HOMOTOPY PRINCIPLES FOR EQUIVARIANT ISOMORPHISMS

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Abstract. Let G be a reductive complex Lie group acting holomorphically on Stein manifolds X and Y. Let $p_X\colon X \to Q_X$ and $p_Y\colon Y \to Q_Y$ be the quotient mappings. When is there an equivariant biholomorphism of X and Y? A necessary condition is that the categorical quotients Q_X and Q_Y are biholomorphic and that the biholomorphism φ sends the Luna strata of Q_X isomorphically onto the corresponding Luna strata of Q_Y. Fix φ. We demonstrate two homotopy principles in this situation. The first result says that if there is a G-diffeomorphism $\Phi\colon X \to Y$, inducing φ, which is G-biholomorphic on the reduced fibres of the quotient mappings, then Φ is homotopic, through G-diffeomorphisms satisfying the same conditions, to a G-equivariant biholomorphism from X to Y. The second result roughly says that if we have a G-homeomorphism $\Phi\colon X \to Y$ which induces a continuous family of G-equivariant biholomorphisms of the fibres ${p_X}^{-1}(q)$ and ${p_Y}^{-1}(\varphi(q))$ for $q \in Q_X$ and if X satisfies an auxiliary property (which holds for most X), then Φ is homotopic, through G-homeomorphisms satisfying the same conditions, to a G-equivariant biholomorphism from X to Y. Our results improve upon those of [KLS15] and use new ideas and techniques.

Contents

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1. Introduction

Let G be a reductive complex Lie group. Let X and Y be Stein manifolds (always taken to be connected) on which G acts holomorphically. We have quotient mappings $p_X \colon X \to Q_X$ and $p_Y \colon Y \to Q_Y$ where Q_X and Q_Y are normal Stein spaces, the categorical quotients of X and Y. Let $q, q' \in Q_X$. We say that *q and q' are in the same Luna stratum of Q_X* if the fibres $X_q = {p_X}^{-1}(q)$ and $X_{q'} = {p_X}^{-1}(q')$ are G-biholomorphic. The fibres are affine G-varieties, not necessarily reduced. The Luna strata form a locally finite stratification of Q_X by locally closed smooth subvarieties. A necessary condition for X and Y to be G-equivariantly biholomorphic is that there is a biholomorphism $\varphi \colon Q_X \to Q_Y$ which preserves the Luna strata, i.e., X_q is G-biholomorphic to $Y_{\varphi(q)}$ for all $q \in Q_X$. Suppose that such a φ exists. Our problem then is to find a G-equivariant biholomorphism $\Phi \colon X \to Y$ inducing $\varphi \colon Q_X \to Q_Y$. It is possible that one has made a poor choice of φ (see Example 4.1) or it could be that no choice of φ admits a lift (see Example 4.2).

Use φ to identify the quotients, and call the common quotient Q with quotient maps $p_X \colon X \to Q$ and $p_Y \colon Y \to Q$. We say that *X and Y have common quotient Q*. More specifically, we replace Y by $\varphi^*Y = \{(q, y) \in Q_X \times Y \mid p_Y(y) = \varphi(q)\}$. Then φ^*Y is a Stein G-manifold whose quotient mapping is projection onto the first factor and $Q = Q_X$ is the common quotient. Our problem then is to find a G-equivariant biholomorphism $\Phi \colon X \to \varphi^*Y$ which induces id_Q, the identity map of Q. So we can always reduce to the case that X and Y have a common quotient Q and our problem is to lift id_Q to a G-biholomorphism of X and Y. In the spirit of Gromov's work [Gro89], we show that there is a G-biholomorphic lift of id_Q if there are appropriate continuous or smooth lifts of id_Q.

Set

$$\mathrm{Iso}(X, Y) = \prod_{q \in Q} \mathrm{Iso}(X_q, Y_q)$$

where $\mathrm{Iso}(X_q, Y_q)$ denotes the set of G-biholomorphisms of X_q and Y_q. Let π denote the natural projection of $\mathrm{Iso}(X, Y)$ to Q. Then $\mathrm{Iso}(X_q, Y_q)$ is a principal homogeneous space for the group $\mathrm{Iso}(X_q, X_q)$ and the global sections of $\mathrm{Iso}(X, Y)$ form a principal homogeneous space for the group of global sections of $\mathrm{Iso}(X, X)$. In general, there is no reasonable structure of complex variety on $\mathrm{Iso}(X, Y)$ or $\mathrm{Iso}(X, X)$ (see [KLS15, Section 3]). However, we can say what the sections of π of various kinds are. Clearly a holomorphic section of $\mathrm{Iso}(X, Y)$ over an open subset $U \subset Q$ should be a G-biholomorphism $\Phi \colon {p_X}^{-1}(U) \to {p_Y}^{-1}(U)$ inducing id_U. We are also able to define what a continuous section of $\mathrm{Iso}(X, Y)$ over U is, which we call a *strong G-homeomorphism* (see Section 3).

Let $\Phi \colon X \to Y$ be a G-diffeomorphism inducing id_Q. We say that Φ is *strict* if it induces a G-biholomorphism of $(X_q)_{\mathrm{red}}$ with $(Y_q)_{\mathrm{red}}$ for all $q \in Q$ where the subscript means that we are considering the reduced structures on the fibres (see Example 3.2). Let $\mathrm{Iso}(X, Y)_{\mathrm{red}}$ denote the product of the $\mathrm{Iso}((X_q)_{\mathrm{red}}, (Y_q)_{\mathrm{red}})$ with the obvious projection to Q. Then the smooth sections of $\mathrm{Iso}(X, Y)_{\mathrm{red}}$ are the strict G-diffeomorphisms. A strict G-diffeomorphism is not necessarily a strong G-homeomorphism (Example 3.2). Our definition of *strict* is more general than in [KLS15]; see Remark 5.9.

Here is our first main result.

Theorem 1.1. *Let X and Y be Stein G-manifolds with common quotient Q. Suppose that there is a strict G-diffeomorphism $\Phi\colon X \to Y$. Then Φ is homotopic, through strict G-diffeomorphisms, to a G-biholomorphism from X to Y.*

The theorem says that a smooth section of $\mathrm{Iso}(X,Y)_{\mathrm{red}}$ is homotopic to a holomorphic section.

There is also a version of the theorem for continuous sections of $\mathrm{Iso}(X,Y)$, but we need an additional assumption. Let D be a vector field on Q. We say that D is *strata preserving* if for all Luna strata Z of Q and $z \in Z$, $D(z) \in T_z(Z)$. We say that X has the *infinitesimal lifting property* if every holomorphic strata preserving vector field D defined on a neighbourhood U of $q \in Q$ has a lift to a G-invariant holomorphic vector field A on $p^{-1}(U')$ where U' is a neighbourhood of q contained in U. This means that $A(p^*f) = p^*(D(f))$ for all $f \in \mathscr{O}(U')$. The infinitesimal lifting property really only depends upon the isomorphism classes of the fibres of $p\colon X \to Q$; equivalently, on the slice representations of X (see Section 2). For most representations of reductive groups, the infinitesimal lifting property holds (Remark 2.1) and for most representations all holomorphic vector fields on the quotient automatically preserve the strata [Sch13].

Here is our second main result.

Theorem 1.2. *Let X and Y be Stein G-manifolds with common quotient Q. Suppose that there is a strong G-homeomorphism $\Phi\colon X \to Y$. If X has the infinitesimal lifting property, then Φ is homotopic, through strong G-homeomorphisms, to a G-biholomorphism from X to Y.*

See Section 3 for the definition of a homotopy of strong G-homeomorphisms. The theorem says that a continuous section of $\mathrm{Iso}(X,Y)$ is homotopic to a holomorphic section, provided that X (equivalently, Y) has the infinitesimal lifting property.

Our proofs of Theorems 1.1 and 1.2 have two steps, where we first reduce our homotopy principles to Oka principles of the form considered by Grauert. Let X, Y and Q be as before. We say that *X and Y are locally G-biholomorphic over Q* if there is an open cover $\{U_i\}$ of Q and G-biholomorphisms $\Phi_i\colon {p_X}^{-1}(U_i) \to {p_Y}^{-1}(U_i)$ inducing the identity on U_i. This condition says that there are no local obstructions to the existence of a global G-biholomorphism $\Phi\colon X \to Y$ inducing id_Q.

Theorem 1.3. *Let X and Y be Stein G-manifolds with common quotient Q. Suppose that one of the following holds.*

(1) *There is a strict G-diffeomorphism from X to Y.*
(2) *There is a strong G-homeomorphism from X to Y and X has the infinitesimal lifting property.*

Then X and Y are locally G-biholomorphic over Q.

Once we have no local obstructions we are able to establish the following versions of Grauert's Oka principle.

Theorem 1.4. *Let X and Y be Stein G-manifolds locally G-biholomorphic over a common quotient Q.*

(1) *Any strict G-diffeomorphism $\Phi\colon X \to Y$ is homotopic, through strict G-diffeomorphisms, to a G-biholomorphism from X to Y.*
(2) *Any strong G-homeomorphism $\Phi\colon X \to Y$ is homotopic, through strong G-homeomorphisms, to a G-biholomorphism from X to Y.*

Note that Theorems 1.3 and 1.4 establish Theorems 1.1 and 1.2. The proof of Theorem 1.4 is along the lines of Grauert's Oka principle for principal bundles of complex Lie groups (Section 10). A main result of our previous paper [KLS15] is a weaker version of Theorem 1.4. In (1) and (2) we were only able to state the existence of a G-biholomorphism, but not that it was homotopic to Φ. Also, we had to assume that X (equivalently Y) is generic, which means that the set of closed orbits with trivial isotropy group is open in X and that the complement (which is a closed G-stable subvariety of X) has codimension at least two.

We briefly mention here the Linearisation Problem. Suppose that $X = \mathbb{C}^n$ and that Y is a G-module such that we have a G-biholomorphism of X and Y. Then the G-action on $\mathbb{C}^n$ is linearisable, i.e., there is a biholomorphic automorphism Φ of $\mathbb{C}^n$ such that $\Phi \circ g \circ \Phi^{-1}$ is linear for every $g \in G$. The problem of linearising actions of reductive groups on $\mathbb{C}^n$ has attracted much attention both in the algebraic and holomorphic settings ([Huc90], [Kra96]). The first counterexamples for the algebraic linearisation problem were constructed by Schwarz [Sch89] for $n \geq 4$. His examples are holomorphically linearisable. Derksen and Kutzschebauch [DK98] showed that for G nontrivial, there is an $N_G \in \mathbb{N}$ such that there are nonlinearisable actions of G on $\mathbb{C}^n$ for all $n \geq N_G$. Their method was to construct actions whose stratified quotients cannot be isomorphic to the stratified quotient of a linear action. In [KLS], we show that, most of the time, a holomorphic G-action on $\mathbb{C}^n$ is linearisable if and only if the stratified quotient is isomorphic to the stratified quotient of a G-module.

Here is a brief summary of the contents of the paper. In Section 2 we review general results about quotients and the Luna stratification. In Section 3 we recall facts about G-finite functions and use them to define the notion of strong G-homeomorphism. Section 4 gives examples showing problems that can arise in finding local or global lifts of strata preserving biholomorphisms of quotients. In Section 5 we establish Theorem 1.3. Here we use two techniques: deforming an automorphism of Q to a liftable automorphism and lifting homotopies on the quotient by lifting associated vector fields. After establishing Theorem 1.3 we are able to assume that X and Y are locally G-biholomorphic over Q. In Section 6 we define a type of G-diffeomorphism from X to Y, those of type $\mathcal{F}$, which roughly are those G-diffeomorphisms inducing id_Q whose restriction to each fibre X_q has a biholomorphic G-equivariant extension to a neighbourhood of X_q. We also define the notion of a G-invariant vector field on X of type $\mathcal{LF}$. These are roughly the smooth G-invariant vector fields, annihilating the G-invariant holomorphic functions, whose restrictions to each fibre X_q extend in a neighbourhood of X_q to a G-invariant holomorphic vector field annihilating the G-invariant holomorphic functions. We establish important properties of the G-diffeomorphisms of type $\mathcal{F}$ (assuming the results of Section 7). In Section 7 we prove several technical results, among them the fact that the G-invariant vector fields of type $\mathcal{LF}$ are closed in the Fréchet space of all smooth G-invariant vector fields on X. In Section 8 we show that any strong G-homeomorphism from X to Y is homotopic, through strong G-homeomorphisms from X to Y, to one of type $\mathcal{F}$. The analogous result for strict G-diffeomorphisms follows similarly. In Sections 9 and 10 we modify the techniques of Cartan [Car58] to show that any G-diffeomorphism from X to Y of type $\mathcal{F}$ is homotopic, through G-diffeomorphisms of type $\mathcal{F}$, to a G-biholomorphism from X to Y, completing the proof of Theorem 1.4.

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2. Background

For details of what follows see [Lun73] and [Sno82, Section 6]. Let X be a normal Stein space with a holomorphic action of a reductive complex Lie group G. The categorical quotient $Q_X = X/\!\!/G$ of X by the action of G is the set of closed orbits in X with a reduced Stein structure that makes the quotient map $p_X\colon X \to Q_X$ the universal G-invariant holomorphic map from X to a Stein space. When X is understood, we drop the subscript X in p_X and Q_X. Since X is normal, Q is normal. If U is an open subset of Q, then p^* induces isomorphisms of $\mathbb{C}$-algebras $\mathscr{O}_X(p^{-1}(U))^G \simeq \mathscr{O}_Q(U)$ and $\mathcal{C}^0(p^{-1}(U))^G \simeq \mathcal{C}^0(U)$. We say that a subset of X is *G-saturated* if it is a union of fibres of p. If X is an affine G-variety, then Q is just the complex space corresponding to the affine algebraic variety with coordinate ring $\mathscr{O}_{\mathrm{alg}}(X)^G$. If V is a G-module and $p\colon V \to V/\!\!/G$ is the quotient mapping, then the fibre $\mathcal{N}(V) = p^{-1}(p(0))$ is the *null cone of V*.

If Gx is a closed orbit in X, then the stabiliser (or isotropy group) G_x is reductive. We say that closed orbits Gx and Gy have the same *isotropy type* if G_x is G-conjugate to G_y. Thus we get the *isotropy type stratification* of Q with strata whose labels are conjugacy classes of reductive subgroups of G.

Assume that X is smooth and connected, and let Gx be a closed orbit. Then we can consider the *slice representation* which is the action of G_x on $T_xX/T_x(Gx)$. We say that closed orbits Gx and Gy have the same *slice type* if they have the same isotropy type and, after arranging that $G_x = G_y$, the slice representations are isomorphic representations of G_x. The stratification by slice type is finer than that by isotropy type, but the slice type strata are unions of irreducible components of the isotropy type strata [Sch80, Proposition 1.2]. Hence if the isotropy type strata are irreducible, the slice type strata and isotropy type strata are the same. This occurs for the case of a G-module [Sch80, Lemma 5.5]. Let $H = G_x$ and $W = T_xX/T_x(Gx)$ be the slice representation as above. Write $W = W^H \oplus W'$ where W' is an H-module. The Zariski tangent space to the fibre $X_{p(x)}$ at x is isomorphic to $\mathfrak{g}/\mathfrak{h} \oplus W'$ as H-module, and

$$\dim W^H = \dim X - \dim G + \dim H - \dim W',$$

so that the fibre determines the slice representation (and vice versa). Hence the Luna stratification of the introduction is the same as the slice type stratification.

There is a unique open stratum $Q_{\mathrm{pr}} \subset Q$, corresponding to the closed orbits with minimal stabiliser. We call this the *principal stratum* and the closed orbits above Q_{pr} are called *principal orbits*. The isotropy groups of principal orbits are called *principal isotropy groups*. By definition, X is generic when the principal isotropy groups are trivial and $p^{-1}(Q \setminus Q_{\mathrm{pr}})$ has codimension at least two in X.

Remark 2.1. If G is simple, then, up to isomorphism, all but finitely many G-modules V with $V^G = 0$ are generic and have the infinitesimal lifting property [Sch95, Corollary 11.6 (1)]. The same result holds for semisimple groups but one needs to assume that every irreducible component of V is a faithful module for the Lie algebra of G [Sch95, Corollary 11.6 (2)]. A "random" $\mathbb{C}^*$-module is generic and has the infinitesimal lifting property, although infinite families of counterexamples exist. More precisely, a faithful n-dimensional $\mathbb{C}^*$-module without zero weights is generic (and has the infinitesimal lifting property) if and only if it has at least two positive weights and at least two negative weights and no $n-1$ weights have a common prime divisor. Finally, X is generic (or has the infinitesimal lifting property) if and only if every slice representation does. Hence these properties only depend upon the Luna stratification of Q. If one is in the situation where all slice representations are orthogonal, then [Sch80, Theorems 3.7 and 6.7] shows that one has the infinitesimal lifting property.

3. G-finite functions and strong homeomorphisms

Let X and Y be Stein G-manifolds. Despite the fact that we can state our main theorems in the case that $\varphi\colon Q_X \to Q_Y$ is the identity, our proofs (especially in Section 5) require us to consider the case that φ is an arbitrary strata preserving biholomorphism.

If U is a subset of Q_X, we denote ${p_X}^{-1}(U)$ by X_U, and Y_U for $U \subset Q_Y$ is defined analogously. The group G acts on $\mathscr{O}(X)$, $f \mapsto g \cdot f$, where $(g \cdot f)(x) = f(g^{-1}x)$, $x \in X$, $g \in G$, $f \in \mathscr{O}(X)$. Let $\mathscr{O}_{\rm fin}(X)$ denote the holomorphic functions f such that the span of $\{g \cdot f \mid g \in G\}$ is finite dimensional. They are called the *G-finite holomorphic functions on X* and obviously form an $\mathscr{O}(Q) = \mathscr{O}(X)^G$-algebra. If X is a smooth affine G-variety, then the techniques of [Sch80, Proposition 6.8, Corollary 6.9] show that for $U \subset Q$ open and Stein we have

$$\mathscr{O}_{\rm fin}(X_U) \simeq \mathscr{O}(U) \otimes_{\mathscr{O}_{\rm alg}(Q)} \mathscr{O}_{\rm alg}(X).$$

Let H be a reductive subgroup of G and let B be an H-saturated neighbourhood of the origin of an H-module W. We always assume that B is Stein, in which case $B/\!\!/H$ is also Stein. Let $G \times^H B$ (or T_B) denote the quotient of $G \times B$ by the (free) H-action sending (g, w) to (gh^{-1}, hw) for $h \in H$, $g \in G$ and $w \in B$. We denote the image of (g, w) in $G \times^H B$ by $[g, w]$. By the slice theorem, X is locally G-biholomorphic to such tubes T_B. If V is an irreducible nontrivial G-module, let $\mathscr{O}(X)_V$ denote the elements of $\mathscr{O}_{\rm fin}(X)$ contained in a copy of V, and similarly define $\mathscr{O}_{\rm alg}(T_W)_V$. Then $\mathscr{O}_{\rm alg}(T_W)_V$ generates $\mathscr{O}(T_B)_V$ over $\mathscr{O}(B)^H$. By Nakayama's Lemma, $f_1, \dots, f_m \in \mathscr{O}(X)_V$ restrict to minimal generators of the $\mathscr{O}(U)$-module $\mathscr{O}(X_U)_V$ for some neighbourhood U of $q \in Q$ if and only if the restrictions of the f_i to X_q form a basis of $\mathscr{O}(X_q)_V = \mathscr{O}_{\rm alg}(X_q)_V$.

Let $\Phi\colon X \to Y$ be a G-biholomorphism inducing $\varphi\colon Q_X \to Q_Y$. Let $q \in Q_X$, let $f_1, \dots, f_m$ be elements of $\mathscr{O}(X)_V$ whose restrictions to $\mathscr{O}(X_q)$ span $\mathscr{O}(X_q)_V$ and let $f_1', \dots, f_n'$ be elements of $\mathscr{O}(Y)_V$ whose restrictions to $\mathscr{O}(Y_{\varphi(q)})$ span $\mathscr{O}(Y_{\varphi(q)})_V$. Then the f_i generate $\mathscr{O}(X)_V$ over a neighbourhood of q and the f_j' generate $\mathscr{O}(Y)_V$ over a neighbourhood of $\varphi(q)$. Over a neighbourhood U of q we have $\Phi^* f_i' = f_i' \circ \Phi = \sum a_{ij} f_j$ where the a_{ij} are in $\mathscr{O}(U) \simeq \mathscr{O}(X_U)^G$. The a_{ij} are generally not unique. However, if the f_i and f_j' are linearly independent when restricted to X_q and $Y_{\varphi(q)}$, respectively, then $m = n$ and the matrix $(a_{ij}(q))$ is unique and invertible. It follows that $(\Phi^{-1})^* f_i =$

$\sum b_{ij} f'_j$ over a neighbourhood of $\varphi(q)$ where the matrix valued function (b_{ij}) equals $(a_{ij} \circ \varphi^{-1})^{-1}$.

Let $\Phi\colon X \to Y$ be a G-equivariant homeomorphism inducing a strata preserving biholomorphism $\varphi\colon Q_X \to Q_Y$. Let V and the f_i, f'_j and q be as above. We say that Φ is *strong for V over φ at q* if $\Phi^* f'_i = \sum a_{ij} f_j$ where the a_{ij} are continuous in a neighbourhood of q, inducing an isomorphism $\mathscr{O}(Y_{\varphi(q)})_V \to \mathscr{O}(X_q)_V$. Note that this condition is independent of our choice of the f_i and f'_j. We say that Φ *is strong over φ at q* if Φ is strong over φ for V at q for all irreducible nontrivial V. One does not actually need to worry about all V. Let $V_1, \dots, V_r$ be irreducible nontrivial G-modules such that $\mathscr{O}_{\mathrm{alg}}(X_q)$ is generated by $\bigoplus_j \mathscr{O}_{\mathrm{alg}}(X_q)_{V_j}$. If Φ is strong over φ for the V_j at q, then it is strong over φ at q. Note that if Φ is strong over φ at q, then it is strong over φ at q' for q' sufficiently close to q and that Φ^{-1} is strong over φ^{-1} at $\varphi(q)$. Finally, we say that Φ is a *strong G-homeomorphism over φ* if it is strong over φ at all $q \in Q_X$. When we omit the phrase "over φ" we mean that $Q_X = Q_Y = Q$ and $\varphi = \mathrm{id}_Q$ as in the introduction. The strong G-homeomorphisms of X form a group under composition. As we saw above, G-biholomorphisms inducing φ are strong.

If $\Phi\colon X \to Y$ is a strong G-homeomorphism, then the various $a_{ij}(q)$ determine G-isomorphisms of the (not necessarily reduced) algebraic G-varieties X_q and Y_q, so one may consider Φ as a continuous family of isomorphisms of the fibres X_q and Y_q, $q \in Q$. If Φ_t is a family of strong G-homeomorphisms for $t \in [0,1]$, we say that the family is a *homotopy of strong G-homeomorphisms* if the corresponding $a_{ij}(t,q)$ can be chosen to be continuous in t and q.

We consider strong G-homeomorphisms to be the continuous analogues of G-biholomorphisms of X and Y. This is especially evident in the case that the automorphism group scheme associated to X exists. For $U \subset Q$ open, let $\mathrm{Aut}_U(X_U)^G$ denote the group of G-biholomorphisms of X_U inducing id_U. Suppose that there is a complex space $\mathfrak{G}$ over Q whose fibres are complex Lie groups such that we have a canonical identification of the holomorphic sections $\Gamma(U, \mathfrak{G})$ with $\mathrm{Aut}_U(X_U)^G$ for all U open in Q. Then we say that $\mathfrak{G}$ *is the group scheme associated to the Stein G-manifold X*. Most of the time $\mathfrak{G}$ does not exist (see [KLS15, Section 3]). However, it does exist in case $p\colon X \to Q$ is flat (compare [KS92, Chapter III, Section 2]).

