Homotopy Perturbation Transform Method for Solving Third Order Korteweg-DeVries (KDV) Equation

Abdelilah Kamal Hassan Sedeeg[1, 2]

[1]Mathematics Department Faculty of Sciences and Arts, Almikwah-Albaha University, Albaha, Saudi Arabia

[2]Mathematics Department Faculty of Education, Holy Quran and Islamic Sciences University, Khartoum, Sudan

Email address:

aelilah63@hotmail.com

Abstract: In this paper, we develop a method to calculate approximate solution of some Third-order Korteweg-de Vries equations with initial condition with the help of a new method called Aboodh transform homotopy perturbation method (ETHPM). This method is a combination of the new integral transform "Aboodh transform" and the homotopy perturbation method. The nonlinear term can be easily handled by homotopy perturbation method. The results reveal that the combination of Aboodh transform and homotopy perturbation method is quite capable, practically well appropriate for use in such problems and can be applied to other nonlinear problems. This method is seen as a better alternative method to some existing techniques for such realistic problems.

Keywords: Aboodh Transform, Homotopy Perturbation Method, Korteweg-DeVries (KDV) Equation

1. Introduction

The exact solutions of nonlinear partial differential equations are difficult, therefore many various approximate methods have recently been developed such as homotopy perturbation method [1-6], Adomian's decomposition method, differential transform method and projected differential transform method to solve linear and nonlinear differential equations. The homotopy perturbation method has the merits of simplicity and easy execution. Homotopy theory becomes a powerful mathematical tool also ; homotopy theory can overcome the difficulties arising in calculation of Adomian's polynomials in Adomian's decomposition method. The KDV equation plays an important role in diverse areas of engineering and scientific applications, and therefore, the enormous amount of research work has been invested in the study of KDV equations [7–12]. The Aboodh transform [13-16] is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. Furthermore, the homotopy perturbation method is also combined with the well-known Aboodh transform method and the variational iteration method to produce a highly effective technique for handling many nonlinear problems.

In this paper, we shall deal with the KDV equation in the following form,

$$u_t + auu_x + bu_{xxx} = 0 \qquad (1)$$

Where $u\,(x, t)$ is the displacement. KDV equation Was first derived by Korteweg and Vries (1895) to the water waves in shallow canal, when the study of water waves was of vital interest for applications in naval architecture and for the knowledge of tides and floods. The purpose of this paper is to extend the (HPTM) for the solution of Korteweg-DeVries (KDV) Equation. The method has been successfully applied for obtaining exact solutions for nonlinear equations.

2. Aboodh Transform

Definition:

A new transform called the Aboodh transform defined for function of exponential order we consider functions in the set a, defined by:

$$A = \{f(t) \colon \exists\, M, k_1, k_2 > 0, |f\,(t)| < M e^{-vt}$$

For a given function in the set M must be finite

number, k_1, k_2 may be finite or infinite.Aboodh transform which is defined by the integral equation

$$A[f(t)] = K(v) = \frac{1}{v}\int_0^\infty f(t)e^{-vt}\,dt, t \geq 0, k_1 \leq v \leq k_2 \quad (2)$$

Aboodh transform of partial derivative:

To obtain Aboodh transform of partial derivative we use integration by parts, and then we have:

$$A\left[\frac{\partial u\,(x,t)}{\partial t}\right] = vK(x,v) - \frac{u\,(x,0)}{v},$$

$$A\left[\frac{\partial^2 u\,(x,t)}{\partial t^2}\right] = v^2 K(x,v) - \frac{1}{v}\frac{\partial u\,(x,0)}{\partial t} - u\,(x,0)$$

Proof: To obtain transforms of partial derivatives we use integration by parts as follows:

$$A\left[\frac{\partial u(x,t)}{\partial t}\right] = \frac{1}{v}\int_0^\infty \frac{\partial u(x,t)}{\partial t}e^{-vt}\,dt =$$

$$\lim_{p\to\infty}\frac{1}{v}\int_0^p \frac{\partial u(x,t)}{\partial t}e^{-vt}\,dt = \lim_{p\to\infty}\left\{\left[\frac{1}{v}u(x,t)e^{-vt}\right]_0^p + \frac{1}{v}\int_0^p u\,(x,t)e^{-vt}\,dt\right\} = vK(x,v) - \frac{u\,(x,0)}{v}(3)$$

We assume that f is piecewise continuous and it is of exponential order.

