><td>123</td><td>8.0</td></tr>
<tr><td>3</td><td>601</td><td>223</td><td>6*</td><td>1.3</td></tr>
<tr><td>4</td><td>671</td><td>518</td><td>6*</td><td>1.9</td></tr>
</table>

As shown by Table 1, the input normal form homotopies can be very efficient. Also there is no need to adjust any parameter to achieve this efficiency (although to obtain the minimum solution of Example 1, some adjustment of the initial point was necessary). However, note that the potential ill conditioning of the input normal form formulation can result in failure (Example 6) or the need for extraordinarily delicate linear algebra (Example 9).

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