Example 3.1. This example is from Kraft-Schwarz [KS92, Chapter III, Section 2]. Let $V = \mathbb{C}^2$ with coordinate functions x and y, and let G be the normaliser of the diagonal matrices in $\mathrm{SL}_2(\mathbb{C})$. Let $s = xy$ and $t = s^2$. Then $\mathscr{O}_{\mathrm{alg}}(Q) = \mathbb{C}[t]$ (so $Q = \mathbb{C}$), and $\mathscr{O}_{\mathrm{alg}}(V)$ is generated by $\mathscr{O}_{\mathrm{alg}}(V)_{V^*}$ which has minimal generators x, y, sx and sy. An element $\Phi \in \mathrm{Aut}_Q(V)^G$ sends x to $\alpha x + \beta s x$ for some α, $\beta \in \mathscr{O}(Q)$ and it sends y to $\alpha y - \beta s y$. Then $\Phi^*(s) = (\alpha^2 - \beta^2 t)s$ where $\Phi^*(s^2) = s^2$. Hence $\alpha^2 - \beta^2 t = \pm 1$. Conversely, given α, $\beta \in \mathscr{O}(Q)$ such that $\alpha^2 - \beta^2 t = \pm 1$, there is a corresponding $\Phi \in \mathrm{Aut}_Q(V)^G$ with $\Phi^* x = (\alpha + s\beta)x$ and $\Phi^* y = (\alpha - s\beta)y$. We can see our automorphisms as sections of a group scheme $\mathfrak{G}$, as follows. As variety,

$$\mathfrak{G} = \{(t, a, b) \in \mathbb{C}^3 \mid a^2 - b^2 t = \pm 1\},$$

with projection $\pi\colon \mathfrak{G} \to \mathbb{C}$ sending (t, a, b) to t. The group structure on $\mathrm{Aut}_Q(V)^G$ induces a group structure on the fibres of π:

$$(t, a', b') \cdot (t, a, b) = (t, aa' + \epsilon bb' t, a'b + \epsilon ab') \text{ where } \epsilon = a^2 - b^2 t.$$

The inverse to (t, a, b) is $(t, \epsilon a, -b)$. The group $\mathrm{Aut}_Q(V)^G$ is isomorphic to the group of holomorphic sections of $\pi\colon \mathfrak{G} \to \mathbb{C}$. The strong G-homeomorphisms of V are the same thing as the continuous sections of π.

Example 3.2. Let (V, G) be as in the previous example. Choose the branch of the square root in a neighbourhood of $1 \in \mathbb{C}$ with $\sqrt{1} = 1$ and let $a(t) = \sqrt{1 + \bar{t}}$, $t \in \mathbb{C}$. Then $a(t)$ is smooth in a neighbourhood of $0 \in \mathbb{C}$. Let Φ be the G-diffeomorphism which sends x to $(a(t) + \bar{s})x$ and y to $(a(t) - \bar{s})y$ for t near 0. Then the corresponding $b(t)$ is $\bar{t}/|t| = \bar{s}/s$ which has no limit at $t = 0$. The fibre of V over 0 is not reduced, and the reduced fibre is $\{xy = 0\}$ on which the restriction of Φ is the identity. Thus Φ is a strict G-diffeomorphism, but it is not a strong G-homeomorphism.

We now establish some differentiability properties of strong G-homeomorphisms inducing a strata preserving biholomorphism φ. Our results are a generalisation of [KLS15, Lemma 24] (see Remark 3.6 below).

Let H, W, B, etc. be as in the beginning of this section. We always assume that B is stable under multiplication by $t \in [0, 1]$. Let $f \in \mathscr{O}_{\mathrm{alg}}(T_W)$. We say that f has degree n if $f([g, tw]) = t^n f([g, w])$, $g \in G$, $w \in W$, $t \in \mathbb{C}$. We denote the elements of degree n by $\mathscr{O}_{\mathrm{alg}}(T_W)_n$. The elements of degree zero are the pullbacks to $\mathscr{O}_{\mathrm{alg}}(T_W)$ of the elements of $\mathscr{O}_{\mathrm{alg}}(Z)$ where $Z \simeq G/H$ is the zero section of T_W. For the moment, we also assume that $W^H = 0$.

By [KLS15, Lemma 23] we have

Lemma 3.3. *$\mathscr{O}_{\mathrm{alg}}(T_W)$ is generated by $\mathscr{O}_{\mathrm{alg}}(T_W)_1$ as an $\mathscr{O}_{\mathrm{alg}}(G/H)$-algebra.*

Let $V_1, \dots, V_r$ be nonisomorphic irreducible nontrivial G-modules which appear in $\mathscr{O}_{\mathrm{alg}}(T_W)_0$ and $\mathscr{O}_{\mathrm{alg}}(T_W)_1$ such that $\mathscr{O}_{\mathrm{alg}}(T_W)$ is generated by

$$M = \mathscr{O}_{\mathrm{alg}}(T_W)_{V_1} \oplus \cdots \oplus \mathscr{O}_{\mathrm{alg}}(T_W)_{V_r}.$$

We may assume that $V_1, \dots, V_r$ are minimal with the above property. Let $f_1, \dots, f_n$ minimally generate M where each f_i is homogeneous and in $\mathscr{O}_{\mathrm{alg}}(T_W)_{V_j}$ for some j.

Definition 3.4. Let $V_1, \dots, V_r$ and $f_1, \dots, f_n$ be as above. We call $\{f_i\}$ a *standard set of generators of $\mathscr{O}_{\mathrm{fin}}(T_B)$* or a *standard set of generators of $\mathscr{O}_{\mathrm{alg}}(T_W)$*.

The span of the f_i is G-stable and the f_i are linearly independent on $F = G \times^H \mathcal{N}(W)$. Let d_i denote the degree of f_i, $i = 1, \dots, n$. We arrange that $d_i = 0$ for $i \leq \ell$, $d_i = 1$ for $\ell < i \leq m$ and $d_i > 1$ for $i > m$.

Let X denote T_B so that $Q = T_B /\!\!/ G \simeq B /\!\!/ H$. Let $q_0 \in Q$ be the image of the point $x_0 = [e, 0] \in X$. Let U be a neighbourhood of q_0 and let $\varphi\colon U \to \varphi(U) \subset Q$ be a strata preserving biholomorphism. Then $\varphi(q_0) = q_0$. Let $\Phi\colon X_U \to X_{\varphi(U)} \subset X$ be a strong G-homeomorphism over φ.

Shrinking U we may assume that there is an $(n \times n)$-matrix (a_{ij}) of continuous invariant functions such that $\Phi^* f_i = \sum a_{ij} f_j$. Since Φ preserves F, it preserves the closed orbit $Z \subset F$. Let $\mathcal{I}$ denote the ideal in $\mathcal{C}^0(X_U)^G$ generated by $\bigoplus_{k \geq 2} \mathscr{O}_{\mathrm{alg}}(T_W)_k^G$. Note that since $W^H = 0$, $\mathscr{O}_{\mathrm{alg}}(T_W)_1^G = 0$.

Lemma 3.5. (1) *The matrix $(a_{ij}(x_0))$ is invertible, $1 \leq i, j \leq \ell$.*

(2) *Perhaps shrinking U, we have $a_{ij} \in \mathcal{I}$ for $\ell < i \leq m$ and $1 \leq j \leq \ell$.*

Proof. If we restrict $f_1, \dots, f_\ell$ and the $\Phi^* f_i$ to Z, then Φ^* is an isomorphism, with matrix $(a_{ij}(x_0))$, and we have (1).

Let ρ denote the Reynolds operator, i.e., the G-equivariant projection from $\mathscr{O}_{\mathrm{alg}}(T_W)$ to $\mathscr{O}_{\mathrm{alg}}(T_W)^G$. It extends to a projection of $\mathcal{C}^0(X_U)^G \cdot \mathscr{O}_{\mathrm{alg}}(T_W)$ to $\mathcal{C}^0(X_U)^G$. Let us assume that some $a_{ij} \neq 0$ for $\ell < i \le m$ and $1 \le j \le \ell$, say $a_{m1} \neq 0$. Then f_m and f_1 correspond to the same irreducible G-module V. We may assume that $f_1, \dots, f_s$ are the polynomials corresponding to V among $f_1, \dots, f_\ell$. Then $a_{mj} = 0$ for $s < j \le \ell$. Now V occurs in $\mathscr{O}_{\mathrm{alg}}(Z) \simeq \mathscr{O}_{\mathrm{alg}}(T_W)_0$ and we have a non-degenerate pairing of $\mathscr{O}_{\mathrm{alg}}(Z)$ with itself, which sends a pair of functions f, f' to $\rho(f \cdot f')$. Thus $\mathscr{O}_{\mathrm{alg}}(Z)_{V^*}$ has a basis $f_1', \dots, f_s'$ of cardinality s. We may assume that $\rho(f_i' \cdot f_j) = \delta_{ij}$ for $1 \le i, j \le s$. There is a homogeneous minimal generating set $f_1', \dots, f_s', f_{s+1}', \dots, f_{s+t}'$ of the $\mathscr{O}_{\mathrm{alg}}(T_W)^G$-module $\mathscr{O}_{\mathrm{alg}}(T_W)_{V^*}$. Then $\deg f_i' > 0$ for $i > s$. There are $b_{ij} \in \mathcal{C}^0(X_U)^G$ such that $\Phi^* f_i' = \sum b_{ij} f_j'$. Let $1 \le i \le s$. Since f_m has degree 1, $\rho(f_i' \cdot f_m) = 0$. Thus

$$0 = \rho(\Phi^* f_i' \cdot \Phi^* f_m) = \sum_{j,\,k} b_{ij} a_{mk} \rho(f_j' \cdot f_k) \in \sum_{j,\,k=1}^{s} b_{ij} a_{mk} \rho(f_j' \cdot f_k) + \mathcal{I} = \sum_{j=1}^{s} b_{ij} a_{mj} + \mathcal{I}$$

Applying (1) to the matrix b_{ij}, $1 \le i, j \le s$, we see that $(b_{ij}(x_0))$ is invertible. In a G-saturated neighbourhood of x_0 we can invert (b_{ij}) and then the equation above shows that $a_{mj} \in \mathcal{I}$, $j = 1, \dots, s$. This gives (2). □

Remark 3.6. In the proof of [KLS15, Lemma 24] we tacitly assumed that Φ^* sends the f_i of degree 1 to $\sum_j a_{ij} f_j$ where $j > \ell$, i.e., to terms only involving the f_j of degree at least 1. This, of course, is not justified. The lemma above is what is required. Lemma 3.7 and Corollary 3.11 below replace [KLS15, Lemma 24].

We may assume that U is the image of a tube $T_{B'}$. Then we have a scalar action of $[0,1]$ on X_U as follows. Let $x = [g, w] \in X_U$. Then $t \cdot [g, w] = [g, tw]$. Let $\Phi_t(x) = t^{-1} \cdot \Phi(t \cdot x)$, $x \in X_U$, $t \in (0,1]$. Let W_g denote the fibre of T_W at $[g, 0] \in Z$. Let $a_{ij}^t(x)$ denote $t^{d_j - d_i} a_{ij}(t \cdot x)$, $t \in (0,1]$, $x \in X_U$, $1 \le i, j \le n$. Finally, let $\mathrm{Aut}_{\mathrm{vb}}(T_W)^G$ denote the complex algebraic group of G-vector bundle automorphisms of T_W (see Corollary 6.31).

Lemma 3.7. *The following hold.*

1. $(\Phi_t^* f_i)(x) = \sum_j a_{ij}^t(x) f_j(x)$, $x \in X_U$, $t \in (0,1]$, $1 \le i \le n$.
2. *The limit as $t \to 0$ of Φ_t acting on the f_i, $\ell < i \le m$, is given by the matrix L with entries $a_{ij}(x_0)$, $\ell < i, j \le m$.*
3. *Φ has a normal derivative $\delta\Phi$ along Z, and $\delta\Phi \in \mathrm{Aut}_{\mathrm{vb}}(T_W)^G$. If φ is the identity, then $\delta\Phi$ fixes $\mathscr{O}_{\mathrm{alg}}(T_W)^G$.*
4. *The limit as $t \to 0$ of the Φ_t is $\delta\Phi$, uniformly on compact subsets of X_U.*

Proof. For (1) we compute that

$$(\Phi_t^* f_i)(x) = f_i(t^{-1}\Phi(t \cdot x)) = t^{-d_i}(\Phi^* f_i)(t \cdot x) = t^{-d_i} \sum_j a_{ij}(t \cdot x) f_j(t \cdot x)$$
$$= \sum_j t^{d_j - d_i} a_{ij}(t \cdot x) f_j(x).$$

Now let $\ell < i \leq m$, so that $d_i = 1$. By Lemma 3.5, for $1 \leq j \leq \ell$, $a_{ij}(t \cdot x) = t^2 d_{ij}(t, x)$ where d_{ij} is continuous. Hence, uniformly on compact subsets of X_U,

$$\lim_{t\to 0} \Phi_t^* f_i(x) = \lim_{t\to 0} \sum_{j>\ell} t^{d_j-1} a_{ij}(t \cdot x) f_j(x) = \sum_{\ell<j\leq m} a_{ij}(x_0) f_j(x), \tag{3.1}$$

giving (2). For now, just consider indices i between $\ell+1$ and m. We have $\Phi(x_0) = [g, 0]$ for some $g \in N_G(H)$, and $\Phi_t(x_0) = [g, 0]$ for all $t \in (0, 1]$. The f_i are sections of T_{W^*} and they span each fibre of T_{W^*}. If $f = \sum c_i f_i$, $c_i \in \mathbb{C}$, vanishes at $[g, 0]$, then $f \circ \Phi_t(x_0) = 0$ for all $t \in (0, 1]$. By (3.1), $\sum c_i a_{ij}(x_0) f_j(x_0) = 0$. Thus the matrix L induces a linear mapping from W_g^* to W_e^*. Let $\delta\Phi(x_0)\colon W_e \to W_g$ be the dual mapping. It follows easily from (3.1) that $\delta\Phi(x_0)$ is the normal derivative of Φ at x_0, and the analogous fact at other points of Z follows by equivariance of Φ. Clearly, $\delta\Phi \in \mathrm{Aut}_{\mathrm{vb}}(T_W)^G$. If φ is the identity, then the Φ_t and $\delta\Phi$ fix the invariants. Hence we have established (3).

We have shown that a^t_{ij} converges uniformly as $t \to 0$ on compact subsets of X_U for $\ell < i \leq m$, and this is trivially true for $1 \leq i \leq \ell$. Since the f_i for $i \leq m$ generate $\mathscr{O}_{\mathrm{alg}}(T_W)$, a subset of them gives local coordinates at any point of T_W. Hence Φ_t converges to $\delta\Phi$ uniformly on compact subsets of X_U as $t \to 0$ and we have (4). □

It is not clear that $\lim_{t\to 0} a^t_{ij}(x)$ exists for all i and j. To remedy this we replace (a_{ij}) by a matrix (b_{ij}) for which $\lim_{t\to 0} b^t_{ij}(x)$ does exist.

Lemma 3.8. *Let $m < u \leq n$. For $1 \leq v \leq n$ there are polynomials*

$$P_{uv}(x, z_{ij}) \in \mathscr{O}(T_W)^G[z_{11}, \dots z_{1n}, \dots, z_{m1}, \dots, z_{mn}]$$

such that

$$\Phi^* f_u = \sum_v P_{uv}(x, a_{ij}) f_v$$

for any strong G-homeomorphism Φ with associated matrix (a_{ij}).

Proof. By Lemma 3.3, for $m < u \leq n$, $f_u = P_u(f_1, \dots, f_m)$, where $P_u(z_1, \dots, z_m)$ is a polynomial with constant coefficients. We may assume that each monomial in $P_u(z_1, \dots, z_m)$ is homogeneous of degree d_u where we give z_i degree 0 for $i \leq \ell$ and degree 1 for $i > \ell$. Now $\Phi^* f_u = P_u(\Phi^* f_1, \dots, \Phi^* f_m)$. Replace $\Phi^* f_i$ by $\sum a_{ij} f_j$, $1 \leq i \leq m$. Expanding and re-expressing in terms of invariants times the f_v we obtain

$$\Phi^* f_u = \sum_v P_{uv}(x, a_{11}, \dots, a_{1n}, \dots, a_{m1}, \dots, a_{mn}) f_v$$

where P_{uv} is independent of the a_{ij} (which we may treat as indeterminants). □

Remark 3.9. By equivariance of Φ, some of the a_{ij} are necessarily zero.

We break up $P_{uv}(x, a_{11}, \dots, a_{mn})$ into a sum of terms $P_{uv\alpha\beta}(x, a_{11}, \dots, a_{mn})$ as follows. Let $\tau = h(x) a_{11}^{\gamma_{11}} \cdots a_{mn}^{\gamma_{mn}}$ be a term in $P_{uv}(x, a_{ij})$. Let $\alpha_0(\tau)$ be the sum of the γ_{ij} for which $d_j - d_i = -1$ and let $\beta_0(\tau)$ be the sum of the $\gamma_{ij}(d_j - d_i)$ for which $d_j - d_i > 0$. Then $h(x)$ is homogeneous of degree $d_u - d_v + \beta_0(\tau) - \alpha_0(\tau)$. For α, $\beta \in \mathbb{Z}^+$ let $P_{uv\alpha\beta}$ be the sum of the terms τ in P_{uv} for which $\alpha_0(\tau) = \alpha$ and $\beta_0(\tau) = \beta$. Then $P_{uv\alpha\beta} \in \mathscr{O}_{\mathrm{alg}}(T_W)^G_{d_u-d_v+\beta-\alpha}[a_{11}, \dots, a_{mn}]$. From Lemma 3.5 we know that $a_{ij}(t \cdot x) = t^2 \tilde{a}_{ij}(t, x)$, $\ell < i \leq m$, $1 \leq j \leq \ell$, where $\tilde{a}_{ij}$ is continuous and invariant. Let $\tilde{a}_{ij}(t, x) = a_{ij}(t \cdot x)$ for $1 \leq i \leq \ell$ or $\ell < i \leq m$ and $\ell < j \leq n$. One easily shows:

Lemma 3.10. *Let* $\alpha,\ \beta \in \mathbb{Z}^+$. *Then*

$$t^{d_v-d_u}P_{uv\alpha\beta}(t\cdot x, a_{ij}(t\cdot x)) = t^{\alpha+\beta}P_{uv\alpha\beta}(x,\tilde{a}_{ij}(t,x))$$

is jointly continuous in $t\in[0,1]$ *and* $x\in X_U$. *Moreover, uniformly on compact subsets of* X_U *we have*

$$\lim_{t\to 0} t^{d_v-d_u}P_{uv\alpha\beta}(t\cdot x, a_{ij}(t\cdot x)) = \begin{cases} 0 & \alpha+\beta>0 \\ P_{uv00}(x,\delta_{d_id_j}a_{ij}(x_0)) & \alpha+\beta=0\end{cases}$$

where $P_{uv00}(x,\delta_{d_id_j}a_{ij}(x_0))\in\mathscr{O}_{\mathrm{alg}}(T_W)^G_{d_u-d_v}$.

Corollary 3.11. *Let* $b_{ij}(x)$ *denote* $P_{ij}(x,a_{11}(x),\dots,a_{mn}(x))$ *for* $i>m$, $1\le j\le n$ *and let* $b_{ij}(x)=a_{ij}(x)$ *for* $i\le m$, $1\le j\le n$. *Then* $\Phi^*f_i=\sum b_{ij}f_j$, $1\le i\le n$. *The* $b^t_{ij}(x)$ *are jointly continuous in* t *and* x *and* $(\delta\Phi)^*f_i=\sum b^0_{ij}(x)f_j$ *where* $b^0_{ij}(x)\in\mathscr{O}_{\mathrm{alg}}(T_W)^G_{d_i-d_j}$, $1\le i,\ j\le n$. *Hence* Φ_t *is a homotopy of strong* G*-homeomorphisms where* Φ_0 *is* G*-biholomorphic.*

We have been considering a special case, where the slice representation of the closed orbit Gx_0 has no nonzero fixed vectors. Now we consider the general case. So the slice representation is of the form $(\mathbb{C}^d\oplus W,H)$ where H acts trivially on $\mathbb{C}^d$ and $W^H=0$. Then the slice theorem says that a neighbourhood of our closed orbit is G-biholomorphic to $S\times T_B$ where T_B is a tube as before and S is an open subset of $\mathbb{C}^d$ which we may take to be Stein and connected.

Definition 3.12. Let S, W, H and B be as above. We call $S\times T_B$ a *standard neighbourhood in* X or a *standard open set in* X.

Note that our standard generators $\{f_i\}$ of $\mathscr{O}_{\mathrm{alg}}(T_W)$, considered as functions on $S\times T_B$, generate $\mathscr{O}_{\mathrm{fin}}(S\times T_B)$ as an $\mathscr{O}(S\times T_B)^G$-algebra. Thus we also call $\{f_i\}$ a standard generating set for $\mathscr{O}_{\mathrm{fin}}(S\times T_B)$. Now let $s_0\in S$ and consider a G-saturated neighbourhood of (s_0,x_0) of the form $S_0\times T_{B'}$ where $B'\subset B$. Let U denote $p(S_0\times T_{B'})$. Then U is a neighbourhood of $q_0=p(s_0,x_0)$. Suppose that we have a strata preserving biholomorphism $\psi\colon U\to\psi(U)\subset S\times T_B$ where $\psi(q_0)=q_0$. We may write ψ as (θ,φ) where θ maps to S and φ to T_B. Further suppose that we have a strong G-homeomorphism $\Psi=(\Theta,\Phi)\colon S_0\times T_{B'}\to S\times T_B$ inducing ψ. Then $\Theta(s,x)=\theta(s,p_{B'}(x))$, $s\in S_0$, $x\in T_{B'}$. Hence Θ is holomorphic. Since Ψ is strong, $\Phi^*f_i=\sum a_{ij}(s,x)f_j(x)$ where the a_{ij} are continuous and invariant. For $t\in[0,1]$ and $(s,x)\in S\times T_B$ let $t\cdot(s,x)$ denote $(s,t\cdot x)$. Then we have the homotopy $\Psi_t(s,x)=(\Theta_t(s,x),\Phi_t(s,x))$ where $\Theta_t(s,x)=\Theta(s,t\cdot x)$. Define $\Psi^s(x)=\Psi(s,x)$ for $(s,x)\in S_0\times T_{B'}$, and similarly define Φ^s and Θ^s. Then applying Lemma 3.7 and Corollary 3.11 and perhaps shrinking our neighbourhoods we have the following:

Corollary 3.13. *Let* U, Ψ, *etc. be as above.*

1. *The limit as* $t\to 0$ *of* Φ^s_t *acting on the* f_i, $\ell<i\le m$, *is given by the matrix* $L(s)$ *with entries* $a_{ij}(s,x_0)$, $\ell<i,\ j\le m$.
2. Φ^s *has a normal derivative* $\delta\Phi^s$ *along* Z, *and* $\delta\Phi^s\in\mathrm{Aut}_{\mathrm{vb}}(T_W)^G$ *depends continuously on* s. *If* φ *is the identity, then* $\delta\Phi^s$ *fixes* $\mathscr{O}_{\mathrm{alg}}(T_W)^G$ *for all* s.
3. *The limit as* $t\to 0$ *of the* Ψ^s_t *is* $(\Theta(s,x_0),\delta\Phi^s)$, *uniformly on compact subsets of* $S_0\times T_{B'}$.