Let $\frac{\partial u\,(x,t)}{\partial t} = g$ then, by using Eq. (2) we have

$$A\left[\frac{\partial^2 u\,(x,t)}{\partial t^2}\right] = A\left[\frac{\partial g\,(x,t)}{\partial t}\right] = vA\big(g(x,t)\big) - \frac{g\,(x,0)}{v} = v^2 K(x,v) - \frac{1}{v}\frac{\partial u\,(x,0)}{\partial t} - u(x,0)(4)$$

We can easily extend this result to the *nth* partial derivative by using mathematical induction.

3. Homotopy Perturbation Method

Let X and Y be the topological spaces. If f and g are continuous maps of the space X into Y, it is said that f is homotopic to g, if thereis continuous map $F: X \times [0,1] \to Y$ such that $F\,(x,0) = f\,(x)$ and $F\,(x,1) = g\,(x)$, foreach $x \in X$, then the map is called homotopy between f and g.

To explain the homotopy perturbation method, we consider a general equation of the type,

$$L\,(U) = 0 \quad (5)$$

Where L is any differential operator, we define a convex homotopy $H\,(U,p)$ by

$$H(U,p) = (1-p)F(U) + pL\,(U) \quad (6)$$

Where $F\,(U)$ is a functional operator with known solution V_0 which can be obtained easily. It is clear that, for

$$H(U,p) = 0 \quad (7)$$

We have: $H(U,0) = F(U), H(U,1) = L\,(U)$.

In topology this show that $H\,(U,P)$ continuously traces an implicitly defined carves from a starting point $H\,(V_0,0)$ to a solution function $H\,(f,1)$. The HPM uses the embed ling

parameter p as a small parameter and write the solution as a power series

$$U = U_0 + pU_1 + p^2 U_2 + p^3 U_3 + \dots \quad (8)$$

If $p \to 1$, then Eq. (8) corresponds to Eq. (6) and becomes the approximate solution of the form,

$$f = \lim_{p\to 1} U = \sum_{i=0}^\infty U_i \quad (9)$$

We assume that Eq. (8) has a unique solution. The comparisons of like powers of p give solutions of various orders, for more details see [17-20].

3.1. Basic Idea

To illustrate the basic idea of this method, we consider a general form of nonlinear non homogeneous partial differential equation as the follow:

$$Du(x,t) + Ru(x,t) + Nu(x,t) = g(x,t) \quad (10)$$

with the following initial conditions

$$u(x,0) = h(x), u_t(x,0) = f\,(x) \quad (11)$$

Where D is the second order linear differential operator $D = \frac{\partial^2}{\partial t^2}$, is the linear differential operator ofless order than D, N represents the general non-linear differential operator and $g\,(x,t)$ is the source term.

Taking Aboodh transform (denoted throughout this paper by $A\,(.)$) on both sides of Eq. (10), to get:

$$A[Du(x,t)] + A[Ru(x,t)] + A[Nu(x,t)] = A[g\,(x,t)](12)$$

Using the differentiation property of Aboodh transform and above initial conditions, we have:

$$A[u(x,t)] = \frac{1}{v^2}A[g(x,t)] + \frac{1}{v^2}h(x) + \frac{1}{v^3}f(x) - \frac{1}{v^2}A\,[Ru\,(x,t) + Nu\,(x,t)](13)$$

Operating with the Aboodh inverse on both sides of Eq. (12) gives:

$$u\,(x,t) = G\,(x,t) - A^{-1}\left[\frac{1}{v^2}A\,[Ru\,(x,t) + Nu\,(x,t)]\right](14)$$

Where $G\,(x,t)$ represents the term arising from the source term and the prescribed initial condition. Now, we apply the homotopy perturbation method

$$u(x,t) = \sum_{n=0}^\infty p^n u_n\,(x,t) \quad (15)$$

And the nonlinear term can be decomposed as:

$$Nu\,(x,t) = \sum_{n=0}^\infty p^n H_n\,(u) \quad (16)$$

Where $H_n\,(u)$ are He's polynomial and given by:

$$H_n\,(u_0, u_1, u_2 \dots u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{n=0}^\infty p^i u_i\right)\right]_{p=0}, n = 0,1,2,\dots(17)$$

Substituting Eqs. (15) and (16) in Eq. (14) we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) - pA^{-1}\left[\frac{1}{v^2}A[R\sum_{n=0}^{\infty}p^n u_n(x,t) + \sum_{n=0}^{\infty}p^n H_n(u)]\right](18)$$

Which is the coupling of the Aboodh transform and the homotopy perturbation method using He's polynomials. Comparing the coefficient of same powers of p , thefollowingapproximations are obtained:

$$p^0 : u_0(x,t) = G(x,t),$$

$$p^1 : u_1(x,t) = -A^{-1}\left[\frac{1}{v^2}A[Ru_0(x,t) + H_0(u)]\right],$$

$$p^2 : u_2(x,t) = -A^{-1}\left[\frac{1}{v^2}A[Ru_1(x,t) + H_1(u)]\right],$$

$$p^3 : u_3(x,t) = -A^{-1}\left[\frac{1}{v^2}A[Ru_2(x,t) + H_2(u)]\right],$$

Then the solution is;

$$u(x,t) = \lim_{p\to 1} u_n(x,t) = u_0 + u_1 + u_2 + \cdots \quad (19)$$

3.2. Applications

In this section, the effectiveness and the usefulness of homotopy perturbation transform method (HPTM) are demonstrated by finding exact solutions of Korteweg-DeVries (KDV) Equation.