(4) *There are invariant* $b_{ij}(s,x)$ *such that* $b_{ij} = a_{ij}$ *for* $i \le m$, $\Phi^* f_i = \sum b_{ij} f_j$ *for* $1 \le i \le n$, *and the* $b^t_{ij}(s,x)$ *are jointly continuous in* $t \in [0,1]$, s *and* x. *Moreover,* $(\delta\Phi)^* f_i(s,x) = \sum b^0_{ij}(s,x) f_j(x)$ *where* $b^0_{ij}(s,x)$ *lies in* $\mathcal{C}^0(S_0) \otimes \mathscr{O}_{\mathrm{alg}}(T_W)^G_{d_i - d_j}$, $1 \le i,\ j \le n$.

We usually denote $\delta\Phi$ by Φ_0. Then Ψ_t has limit $\Psi_0 = (\Theta_0, \Phi_0)$ where $\Theta_0(s,x) = \Theta(s,x_0)$ for $(s,x) \in S_0 \times T_{B'}$.

Corollary 3.14. *Let* $\Psi = (\Theta, \Phi)$ *be a strong* G*-homeomorphism inducing* $\psi = (\theta, \varphi)$ *as above. Then the family* $t \mapsto \Psi_t$ *is a homotopy of strong* G*-homeomorphisms inducing a family of biholomorphisms* ψ_t. *There is a continuous map* $\sigma\colon S \to \mathrm{Aut}_{\mathrm{vb}}(T_W)^G$ *such that* $\Phi_0(s,x) = \sigma(s)(x)$ *for* $(s,x) \in S_0 \times T_{B'}$. *Moreover,* $\Theta_0(s,x) = \Theta(s,x_0)$ *is a holomorphic map of* $S_0 \times T_{B'}$ *into* S *which is biholomorphic when restricted to* $S_0 \times \{x_0\}$. *If* ψ *is the identity, then* Ψ_0 *fixes* $\mathscr{O}(S_0 \times T_{B'})^G$.

Remark 3.15. Suppose that the $a_{ij}(s,x)$ are smooth (resp. holomorphic). Then Ψ is smooth (resp. holomorphic). By Lemma 3.5 we have:

(1) For $\ell < i \le m$ and $1 \le j \le \ell$, there are smooth functions $\tilde{a}_{ij}(t,s,x)$ (resp. holomorphic in s and x) such that $a_{ij}(s, t\cdot x) = t^2 \tilde{a}_{ij}(t,s,x)$.

Let $\tilde{a}_{ij}(t,s,x) = a_{ij}(s, t\cdot x)$ for $1 \le i \le \ell$ or $\ell < j \le n$. Let the $P_{uv}(x, a_{ij})$ be the polynomials of Lemma 3.8. For $1 \le v \le n$, set $b_{uv} = a_{uv}$ if $u \le m$ and set $b_{uv}(s,x) = P_{uv}(x, a_{ij}(s,x))$ for $u > m$. Then $\Phi^* f_i = \sum b_{ij} f_j$, $1 \le i \le n$. The functions $b^t_{ij}(s,x)$ are polynomials in t, x and the $\tilde{a}_{ij}(t,s,x)$. Hence:

(2) The functions $b^t_{ij}(s,x)$ are smooth in t, s and x (resp. holomorphic in s and x) and we have $\Phi^*_t f_i(s,x) = \sum b^t_{ij}(s,x) f_j(x)$, $1 \le i,\ j \le n$.

4. Examples

First we exhibit a biholomorphism of quotients which does not have local G-equivariant lifts.

Example 4.1. (See [Sch14, Example 4.4].) Let $G = \mathrm{SL}_2(\mathbb{C})$ and $H = \mathrm{SL}_4(\mathbb{C})$ and consider the canonical action of $G \times H$ on $V = (\mathbb{C}^4)^* \otimes \mathbb{C}^2$. Then the G-invariant functions of V are generated by determinants d_{ij}, $1 \le i < j \le 4$, with the relation

$$d_{12}d_{34} - d_{13}d_{24} + d_{14}d_{23} = 0.$$

We may identify the quotient with the set of 2-forms $\omega = \sum d_{ij} e_i \wedge e_j \in \wedge^2(\mathbb{C}^4)$ with the property that $\omega \wedge \omega \in \wedge^4(\mathbb{C}^4) \simeq \mathbb{C}$ vanishes. Now $\omega \mapsto \omega \wedge \omega \in \mathbb{C}$ is the $\mathrm{SO}_6(\mathbb{C}) \simeq (\mathrm{SL}_4(\mathbb{C})/\{\pm I\})$-invariant bilinear form on $\mathbb{C}^6 \simeq \wedge^2(\mathbb{C}^4)$. Hence Q can be identified with the null cone of the action of $L^0 = \mathrm{SO}_6(\mathbb{C})$ on $\mathbb{C}^6$. There is an action of $L = \mathrm{O}_6(\mathbb{C})$ on Q as well and $L \setminus L^0$ acts by outer automorphisms on L^0. Suppose that some $\ell \in L \setminus L^0$ has a local biholomorphic lift Φ to V. Then $\Phi'(0)$ is a lift of ℓ, induces an outer automorphism of H and is in the normaliser of G [Sch14, Proposition 2.9]. But G has no outer automorphisms, hence changing $\Phi'(0)$ by an element of G one can assume that $\Phi'(0)$ lies in $\mathrm{GL}(V)^G = \mathrm{GL}_4(\mathbb{C})$. But no element of $\mathrm{GL}_4(\mathbb{C})$ induces an outer automorphism of H. Hence we have a contradiction, and ℓ has no local lift.

By modifying the example above, we find X and Y such that Q_X and Q_Y have a unique strata preserving biholomorphism which does not have local lifts (let alone a global lift).

Example 4.2. Let V, G and Q be as above. Let $Q_0 = Q \cap \Delta$ where Δ is the open ball of radius one in $\mathbb{C}^6$. Since Δ is hyperbolic, so is Q_0, and this implies that the automorphism group of Q_0 is a real Lie group H with the property that every isotropy group is compact [Kob98, Theorem 5.4.2]. Since the origin is the unique singular point of Q_0, it is fixed by H and H is compact. We show that $H = \mathrm{O}_6(\mathbb{R})$. We also show that there is a homogeneous complex polynomial f of degree 3 on $\mathbb{C}^6$ which does not vanish everywhere on Q_0 and is fixed only by the identity of H. Let $0 \neq z_0 \in Q_0$ such that $f(z_0) = c \neq 0$ and let Q_0' denote the complement in Q_0 of $\{z \in Q_0 \mid f(z) = c\}$. Any holomorphic automorphism φ of Q_0' extends to Q_0 and is an element $h \in H$. Suppose that $\{z \in Q_0 \mid (h \cdot f)(z) = c\} = \{z \in Q_0 \mid f(z) = c\}$ (and $h \neq e$). Then $\{z \in Q_0 \mid f(z) = c\}$ is a union of irreducible components of $\{z \in Q_0 \mid (h \cdot f)(z) - f(z) = 0\}$ which are cones in Q passing through the origin. Then $f(0) = c$, which is absurd. Thus $h = e$. Now let X denote $p^{-1}(Q_0') \subset V$ and let Y denote $p^{-1}(\varphi(Q_0'))$ where $\varphi \in \mathrm{O}_6(\mathbb{R}) \setminus \mathrm{SO}_6(\mathbb{R})$. Note that φ preserves Q_0. Now we have a unique biholomorphism $\varphi\colon Q_X = Q_0' \to Q_Y = \varphi(Q_0')$ and by the previous example there is no lift of φ near the origins.

We show that $H = \mathrm{O}_6(\mathbb{R})$. We have the representation of H on the Zariski tangent space of Q_0 at 0, which is $\mathbb{C}^6$. Let $G = H_{\mathbb{C}}$ be the complexification of H. Then G acts on Q. By the slice theorem, the kernel of $G \to \mathrm{GL}(T_0(Q)) = \mathrm{GL}_6(\mathbb{C})$ acts trivially in a neighbourhood of 0, hence trivially on Q. But G has to act faithfully on Q since H does. Hence $G \to \mathrm{GL}_6(\mathbb{C})$ is faithful and G acts linearly on $\mathbb{C}^6$, and it clearly has to lie in $\mathrm{O}_6(\mathbb{C})$. Now the maximal compact subgroups of $\mathrm{O}_6(\mathbb{C})$ are all conjugate to $\mathrm{O}_6(\mathbb{R})$ where $\mathrm{O}_6(\mathbb{R}) \subset H$. Hence $H = \mathrm{O}_6(\mathbb{R})$.

We now show that there is an f as claimed above. As a real representation of H, the space of polynomials of degree three contains two copies of $S^3(\mathbb{R}^6)$. We show that the principal isotropy group for the action on one copy of $S^3(\mathbb{R}^6)$ is trivial. The isotropy group of the vector e_1^3 in $S^3(\mathbb{R}^6)$ is a copy of $\mathrm{O}_5(\mathbb{R})$ whose slice representation is $S^3(\mathbb{R}^5) \oplus S^2(\mathbb{R}^5) \oplus \mathbb{R}$ with trivial action on $\mathbb{R}$ and the obvious actions on $S^3(\mathbb{R}^5)$ and $S^2(\mathbb{R}^5)$. The principal isotropy group of $S^2(\mathbb{R}^5)$ is finite (a product of copies of $\mathbb{Z}/2\mathbb{Z}$), and its action on $S^3(\mathbb{R}^5)$ is faithful. Hence the principal isotropy group is trivial. This means that there is an open set of homogenous polynomials f of degree 3 on $\mathbb{C}^6$ whose H-isotropy is trivial, and we can choose such an f which does not vanish on Q_0. Then as above one constructs a Stein open set $Q_0' \subset Q_0$ with trivial holomorphic automorphism group.

Example 4.3. Let $G = \mathbb{C}^*$ and $V = \mathbb{C}^4$ with basis $\{a_1, b_1, a_2, b_2\}$ where the a_i have weight 1 and the b_i have weight -1. Let $\Phi\colon V \to V$ sending (a_1, b_1, a_2, b_2) to (b_1, a_1, b_2, a_2). Then $\Phi(\lambda v) = \lambda^{-1}\Phi(v)$ for $\lambda \in G$ and $v \in V$. Hence Φ is not equivariant in the usual sense. Let φ be the automorphism of Q induced by Φ. If Ψ is a lift of φ, then so is its derivative $\Psi'(0)$ and one can show that $\Psi'(0)$ has to have the same equivariance property as Φ does. Hence φ has a lift, but there is no lift which is equivariant in the usual sense.

The G-modules in the examples above have the infinitesimal lifting property. Here is an example where this property fails.

Example 4.4. This is due to H. M. Meyer. Let $G = \mathbb{C}^*$ act on $V = \mathbb{C}^3$ with coordinate functions x, y, z of weights -1, 1, 2, respectively. Let $u = xy$ and $w = x^2z$. Then u and w are coordinate functions on $Q = \mathbb{C}^2$ and the Luna strata are $\{0\}$ and $\mathbb{C}^2 \setminus \{0\}$. The vector field $u\partial/\partial w$ is strata preserving. A lift would have to send x^2z to xy but no smooth vector field with this property exists.

5. Local lifting of automorphisms

Let X and Y be Stein G-manifolds with common quotient Q. We establish Theorem 1.3 which gives sufficient conditions for the existence of local G-biholomorphic lifts of id_Q. The main idea behind the proof is the lifting of vector fields defined on the quotients. We begin with results about this problem. For the moment we only deal with X.

For U open in Q, let $\mathrm{Der}(U)$ denote the holomorphic vector fields on U, i.e., the derivations of $\mathscr{O}(U)$. Let $\mathrm{Der}(X_U)$ denote the holomorphic vector fields on X_U. Then we have an $\mathscr{O}(U)$-module homomorphism $p_*\colon \mathrm{Der}(X_U)^G \to \mathrm{Der}(U)$ which is just restriction of $A \in \mathrm{Der}(X_U)^G$ to $\mathscr{O}(X_U)^G \simeq \mathscr{O}(U)$. If $D = p_*A$, we say that *A is a lift of D*.

Remark 5.1. Recall that an element $D \in \mathrm{Der}(U)$ is strata preserving if for each Luna stratum Z of U, $D(z) \in T_zZ$ for every $z \in Z$. Equivalently, D preserves the ideals of the closures of the strata. Recall that X has the infinitesimal lifting property if any strata preserving $D \in \mathrm{Der}(U)$ has local lifts to $\mathrm{Der}(X_U)^G$. It is easy to see that if $A \in \mathrm{Der}(X_U)^G$, then p_*A is strata preserving. Hence, when we have the infinitesimal lifting property, the locally liftable vector fields are precisely the strata preserving vector fields.

Lemma 5.2.
(1) *The sheaves of $\mathscr{O}_Q$-modules $U \mapsto \mathrm{Der}(X_U)^G$ and $U \mapsto \mathrm{Der}(U)$ are coherent.*
(2) *Suppose that $U \subset Q$ is Stein, $D \in \mathrm{Der}(U)$ and $\{U_i\}$ is an open cover of U such that $D|_{U_i}$ lifts to $\mathrm{Der}(X_{U_i})^G$ for all i. Then D lifts to $\mathrm{Der}(X_U)^G$.*
(3) *Let W be an H-module where H is a reductive subgroup of G. Then T_W has the infinitesimal lifting property if and only if W does.*

Proof. By [Rob86], $U \mapsto \mathrm{Der}(X_U)^G$ is coherent and by [Fis76, Chapter 2], so is $U \mapsto \mathrm{Der}(U)$. Hence we have (1).

Let $\mathrm{Der}_U(X_U)^G$ denote the kernel of $p_*\colon \mathrm{Der}(X_U)^G \to \mathrm{Der}(U)$. Then the corresponding sheaf of $\mathscr{O}_Q$-modules is coherent and part (2) follows from Cartan's Theorem B.

Since $G \times W \to T_W$ is a principal H-bundle, hence locally trivial, the mapping $\mathrm{Der}(G \times W)^{G\times H} \to \mathrm{Der}(T_W)^G$ is surjective. Now

$$\mathrm{Der}(G \times W)^{G\times H} = (\mathfrak{g} \otimes \mathscr{O}(W))^H \oplus (1_G \otimes \mathrm{Der}(W)^H)$$

where 1_G denotes the constant function 1 on G and $\mathfrak{g}$ denotes the Lie algebra of G. Since $(\mathfrak{g} \otimes \mathscr{O}(W))^H$ obviously acts trivially on $\mathscr{O}(G \times W)^{G\times H}$, we see that the images of $\mathrm{Der}(T_W)^G$ and $\mathrm{Der}(W)^H$ in $\mathrm{Der}(Q)$ are the same. The same reasoning works for Q replaced by an open subset $U \subset Q$. Hence we have (3). $\square$

Remark 5.3. We say that T_W (resp. W) has the *smooth infinitesimal lifting property* if elements in $\mathrm{Der}(U)$, U open in Q, have local lifts to smooth invariant vector fields on open sets of T_W (resp. W). Then (3) above holds with "infinitesimal" replaced by "smooth infinitesimal."

We now show that the smooth infinitesimal lifting property implies the infinitesimal lifting property.

Lemma 5.4. *Let W be an H-module where H is a reductive subgroup of G. Let $D \in \mathrm{Der}(U)$ where U is open and Stein in $Q = W/\!\!/H$. Suppose that D has a smooth H-invariant lift on $p^{-1}(U)$. Then D has an H-invariant holomorphic lift over U.*

Proof. Let $p = (p_1, \dots, p_m)\colon W \to \mathbb{C}^m$ where the p_i are homogeneous generators of $\mathcal{O}_{\mathrm{alg}}(W)^H$. Then we may identify Q with $\operatorname{Im} p$. By the slice theorem, Lemma 5.2 and Remark 5.3 it is enough to consider the case that U is a neighbourhood of $0 \in Q$ and to produce a local lift over a perhaps smaller neighbourhood of 0. Give $\mathcal{O}_{\mathrm{alg}}(Q) \simeq \mathcal{O}_{\mathrm{alg}}(W)^H$ the grading of $\mathcal{O}_{\mathrm{alg}}(W)^H$. The $\mathcal{O}_{\mathrm{alg}}(Q)$-module $\mathrm{Der}_{\mathrm{alg}}(Q)$ is graded where $E \in \mathrm{Der}_{\mathrm{alg}}(Q)$ has degree k if it sends polynomials of degree s to ones of degree $s+k$ for all s. By a Taylor series argument, one sees that $E \in \mathrm{Der}_{\mathrm{alg}}(Q)$ has a local H-invariant holomorphic lift if and only it has a global H-invariant polynomial lift. Let $\widehat{\mathcal{O}}_{Q,0}$ denote the completion of the local ring $\mathcal{O}_{Q,0}$. If E is a derivation of $\widehat{\mathcal{O}}_{Q,0}$ and E_j is the component of E of degree j, then E has a lift to an H-invariant formal power series vector field on W if and only if the same is true for each E_j. The $\mathcal{O}_{\mathrm{alg}}(Q)$-submodule of $\mathrm{Der}_{\mathrm{alg}}(Q)$ of algebraically liftable vector fields is homogeneous and finitely generated, say by the homogeneous elements $D_1, \dots, D_k$.

Now let A be a smooth H-invariant lift of D. Then

$$A = \sum_{j=1}^{n} a_j(z, \bar{z})\partial/\partial z_j + b_j(z, \bar{z})\partial/\partial \bar{z}_j,$$

where $n = \dim W$ and the z_j, $\bar{z}_j$ are the usual holomorphic and antiholomorphic coordinate functions. For any H-invariant holomorphic function f, $A(f)$ is holomorphic, hence

$$A(f) = \sum_{j=1}^{n} a_j(z, 0)\partial f/\partial z_j.$$

Let $\widehat{A} = \sum_{j=1}^{n} \hat{a}_j(z, 0)\partial/\partial z_j$ where $\hat{a}_j(z, 0)$ is the Taylor series of $a_j(z, 0)$ at 0. Then $\widehat{A}$ is an H-invariant formal vector field lifting D. Any such $\widehat{A}$ is of the form $\sum \hat{a}_i A_i$ where the A_i are homogeneous $\mathcal{O}_{\mathrm{alg}}(W)^H$-module generators of $\mathrm{Der}_{Q,\mathrm{alg}}(W)^H$ and the $\hat{a}_i$ are invariant formal power series. Thus $D = p_*\widehat{A} = \sum_i \hat{b}_i p_* A_i$ where the $\hat{b}_i$ are elements of $\widehat{\mathcal{O}}_{Q,0}$ which pull back to the $\hat{a}_i$. Since the $p_* A_j$ are obviously liftable, we see that D is in the $\widehat{\mathcal{O}}_{Q,0}$-module generated by the D_i. Hence $D = \sum_{i=1}^{k} d_i D_i$ where the $d_i \in \mathcal{O}_{Q,0}$. This follows from the fact that the inclusion of $\mathcal{O}_{Q,0}$ into $\widehat{\mathcal{O}}_{Q,0}$ is faithfully flat. Hence D has a holomorphic lift over a neighbourhood of $0 \in Q$. □

We now use some parts of the theory of topological tensor products. Let N and P be smooth manifolds. Then $\mathcal{C}^\infty(N \times P)$ is isomorphic to a completion of $\mathcal{C}^\infty(N) \otimes \mathcal{C}^\infty(P)$. The completed tensor product is denoted by $\mathcal{C}^\infty(N) \widehat{\otimes} \mathcal{C}^\infty(P)$. Since both factors are nuclear, the completion is the same for either of the two main topologies on the tensor product (π and ε) [Trè67, Theorem 44.1 and Theorem 50.1]. If P is a complex manifold, then the space of functions smooth on $N \times P$ and holomorphic for fixed $x \in N$ is isomorphic to $\mathcal{C}^\infty(N) \widehat{\otimes} \mathscr{O}(P)$ and is Fréchet and nuclear.

Proposition 5.5. *Let $U \subset Q = Q_X$ be open and let D_t be a smooth vector field on $[0,1] \times U$ which is in $\operatorname{Der}(U)$ for each fixed t. If each D_t is liftable, then there is a smooth vector field A_t on $[0,1] \times X_U$ such that, for each fixed t, $A_t \in \operatorname{Der}(X_U)^G$ and $p_* A_t = D_t$.*

Proof. The sections of coherent sheaves of $\mathscr{O}_Q$-modules have natural Fréchet space structures [GR79, Ch. V, §6]. Let $\operatorname{Der}_\ell(U) = p_* \operatorname{Der}(X_U)^G$ denote the liftable vector fields. Then they form a closed subspace of $\operatorname{Der}(U)$ [GR79, p. 169, Closedness Theorem]. Now our smooth vector field D_t is the same thing as an element of $\mathcal{C}^\infty([0,1], \operatorname{Der}(U))$, the smooth mappings of $[0,1]$ into $\operatorname{Der}(U)$ [Trè67, Sec. 44]. Since each D_t is liftable, we are actually in $\mathcal{C}^\infty([0,1], \operatorname{Der}_\ell(U)) \simeq \mathcal{C}^\infty([0,1]) \widehat{\otimes} \operatorname{Der}_\ell(U)$ [Trè67, Theorem 44.1]. We have a surjection $\operatorname{Der}(X_U)^G \to \operatorname{Der}_\ell(U)$, hence

$$\mathcal{C}^\infty([0,1]) \widehat{\otimes} \operatorname{Der}(X_U)^G \to \mathcal{C}^\infty([0,1]) \widehat{\otimes} \operatorname{Der}_\ell(U)$$

is surjective [Trè67, Proposition 43.9]. It follows that there is a smooth family A_t in $\operatorname{Der}(X_U)^G$ covering D_t. □

Let $U \subset Q$ and let $\psi\colon U \to \psi(U) \subset Q$ be a strata preserving biholomorphic map. We say that a G-diffeomorphism $\Psi\colon X_U \to X_{\psi(U)}$ inducing ψ is a *strict G-diffeomorphism over ψ* if it induces a biholomorphism on reduced fibres. As before, when we say that Ψ is a strict G-diffeomorphism, we mean that it is over (or induces) the identity on the quotients. We will prove Theorem 1.3 using the following result.

Theorem 5.6. *Let ψ be as above. Suppose that there is a strict G-diffeomorphism $\Psi\colon X_U \to X_{\psi(U)}$ over ψ or that there is a strong G-homeomorphism $\Psi\colon X_U \to X_{\psi(U)}$ over ψ and X has the infinitesimal lifting property. Then for every $q_0 \in U$ there is a neighbourhood U' of q_0 and a G-biholomorphism $\widetilde{\Psi}\colon X_{U'} \to X_{\psi(U')}$ inducing $\psi|_{U'}$.*

Assume the theorem for the moment. Then we have:

Proof of Theorem 1.3. Let $q_0 \in Q$. Then X_{q_0} and Y_{q_0} are G-isomorphic and by the slice theorem there is a neighbourhood U of q_0 and a G-biholomorphism $\Psi\colon X_U \to Y_{\psi(U)}$ where Ψ induces $\psi\colon U \to \psi(U)$ and $\psi(q_0) = q_0$. Suppose that we have a strict G-diffeomorphism $\Psi'\colon X \to Y$. Then $(\Psi')^{-1} \circ \Psi$ is a strict G-diffeomorphism of X_U with $X_{\psi(U)}$ over ψ. By Theorem 5.6 we can find a G-biholomorphism $\widetilde{\Psi}\colon X_{U'} \to X_{\psi(U')}$ covering ψ where $q_0 \in U'$. Then $\Psi \circ \widetilde{\Psi}^{-1}$ is a G-biholomorphism of $X_{\psi(U')}$ with $Y_{\psi(U')}$ inducing the identity on $\psi(U')$. Hence X and Y are locally G-biholomorphic over Q. One gets the same conclusion if Ψ' is a strong G-homeomorphism and X has the infinitesimal lifting property. □

To prove Theorem 5.6 we may assume that $X = S \times T_B$ is a standard open set (Definition 3.12). Abusing notation a bit, we will let Q denote $T_W/\!\!/G$. As before, let $p_1, \dots, p_m$ be homogeneous generators of $\mathscr{O}_{\mathrm{alg}}(T_W)^G \simeq \mathscr{O}_{\mathrm{alg}}(W)^H$ and let $p = (p_1, \dots, p_m)\colon T_W \to \mathbb{C}^m$. Then $p\colon T_W \to Q = \operatorname{Im} p \subset \mathbb{C}^m$ is the quotient mapping for T_W. We may assume that $q_0 = (s_0, 0)$ for some $s_0 \in S$. Then U is a neighbourhood of q_0 which is sent to the neighbourhood $\psi(U)$ of q_0 which we may assume still lies in $S \times Q_B$. Let F denote $G \times^H \mathcal{N}(W)$. As we go along we will keep shrinking U which we can always take to be of the form $S_0 \times U_0$ where S_0 is a neighbourhood of $s_0 \in S$ and U_0 is a neighbourhood of $0 \in Q$. Write $\psi = (\theta, \varphi)$ as before.

Lemma 5.7. *To prove Theorem 5.6 we may reduce to the case that $\theta(s, q) = s$ for $(s, q) \in U$.*

Proof. Since ψ is strata preserving, θ induces a biholomorphism of $S_0 \times \{0\}$ into $S \times \{0\}$ fixing $(s_0, 0)$. Let $\tilde{\psi}(s, q) = (\theta(s, q), q)$ for $(s, q) \in U$. Then $\tilde{\psi}$ is a biholomorphism in a neighbourhood of $(s_0, 0)$ and $\tilde{\psi}^{-1}$ has the form $(s, q) \mapsto (\tilde{\theta}(s, q), q)$ where $\tilde{\theta}(s_0, 0) = s_0$. We have an equivariant biholomorphism $\widetilde{\Psi}$ defined over a G-saturated neighbourhood Ω of $\{s_0\} \times F$ given by $\widetilde{\Psi}(s, x) = (\tilde{\theta}(s, p(x)), x)$ for $(s, x) \in \Omega$. Then $\widetilde{\Psi}$ induces $\tilde{\psi}^{-1}$ so we may replace ψ by $\psi \circ \tilde{\psi}^{-1}$ to reduce to the case that $\theta(s, \cdot) = s$. □

For $s \in S_0$ we now have that $\varphi(s, \cdot)$ is a biholomorphism defined on U_0 such that $\varphi(s, 0) = 0$. We also denote $\varphi(s, \cdot)$ by $\varphi^s(\cdot)$. We have a $\mathbb{C}^*$-action on Q induced by the scalar $\mathbb{C}^*$-action on T_W. The action of $t \in \mathbb{C}^*$ sends $(q_1, \dots, q_m) \in Q$ to $(t^{e_1} q_1, \dots, t^{e_m} q_m)$ where e_i is the degree of p_i, $i = 1, \dots, m$. We have automorphisms $\varphi^s_t(q) = t^{-1} \cdot \varphi^s(t \cdot q)$ which we may assume are defined for $t \in (0, 1]$, $s \in S_0$ and $q \in U_0$. We want to show that $\varphi^s_0 = \lim_{t \to 0} \varphi^s_t$ exists. If φ^s_0 exists, then it commutes with the action of $\mathbb{R}^*$, hence also the action of $\mathbb{C}^*$, so we say that it is *quasilinear*. Let $\operatorname{Aut}_{\mathrm{q}\ell}(Q)$ denote $\operatorname{Aut}(Q)^{\mathbb{C}^*}$, the group of quasilinear automorphisms of Q. Since $Q/\!\!/\mathbb{C}^*$ is a point, $\operatorname{Aut}_{\mathrm{q}\ell}(Q)$ is a linear algebraic group. As before, $\operatorname{Aut}_{\mathrm{vb}}(T_W)^G$ denotes the group of G-vector bundle automorphisms of T_W. Then any $\ell \in \operatorname{Aut}_{\mathrm{vb}}(T_W)^G$ is a G-biholomorphism which commutes with $\mathbb{C}^*$, hence it induces an element $p_* \ell \in \operatorname{Aut}_{\mathrm{q}\ell}(Q)$.

Let $\gamma \colon U_0 \to \gamma(U_0)$ be a strata preserving biholomorphism (e.g., one of the φ^s) and suppose that γ has a lift $\Gamma \colon p^{-1}(U_0) \to p^{-1}(\gamma(U_0))$ which is a strict G-diffeomorphism over γ. Define $\Gamma_t = t^{-1} \circ \Gamma \circ t$ where $t \in (0, 1]$ acts by scalar multiplication on the fibres of T_W. We may assume that $p^{-1}(U_0)$ is stable under the scalar action restricted to $[0, 1]$. Since Γ preserves the fibre over q_0, it preserves the closed orbit $Z \simeq G/H$. Since Γ is smooth, $\Gamma_0 = \lim_{t \to 0} \Gamma_t$ exists and is the normal derivative $\delta\Gamma$ of Γ along Z.

Lemma 5.8. *Let Γ be a strict G-diffeomorphism as above. Then $\delta\Gamma \in \operatorname{Aut}_{\mathrm{vb}}(T_W)^G$.*

Proof. We have to show that $\delta\Gamma$ is complex linear. Now $\Gamma([e, 0]) = [g, 0]$ for some $g \in N_G(H)$. Let W_e and W_g denote the fibres of T_W at $[e, 0]$ and $[g, 0]$, respectively. Then $\delta\Gamma$ restricts to an H-equivariant real linear map of $W \simeq W_e$ to W_g where the H-action on W_g is twisted: $h \cdot [g, w] = [g, (g^{-1} h g) w]$. Write $W_e \simeq W = W_1 \oplus W_2$ where W_2 is the fixed space of H^0 and W_1 is an H-stable complement to W_2. We have a corresponding direct sum decomposition of W_g. Any H-equivariant real linear map of W_e to W_g has to preserve the direct sum decompositions. It is easy to see that the Zariski tangent space of $\mathcal{N}(W)_{\mathrm{red}}$ at 0 is W_1. Since Γ restricted to $G \times^H \mathcal{N}(W)_{\mathrm{red}}$ is G-equivariant and biholomorphic, $\delta\Gamma$ is complex linear on W_1. Hence we only need to show that $\delta\Gamma$ is complex linear on W_2. Let H_2 denote the image of H in $\mathrm{GL}(W_2)$. Then H_2 is finite and the set of principal orbits $W_{2,\mathrm{pr}}$ relative to the H_2-action is open and dense in W_2. Let H_0 denote the kernel of $H \to H_2$ and let S denote the stratum of Q corresponding to H_0. Then $W_{2,\text{pr}} \to S$ is a covering map onto its (open) image U and $G \times^H W_{2.\text{pr}} \to U$ is a fibre bundle with fibre G/H_0. Then Γ is holomorphic on $G \times^H W_{2,\text{pr}}$ since it is equivariant and covers the holomorphic map γ restricted to U. It follows that $\delta\Gamma$ is complex linear on W_2. Hence $\delta\Gamma \in \text{Aut}_{\text{vb}}(T_W)^G$. □

Remark 5.9. Our definition of strict G-diffeomorphism is more general than the one in [KLS15]. The main use of strictness in [KLS15] is to show that if Γ is strict, then $\delta\Gamma$ is complex linear. Using Lemma 5.8 one can substitute our definition of strict G-diffeomorphism for the one in [KLS15] and obtain the same theorems.

Lemma 5.10. *Let γ be one of the φ^s, $s \in S_0$. Suppose that $\Gamma\colon p^{-1}(U_0) \to p^{-1}(\gamma(U_0))$ is a strict G-diffeomorphism over γ or a strong G-homeomorphism over γ. Then $\gamma_0 = \lim_{t\to 0} \gamma_t$ exists and is an element of $p_*(\text{Aut}_{\text{vb}}(T_W)^G)$.*

Proof. Let Γ_t and γ_t be as above. Then Γ_t covers γ_t and by Lemmas 5.8 and 3.7, $\lim_{t\to 0} \Gamma_t = \delta\Gamma \in \text{Aut}_{\text{vb}}(T_W)^G$. Then $p \circ \delta\Gamma$ is the limit of $p \circ \Gamma_t$ as $t \to 0$, hence γ_0 exists and is covered by $\delta\Gamma$. □

Proof of Theorem 5.6. We have reduced to the case that $\Psi(s,x) = (s, \Phi(s,x))$ for (s,x) in the set $S_0 \times U_0$. Let $y_1, \dots, y_m$ be the coordinate functions on $\mathbb{C}^m$. Then near $0 \in Q$,

$$(\varphi^s_t)^* y_i = \sum_{\alpha \in \mathbb{N}^m} t^{-e_i} t^{|\alpha|} c_\alpha(s) y^\alpha$$

where the $c_\alpha(s)$ are holomorphic in $s \in S_0$ and $|\alpha| = \sum e_i \alpha_i$ for $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$. Since $\lim_{t\to 0} \varphi^s_t$ exists, we must have that $\sum_{|\alpha|=e} c_\alpha(s) y^\alpha$ vanishes on Q for each $e < e_i$. Hence we can throw away these terms and reduce to a sum over the α with $|\alpha| \geq e_i$. Thus the family $\varphi^s_t(q)$ is holomorphic where defined in s, q and $t \in \mathbb{C}$. Now $p_*\colon \text{Aut}_{\text{vb}}(T_W)^G \to \text{Aut}_{\text{q}\ell}(Q)$ is a homomorphism of linear algebraic groups, hence the image Λ is an algebraic subgroup of $\text{Aut}_{\text{q}\ell}(Q)$. By Lemma 5.10, the φ^s_0 are in Λ. The surjection $p_*\colon \text{Aut}_{\text{q}\ell}(T_W)^G \to \Lambda$ has local holomorphic sections, so we can lift the holomorphic family φ^s_0 near $s_0 \in S_0$ to a holomorphic family in $\text{Aut}_{\text{vb}}(T_W)^G$. Hence, shrinking S_0, we can reduce to the case that φ^s_0 is the identity for all $s \in S_0$. Then we have a homotopy φ_t with domain $U = S_0 \times U_0$, starting at the identity. Let

$$\Delta = \{(t,s,q) \in [0,1] \times U \mid (s,q) \in \varphi_t(U)\}.$$

Then Δ is open in $[0,1] \times U$ and contains $[0,1] \times \{(s_0, 0)\}$. Let $U' = S'_0 \times U'_0$ be a neighbourhood of $(s_0, 0)$ such that $[0,1] \times U' \subset \Delta$. Our local homotopy φ_t is obtained by integrating a time dependent vector field D_t and the domain of D_t contains $[0,1] \times U'$. Since the family φ_t is strata preserving, the D_t are strata preserving. If X has the infinitesimal lifting property, then Proposition 5.5 shows that we have a smooth family $A_t \in \text{Der}(X_{U'})^G$ which lifts D_t.

In the case that we have a strict G-diffeomorphism $\Psi(s,\cdot) = (s, \Phi(s,\cdot))$ covering ψ, the family $\Phi^s_t(x)$ is smooth in t, s and x and covers φ^s_t. Since each Φ^s_0 is in $\text{Aut}_{\text{vb}}(T_W)^G$ and φ^s_0 is the identity, we may replace each Φ^s by $(\Phi^s_0)^{-1} \circ \Phi^s$ in which case we have reduced to the case that Φ_t is a smooth homotopy of strict G-diffeomorphisms over φ_t starting at the identity. We have $\Phi_t(X_{U'}) \subset X_U$ for $t \in [0,1]$ where U' is as above. Now Φ_t is obtained by integrating a smooth time dependent vector field B_t, and B_t is defined on $[0,1]\times X_{U'}$. Moreover, B_t lifts D_t for each t. By Lemma 5.4 each D_t is holomorphically liftable, and by Proposition 5.5 we can find a smooth family $A_t \in \operatorname{Der}(X_{U'})^G$ which lifts D_t.

Let

$$\Delta' = \{(t, s, q) \in [0, 1] \times U' \mid \varphi_t(s, q) \in U'\}.$$

Then Δ' is again a neighbourhood of $[0,1] \times \{(s_0, 0)\}$ and we can find a neighbourhood U'' of $(s_0, 0)$ such that $\varphi_t(U'') \subset U'$ for all $t \in [0, 1]$. Starting at any point $(s, q) \in U''$, the flow of D_t at time 1 gives $\varphi(s, q)$. By [Sch14, proof of Theorem 3.4] or [KLS, Lemma 3.1, Corollary 3.3] the flow $\widetilde{\Phi}_t(s, x)$ of A_t exists for $(s, x) \in X_{U''}$ and $t \in [0, 1]$. Then $(s, x) \mapsto (s, \widetilde{\Phi}_1(s, x))$ is a lift of ψ to a G-biholomorphism of $X_{U''}$ with $X_{\psi(U'')}$. □

Now that we have Theorem 1.3, so that we may assume that X and Y are locally G-biholomorphic over Q, we can give a cohomological interpretation of Y and G-biholomorphisms over id_Q. Let $\mathscr{A}$ be the sheaf of groups on Q such that $\mathscr{A}(U) = \operatorname{Aut}_U(X_U)^G$ for U open in Q. Let $\{U_i\}$ be an open cover of Q such that we have G-biholomorphisms $\Psi_i \colon X_{U_i} \to Y_{U_i}$, inducing the identity on U_i. Let $\Phi_{ij} = {\Psi_i}^{-1} \circ \Psi_j \in \mathscr{A}(U_i \cap U_j)$. Then Φ_{ij} is a cocycle giving an element of $H^1(Q, \mathscr{A})$. If Y' is also locally G-biholomorphic to X over Q and has cocycle Φ'_{ij}, then Y and Y' are G-biholomorphic over Q if and only if the two cocycles represent the same cohomology class. Given a cocycle Φ_{ij}, one constructs a complex G-manifold Y by glueing X_{U_i} and X_{U_j} over $U_i \cap U_j$ using Φ_{ij}. However, it is not clear that Y is Stein.

Theorem 5.11. *Let Y be a complex G-manifold locally G-biholomorphic to X over Q. Then Y is Stein. Hence there is a one-to-one correspondence between isomorphism classes of Stein G-manifolds locally G-biholomorphic to X over Q and the cohomology set $H^1(Q, \mathscr{A})$.*

Proof. Consider the sheaf of $\mathscr{O}_Q$-modules given by $U \mapsto \mathscr{O}(Y_U)_V$ where V is an irreducible G-module. The sheaf is coherent since it is locally isomorphic to the corresponding sheaf of G-finite holomorphic functions on X.

Let $q \in Q$ and let $f_1, \dots, f_m$ be G-finite functions on Y whose restrictions to Y_q generate $\mathscr{O}_{\mathrm{alg}}(Y_q)$. Then the f_i generate the G-finite functions on an open set Y_U, as a module over $\mathscr{O}(U)$, where U is a sufficiently small neighbourhood of q. Shrink U so that it is Runge in Q and so that $Y_U \simeq X_U$. The G-finite functions on Y_U are dense in $\mathscr{O}(Y_U)$. Since the G-finite functions are generated by the f_i as $\mathscr{O}(U)$-algebra and since $\mathscr{O}(Y)^G = \mathscr{O}(Q)$ is dense in $\mathscr{O}(U)$, the algebra $\mathscr{O}(Q)[f_1, \dots, f_m]$ is dense in $\mathscr{O}(Y_U)$. Hence $\mathscr{O}(Y)$ is dense in $\mathscr{O}(Y_U)$. Moreover, for U' a relatively compact neighbourhood of q in U, there are $h_1, \dots, h_n$ in $\mathscr{O}(Q)$ such that the mapping

$$\rho = (h_1, \dots, h_n, f_1, \dots, f_m) \colon {p_Y}^{-1}(\overline{U'}) \to \mathbb{C}^{n+m}$$

is a closed embedding.

Let $K \subset Y$ be compact. Let $y_n \in \widehat{K}$, the holomorphically convex hull of K in Y. Then $\{p_Y(y_n)\}$ has a convergent subsequence, so we can assume that $p_Y(y_n) \to q$ where q sits inside our open set U' as above. The f_i are bounded on K, hence the $f_i(y_n)$ are bounded, which implies that there is a subsequence of $\{y_n\}$ such that $f_i(y_n)$ converges for all i. We already know that the $h_j(y_n)$ converge. Hence $\rho(y_n)$ converges which implies that y_n converges to a point in Y_q. Hence $\widehat{K}$ is compact and it follows that Y is Stein. □

Let $\Psi\colon S\times T_B \to S\times T_B$ be a strong G-homeomorphism or strict G-diffeomorphism. Then $\Psi(s,x) = (s,\Phi(s,x))$ where $\Phi(s,x)\colon S\times T_B\to T_B$. From now on we will identify Ψ with the family $\Phi(s,\cdot)$ and will say that Φ is a strong G-homeomorphism (or strict G-diffeomorphism).

6. Sections of type $\mathcal{F}$

In Theorem 1.4 we are assuming that X and Y are locally G-biholomorphic over Q and that there is a strict G-diffeomorphism $\Phi\colon X\to Y$ or that there is a strong G-homeomorphism $\Phi\colon X\to Y$. In this section we define the notion of a G-diffeomorphism from X to Y of type $\mathcal{F}$. In Section 8 we show that our (strict or strong) Φ is homotopic to a G-diffeomorphism of type $\mathcal{F}$. In Sections 9 and 10 we will show that any Φ of type $\mathcal{F}$ can be deformed to a G-biholomorphism. In this section we investigate the local structure of G-diffeomorphisms of type $\mathcal{F}$ and vector fields of type $\mathcal{LF}$ (defined below). The main result is Theorem 6.36 which says that a G-diffeomorphism of type $\mathcal{F}$, which is sufficiently close to the identity over a neighbourhood U of a compact subset $K\subset Q$, has a canonically associated vector field $D=\log\Phi$ of type $\mathcal{LF}$, defined over a smaller relatively compact neighbourhood U' of K, such that $\exp D=\Phi$ over U'. The definition of $\log\Phi$ depends upon a choice of standard generating set for $\mathscr{O}_{\rm fin}(X_U)$ (Definition 6.12). We will use Theorem 7.1 of the next section which says that the vector fields of type $\mathcal{LF}$ are closed in the space of smooth vector fields on X and that, for any irreducible G-module V, the $\mathcal{C}^\infty(X)^G$-module generated by $\mathscr{O}(X)_V$ is closed in $\mathcal{C}^\infty(X)$.

Definition 6.1. Let $\Phi\colon X\to Y$ be a G-diffeomorphism inducing id_Q. We say that Φ *is of type* $\mathcal{F}$ if for every $x_0\in X$ there is a G-saturated neighbourhood U of x_0 and a map $\Psi\colon U\times U\to Y$ such that:

1. For $x\in U$ fixed, $\Psi(x,y)$ is a biholomorphic G-equivariant map of $\{x\}\times U$ into Y, inducing the identity on the quotient.
2. Ψ is smooth in x and y and G-invariant in x.
3. $\Phi(x)=\Psi(x,x)$, $x\in U$.

We call Ψ a *local holomorphic extension of* Φ.

Note that if Φ is holomorphic, then it is of type $\mathcal{F}$ by setting $\Psi(x,y)=\Phi(y)$. A G-diffeomorphism of X of type $\mathcal{F}$ is obviously strict, and in Proposition 6.8 we will see that it is also strong. For an open subset $U\subset Q$ let $\mathcal{F}(U)$ denote the G-diffeomorphisms of type $\mathcal{F}$ on X_U. Then $\mathcal{F}$ is a sheaf of groups. The corresponding sheaf of Lie algebras is the sheaf of smooth vector fields of type $\mathcal{LF}$, as follows. Let $\mathrm{Der}_Q(X)^G$ denote the Lie algebra of holomorphic G-invariant vector fields on X which annihilate $\mathscr{O}(X)^G$. They are the kernel of the push-forward mapping $p_*\colon \mathrm{Der}(X)^G\to\mathrm{Der}(Q)$. Define $\mathcal{C}^\infty(X)^G\cdot\mathrm{Der}_Q(X)^G$ to be the Lie algebra of vector fields on X which are locally finite sums $\sum a_i(x)A_i(x)$ where the A_i are in $\mathrm{Der}_Q(X)^G$ and the a_i are locally defined G-invariant smooth functions.

Definition 6.2. A vector field D *is of type* $\mathcal{LF}$ if $D\in\mathcal{C}^\infty(X)^G\cdot\mathrm{Der}_Q(X)^G$.

For $U \subset Q$ an open set, we define $\mathcal{LF}(U)$ to be the Lie algebra of vector fields of type $\mathcal{LF}$ on X_U. If $D \in \mathcal{LF}(U)$, then D is locally of the form $\sum a_i(x)A_i(x)$, and we automatically get a family $D(x,x') = \sum a_i(x)A_i(x')$ where for x fixed, $D(x,x')$ is an element of $\operatorname{Der}_Q(X)^G$. This hints why the sections of $\mathcal{LF}$ should be thought of as the Lie algebra of the sections of $\mathcal{F}$.

Remark 6.3. Since $\mathcal{LF}(Q)$ is closed in the space of $\mathcal{C}^\infty$ vector fields on X (Theorem 7.1), it is a Fréchet space. We doubt that the analogous result is true if we replace $\mathcal{C}^\infty$ by $\mathcal{C}^0$ in Definition 6.2. This explains our need for smoothness assumptions.

Recall that a vector field D is *complete* if the flow $\exp(tD)$ exists for all $t \in \mathbb{R}$.

Lemma 6.4. *Let D be a G-invariant smooth vector field on X which is tangent to the fibres of $p\colon X \to Q$. Then D is complete. In particular, vector fields of type $\mathcal{LF}$ are complete.*

Proof. Let F be a fibre of p. Then D is tangent to F and gives an element of the Lie algebra of $\operatorname{Aut}(F)^G$, which is a linear algebraic group. Hence $D|_F$ can be integrated for all time. □

Corollary 6.5. *Let $U \subset Q$ be open and $D \in \mathcal{LF}(U)$. Then* $\exp(D) \in \mathcal{F}(U)$.

Example 6.6. Let D be a smooth vector field on the G-module V such that D annihilates $\mathscr{O}(V)^G$ and lies in $\mathcal{C}^\infty(V)^G \cdot \operatorname{Der}_{\mathrm{alg}}(V)^G$. One can show that if all the isotropy groups of closed orbits of V are connected, then D is of type $\mathcal{LF}$. Here is an example where D is not of type $\mathcal{LF}$.