Example 3.1.

Consider the following linear homogeneous KDV Equation;

$$u_t - 6uu_x + u_{xxx} = 0 \quad (20)$$

With the initial condition;

$$u(x,0) = 6x \quad (21)$$

Applying the Aboodh transform of both sides of Eq. (20),

$$A[u_t] = -A[u_{xxx} - 6uu_x] \quad (22)$$

Using the differential property of Aboodh transform Eq. (22) can be written as:

$$vA[u(x,t)] - \frac{1}{v}u(x,0) = -A[u_{xxx} - 6uu_x] \quad (23)$$

Using initial condition Eq. (21), Eq. (23) can be written as:

$$A[u(x,t)] = \frac{6x}{v^2} - \left[\frac{1}{v}A[u_{xxx} - 6uu_x]\right] \quad (24)$$

The inverse Aboodh transform implies that:

$$u(x,t) = 6x - A^{-1}\left[\frac{1}{v}A[u_{xxx} - 6uu_x]\right] \quad (25)$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 6x - pA^{-1}\left[\frac{1}{v}A(\sum_{n=0}^{\infty}p^n u_n(x,t))_{xxx} - \sum_{n=0}^{\infty}p^n H_n(u)\right](26)$$

Where $H_n(u)$ are He's polynomials that represent the nonlinear terms.

The first few components of He's polynomials are given by

$$H_0(u) = u_0 u_{0x}$$

$$H_1(u) = u_0 u_{1x} + u_1 u_{0x}$$

$$H_2(u) = u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x}$$

Comparing the coefficient of same powers of p , the following approximations are obtained ;

$$p^0 : u_0(x,t) = 6x$$

$$p^1 : u_1(x,t) = -A^{-1}\left[\frac{1}{v}A[u_0(x,t)_{xxx} - 6H_0(u)]\right]$$

$$= -A^{-1}\left[\frac{1}{v}A[-6u_0 u_{0x}]\right]$$

$$= -A^{-1}\left[\frac{1}{v}A[-6(6x)(6)]\right]$$

$$= -A^{-1}\left[-\frac{6^3 x}{v^3}\right] = 6^3 xt$$

$$p^2 : u_2(x,t) = -A^{-1}\left[\frac{1}{v}A[u_1(x,t)_{xxx} - 6H_1(u)]\right]$$

$$= -A^{-1}\left[\frac{1}{v}A[-6(u_0 u_{1x} + u_1 u_{0x})]\right]$$

$$= -A^{-1}\left[\frac{1}{v}A[-6(6^4 xt + 6^4 xt)]\right]$$

$$= -A^{-1}\left[-2\frac{6^5 x}{v^4}\right] = 6^5 xt^2$$

$$p^3 : u_3(x,t) = -A^{-1}\left[\frac{1}{v}A[-6(u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x})]\right]$$

$$= 6^7 xt^3$$

Therefore the solution $u(x,t)$ is given by:

$$u(x,t) = 6x(1 + 36t + (36t)^2 + (36t)^3 + \cdots) \quad (27)$$

In series form,

$$u(x,t) = \frac{6x}{1-36t}, |36t| < 1 \quad (28)$$

Example 3.2.

Consider the following linear homogeneous KDV Equation;

$$u_t + uu_x + u_{xxx} = 0 \quad (29)$$

With the initial condition;

$$u(x,0) = 1 - x \quad (30)$$

Applying the Aboodh transform of both sides of Eq. (29),

$$A[u_t] = -A[u_{xxx} + uu_x] \quad (31)$$

Using the differential property of Aboodh transform Eq.(31) can be written as:

$$vA[u(x,t)] - \frac{1}{v}u(x,0) = -A[u_{xxx} + uu_x] \quad (32)$$

Using initial condition Eq. (30), Eq. (32) can be written as:

$$A[u(x,t)] = \frac{1-x}{v^2} - \left[\frac{1}{v}A[u_{xxx} + uu_x]\right] \quad (33)$$