Let V be the direct sum of two copies of $\mathbb{C}^2$ with the diagonal action of the group G of Example 3.1. We have $V = \mathbb{C}^4$ with coordinate functions x_1, x_2, y_1, y_2 where the x_i have weight 1 and the y_i have weight -1 for the action of $G^0 = \mathbb{C}^*$. Then G is generated by G^0 and the element sending x_i to y_i and y_i to $-x_i$, $i = 1, 2$. The G-invariant linear vector fields have as basis the fields

$$X_{ij} = x_i\partial/\partial x_j + y_i\partial/\partial y_j,\ 1 \le i,j \le 2$$

and the polynomials transforming by the sign representation of G are generated by

$$f_{ij} = x_iy_j + x_jy_i,\ 1 \le i \le j \le 2.$$

Up to scalars, there is one quadratic invariant $f = x_1y_2 - x_2y_1$. Let A denote the generator of the Lie algebra of $\mathbb{C}^*$. Then

$$A = x_1\partial/\partial x_1 - y_1\partial/\partial y_1 + x_2\partial/\partial x_2 - y_2\partial/\partial y_2$$

and A transforms by the sign representation of G. Thus $\operatorname{Der}_{Q,\mathrm{alg}}(V)^G$ is generated by the $f_{ij}A$. We have the curious relation

$$fA = f_{12}X_{11} + f_{22}X_{12} - f_{11}X_{21} - f_{12}X_{22}.$$

Let $h(\bar{z})$ be an antiholomorphic quadratic polynomial which transforms by the sign representation of G. Let $D = h(\bar{z})fA$. Using the curious relation we may express D as a sum of real invariant polynomial functions times the X_{ij}. Hence $D \in \mathcal{C}^\infty(V)^G \cdot$

$\mathrm{Der}_{\mathrm{alg}}(V)^G$ and D annihilates the G-invariant holomorphic functions. If D were of type $\mathcal{LF}$ then we would have

$$h(\bar z)fA = \sum_{1\le i\le j\le 2} k_{ij}(\bar z)f_{ij}A$$

where the k_{ij} are quadratic and invariant. This is clearly not possible. Hence D is not of type $\mathcal{LF}$.

A simple but useful result:

Lemma 6.7. *Let X be a Stein G-manifold and let $\{X_\alpha\}$ be a cover of X by G-saturated open sets. Then there is a partition of unity by smooth G-invariant functions f_α where $\mathrm{supp} f_\alpha \subset X_\alpha$.*

Proof. It is enough to do the case of a G-module V. Let $p\colon V \to \mathbb{C}^d$ be the quotient morphism with image Q. Then the $p(V_\alpha)$ give an open cover of Q. Let U_α be open in $\mathbb{C}^d$ such that $U_\alpha \cap Q = p(V_\alpha)$. Let $\{f_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then $\{p^* f_\alpha\}$ is the required partition of unity on V. □

Let $S \times T_B$ be a standard neighbourhood in X (which we now just assume is X) and let $\{f_i\}_{i=1}^n$ be standard generators of $\mathscr{O}_{\mathrm{fin}}(T_B)$ corresponding to the distinct irreducible nontrivial G-modules $V_1, \dots, V_r$ (Definition 3.4). Then the f_i are linearly independent on $F = G \times^H \mathcal{N}(W)$. Let $p'\colon T_B \to Q'$ denote the quotient mapping so that $p(s, x) = (s, p'(x))\colon S \times T_B \to Q = S \times Q'$ is the quotient mapping of X.

Proposition 6.8. *Let $\Phi \in \mathcal{F}(S \times Q')$ where the f_i, etc. are as above. Then there are G-invariant smooth functions $a_{ij}(s, x)$ such that*

$$(\Phi^* f_i)(s, x) = \sum_j a_{ij}(s, x) f_j(x),\ \ s \in S,\ \ x \in T_B.$$

Hence Φ is strong. If Φ is the identity on $\{s\} \times F$, then the matrix $(a_{ij}(s, x))$ is the identity for $x \in F$.

Proof. Since the f_i are linearly independent on F the last claim is clear. First suppose that S is a point. If we can produce smooth a_{ij} locally over Q, then using a partition of unity we can produce the desired a_{ij} globally. Thus we may shrink Q' in our proof. Let $\mathcal{M}$ denote

$$\mathscr{O}(Q) \cdot f_1 + \cdots + \mathscr{O}(Q) \cdot f_n \simeq \mathscr{O}(X)_{V_1} \oplus \cdots \oplus \mathscr{O}(X)_{V_r}.$$

Then $\mathcal{M}$ is a Fréchet space [GR79, Ch. V, §6]. We have a surjection π from $\mathscr{O}(Q)^n$ onto $\mathcal{M}$ sending $(a_1, \dots, a_n)$ to $\sum_i a_i f_i$. Let $q \in Q$. Then over a neighbourhood of X_q we have a local holomorphic extension Ψ of Φ as in Definition 6.1. Replacing Q by a neighbourhood of q, we may assume that Ψ is defined on $X \times X$. Then $(\Psi^* f_i)(x, y)$ is smooth and lies in $\mathcal{M}$ for each fixed x. Hence $(\Psi^* f_i)(x, y)$ is an element of $\mathcal{C}^\infty(X)^G \mathbin{\widehat{\otimes}} \mathcal{M}$. The spaces $\mathcal{C}^\infty(X)^G$, $\mathscr{O}(Q) \simeq \mathscr{O}(X)^G$ and $\mathcal{M}$ are all Fréchet and nuclear since closed subspaces of Fréchet nuclear spaces are Fréchet and nuclear [Trè67, Proposition 50.1]. Since $\pi\colon \mathscr{O}(Q)^n \to \mathcal{M}$ is surjective, the induced mapping of $\mathcal{C}^\infty(X)^G \mathbin{\widehat{\otimes}} \mathscr{O}(Q)^n$ to $\mathcal{C}^\infty(X)^G \mathbin{\widehat{\otimes}} \mathcal{M}$ is surjective [Trè67, Proposition 43.9]. Hence there are smooth functions $a_{ij}(x,y)$ on $X \times X$ which are $(G \times G)$-invariant and holomorphic in y such that

$$(\Psi^* f_i)(x,y) = \sum_j a_{ij}(x,y) f_j(y),\ i = 1, \dots, n.$$

Set $a_{ij}(x) = a_{ij}(x,x)$. This gives the case where S is a point. In the general case, since Φ has to induce the identity on $S \times Q'$, it gives a smooth family of G-diffeomorphisms of T_B of type $\mathcal{F}$ parameterised by S. One now uses another topological tensor product argument. □

Remark 6.9. Suppose that Φ is holomorphic. Then we don't have to introduce a Ψ and the proof above shows that $\Phi^* f_i = \sum a_{ij} f_j$ where the $a_{ij} \in \mathscr{O}(Q)$.

We will consider the a_{ij} both as functions on X and as functions on Q. As functions on Q they may not be smooth, however. See Example 7.8.

If (a_{ij}) is near the identity, we can take its logarithm (d_{ij}). We want to have a G-invariant vector field D, of type $\mathcal{LF}$, such that $D(f_i) = \sum d_{ij} f_j$. Then $\exp(D) = \Phi$. Note that D, if it exists, is uniquely determined by the d_{ij}. We show that D exists if (a_{ij}) is close to the identity in the $\mathcal{C}^\infty$-topology (not just in the $\mathcal{C}^0$-topology).

We consider explicit seminorms on $\mathcal{C}^\infty(X)$ for X a general Stein G-manifold. Let $\{M_i\}$ be a locally finite collection of compact sets which cover X such that $M_i \subset U_i$ where U_i is the domain of a coordinate chart. Let M be a compact subset of X and k a non-negative integer. For $f \in \mathcal{C}^\infty(X)$ we define $||f||_{M,k}$ to be the maximum of the partial derivatives of f up to order k on $M \cap M_i$ relative to the coordinate functions of U_i. We will abbreviate $||\cdot||_{M,0}$ by $||\cdot||_M$. A sequence $f_n \in \mathcal{C}^\infty(X)$ converges to $f \in \mathcal{C}^\infty(X)$ if and only if $||f_n - f||_{M,k} \to 0$ as $n \to \infty$ for all M and k. This is the $\mathcal{C}^\infty$-topology, which does not depend upon the choices made. We give the closed subspace $\mathcal{C}^\infty(X)^G \subset \mathcal{C}^\infty(X)$ the induced topology. We give spaces of smooth functions $f\colon X \to \mathbb{C}^m$ the topology of convergence of each component function in the $\mathcal{C}^\infty$-topology. We fix norms $|\cdot|_n$ on $\mathrm{M}(n,\mathbb{C})$, $n \in \mathbb{N}$, with the property that $|CD|_n \le |C|_n \cdot |D|_n$ for C, $D \in \mathrm{M}(n,\mathbb{C})$ and such that the identity matrix has norm 1. We usually drop the subscript n and just write $|\cdot|$. If (f_{ab}) is a square matrix of smooth functions on X, then we define the seminorm $||(f_{ab})||_{M,k}$ to be the maximum of $|\cdot|$ applied to the partial derivatives of (f_{ab}) up to order k on $M \cap M_i$ relative to the coordinate functions of U_i. Again, we abbreviate $||(f_{ab})||_{M,0}$ by $||(f_{ab})||_M$. Since X is Stein, it can be embedded as a closed complex submanifold $\widetilde{X}$ of some $\mathbb{C}^m$. Then we may consider diffeomorphisms Φ of X as mappings $\widetilde{\Phi}$ of $X \to \widetilde{X} \subset \mathbb{C}^m$. We say that a sequence of diffeomorphisms Φ_n converges to the diffeomorphism Φ if the mappings $\widetilde{\Phi}_n$ converge to $\widetilde{\Phi}$ in the $\mathcal{C}^\infty$-topology. Again, this topology on the diffeomorphisms does not depend upon the choices we have made. Similarly, smooth vector fields can be considered as mappings of $X \to \mathbb{C}^m$, giving the $\mathcal{C}^\infty$-topology on smooth vector fields.

The following lemma will come in handy.

Lemma 6.10. *Let $\alpha(z) = \sum_{i=1}^\infty a_i z^i$ be a power series without constant term and radius of convergence $R > 0$. Let A be a square matrix of elements of $\mathcal{C}^\infty(X)$. Then*

(1) *For $||A||_M < R$, $\alpha(A)$ converges absolutely and uniformly on a neighbourhood of M to a matrix of smooth functions.*

(2) *For $k > 0$,*

$$||\alpha(A)||_{M,k} \leq \sum_{i=1}^{k} \beta_i(||A||_M) \cdot (||A||_{M,k})^i$$

where each β_i is a series with radius of convergence R.

(3) *Given $k \geq 0$ and $\epsilon > 0$ there is a $\delta > 0$ such that $||A||_{M,k} < \delta$ implies that $||\alpha(A)||_{M,k} < \epsilon$.*

Remark 6.11. Suppose that $A \in \mathrm{M}(n, \mathbb{C})$. If $|A| < \log 2$, then $|\exp(A) - I| < 1$ and $\log \exp A = A$. If $|A - I| < 1/2$, then $|\log A| < \log 2$ and $\log A$ is the unique matrix of norm less than $\log 2$ whose exponential is A.

Definition 6.12. Let $V_1, \dots, V_r$ be nonisomorphic irreducible nontrivial G-modules appearing in $\mathscr{O}_{\mathrm{fin}}(X)$ such that the $\mathscr{O}(X)_{V_j}$ generate $\mathscr{O}_{\mathrm{fin}}(X)$ as $\mathscr{O}(Q)$-algebra. Further suppose that the $\mathscr{O}(X)_{V_j}$ are finitely generated $\mathscr{O}(Q)$-modules and that $f_1, \dots, f_n$ are a minimal generating set of $\oplus \mathscr{O}(X)_{V_j}$ with each f_i in some $\mathscr{O}(X)_{V_j}$. Then we call $\{f_i\}$ a *standard set of generators of* $\mathscr{O}_{\mathrm{fin}}(X)$. When $X = T_B$, as before, we always assume that our standard generators are in $\mathscr{O}(T_W)$ and are homogeneous.

Assume that we have a standard generating set $\{f_i\}_{i=1}^n$ for $\mathscr{O}_{\mathrm{fin}}(X)$ corresponding to irreducible G-representations $V_1, \dots, V_r$. The span of the f_i contains c_1 copies of V_1, ..., c_r copies of V_r for some $c_j \in \mathbb{N}$. Assume that $f_1, \dots, f_k$ are a basis for the c_1 copies of V_1. Then we have a mapping $\gamma_1 \colon X \to (V_1^{\oplus c_1})^*$ where $\gamma_1(x)$ is the element of $(V_1^{\oplus c_1})^*$ that sends a linear combination f of $f_1, \dots, f_k$ to $f(x)$. Similarly there are $\gamma_j \colon X \to (V_j^{\oplus c_j})^*$, $j = 2, \dots, r$. Let γ denote the product of the γ_j. Then $\gamma \colon X \to V^*$ where $V = \oplus V_j^{\oplus c_j}$. Since $\mathscr{O}_{\mathrm{fin}}(X)$ is dense in $\mathscr{O}(X)$, the mapping $(p, \gamma) \colon X \to Q \times V^*$ is an equivariant embedding. It follows that any slice representation of X is a subrepresentation of a slice representation of V^*, hence X has finitely many slice types (equivalently, a finite Luna stratification). By [Hei88, Einbettungssatz I] this implies that X equivariantly embeds into a G-module, hence Q has a closed embedding $\sigma \colon Q \to \mathbb{C}^m$ for some m. Then γ and $\sigma \circ p \colon X \to \mathbb{C}^m$ give us an equivariant closed embedding

$$\Gamma \colon X \to \mathbb{C}^m \times V^*. \tag{6.1}$$

Conversely, given a closed equivariant embedding Γ of X into a G-module (which we allow to include the trivial representation), it is clear that $\mathscr{O}_{\mathrm{fin}}(X)$ has a standard generating set.

Lemma 6.13. *Let X be a Stein G-manifold and let U be a relatively compact Stein domain in Q. Then there is an equivariant closed embedding $X_U \to V^*$ for some G-module V. Hence $\mathscr{O}_{\mathrm{fin}}(X_U)$ has a standard generating set.*

Proof. Since the Luna stratification of Q is locally finite, U intersects only finitely many Luna strata of Q. Then X_U admits a proper holomorphic G-equivariant embedding in a G-module [Hei88, Einbettungssatz I]. □

We continue to assume that $\mathscr{O}_{\mathrm{fin}}(X)$ has a standard generating set. We define two topologies on $\mathcal{F}(Q)$ equivalent to the $\mathcal{C}^\infty$-topology.

Let $\Phi \in \mathcal{F}(Q)$. Then it follows from Proposition 6.8 that $\Phi^* f_i = \sum a_{ij} f_j$ where the $a_{ij} \in \mathcal{C}^\infty(X)^G$ and $a_{ij} = 0$ if f_i and f_j do not correspond to the same V_ℓ. Let

$$\mathcal{M} = \mathcal{C}^\infty(X)^G \cdot \mathscr{O}(X)_{V_1} \oplus \cdots \oplus \mathcal{C}^\infty(X)^G \cdot \mathscr{O}(X)_{V_r} \subset \mathcal{C}^\infty(X)^r.$$

Then each $\mathcal{C}^\infty(X)^G \cdot \mathscr{O}(X)_{V_j}$ is generated over $\mathcal{C}^\infty(X)^G$ by the f_i transforming by the representation V_j. By Theorem 7.1, $\mathcal{M}$ is closed in $\mathcal{C}^\infty(X)^r$. Let E be the space of endomorphisms of $\mathcal{M}$ as $\mathcal{C}^\infty(X)^G$-module and as G-module. Then $\alpha \in E$ is determined by $(\alpha(f_i)) \in \mathcal{M}^n$. This gives a topology on E and it is easy to see that the image of E in $\mathcal{M}^n$ is closed. Hence E is a Fréchet space. Let E_0 denote the space of $(n \times n)$-matrices (a_{ij}) of elements of $\mathcal{C}^\infty(X)^G$ such that $f_i \mapsto \sum a_{ij} f_j$ is an element of E. Then E_0 is closed in the space of $(n \times n)$-matrices with entries in $\mathcal{C}^\infty(X)^G$ and obviously maps onto E.

Proposition 6.14. *Let $f_1, \dots, f_n$, X, etc. be as above. Let $\Phi \in \mathcal{F}(Q)$ with corresponding matrix $(a_{ij}) \in E_0$. Then the following give equivalent neighbourhood bases of Φ in $\mathcal{F}(Q)$.*

(1) *The neighbourhoods of Φ in the $\mathcal{C}^\infty$-topology.*
(2) *The set of all Φ' such that $(\Phi')^*$ is in a neighbourhood of Φ^* in E.*
(3) *The set of all Φ' such that some choice of corresponding matrix (a'_{ij}) is in a neighbourhood of (a_{ij}) in E_0.*

Proof. If Φ' is close to Φ in the $\mathcal{C}^\infty$-topology, then the $(\Phi')^* f_i$ are close to the $\Phi^* f_i$, i.e., $(\Phi')^*$ is close to Φ^* in E. Now suppose that $(\Phi')^*$ lies in a neighbourhood U of Φ^* in E. Let U_0 be a neighbourhood of $(a_{ij}) \in E_0$ which maps into $U \subset E$. By the open mapping theorem, the image of U_0 is a neighbourhood of Φ^* in E. Hence if $(\Phi')^*$ is close to Φ^* we can choose (a'_{ij}) close to (a_{ij}). Finally, suppose that (a'_{ij}) is close to (a_{ij}). Let $\Gamma\colon X \to \widetilde{X} \subset \mathbb{C}^m \times V^*$ be the embedding of (6.1). Then Φ acts on $\widetilde{X}$ by

$$\tilde{x} = (z, v_1^*, \dots, v_n^*) \mapsto (z, \sum_j a_{j1}(\tilde{x}) v_j^*, \dots, \sum_j a_{jn}(\tilde{x}) v_j^*)$$

where $z \in \mathbb{C}^m$, $(v_1^*, \dots, v_n^*) \in V^*$. Since (a'_{ij}) is close to (a_{ij}), we see that Φ' is close to Φ in the $\mathcal{C}^\infty$-topology. Thus the three neighbourhood bases of Φ are equivalent. □

Let $D \in \mathcal{LF}(Q)$. Then $D(f_i) = \sum d_{ij} f_j$ with $d_{ij} \in \mathcal{C}^\infty(X)^G$. Thus D gives us an element D^* of E and an element $(d_{ij}) \in E_0$. As above we have the following result.

Proposition 6.15. *Let $f_1, \dots, f_n$, X, etc. be as above. Let $D \in \mathcal{LF}(Q)$ with corresponding matrix $(d_{ij}) \in E_0$. Then the following give equivalent neighbourhood bases of D in $\mathcal{LF}(Q)$.*

(1) *The neighbourhoods of D in the $\mathcal{C}^\infty$-topology.*
(2) *The set of all D' such that $(D')^*$ is in a neighbourhood of D^* in E.*
(3) *The set of all D' such that some choice of corresponding matrix (d'_{ij}) is in a neighbourhood of (d_{ij}) in E_0.*

We now come to some key definitions.

Definition 6.16. Let X be a Stein G-manifold with a standard generating set $f_1, \dots, f_n$ of $\mathscr{O}_{\mathrm{fin}}(X)$. Let $q \in Q$. We say that the fibre X_q *has property (LF)* if there is an open neighbourhood U_0 of $q \in Q$ with the following property. For every open neighbourhood U of q contained in U_0 there is a smaller neighbourhood U' of q with compact closure in U and a neighbourhood Ω of the identity in $\mathcal{F}(U)$ such that for $\Phi \in \Omega$ we have:

(1) Φ corresponds to a matrix (a_{ij}) with $||(a_{ij}) - I||_{\overline{U'}} < 1/2$.
(2) There is a $D \in \mathcal{LF}(U')$ such that $D(f_i) = \sum d_{ij} f_j$ on $X_{U'}$ where $(d_{ij}) = \log(a_{ij})$.

As shorthand for conditions (1) and (2) we say that Φ *admits a logarithm in* $\mathcal{LF}(U')$. Note that (a_{ij}) is not unique. The condition is that some (a_{ij}) corresponding to Φ satisfies (1) and (2).

Of course, we have to show that the choice of standard generating set does not matter. First we make some remarks. In what follows, when we write (a_{ij}) we will always mean that (a_{ij}) corresponds to the Φ or Ψ of type $\mathcal{F}$ that we are considering.

Remark 6.17. The condition (1) above can always be arranged. Choose a compact subset $M \subset X$ whose image in Q is $\overline{U'}$. Then $\{\Phi \in \mathcal{F}(U) : ||(a_{ij}) - I||_M < 1/2\}$ is a neighbourhood of the identity in $\mathcal{F}(U)$. Moreover, by Lemma 6.10, the series $\log(a_{ij})$ converges uniformly on compact subsets of a G-saturated neighbourhood of M to a matrix of smooth invariant functions.

Remark 6.18. The formal series $\log \Phi^*$, when applied to any f_i, converges to $D(f_i)$. Hence D is independent of the choice of (a_{ij}).

Remark 6.19. Properties (1) and (2) imply that $\exp D = \Phi$ over U'. Note that $||(d_{ij})||_{\overline{U'}} < \log 2$ and (d_{ij}) is the unique matrix satisfying this property whose exponential is (a_{ij}). See Remark 6.11.

Lemma 6.20. *Let X, etc. be as above. Let $f'_1, \dots, f'_m$ be a second standard generating set of $\mathscr{O}_{\text{fin}}(X)$. If (1) and (2) of Definition 6.16 hold for the f'_j, then they hold for the f_i.*

Proof. Let Ω be the neighbourhood of the identity in $\mathcal{F}(U)$ such that (1) and (2) hold for the f'_j. By Propositions 6.14 and 6.15, making Ω smaller we can guarantee that any $\Phi \in \Omega$ has a matrix (a_{ij}) relative to the f_i such that $||(a_{ij}) - I||_{\overline{U'}} < 1/2$ and we can guarantee that D satisfies $D(f_i) = \sum d'_{ij} f_j$ where $||(d'_{ij})||_{\overline{U'}} < \log 2$. Then since $\exp D = \Phi$, the series $(\log \Phi^*) f_i$ converges to $D(f_i) = \sum d'_{ij} f_j$ as well as to $\sum d_{ij} f_j$ where $\log(a_{ij}) = (d_{ij})$. Hence (1) and (2) hold for the f_i. □

Proposition 6.21. *Let X be a Stein G-manifold and let $K \subset Q$ be compact. Suppose that every fibre of X has property (LF). Let U be a neighbourhood of K such that we have a standard generating set $\{f_i\}$ for $\mathscr{O}_{\text{fin}}(X_U)$. Then for any neighbourhood U' of K with compact closure in U there is a neighbourhood Ω of the identity in $\mathcal{F}(U)$ such that every $\Phi \in \Omega$ admits a logarithm in $\mathcal{LF}(U')$.*

Proof. We may assume that U is Stein and we can replace X by X_U. By Remark 6.17 we may choose Ω such that every $\Phi \in \Omega$ satisfies (1) of Definition 6.16. Let $q \in \overline{U'}$. Since X_q has property (LF), there are neighbourhoods $U_q \supset U'_q$ of q and a neighbourhood Ω_q of the identity in $\mathcal{F}(U_q)$ such that if the restriction of Φ to X_{U_q} lies in Ω_q, then Φ admits a logarithm $D_q \in \mathcal{LF}(U'_q)$. By Remark 6.18, D_q is uniquely determined by Φ restricted to $X_{U'_q}$. Since $\overline{U'}$ can be covered by finitely many U'_{q_i}, we can shrink Ω so that it maps into Ω_{q_i} for all i under the continuous restriction maps $\mathcal{F}(U) \to \mathcal{F}(U_{q_i})$. Then every $\Phi \in \Omega$ admits a logarithm in $\mathcal{LF}(U')$. $\square$

Our goal now is to prove Proposition 6.35 which says that every fibre of every Stein G-manifold X has property (LF).

Remark 6.22. Suppose that we have another Stein G-manifold X_0 with quotient Q_0 and that we have a G-equivariant biholomorphism $\Psi\colon X \to X_0$ inducing $\psi\colon Q \to Q_0$. Then Ψ induces an isomorphism of $\mathrm{Aut}_Q(X)^G$ with $\mathrm{Aut}_{Q_0}(X_0)^G$, sending $\Phi \in \mathrm{Aut}_Q(X)^G$ to $\Psi \circ \Phi \circ \Psi^{-1}$. Similarly, we obtain isomorphisms of $\mathrm{Der}_Q(X)^G$ and $\mathrm{Der}_{Q_0}(X_0)^G$, of $\mathcal{F}(Q)$ and $\mathcal{F}(Q_0)$, etc. It follows that a fibre X_q has property (LF) in X if and only if $\Psi(X_q)$ has property (LF) in X_0. Hence by the slice theorem we are reduced to showing that fibres $G \times^H \mathcal{N}(W)$ in G-vector bundles T_W have property (LF).

Lemma 6.23. *Let S be a Stein manifold with trivial G-action and let $\{f_i\}$ be a standard set of generators of $\mathcal{O}_{\mathrm{fin}}(X)$. Let X_q be a fibre of X with property (LF) and let $s_0 \in S$. Then $\{s_0\} \times X_q$ has property (LF) in $S \times X$.*

Proof. There are arbitrarily small neighbourhoods of (s_0, q) of the form $U_S \times U_q$ where U_S is a neighbourhood of s_0 in S and U_q is a neighbourhood of $q \in Q$. Any $\Phi \in \mathcal{F}(U_S \times U_q)$ gives us a smooth family $\Phi_s \in \mathcal{F}(U_q)$, $s \in U_S$. Since X_q has property (LF), there is a neighbourhood U' of q with compact closure in U_q and a neighbourhood Ω of the identity in $\mathcal{F}(U_q)$ such that any $\Phi \in \Omega$ admits a logarithm in $\mathcal{LF}(U')$. Let U'_S be a neighbourhood of s_0 with compact closure in U_S. Let $\widetilde{\Omega} = \{\Phi \in \mathcal{F}(U_S \times U_q) \mid \Phi_s \in \Omega \text{ for all } s \in \overline{U'_S}\}$. Then $\widetilde{\Omega}$ is a neighbourhood of the identity in $\mathcal{LF}(U_S \times U_q)$. Shrinking $\widetilde{\Omega}$ we can arrange that any $\Phi \in \widetilde{\Omega}$ has a matrix (a_{ij}) with $||(a_{ij}) - I||_{\overline{U'_S} \times \overline{U'}} < 1/2$. By the choice of $\widetilde{\Omega}$, for each $s \in U'_S$ there is a $D_s \in \mathcal{LF}(U')$ with matrix $\log(a_{ij}(s, \cdot))$ where the $a_{ij}(s, \cdot)$ are smooth by Proposition 6.8. Thus D_s varies smoothly in s and we have an element in $\mathcal{LF}(U'_S \times U')$ which is the logarithm of Φ. $\square$

We are reduced to the case that $F = G \times^H \mathcal{N}(W)$ where $W^H = 0$ and $X = T_W$.

Lemma 6.24. *Let F and T_W be as above. If F is a principal fibre, then it has property (LF).*

Proof. Let $\{f_i\}$ be a standard generating set for $\mathcal{O}_{\mathrm{alg}}(T_W)$ such that the f_i are linearly independent when restricted to F. Since F is principal, the only closed orbit in the H-module W is the origin, hence $X = F = T_W$ and Q is a point. Thus $\mathcal{F}(Q) = \mathrm{Aut}(F)^G$. By choice of the f_i, any $\Phi \in \mathrm{Aut}(F)^G$ has a unique matrix (a_{ij}). Any sufficiently small neighbourhood Ω of the identity in $\mathrm{Aut}(F)^G$ is the isomorphic image of a neighbourhood of the origin in its Lie algebra via the exponential map. We may define such an Ω by the condition that $\Phi^* f_i = \sum a_{ij} f_j$ and $|(a_{ij}) - I| < \epsilon \leq 1/2$ for ϵ sufficiently small. $\square$

Let W and H be as above. Even though we have reduced to considering the fibre $F \in T_B \subset T_W$, it is useful to prove lemmas concerning the case that $X = S \times T_B \subset S \times T_W$. Recall that for $\Phi \in \mathcal{F}(S \times Q_B)$, $\Phi_t(s, x) = t^{-1} \cdot \Phi(s, t \cdot x)$, $t \in (0, 1]$, $(s, x) \in S \times T_B$. Using the argument of Remark 6.22 it is easy to see that each Φ_t is of type $\mathcal{F}$, $0 < t \leq 1$, and $\Phi_0 = \lim_{t \to 0} \Phi_t$ is a smooth family of G-biholomorphisms, hence of type $\mathcal{F}$. Let

$\mathcal{F}([0,1]\times S\times Q_B)$ denote smooth families $\Phi(t,s,x)$ of elements of $\mathcal{F}(Q_B)$ parameterised by $[0,1]\times S$.

Let $\{f_1,\dots,f_n\}$ be a standard generating set for $\mathscr{O}_{\rm alg}(T_W)$ as in Definition 3.4. The f_i are linearly independent when restricted to F. Let d_i be the degree of f_i, $i=1,\dots,n$. We assume that $d_i=0$ for $1\le i\le \ell$, $d_i=1$ for $\ell<i\le m$ and $d_i>1$ for $i>m$. Let $\Phi\in\mathcal{F}(S\times Q_B)$ correspond to the matrix (a_{ij}). By Remark 3.15 we know that

$$a_{ij}(s,t\cdot x)=t^2\tilde a_{ij}(t,s,x),\ \ell<i\le m,\ 1\le j\le \ell,\ s\in S,\ x\in T_B,\ t\in[0,1] \tag{6.2}$$

where $\tilde a_{ij}(t,s,x)$ is smooth in t, s and x. We also know that $\Phi_t^*f_i=\sum b_{ij}^t f_j$ where the $b_{ij}^t(s,x)$ are smooth in t, s and x.

Lemma 6.25. *The mapping from $\mathcal{F}(S\times Q_B)$ to $\mathcal{F}([0,1]\times S\times Q_B)$ sending Φ to Φ_t is continuous.*

Proof. The reasoning above shows that $\Phi_t\in\mathcal{F}([0,1]\times S\times Q_B)$. From Lemma 3.10 and Corollary 3.11 we know that the b_{ij}^t are polynomials, with coefficients in $\mathscr{O}_{\rm alg}(T_W)^G[t]$, in the $\tilde a_{ij}(s,t,x)$, $1\le i\le m$, where the $\tilde a_{ij}(t,s,x)$ are as given above or are just simply $a_{ij}(s,t\cdot x)$. Thus to show that $\Phi\mapsto\Phi_t$ is continuous it is enough to show that $a_{ij}(s,x)\mapsto t^{-2}a_{ij}(s,t\cdot x)$ is continuous in the $\mathcal{C}^\infty$-topologies for $\ell<i\le m$ and $1\le j\le m$. But this follows from Taylor's theorem. □

Corollary 6.26. *Let Ω be a neighbourhood of the identity in $\mathcal{F}(S\times Q_B)$. Then there is a neighbourhood Ω_0 of the identity such that $\Phi_t\in\Omega$, $t\in[0,1]$, for all $\Phi\in\Omega_0$.*

Lemma 6.27. *Let K be a compact subset of $S\times Q_B$. Then there is a neighbourhood Ω of the identity in $\mathcal{F}(S\times Q_B)$ such that any $\Phi\in\Omega$ corresponds to a matrix (b_{ij}) that, as a matrix function on $S\times Q_B$, satisfies $||(b_{ij}^t)-I||_K<1/2$, $0\le t\le 1$.*

Proof. Let E_0 be as in the discussion before Proposition 6.14 with $X=S\times T_B$. Let E_0' denote the closed linear subspace of E_0 consisting of matrices (a_{ij}) such that (6.2) holds. Construct (b_{ij}) as in Corollary 3.13. Then by the proof of Lemma 6.25 the corresponding function $\rho\colon(a_{ij})\mapsto(b_{ij}^t)$ is continuous for the $\mathcal{C}^\infty$-topologies on E_0' and $E_0\,\widehat{\otimes}\,\mathcal{C}^\infty([0,1])$ and, by construction, $\rho(I)=I$. Let

$$\Gamma=\{(b_{ij}^t)\in E_0\,\widehat{\otimes}\,\mathcal{C}^\infty([0,1]) : ||(b_{ij}^t)-I||_K<1/2,\ 0\le t\le 1\}.$$

Then there is a neighbourhood Δ' of the identity in E_0' such that $\rho(\Delta')\subset\Gamma$. Let Δ be a neighbourhood of the identity in E_0 such that $\Delta\cap E_0'\subset\Delta'$. By the open mapping theorem (see the proof of Proposition 6.14), if $\Phi\in\mathcal{F}(S\times Q_B)$ is sufficiently close to the identity, then it has a corresponding matrix $(a_{ij})\in\Delta$. Since (a_{ij}) is automatically in E_0', we are in the open set Δ' and (b_{ij}^t) satisfies the desired inequality. □

The graded $\mathscr{O}_{\rm alg}(Q)$-module ${\rm Der}_{Q,{\rm alg}}(T_W)^G$ has a generating set $A_1,\dots,A_k$ such that $A_i(t\cdot x)=t^{n_i}A_i(x)$ for some $n_i\in\mathbb{Z}^+$, $i=1,\dots,k$, $x\in T_W$, $t\in\mathbb{C}^*$. If $n_i=0$, then $A_i([e,w])=A_i([e,0])$ is fixed by H, hence A_i belongs to the Lie algebra of the centraliser $C_G(H)$ where $c\in C_G(H)$ acts on T_W sending $[g,w]$ to $[gc,w]$. Since the $C_G(H)$-action commutes with the $\mathbb{C}^*$-action, A_i is fixed by $\mathbb{C}^*$. For $n_i\ge 1$, A_i acts as an endomorphism of $\mathscr{O}_{\rm alg}(T_W)$ of degree n_i-1. Define m_i to be 0 if $n_i=0$, otherwise set $m_i=n_i-1$. Then $t^{-1}\cdot A_i(t\cdot x)=t^{m_i}A_i(x)$, $t\in\mathbb{C}^*$, $x\in T_W$, $i=1,\dots,k$.

Let $p = (p_1, \dots, p_d)$ where the p_i are generators of $\mathscr{O}_{\mathrm{alg}}(T_W)^G$ and p_i is homogeneous of degree e_i, $i = 1, \dots, d$. Then we can identify $Q = T_W/\!\!/G$ with the image of p. Let $K = \{(q_1, \dots, q_d) \in Q \mid \sum_i |q_i|^{2c_i} = 1\}$ where we choose the $c_i \in \mathbb{N}$ such that $e = c_i e_i$ is independent of i. Then K is compact and the points $t \cdot q$, $q \in K$, $t \in [0,1)$, form a neighbourhood U' of 0 in Q where $\overline{U'}$ is compact. Changing the p_i by scalars we may arrange that $\overline{U'} \subset Q_B$. Let U be a relatively compact neighbourhood of the origin in Q_B which contains $\overline{U'}$. By induction we can assume that all fibres of $X \setminus F$ have property (LF). Let U'' denote $U' \setminus \{0\}$.

Lemma 6.28. *Let $X = T_B$, etc. be as above. Then there is a neighbourhood Ω of the identity in $\mathcal{F}(Q_B)$ such that any $\Phi \in \Omega$ admits a logarithm in $\mathcal{LF}(U'')$.*

Proof. Let U_0 be a neighbourhood of K with compact closure in U. By Proposition 6.21, there is a neighbourhood Ω_0 of the identity in $\mathcal{F}(Q_B)$ such that any $\Phi \in \Omega_0$ admits a logarithm in $\mathcal{LF}(U_0)$. By Corollary 6.26 and Lemma 6.27 there is a neighbourhood Ω of the identity in $\mathcal{F}(Q_B)$ such that for $\Phi \in \Omega$, $\Phi_t \in \Omega_0$, $t \in [0,1]$, and $||(b_{ij}^t) - I||_{\overline{U}} < 1/2$, $t \in [0,1]$, where (b_{ij}) is a matrix corresponding to Φ. We have a smooth family $D_t \in \mathcal{C}^\infty([0,1]) \widehat{\otimes} \mathcal{LF}(U_0)$ where $D_t = \log \Phi_t$. Hence $D_t = \sum a_i(t,x) A_i(x)$, $t \in [0,1]$, $x \in X_{U_0}$, where the a_i are smooth and invariant. Let $(d_{ij}) = \log(b_{ij})$. Then a straightforward calculation shows that the family (d_{ij}^t) is the logarithm of the family (b_{ij}^t), $0 \le t \le 1$. Since $||(b_{ij}^t) - I||_{\overline{U}} < 1/2$, the vector field D_t has matrix (d_{ij}^t).

Let $t \in (0,1]$ and define $U''_t = \{q \in U'' \mid t^{-1} \cdot q \in U_0\}$. For $x \in X_{U''_t}$

$$D_0(x) := t \cdot D_t(t^{-1}x) = \sum a_i(t, t^{-1} \cdot x)(t^{-1})^{m_i} A_i(x)$$

is of type $\mathcal{LF}$. Since D_t has matrix (d_{ij}^t), the vector field D_0 has matrix (d_{ij}). Hence Φ admits a logarithm on U''_t with matrix (d_{ij}). Since U'' is the union of the U''_t it follows that Φ admits a logarithm $D \in \mathcal{LF}(U'')$. $\square$

We will have shown that all fibres of any X have property (LF) if we can extend our $D \in \mathcal{LF}(U'')$ to $\mathcal{LF}(U')$. Now there is one case where this is automatic. Suppose that our Φ is holomorphic. Then $D \in \mathrm{Der}_{U''}(X_{U''})^G$. Near any point of F, there is a system of local coordinates consisting of invariant functions (which D annihilates) and some of the f_i (which D sends into bounded linear combinations of the f_j). By the Riemann extension theorem, D extends to an element of $\mathrm{Der}(X_{U'})$, which actually has to be in $\mathrm{Der}_{U'}(X_{U'})^G$. Now the $\mathcal{C}^0$-topology and the $\mathcal{C}^\infty$-topology are the same on $\mathrm{Aut}_Q(X)^G$. Hence we have the following result.

Theorem 6.29. *Let X be a Stein G-manifold and assume that we have a standard generating set $\{f_i\}$ for $\mathscr{O}_{\mathrm{fin}}(X)$. Let U be a neighbourhood of the compact subset $K \subset Q$ and let U' be a neighbourhood of K with compact closure in U. Then there is a compact subset $K' \subset U$ and $0 < \epsilon < 1/2$, with the following property. If $\Phi \in \mathrm{Aut}_U(X_U)^G$ has an associated matrix (a_{ij}) satisfying $||(a_{ij}) - I||_{K'} < \epsilon$, then there is a $D \in \mathrm{Der}_{U'}(X_{U'})^G$ with associated matrix $\log(a_{ij})$ such that $\exp D = \Phi$.*

Let $\Phi \in \Omega \subset \mathcal{F}(T_B)$ as in Lemma 6.28 so that Φ has a (unique) logarithm $D \in \mathcal{LF}(U' \setminus \{0\})$. We now show that if Φ is sufficiently close to the identity, then Φ has a logarithm over *some* neighbourhood of $0 \in U'$. By uniqueness of logarithms, this shows that D extends to $\mathcal{LF}(U')$ and we will have shown that F has property (LF).

We need to study various G-automorphisms of $F = G \times^H \mathcal{N}(W)$. As before $\mathrm{Aut}_{\mathrm{vb}}(T_W)^G$ denotes the G-vector bundle automorphisms of T_W. Let $\rho\colon H \to \mathrm{GL}(W)$ be the representation associated to W. Then one easily shows:

Lemma 6.30. *Let $\Psi \in \mathrm{Aut}_{\mathrm{vb}}(T_W)^G$. Then there is a $g_0 \in N_G(H)$ and $\gamma \in \mathrm{GL}(W)$ normalising $\rho(H)$ such that:*

1. *For $g \in G$ and $w \in W$, $\Psi([g,w]) = [gg_0, \gamma(w)]$.*
2. *For $h \in H$, $\rho({g_0}^{-1} h g_0) = \gamma \circ \rho(h) \circ \gamma^{-1}$.*

Let L_{vb} denote the subgroup of $\mathrm{Aut}_{\mathrm{vb}}(T_W)^G$ fixing $\mathscr{O}_{\mathrm{alg}}(W)^H$.

Corollary 6.31. *The groups $\mathrm{Aut}_{\mathrm{vb}}(T_W)^G$ and L_{vb} are reductive.*

Proof. We have a homomorphism $\sigma\colon \mathrm{Aut}_{\mathrm{vb}}(T_W)^G \to N_G(H)/H$, $(g_0,\gamma) \mapsto g_0H$. The image clearly contains $C_G(H)H/H$, which in turn contains the identity component of $N_G(H)/H$. Hence $\mathrm{Im}\,\sigma \subset N_G(H)/H$ is reductive. The kernel of σ is $\rho(H) \cdot \mathrm{GL}(W)^H$ which is isomorphic to $\rho(H) \times \mathrm{GL}(W)^H$ divided by the centre of $\rho(H)$. Hence $\mathrm{Aut}_{\mathrm{vb}}(T_W)^G$ is reductive.

Let (g_0,γ) represent an element Ψ of $\mathrm{Aut}_{\mathrm{vb}}(T_W)^G$. Then Ψ acts on the homogeneous elements of $\mathscr{O}_{\mathrm{alg}}(W)^H$ of degree d, sending f to $f \circ \gamma^{-1}$. This gives us representations σ_d of $\mathrm{Aut}_{\mathrm{vb}}(T_W)^G$ such that L_{vb} is the joint kernel of (finitely many of) the σ_d. Hence L_{vb} is reductive. □

Let L denote $\mathrm{Aut}(F)^G = \mathrm{Aut}(\mathscr{O}_{\mathrm{alg}}(F))^G$, a linear algebraic group. Let $\ell \in L$. Then ℓ preserves the closed orbit in F, which is Z. We have the deformation $\ell_t \in L$ where $\ell_t(x) = t^{-1} \cdot \ell(t \cdot x)$, $t \in (0,1]$, $x \in F$. The limit as $t \to 0$ is the normal derivative $\delta\ell$ of ℓ along Z, so $\delta\ell \in \mathrm{Aut}_{\mathrm{vb}}(T_W)^G$. Let $\ell' \in L$. Since $\ell'_t \circ \ell_t = (\ell' \circ \ell)_t$, the map $\delta\colon L \to \mathrm{Aut}_{\mathrm{vb}}(T_W)^G$ is a homomorphism of algebraic groups. If $\ell \in \mathrm{Aut}_{\mathrm{vb}}(T_W)^G$, then $\delta(\ell|_F) = \ell$.

Let L_{hr} denote the subgroup of automorphisms that extend to be G-equivariant biholomorphisms over the identity in a G-saturated neighbourhood of F (the "holomorphically reachable points"). Then $L_{\mathrm{vb}} \subset L_{\mathrm{hr}} \subset L$. Let $A_1,\dots,A_k$ be our minimal homogeneous generators of $\mathrm{Der}_{Q,\mathrm{alg}}(T_W)^G$ as before, where $t^{-1} \cdot A_i(t \cdot x) = t^{m_i} A_i(x)$, $t \in \mathbb{C}^*$, $x \in T_W$ and $m_i \in \mathbb{Z}^+$. Since L_{vb} is reductive and acts by conjugation in a degree preserving way on $\mathrm{Der}_{Q,\mathrm{alg}}(T_W)^G$, we may assume that L_{vb} preserves the span of the A_i for any fixed m_i. It is easy to see that the Lie algebra of L_{vb} is the span of the A_i with $m_i = 0$. Each A_i acts on $\mathscr{O}_{\mathrm{alg}}(T_W)$ increasing degree by m_i. The restrictions $\widetilde{A}_i$ of the A_i to F are linearly independent (by minimality). Let $\tilde{\mathfrak{r}}$ be the span of the $\widetilde{A}_i$ with $m_i > 0$. Let s be the maximum degree of the f_i. Then any element $\widetilde{A}$ of $\tilde{\mathfrak{r}}$ acts nilpotently on $\mathscr{O}_{\mathrm{alg}}(F)$ since the $(s+1)$st power annihilates all the f_i. It follows that $\exp\widetilde{A}$ is a unipotent algebraic G-automorphism of F. Let R denote $\exp(\tilde{\mathfrak{r}})$, an algebraic subgroup of L. By our choice of the A_i, the conjugation action of L_{vb} preserves $\tilde{\mathfrak{r}}$. We have a morphism σ of algebraic groups

$$L_{\mathrm{vb}} \ltimes R \ni (\ell, \exp\widetilde{A}) \mapsto \ell\exp\widetilde{A} \in L$$

with image in L_{hr}.

Theorem 6.32. *The homomorphism $\sigma\colon L_{\rm vb}\ltimes R\to L$ is injective with image $L_{\rm hr}$. Hence $L_{\rm hr}$ is an algebraic subgroup of L and its Lie algebra is the restriction of $\operatorname{Der}_{Q,\rm alg}(T_W)^G$ to F.*

Proof. Let $\Phi\in L_{\rm hr}$. We want to show that Φ is in the image of σ. Now Φ is the restriction of some $\Psi\in\operatorname{Aut}_U(X_U)^G$ where U is a neighbourhood of the origin in Q. We have the deformation Ψ_t with limit $\Psi_0\in L_{\rm vb}$. Replacing Ψ by ${\Psi_0}^{-1}\circ\Psi$ and Φ by ${\Phi_0}^{-1}\circ\Phi$ we may reduce to the case that Ψ_0 and Φ_0 are the identity. Since Ψ_0 is the identity, for $t_0\in(0,1]$ sufficiently close to 0, Ψ_{t_0} admits a logarithm D_{t_0} over a neighbourhood U' of $0\in Q_B$ (Theorem 6.29). The restriction E of D_{t_0} to F is a sum $\sum a_i\widetilde{A}_i$. Then $\Phi_{t_0t}=\exp E_t=\exp(\sum t^{m_i}a_i\widetilde{A}_i)$. Letting t tend to zero we see that $\sum_{m_i=0}a_i\widetilde{A}_i$ exponentiates to the identity. Since exp is injective on a neighbourhood of 0 in the Lie algebra of $L_{\rm vb}$, we see that all the a_i for which $m_i=0$ vanish, provided they are sufficiently small, which we can always arrange. Hence $\Phi=\exp(\sum_{m_i>0}t_0^{-m_i}a_i\widetilde{A}_i)$. Thus $\Phi\in R$, the image of σ is $L_{\rm hr}$ and $L_{\rm hr}$ is algebraic. Finally, σ is injective, since $\delta\circ\sigma(\ell,\exp\widetilde{A})=\ell$ for $\ell\in L_{\rm vb}$ and $\widetilde{A}\in\tilde{\mathfrak{r}}$. □

Corollary 6.33. *There is a group homomorphism $\nu\colon L_{\rm hr}\to\operatorname{Aut}_Q(T_W)^G$ such that $\ell=\nu(\ell)|_F$ for all $\ell\in L_{\rm hr}$.*

Proof. Let $\mathfrak{r}$ be the span of the A_i such that $m_i>0$. By our choice of the A_i, $\mathfrak{r}$ is stable under conjugation by $L_{\rm vb}$. Let $\ell\in L_{\rm vb}$ and let $\widetilde{A}\in\tilde{\mathfrak{r}}$ be the restriction of $A\in\mathfrak{r}$. Let $\nu(\ell\cdot\exp\widetilde{A})=\ell\cdot\exp A$. Then ν has the required properties. □

Remark 6.34. Let $S\times T_B$ be a standard neighbourhood in X and let F denote the fiber $(s_0,G\times^H\mathcal{N}(W))$ for some $s_0\in S$. Let U denote $S\times Q_B$. Then we have an extension mapping from $L_{\rm hr}$ to $\operatorname{Aut}_U(X_U)^G$ where the action of $\ell\in L_{\rm hr}$ on $(s,x)\in S\times T_B$ gives $(s,\nu(\ell)(x))$.