The inverse Aboodh transform implies that:

$$u(x,t) = 1 - x - A^{-1}\left[\frac{1}{v}A[u_{xxx} + uu_x]\right] \quad (34)$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 1 - x - pA^{-1}\left[\frac{1}{v}A(\sum_{n=0}^{\infty} p^n u_n(x,t))_{xxx} + \sum_{n=0}^{\infty} p^n H_n(u)\right](35)$$

Comparing the coefficient of same powers of p, the following approximations are obtained:

$$p^0 : u_0(x,t) = 1 - x$$

$$p^1 : u_1(x,t) = -A^{-1}\left[\frac{1}{v}A[u_0(x,t)_{xxx} + H_0(u)]\right]$$

$$= -A^{-1}\left[\frac{1}{v}A[u_0 u_{0x}]\right]$$

$$= -A^{-1}\left[\frac{1}{v}A[x-1]\right] = -A^{-1}\left[\frac{(x-1)}{v^3}\right]$$

$$= (1-x)t$$

$$p^2 : u_2(x,t) = -A^{-1}\left[\frac{1}{v}A[u_1(x,t)_{xxx} + H_1(u)]\right]$$

$$= -A^{-1}\left[\frac{1}{v}A[(u_0 u_{1x} + u_1 u_{0x})]\right]$$

$$= -A^{-1}\left[\frac{1}{v}A[(x-1)t + (x-1)t)]\right]$$

$$= -A^{-1}\left[2\frac{(x-1)}{v^4}\right] = (1-x)t^2$$

$$p^3 : u_3(x,t) = -A^{-1}\left[\frac{1}{v}A[(u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x})]\right]$$

$$= -A^{-1}\left[\frac{1}{v}\left[6\frac{(x-1)}{v^5}\right]\right] = (1-x)t^3$$

Therefore the solution $u(x,t)$ is given by:

$$u(x,t) = (1-x)(1 + t + t^2 + t^3 + \cdots) \quad (36)$$

In series form,

$$u(x,t) = \frac{1-x}{1-t} \quad (37)$$

Example 3.3.
Consider the following linear homogeneous KDV Equation;

$$u_t - 6uu_x + u_{xxx} = 0 \quad (38)$$

With the initial condition,

$$u(x,0) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2} \quad (39)$$

Applying the Aboodh transform of both sides of Eq. (38),

$$A[u_t] = -A[u_{xxx} - 6uu_x] \quad (40)$$

Using the differential property of Aboodh transform Eq. (40) can be written as:

$$vA[u(x,t)] - \frac{1}{v}u(x,0) = -A[u_{xxx} - 6uu_x] \quad (41)$$

Using initial condition Eq. (39), Eq. (41) can be written as:

$$A[u(x,t)] = \frac{-2\frac{k^2 e^{kx}}{(1+e^{kx})^2}}{v^2} - \left[\frac{1}{v}A[u_{xxx} - 6uu_x]\right] \quad (42)$$

The inverse Aboodh transform implies that:

$$u(x,t) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2} - A^{-1}\left[\frac{1}{v}A[u_{xxx} - 6uu_x]\right] \quad (43)$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2} - $$
$$pA^{-1}\left[\frac{1}{v}A(\sum_{n=0}^{\infty} p^n u_n(x,t))_{xxx} - \sum_{n=0}^{\infty} p^n H_n(u)\right] \quad (44)$$

Comparing the coefficient of same powers of p, the following approximations are obtained:

$$p^0 : u_0(x,t) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2}$$

$$p^1 : u_1(x,t) = -2\frac{k^5 e^{kx}(e^{kx}-1)}{(1+e^{kx})^3}t$$

$$p^2 : u_2(x,t) = -2\frac{k^8 e^{kx}(e^{2kx}-4e^{kx}+1)}{(1+e^{kx})^4}t^2$$

Therefore the solution $u(x,t)$ is given by:

$$u(x,t) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2} - 2\frac{k^5 e^{kx}(e^{kx}-1)}{(1+e^{kx})^3}t - $$
$$2\frac{k^8 e^{kx}(e^{2kx}-4e^{kx}+1)}{(1+e^{kx})^4}t^2 - \quad (45)$$

In series form,

$$u(x,t) = -2\frac{k^2 e^{k(x-k^2 t)}}{(1+e^{k(x-k^2 t)})^2} \quad (46)$$

4. Conclusions

In this paper, we have applied the homotopy perturbation transform method to Korteweg-DeVries (KDV) Equation. It can be concluded that the HPTM is a very powerful and efficient technique in finding exact and approximates solutions for nonlinear problems. By using this method we

obtain a new efficient recurrent relation to solve (KDV) Equation.

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