Proposition 6.35. *Let X be a Stein G-manifold. Then all fibres have property (LF).*

Proof. We continue with the situation of Lemma 6.28. We may assume that our standard generators f_i are linearly independent when restricted to F. We have only to show that, for some neighbourhood Ω of the identity of $\mathcal{F}(Q_B)$, any $\Phi\in\Omega$ is $\exp D$, $D\in\mathcal{LF}(U)$, where U is a neighbourhood of 0 (which may depend upon Φ). By choosing Ω small, we may assume that the restriction of Φ to F is the restriction to F of $\exp D'$ where $D'\in\operatorname{Der}_{Q,\rm alg}(T_W)^G$. We may assume that the corresponding matrix (d'_{ij}) has norm at most $(\log 2)/2$ over a neighbourhood of $0\in Q$. Thus, replacing Φ by $\exp(-D')\Phi$, we may reduce (temporarily) to the case that Φ restricted to F is the identity.

Since Φ is of type $\mathcal{F}$, there is a holomorphic extension $\Psi(x,y)$ of Φ on $T_{B'}\times T_{B'}$ where $0\in B'\subset B$. Then Ψ is a smooth family Ψ^x of elements of $\operatorname{Aut}_{Q_{B'}}(T_{B'})^G$, $x\in T_{B'}$. The restriction of Ψ^{x_0} to F is the identity. We have the deformations Ψ^x_t, $t\in[0,1]$, where $\Psi^{x_0}_0$ is the identity. Since $\Psi^x_t(y)$ is smooth in t, x, y and holomorphic in y, the corresponding matrix $(b^{t,x}_{ij}(y))$ is smooth in t, x and y, holomorphic in y and $(G\times G)$-invariant. Since $(b^{t,x_0}_{ij}(x_0))=I$, $0\le t\le 1$, there is a $(G\times G)$-saturated neighbourhood of (x_0,x_0) on which we have $|b^{t,x}_{ij}(y)-I|<1/2$, $0\le t\le 1$. Shrinking B' we may thus assume that $||(b^{t,x}_{ij}(y))-I||<1/2$, $0\le t\le 1$, for $(x,y)\in T_{B'}\times T_{B'}$. Since $(b^{t,x}_{ij}(y))$ is G-invariant in y, we may also write $(b^{t,x}_{ij}(q))$. Let U' be a neighbourhood of $0\in Q$

with compact closure in $Q_{B'}$. By Theorem 6.29 there is an $\epsilon > 0$ and a compact subset $K \subset Q_{B'}$ such that $||(b_{ij}^{t,x}(q)) - I||_K < \epsilon$ implies that $\Psi_t^x(y)$ admits a logarithm in $\mathrm{Der}_{U'}(X_{U'})^G$. Since the inequality is true for $t = 0$ and $x = x_0$, there is a neighbourhood Λ of $(0, x_0) \in [0,1] \times T_{B'}$ such that Ψ_t^x admits a logarithm in $\mathrm{Der}_{U'}(X_{U'})^G$ for $(t,x) \in \Lambda$. Now Λ contains a subset of the form $[0,t_0] \times \Delta$ where $t_0 > 0$ and Δ is a neighbourhood of x_0 in $T_{B'}$. Let U_0 denote $t_0 \cdot U'$. Then the usual trick shows that Ψ^x admits a logarithm in $\mathrm{Der}_{U_0}(X_{U_0})^G$ for all $x \in \Delta$. Since $\Psi(x,y)$ is G-invariant in x, we may assume that Δ is G-saturated. Shrinking Δ we arrive at the situation where $\Psi^x(y)$ admits a logarithm $D^x(y)$ for $(x,y) \in \Delta \times \Delta$. Then $\Phi(x) = \Psi^x(x) = \exp D^x(x)$ where $D'' := D^x(x)$ is of type $\mathcal{LF}$ on Δ. Since Φ is the identity on F, D'' vanishes on F.

Now D'' corresponds to (d''_{ij}) where $||(d''_{ij})|| < (\log 2)/2$ on a neighbourhood of $0 \in Q$. We have shown that our original $\Phi = \exp D' \exp D''$. Then $\Phi = \exp D$ where D is given by the Campbell-Hausdorff series

$$D = D' + D'' + \frac{1}{2}[D', D''] + \frac{1}{12}([D', [D', D'']] + [D'', [D'', D']]) - \dots$$

Since the coefficient matrices (d'_{ij}) and (d''_{ij}) have norm at most $(\log 2)/2$ in a neighbourhood of F, the series converges there. It is a series of elements of type $\mathcal{LF}$, hence the limit is of type $\mathcal{LF}$ by Theorem 7.1. The coefficient matrix (d_{ij}) of D has norm less than $\log 2$ in a neighbourhood of F, which we may assume is $T_{B'}$, and Φ has matrix (a_{ij}) where $|(a_{ij}(x)) - I| < 1/2$ for $x \in T_{B'}$. Since $\log \exp(d_{ij}) = (d_{ij})$ on $T_{B'}$, the series $\log \Phi^*$, applied to f_i, gives $\sum d_{ij} f_j$. But $(\log \Phi^*) f_i$ is also $\sum e_{ij} f_j$ where $e_{ij} = \log(a_{ij})$. Hence D also has matrix (e_{ij}) and is indeed the logarithm of Φ. Hence, in the situation of Lemma 6.28, choosing Ω sufficiently small, for all $\Phi \in \Omega$, the vector field D constructed there extends to an element of $\mathcal{LF}(U')$. Thus F has property (LF). □

Combining our results so far (including Lemma 6.10) we have:

Theorem 6.36. *Let X be a Stein G-manifold, let $K \subset Q$ be compact and choose a standard generating set of $\mathscr{O}_{\mathrm{fin}}(X_U)$ for U a neighbourhood of K. Let $U' \subset U$ be a neighbourhood of K with compact closure in U. Then there is a neighbourhood Ω of the identity in $\mathcal{F}(U)$ such that any $\Phi \in \Omega$ admits a (unique) logarithm in $\mathcal{LF}(U')$. Moreover, the mapping $\log\colon \Omega \to \mathcal{LF}(U')$ is continuous.*

Corollary 6.37. *Let $\{\Phi_n\}$ be a sequence of G-diffeomorphisms of X of type $\mathcal{F}$ and suppose that Φ_n converges to a G-diffeomorphism Φ. Then Φ is of type $\mathcal{F}$.*

Proof. Since this is a local question, we can assume that we have a standard generating set $\{f_i\}$ for $\mathscr{O}_{\mathrm{fin}}(X)$. Let $q \in Q$ and let U be a neighbourhood of q with compact closure. Then there is a neighbourhood Ω of the identity in $\mathcal{F}(Q)$ such that any $\Psi \in \Omega$ admits a logarithm in $\mathcal{LF}(U)$. Let Ω_0 be a smaller neighbourhood of the identity with $\overline{\Omega}_0 \subset \Omega$. There is an $N \in \mathbb{N}$ such that $n \geq N$ implies that ${\Phi_N}^{-1}\Phi_n \in \Omega_0$, hence $\log({\Phi_N}^{-1}\Phi_n) = D_n \in \mathcal{LF}(U)$, and D_n converges to a vector field D which is in $\mathcal{LF}(U)$ by Theorem 7.1. Since $\exp D_n = {\Phi_N}^{-1}\Phi_n$ over U, we have $\exp D = {\Phi_N}^{-1}\Phi$ over U. Hence $\Phi = \Phi_N \exp D$ is of type $\mathcal{F}$ over U. □

7. Topology

Let X be a Stein G-manifold with quotient Q. Let R be a G-module where $R = \oplus R_i^{\oplus c_i}$ and the R_i are irreducible. Define $\mathscr{O}(X)_R$ to be $\oplus(\mathscr{O}(X)_{R_i})^{\oplus c_i}$ which is an $\mathscr{O}(X)^G$-submodule of $\mathscr{O}(X)^c$, $c = \sum c_i$. We have the space $\mathcal{C}^\infty(X)^G \cdot \mathscr{O}(X)_R \subset \mathcal{C}^\infty(X)^c$ of smooth functions which are locally (over Q) finite sums $\sum a_j f_j$ where the a_j are smooth and invariant and the $f_j \in \mathscr{O}(X)_R$. The main point of this section is Theorem 7.1 which shows that $\mathcal{C}^\infty(X)^G \cdot \mathscr{O}(X)_R$ is closed in $\mathcal{C}^\infty(X)^c$ and that $\mathcal{C}^\infty(X)^G \cdot \mathrm{Der}_Q(X)^G$ is closed in $\mathrm{Der}_Q^\infty(X)^G$, the space of smooth G-invariant vector fields on X which annihilate $\mathscr{O}(X)^G$. Then it follows that our spaces are Fréchet and nuclear (see [Trè67, Sections 10, 50]). Theorem 7.1 is a technical underpinning of our results in Section 6.

Theorem 7.1. *Let X be a Stein G-manifold and let $R = \oplus R_i^{\oplus c_i}$ be a G-module where the R_i are irreducible.*

(1) *The space $\mathcal{LF}(Q) = \mathcal{C}^\infty(X)^G \cdot \mathrm{Der}_Q(X)^G$ is closed in $\mathrm{Der}_Q^\infty(X)^G$.*
(2) *The space $\mathcal{C}^\infty(X)^G \cdot \mathscr{O}(X)_R$ is closed in $\mathcal{C}^\infty(X)^c$ where $c = \sum c_i$.*

If the quotient mapping $p\colon X \to Q$ were proper, then the theorem would be a consequence of theorems of Bierstone and Milman [BM87a], Theorems A, C, D; [BM87b]. Unfortunately, $p\colon X \to Q$ is proper only when G is finite. Note that (1) and (2) above are equivalent to the subspaces being complete in the induced topology.

Let $R = \oplus c_i R_i$ and $c = \sum c_i$ be as above. Define $\mathcal{C}^\infty(X)_{R_i}$ to be the sum of the elements of $\mathcal{C}^\infty(X)$ which transform by the representation R_i of G and let $\mathcal{C}^\infty(X)_R = \oplus(\mathcal{C}^\infty(X)_{R_i})^{\oplus c_i}$. Then $\mathcal{C}^\infty(X)_R$ is closed in $\mathcal{C}^\infty(X)^c$, hence Fréchet, and $\mathcal{C}^\infty(X)_R$ contains $\mathcal{C}^\infty(X)^G \cdot \mathscr{O}(X)_R$. Let $\mathrm{Mor}(X,R)$ (resp. $\mathrm{Mor}^\infty(X,R)$) denote the holomorphic (resp. smooth) maps of X to R. We have the G-equivariant maps $\mathrm{Mor}(X,R)^G$ and $\mathrm{Mor}^\infty(X,R)^G$. Let $\sigma\colon \mathcal{C}^\infty(X)_R \to \mathrm{Mor}^\infty(X,R^*)^G$ where $\sigma(f)$, $f \in \mathcal{C}^\infty(X)_R$, sends $x \in X$ to the element of R^* sending f to $f(x)$. Let $\Gamma \in \mathrm{Mor}^\infty(X,R^*)$. Define $\tau(\Gamma) \in \mathcal{C}^\infty(X)_R$ by $\tau(\Gamma)(x) = \Gamma(x)(r)$, $x \in X$, $r \in R$.

Lemma 7.2. (1) *The maps σ and τ are inverses of each other, hence $\mathcal{C}^\infty(X)_R$ and $\mathrm{Mor}^\infty(X,R^*)^G$ are isomorphic Fréchet spaces.*
(2) *The isomorphisms σ and τ restrict to isomorphisms of $\mathcal{C}^\infty(X)^G \cdot \mathscr{O}(X)_R$ and $\mathcal{C}^\infty(X)^G \cdot \mathrm{Mor}(X,R^*)^G$.*

Hence $\mathcal{C}^\infty(X)^G \cdot \mathscr{O}(X)_R$ is complete if and only if $\mathcal{C}^\infty(X)^G \cdot \mathrm{Mor}(X,R^*)^G$ is complete.

We say that X is *good* if Theorem 7.1 holds for X and every G-module R.

Proposition 7.3. *Suppose that every slice representation of X is good. Then X is good.*

Proof. Since we have invariant partitions of unity (Lemma 6.7), it is clear that goodness is a local condition over Q. Take an open cover of X by tubes $X_i = T_{B_i}$ corresponding to slice representations (W_i, H_i) of X. If each T_{W_i} is good, it is clear that each T_{B_i} is good, hence X is good. Thus we only need to show that $X = T_W = G \times^H W$ is good if W is a good H-module and H is a reductive subgroup of G. Note that $\mathcal{C}^\infty(X)^G \simeq \mathcal{C}^\infty(W)^H$.

First we show that $\mathcal{C}^\infty(X)^G \cdot \mathrm{Mor}(X,R^*)^G$ is complete. Let $\rho\colon \mathrm{Mor}^\infty(X,R^*)^G \to \mathrm{Mor}^\infty(W,R_H^*)^H$ be the restriction mapping where R_H^* is R^* considered as an H-module and $W = W_e$ is the fibre of T_W at $[e,0]$. We have an inverse η to ρ where $\eta(\Gamma)([g,w]) =$

$g\cdot\Gamma(w)$ for $\Gamma \in \mathrm{Mor}^\infty(W, R_H^*)^H$, $g \in G$ and $w \in W$. Clearly η and ρ induce isomorphisms of $\mathcal{C}^\infty(X)^G\cdot\mathrm{Mor}(X,R^*)^G$ and $\mathcal{C}^\infty(W)^H\cdot\mathrm{Mor}(W,R_H^*)^H$. Since the latter space is complete by hypothesis, we have that $\mathcal{C}^\infty(X)^G\cdot\mathrm{Mor}(X,R^*)^G$ is complete.

Now $\mathrm{Der}_Q^\infty(X)^G$ is the image of

$$\mathrm{Der}_Q^\infty(G\times W)^{G\times H} \simeq (\mathfrak{g}\otimes\mathcal{C}^\infty(W))^H \oplus \mathrm{Der}_Q^\infty(W)^H$$

with kernel the $\mathcal{C}^\infty(G\times W)$-multiples of $\mathfrak{h}$ which are $(G\times H)$-invariant where $\mathfrak{h}$ sits diagonally inside $\mathrm{Der}_Q(G\times W)$ (since the action of H is on both factors of $G\times W$). We may write $\mathfrak{g}$ as a direct sum $\mathfrak{h}\oplus\mathfrak{m}$ where $\mathfrak{m}$ is H-stable, and then $\mathrm{Der}_Q^\infty(X)^G$ is the isomorphic image of the direct sum $(\mathfrak{m}\otimes\mathcal{C}^\infty(W))^H\oplus\mathrm{Der}_Q^\infty(W)^H$ where $(\mathfrak{m}\otimes\mathcal{C}^\infty(W))^H\simeq \mathcal{C}^\infty(W)_{(\mathfrak{m}^*)}$. We have an induced direct sum decomposition

$$\mathcal{C}^\infty(X)^G\cdot\mathrm{Der}_Q(X)^G \simeq \mathcal{C}^\infty(W)^H\cdot\mathscr{O}(W)_{(\mathfrak{m}^*)}\oplus\mathcal{C}^\infty(W)^H\cdot\mathrm{Der}_Q(W)^H.$$

Since W is good, both $\mathcal{C}^\infty(W)^H\cdot\mathscr{O}(W)_{(\mathfrak{m}^*)}$ and $\mathcal{C}^\infty(W)^H\cdot\mathrm{Der}_Q(W)^H$ are complete, hence $\mathcal{C}^\infty(X)^G\cdot\mathrm{Der}_Q(X)^G$ is complete and X is good. □

We want to show that every representation of every reductive group is good. So we need to show that every G-module V is good. Our inductive assumption will be that every proper slice representation (W,H) of V is good. We first reduce to the case that $V^G=0$.

Let N and P be smooth manifolds as before and let N_0 be a closed subset of N. Let $\mathcal{C}^\infty(N,N_0)$ denote the smooth functions on N which are flat on N_0, i.e., have vanishing Taylor series on N_0. It is easy to see that $\mathcal{C}^\infty(N\times P, N_0\times P)\simeq\mathcal{C}^\infty(N,N_0)\,\widehat{\otimes}\,\mathcal{C}^\infty(P)$.

Lemma 7.4. *Write $V=V^G\oplus V'$ where V' is a G-submodule of V. If V' is good, then V is good.*

Proof. Note that $\mathcal{C}^\infty(V)^G\simeq\mathcal{C}^\infty(V^G)\,\widehat{\otimes}\,\mathcal{C}^\infty(V')^G$ and that

$$\mathrm{Der}_{Q_V}(V)^G\simeq\mathscr{O}(V^G)\,\widehat{\otimes}\,\mathrm{Der}_{Q_{V'}}(V')^G.$$

If $\mathcal{C}^\infty(V')^G\cdot\mathrm{Der}_{Q_{V'}}(V')^G$ is complete, then so is $\mathcal{C}^\infty(V)^G\cdot\mathrm{Der}_{Q_{V'}}(V')^G$ since we are just taking completed tensor product with $\mathcal{C}^\infty(V^G)$. Similarly, for R a G-module,

$$\mathscr{O}(V)_R\simeq\mathscr{O}(V^G)\,\widehat{\otimes}\,\mathscr{O}(V')_R.$$

If $\mathcal{C}^\infty(V')^G\cdot\mathscr{O}(V')_R$ is complete, then

$$\mathcal{C}^\infty(V^G)\,\widehat{\otimes}\,(\mathcal{C}^\infty(V')^G\cdot\mathscr{O}(V')_R)$$

is complete. □

The slice representation at a principal orbit of V has the form (W,H) where the closed H-orbits in W are fixed points. Hence we start our induction with the following.

Lemma 7.5. *Let V be a G-module whose closed orbits are all fixed points. Then V is good.*

Proof. By Lemma 7.4 we may reduce to the case that Q_V is a point. Then $\mathcal{C}^\infty(V)^G=\mathbb{C}$ and there is nothing to prove. □

For the next few results, let $V_{\mathbb{R}}$ denote V considered as a real vector space and let $G_{\mathbb{R}}$ denote G considered as a real Lie group.

Lemma 7.6. *The algebra $\mathbb{R}[V_{\mathbb{R}}]^{G_{\mathbb{R}}}$ is finitely generated.*

For lack of a good reference, we give the following:

Proof. It is enough to show that the invariants of $(G^0)_{\mathbb{R}}$ are finitely generated. Now G^0 is a product G_sT where G_s is connected semisimple, T is a torus and the two factors commute. The invariants of $(G_s)_{\mathbb{R}}$ are the same as those of $(\mathfrak{g}_s)_{\mathbb{R}}$ which is a real semisimple Lie algebra. Since any representation of such a Lie algebra is completely reducible, we can assume that $G = T \simeq (\mathbb{C}^*)^d$ for some $d \geq 1$.

Now $T_{\mathbb{R}}$ is a product $(S^1)^d \times (\mathbb{R}^{>0})^d$. Since $(S^1)^d$ is compact, it acts completely reducibly, so we need only consider the factor $(\mathbb{R}^{>0})^d$. Let W be an irreducible subspace of V. Then T acts on $W \simeq \mathbb{C}$ by a character χ where $\chi(z_1, \dots, z_d) = z_1^{a_1} \cdots z_d^{a_d}$ for $(z_1, \dots, z_d) \in T$ and some $(a_1, \dots, a_d) \in \mathbb{Z}^d$. It follows that $(r_1, \dots, r_d) \in (\mathbb{R}^{>0})^d$ acts on $W_{\mathbb{R}} \simeq \mathbb{R}^2$ as multiplication by $\prod r_i^{a_i}$. It is now clear that $(\mathbb{R}^{>0})^d$ acts completely reducibly on $\mathbb{R}[V_{\mathbb{R}}]$. □

We need to consider a different quotient space $\widetilde{Q}$ for our consideration of smooth invariant functions on V. We have homogeneous generators $p_1, \dots, p_m$ of $\mathscr{O}_{\mathrm{alg}}(V)^G$. Let $\tilde{p}_1, \dots, \tilde{p}_{2m}$ be the real and imaginary parts of $p_1, \dots, p_m$. Choose homogeneous real invariant polynomials $\tilde{p}_{2m+1}, \dots, \tilde{p}_n$ such that $\mathbb{R}[V_{\mathbb{R}}]^{G_{\mathbb{R}}}$ is generated by the $\tilde{p}_j$. Then we have a quotient mapping

$$\tilde{p} = (\tilde{p}_1, \dots, \tilde{p}_n) \colon V \to \mathbb{R}^n$$

with image $\widetilde{Q}$.

Remark 7.7. Luna [Lun76] shows that $\tilde{p}^*\mathcal{C}^\infty(\mathbb{R}^n)$ is the set of smooth functions on V constant on the fibres of $\tilde{p}$. In our case the fibres of $\tilde{p}$ and p are the same (as sets) and are of the form $G \times^H \mathcal{N}(W)$. Any G-invariant continuous function must be constant on such a set. Hence $\tilde{p}^*\mathcal{C}^\infty(\mathbb{R}^n) = \mathcal{C}^\infty(V)^G$.

Example 7.8. Let $V = \mathbb{C}$ and $G = \{\pm 1\}$ acting by scalar multiplication. Then $p = z^2 \colon V \to \mathbb{C}$. In real coordinates the real and imaginary parts of z^2 are $x^2 - y^2$ and $2xy$. The invariant smooth function $x^2 + y^2$ is not of the form $h(x^2 - y^2, 2xy)$ where h is smooth, so the real and imaginary parts of p are not enough. One can take $\tilde{p}$ to be

$$(x, y) \mapsto (x^2 - y^2, 2xy, x^2 + y^2) \colon V \to \mathbb{R}^3.$$

We need to use a form of polar coordinates on $\mathbb{R}^n$ (see [Lun76]). Let e_i be the degree of $\tilde{p}_i$ and choose $b_i \in \mathbb{N}$ such that $b_i e_i = e$ is independent of $i = 1, \dots, n$. Let $P = \{y \in \mathbb{R}^n \mid \sum y_i^{2b_i} = 1\}$. Then P is a compact smooth submanifold of $\mathbb{R}^n$. Consider the mapping $\tau \colon \mathbb{R}^+ \times P \to \mathbb{R}^n$ sending (t, y) to $t \cdot y$ where, as usual, $t \cdot y = (t^{e_1} y_1, \dots, t^{e_n} y_n)$. Then $\mathbb{R}^+ \cdot P = \mathbb{R}^n$. From [Lun76, Lemma 1.5] we have:

Lemma 7.9. *Pullback by τ gives an isomorphism $\mathcal{C}^\infty(\mathbb{R}^n, 0) \simeq \mathcal{C}^\infty(\mathbb{R}^+, 0) \mathbin{\widehat{\otimes}} \mathcal{C}^\infty(P)$.*

Let Υ denote $\tilde{p}^{-1}(P)$. Then Υ is a smooth G-submanifold of V since the differential of the homogeneous polynomial $\sum \tilde{p}_i^{2e_i}$ does not vanish on Υ. We have a kind of polar coordinates on V given by $\sigma \colon \mathbb{R}^+ \times \Upsilon \to V$, $(t, v) \mapsto tv$.

Now we need another version of surjectivity of $\tilde{p}^*$ which can be found in [Lun76, Section 3.3].

Lemma 7.10. *We have the equality*

$$(\mathrm{id}\times\tilde{p}|_{\Upsilon})^*(\mathcal{C}^\infty(\mathbb{R}^+,0)\,\widehat{\otimes}\,\mathcal{C}^\infty(P))=\mathcal{C}^\infty(\mathbb{R}^+,0)\,\widehat{\otimes}\,\mathcal{C}^\infty(\Upsilon)^G.$$

Corollary 7.11. *Pullback by σ induces an isomorphism*

$$\sigma^*\colon \mathcal{C}^\infty(V,\mathcal{N}(V))^G\to\mathcal{C}^\infty(\mathbb{R}^+,0)\,\widehat{\otimes}\,\mathcal{C}^\infty(\Upsilon)^G.$$

Proof. It is clear that σ^* is injective. Let $f\in\mathcal{C}^\infty(\mathbb{R}^+,0)\,\widehat{\otimes}\,\mathcal{C}^\infty(\Upsilon)^G$. Then f is the pullback of some $h\in\mathcal{C}^\infty(\mathbb{R}^+,0)\,\widehat{\otimes}\,\mathcal{C}^\infty(P)$. By Lemma 7.9, h corresponds to an element of $\mathcal{C}^\infty(\mathbb{R}^n,0)$ which in turn pulls back to an element of $\mathcal{C}^\infty(V,\mathcal{N}(V))^G$. Hence σ^* is surjective. □

We need a version of E. Borel's lemma [GG73, Chapter IV, proof of Lemma 2.5].

Lemma 7.12. *Let $f_0,\ f_1,\dots$ be a sequence of smooth functions on P. Let ρ in $\mathcal{C}^\infty([0,1))$ be real valued, have compact support and equal 1 on a neighbourhood of 0 in $[0,1)$. Then there are positive increasing numbers λ_j, $\lim_{j\to\infty}\lambda_j=\infty$, such that for $t\in\mathbb{R}^+$ and $x\in P$,*

$$\sum_{j=0}^{\infty}\rho(\lambda_j t)\frac{t^j}{j!}f_j(x)$$

converges to a smooth function f on $\mathbb{R}^+\times P$ such that

$$\frac{\partial^j f}{\partial t^j}(0,x)=f_j(x),\ \ j=0,1,\dots$$

Proposition 7.13. *Let V be a G-module all of whose proper slice representations are good. Then V is good.*

Proof. We may assume that $V^G=0$. Let $A_1,\dots,A_k$ generate the $\mathscr{O}_{\mathrm{alg}}(Q_V)$-module $\mathrm{Der}_{Q_V,\mathrm{alg}}(V)^G$. Then they generate $\mathrm{Der}_{Q_V}(V)^G$. We may assume that the A_i are homogeneous. We first show that the set of all sums $\sum_i h_i(v)A_i(v)$ is complete, $v\in\Upsilon$, $h_i\in\mathcal{C}^\infty(\Upsilon)^G$. There is a retraction η from $V\setminus\mathcal{N}(V)$ to Υ which sends v to $v/\rho(v)$ where $\rho(v)^{2d}=\sum_i\tilde{p}_i(v)^{2e_i}$ and we choose $\rho(v)$ as the positive $2d$th root. If $\sum_i h_{i,\ell}(v)A_i(v)$ for $v\in\Upsilon$ has a limit A as $\ell\to\infty$, then we see that $A\circ\eta$ is the limit of the $\sum_i h_{i,\ell}(v/\rho(v))\rho(v)^{-n_i}A_i(v)$ for $v\in V\setminus\mathcal{N}(V)$ where n_i is the degree of homogeneity of A_i, $i=1,\dots,k$. Since all the slice representations of $V\setminus\mathcal{N}(V)$ are good, we find that $A\circ\eta$ is a sum $\sum f_i(v)A_i(v)$ where $v\in V\setminus\mathcal{N}(V)$. Restricting back to Υ we see that our original A can be expressed in the form $\sum_i h_iA_i$. Let D_1 denote the restriction of $\mathrm{Der}_{Q_V}(V)^G$ to Υ. We have shown that $\mathcal{C}^\infty(\Upsilon)^G\cdot D_1$ is complete. It follows that $\mathcal{C}^\infty(\mathbb{R}^+)\,\widehat{\otimes}\,\mathcal{C}^\infty(\Upsilon)^G\cdot D_1$ is complete.

Consider an element A in the closure of $\mathcal{C}^\infty(V)^G\cdot\mathrm{Der}_{Q_V}(V)^G$ in $\mathrm{Der}^\infty_{Q_V}(V)^G$. Then the Taylor series of A at 0 is in the formal power series module generated by the A_i over $\mathbb{R}[[V_\mathbb{R}]]^G$. Thus replacing A by $A-\sum f_iA_i$ for appropriate $f_1,\dots,f_k\in\mathcal{C}^\infty(V)^G$ we can reduce to the case that A is flat at zero. Now $A_i\circ\sigma(t,v)=t^{n_i}A_i(v)$ for $t\in\mathbb{R}^+$ and $v\in\Upsilon$. Hence $A\circ\sigma$ lies in the closure of sums of the form $\sum a_i(t,v)A_i(v)$, $v\in\Upsilon$, where $a_i(t,v)\in\mathcal{C}^\infty(\mathbb{R}^+)\,\widehat{\otimes}\,\mathcal{C}^\infty(\Upsilon)^G$. By what we just showed, this space is complete, hence $A\circ\sigma$ has the form $\sum_i a_i(t,v)A_i(v)$ where the $a_i(t,v)$ are in $\mathcal{C}^\infty(\mathbb{R}^+)\,\widehat{\otimes}\,\mathcal{C}^\infty(\Upsilon)^G$.

Now $A\circ\sigma$ is flat on $\{0\}\times\Upsilon$ (since A was flat at 0). Hence

$$\sum_i \frac{\partial^j a_i}{\partial t^j}(0,v)A_i(v)=0,\ v\in\Upsilon,\ j=0,1,\dots.$$

Let $h_i^j\in\mathcal{C}^\infty(P)$ such that $(\tilde p|_\Upsilon)^*(h_i^j)(v)=(\partial^j a_i/\partial t^j)(0,v)$, $v\in\Upsilon$. Let ρ and the λ_j be as in Lemma 7.12 such that

$$\sum_{j=0}^\infty h_i^j(t,y)\text{ converges to }h_i(t,y)\in\mathcal{C}^\infty(\mathbb{R}^+\times P),\ i=1,\dots,k,$$

where

$$h_i^j(t,y)=\rho(\lambda_j t)\frac{t^j}{j!}h_i^j(y).$$

For each j, the $h_i^j(t,y)$ pull back to a relation of the A_i, hence the h_i pull back to functions k_i such that $\sum k_iA_i=0$. By construction, k_i and a_i have the same Taylor series on $0\times\Upsilon$, $1\le i\le k$. Hence we can reduce to the case that

$$A\circ\sigma=\sum a_i(t,v)A_i(v),\ \text{where }a_i(t,v)\in\mathcal{C}^\infty(\mathbb{R}^+,0)\,\widehat{\otimes}\,\mathcal{C}^\infty(\Upsilon)^G,\ i=1,\dots,k.$$

By Corollary 7.11 the $t^{-n_i}a_i(t,v)$ are pullbacks of functions $f_i\in\mathcal{C}^\infty(V,\mathcal{N}(V))^G$, and we have that $A=\sum_i f_iA_i$. Hence $\mathcal{C}^\infty(V)^G\cdot\operatorname{Der}_{Q_V}(V)^G$ is complete. The argument for $\mathcal{C}^\infty(V)^G\cdot\mathscr{O}(V)_R$ is similar. Hence V is good. □

Proof of Theorem 7.1. This follows immediately from Proposition 7.3, Lemma 7.5 and Proposition 7.13. □

8. Reduction to type $\mathcal{F}$

We have Stein G-manifolds X and Y which are locally G-biholomorphic over Q. We show that one can deform a strong G-homeomorphism $\Phi\colon X\to Y$ to a G-diffeomorphism of type $\mathcal{F}$ by making small deformations locally. The same process works if Φ is a strict G-diffeomorphism. Note that composition with G-biholomorphisms inducing id_Q preserves the property of being of type $\mathcal{F}$. This will allow us to reduce locally to the case that $X=Y$. First another result [Lee13, Corollary 6.27].

Lemma 8.1. *Let N and P be smooth manifolds. Let A be a closed subset of N and $f\colon N\to P$ a continuous map which is smooth on A. Then there is a homotopy f_t of f with $f_t=f$ on A, $f_1=f$ and $f_0\colon N\to P$ smooth.*

Above, f smooth on A means that $f|_A$ has local extensions to smooth maps from open subsets of N to P.

We work locally on X and Y, so that we may assume that $X=Y=S\times T_B$ are standard neighbourhoods. We consider what transpires in a neighbourhood of (s_0,x_0) where $s_0\in S$ and $x_0=[e,0]\in T_B$. Let L_{vb} denote the group of G-vector bundle automorphisms of T_W inducing the identity on the quotient. Then $L_{\mathrm{vb}}\subset\operatorname{Aut}_{Q_B}(T_B)^G$. We have the action of $t\in[0,1]$ on $x=(s,[g,w])$ where $t\cdot x=(s,[g,tw])$, $s\in S$, $[g,w]\in T_B$. This induces an action $z\mapsto t\cdot z$ on the quotient. Let $\Phi\colon X\to X$ be a strong G-homeomorphism (or a G-diffeomorphism of type $\mathcal{F}$). Let $\Phi_t(x)=t^{-1}\cdot\Phi(t\cdot x)$, $x\in X$, $t\in(0,1]$. By Corollary 3.13, $\Phi_0=\lim_{t\to 0}\Phi_t$ exists. From Lemma 3.7, Corollary 3.14 and Lemma 6.25 we obtain the following.

Lemma 8.2. *Let $X = S \times T_B$ as above.*

(1) *Let $\Phi\colon X \to X$ be a strong G-homeomorphism. Then the family Φ_t is a homotopy of strong G-homeomorphisms. Moreover, there is a continuous map $\sigma\colon S \to L_{\mathrm{vb}}$ such that $\Phi_0(s,[g,w]) = \sigma(s)([g,w])$, $s \in S$, $[g,w] \in T_B$.*

(2) *Let $\Phi\colon X \to X$ be a G-diffeomorphism of type $\mathcal{F}$. Then the family Φ_t is a smooth homotopy of G-diffeomorphisms of type $\mathcal{F}$. Moreover, there is a smooth map $\sigma\colon S \to L_{\mathrm{vb}}$ such that $\Phi_0(s,[g,w]) = \sigma(s)([g,w])$, $s \in S$, $[g,w] \in T_B$.*

Remark 8.3. Let $\sigma\colon S \to L_{\mathrm{vb}}$ be continuous (resp. smooth). Define $\Phi(s,[g,w]) = \sigma(s)([g,w])$. We leave it to the reader to show that Φ is a strong G-homeomorphism (resp. G-diffeomorphism of type $\mathcal{F}$). Given a homotopy $\sigma_t\colon [0,1]\times S \to L_{\mathrm{vb}}$ of continuous maps, the corresponding family Φ_t is a homotopy of strong G-homeomorphisms. Using Lemma 8.1 we will be able to pass from the case of continuous σ to the case of σ smooth, i.e., to the case of G-diffeomorphisms of type $\mathcal{F}$.

We call a strong G-homeomorphism *special* if it corresponds to a continuous map $\sigma\colon S \to L_{\mathrm{vb}}$.

The next few results show that we can locally construct homotopies of a strong G-homeomorphism Φ so that it becomes a G-diffeomorphism of type $\mathcal{F}$ over neighbourhoods of larger and larger compact subsets of a stratum of the quotient.

Let $p_B\colon T_B \to Q_B$ denote the quotient mapping and let p denote the quotient mapping of $S \times T_B$. Let $q_0 = p_B([e,0])$. We shall cut off the homotopy of Φ so that it is constant outside a compact set. Let $K \subset S$ be compact. Let $\rho\colon S \times Q_B \to [0,1]$ be a smooth function which is 1 on a neighbourhood of $K \times \{q_0\}$ and has compact support M. Let $\tau(t,z) = 1 + (t-1)\rho(z)$ for $t \in [0,1]$ and $z \in S \times Q_B$. Then $\tau(t,z) = 1$ outside of M, and $\tau(t,z) = t$ for z in a neighbourhood U of $K \times \{q_0\}$. We have the map $x \mapsto \tau(t,z)\cdot x$ where $z = p(x)$.

Corollary 8.4. *Let ρ, etc. be as above and let $\Phi\colon S \times T_B \to S \times T_B$ be a strong G-homeomorphism. Let $\Phi_t^\rho(x) = \tau(t,z)^{-1}\cdot\Phi(\tau(t,z)\cdot x)$ for $x \in S \times T_B$ and $t \in (0,1]$. Set $\Phi_0^\rho = \lim_{t\to 0}\Phi_t^\rho$. The family Φ_t^ρ, $t \in [0,1]$, is a homotopy of strong G-homeomorphisms joining Φ to Φ_0^ρ. Moreover, for each $t \in [0,1]$, Φ_t^ρ equals Φ over the complement of M, and $\Phi_0^\rho = \Phi_0$ is special over a neighbourhood of $K \times \{q_0\}$.*

Proof. Let $\{f_i\}$ be a standard generating set for $\mathscr{O}_{\mathrm{fin}}(T_B)$ which we may also consider as a standard generating set for $\mathscr{O}_{\mathrm{fin}}(S \times T_B)$. Since Φ_t is a homotopy of strong G-homeomorphisms,

$$\Phi_t^* f_i(x) = \sum b_{ij}^t(x) f_j(x),\ x \in S \times T_B,\ t \in [0,1],$$

where the $b_{ij}^t(x)$ are continuous and G-invariant in x. It follows that

$$(\Phi_t^\rho)^* f_i(x) = \sum b_{ij}^{\tau(t,z)}(x) f_j(x),\ x \in S \times T_B,\ t \in [0,1]$$

where the $b_{ij}^{\tau(t,z)}(x)$ are continuous. Hence Φ_t^ρ is a homotopy of strong G-homeomorphisms. □

Lemma 8.5. *In the situation of Corollary 8.4, suppose that Φ is a G-diffeomorphism of type $\mathcal{F}$ on $p^{-1}(\Omega)$ where Ω is a neighbourhood in $S \times Q_B$ of a closed subset $E \times \{q_0\} \subset$*

$S\times\{q_0\}$. Then there is a smaller neighbourhood Ω' of $E\times\{q_0\}$ such that $(t,z)\mapsto\tau(t,z)\cdot z$ sends $[0,1]\times\Omega'$ into Ω. All the strong G-homeomorphisms Φ_t^ρ are G-diffeomorphisms of type $\mathcal{F}$ when restricted to $p^{-1}(\Omega')$. On the complement of M we can arrange that $\Omega=\Omega'$.

Proof. Let α_t, $t\in[0,1]$, be the endomorphism of $S\times Q_B$ which is induced by the multiplicative action of $\tau(t,z)=1+(t-1)\rho(z)$. Then α_t is the identity outside of M. The α_t are also the identity on $S\times\{q_0\}$. Now Ω is a neighbourhood of the compact set $E_0=M\cap(E\times\{q_0\})$, and the α_t are the identity on E_0. Hence there is a neighbourhood Ω' of E_0 inside Ω such that all the α_t send Ω' into Ω. Of course, we can arrange that $\Omega=\Omega'$ outside M. Now consider the restriction of Φ to $p^{-1}(\Omega')$. It follows from Remark 3.15 and the argument of Corollary 8.4 that $\Phi_t^\rho(x)$ is smooth in (t,x). Thus it is easy to see that each Φ_t^ρ is a G-diffeomorphism inducing the identity on $S\times Q_B$ (with inverse constructed from Φ^{-1}). From the definition it is easy to see that Φ_t^ρ is of type $\mathcal{F}$ for $t\neq 0$. For $t=0$, fix a point $x\in p^{-1}(\Omega')$. We may assume that Φ has a local holomorphic extension $\Psi(x',y)$ for x' and y in a neighbourhood of $\tau(0,z)\cdot x$. Then $\Psi_t(x',y)=t^{-1}\cdot\Psi(t\cdot x',t\cdot y)$ corresponds to a matrix $(b_{ij}^t(x',y))$ which is smooth in (t,x',y) and holomorphic in y. It follows that $\Psi_{\tau(0,z')}(x',y)$ is a local holomorphic extension of $\Phi_{\tau(0,z')}(x')$ since $(b_{ij}^{\tau(0,z')}(x',y))$ is holomorphic in y. Thus Φ_t^ρ is a G-diffeomorphism of type $\mathcal{F}$ over $p^{-1}(\Omega')$ for all $t\in[0,1]$. □

Lemma 8.6. *Let $K\subset S$ be compact. Suppose that Φ is a G-diffeomorphism of type $\mathcal{F}$ over a neighbourhood Ω of the closed subset $E\times\{q_0\}$ of $S\times Q_B$. Also suppose that Φ is special over a neighbourhood $U=U'\times U''$ of $K\times\{q_0\}$ where $\overline{U}$ is compact. Then, perhaps shrinking U' and U'', there is a homotopy Φ_t of Φ with the following properties.*

1. *$\Phi_t(x)=\Phi(x)$ for all t if x is off the inverse image of a compact subset M of U.*
2. *Over a neighbourhood of the set $(K\cup(E\cap\overline{U'}))\times U''$, Φ_0 is a G-diffeomorphism of type $\mathcal{F}$.*
3. *$\Phi_t=\Phi$ over a neighbourhood of $E\times\{q_0\}$ for all t. Hence Φ_0 is a G-diffeomorphism of type $\mathcal{F}$ over a neighbourhood Ω' of $(E\cup K)\times\{q_0\}$.*
4. *$\Omega'=\Omega$ on the complement of M.*

Proof. We may assume that Φ is a G-diffeomorphism of type $\mathcal{F}$ over a closed neighbourhood of the set $(E\cap\overline{U'})\times\{q_0\}$. Since $E\cap\overline{U'}$ is compact, we may assume that the closed neighbourhood is of the form $E'\times\overline{U''}$. Since Φ is special over U, there is a corresponding continuous $\sigma\colon U'\to L_{\mathrm{vb}}$ which is smooth on $E'\cap U'$. By Lemma 8.1 there is a homotopy $\sigma(t,s)$ of σ, $t\in[0,1]$, $s\in U'$, such that $\sigma(1,s)=\sigma(s)$, $\sigma(t,s)=\sigma(s)$ on $E'\cap U'$ and $\sigma(0,s)$ is smooth. Now choose a smooth function $\alpha(t,s,q)$, $t\in[0,1]$, such that $\alpha(t,s,q)=t$ for (s,q) in a neighbourhood of $K\times\{q_0\}$ and such that $\alpha(t,s,q)=1$ for (s,q) off of a compact subset M of $U'\times U''$. Let $(s,x)\in S\times T_B$. Then $\Phi_t(s,x)=(s,\sigma(\alpha(t,s,p_B(x)),s)(x))$ is a homotopy of strong G-homeomorphisms over $U'\times U''$, where $\Phi_1=\Phi$, $\Phi_t=\Phi$ off of the inverse image of M, and Φ_0 is a G-diffeomorphism of type $\mathcal{F}$ over the interior of $E'\times U''$ and over a neighbourhood of $K\times\{q_0\}$, because of smoothness of the corresponding σ. We set $\Phi_t=\Phi$ off of the inverse image of M. By construction, $\Phi_t=\Phi$ over $(E'\cap U')\times U''$ and off the inverse image of M, hence $\Phi_t=\Phi$ over a neighbourhood of $E\times\{q_0\}$ which we can take to be the same as Ω when intersected with the complement of M. □

Theorem 8.7. *Let X and Y be Stein G-manifolds locally G-biholomorphic over a common quotient Q. Let $\Phi\colon X \to Y$ be a strong G-homeomorphism. Then there is a homotopy of Φ, through strong G-homeomorphisms, to a G-diffeomorphism of type $\mathcal{F}$.*

Proof. Consider the stratification of Q by the connected components of the Luna strata. There are at most countably many strata. Let Z_k denote the union of the strata of dimension k. We will inductively find homotopies of Φ such that it becomes a G-diffeomorphism of type $\mathcal{F}$ over a neighbourhood Ω of $Z_0 \cup \cdots \cup Z_k$. Each step of the finite induction will be done by a countable induction.

Let $k \geq 0$. Then $R_0 = Z_0 \cup \cdots \cup Z_{k-1}$ is closed (and perhaps empty). Suppose by induction we have shown that, modulo a homotopy of strong G-homeomorphisms, Φ is a G-diffeomorphism of type $\mathcal{F}$ over a neighbourhood Ω_0 of R_0. Let $K_1, K_2, \ldots$ be a locally finite collection of compact connected subsets of Z_k whose union is $(R_0 \cup Z_k) \setminus \Omega_0$. Let U_j be a neighbourhood of K_j in Q. We may assume that $X_{U_j} \simeq Y_{U_j} \simeq S_j \times T_{B_j}$ is a standard neighbourhood. Let $p_j\colon T_{B_j} \to Q_{B_j}$ denote the quotient mapping and let $q_j = p_j([e, 0])$. We may assume that $S_j \times \{q_j\}$ is the stratum of $S_j \times Q_{B_j}$ containing K_j. We may assume that we have a G-biholomorphism of X_{U_j} and Y_{U_j} inducing the identity on U_j. So we consider the restriction of Φ to X_{U_j} to be a strong G-homeomorphism of X_{U_j}. We may assume that the $\overline{U_j}$ are locally finite on $R_\infty = R_0 \cup Z_k$ and that no $\overline{U}_j$ intersects R_0. By induction assume that Φ is a G-diffeomorphism of type $\mathcal{F}$ over a neighbourhood Ω_{n-1} (in Q) of $R_{n-1} = R_0 \cup K_1 \cup \cdots \cup K_{n-1}$. We have the first step of the induction with Ω_0 our neighbourhood of R_0. Using Corollary 8.4 and Lemmas 8.5 and 8.6 we can find a homotopy Φ_t of Φ which equals Φ off of the inverse image of a compact subset $M_n \subset U_n$ such that Φ_0 is a G-diffeomorphism of type $\mathcal{F}$ over a neighbourhood Ω_n of R_n, where $\Omega_n = \Omega_{n-1}$ on the complement of M_n. We can consider our homotopy as taking place in the space of strong G-homeomorphisms of X and Y. Clearly, by local finiteness of the $\overline{U}_m$, in the limit we construct a homotopy Φ_t of Φ where Φ_0 is a G-diffeomorphism of type $\mathcal{F}$ over a neighbourhood of R_∞. This completes the induction. □

Our procedure of deforming strong G-homeomorphisms to G-diffeomorphisms of type $\mathcal{F}$, applied to a strict G-diffeomorphism, gives a homotopy of strict G-diffeomorphisms to a G-diffeomorphism of type $\mathcal{F}$. The key technical point is Lemma 5.8. Hence we have:

Theorem 8.8. *Let X and Y be Stein G-manifolds locally G-biholomorphic over a common quotient Q. Let $\Phi\colon X \to Y$ be a strict G-diffeomorphism. Then there is a homotopy of Φ, through strict G-diffeomorphisms, to a G-diffeomorphism of type $\mathcal{F}$.*

9. NHC-sections

We work towards proving the following theorem which, in light of Theorems 8.7 and 8.8, completes our proof of Theorem 1.4.

Theorem 9.1. *Let X and Y be Stein G-manifolds locally G-biholomorphic over a common quotient Q. Suppose that $\Phi\colon X \to Y$ is a G-diffeomorphism of type $\mathcal{F}$. Then there is a homotopy Φ_t of G-diffeomorphisms of type $\mathcal{F}$ where $\Phi_0 = \Phi$ and $\Phi_1\colon X \to Y$ is a G-biholomorphism.*

We now consider parameterised families of automorphisms of G-saturated open subsets of X. We use the notation of [Car58]. Let C be a compact Hausdorff space with closed subsets $N \subset H \subset C$. We define a corresponding sheaf $\mathfrak{F}$ on Q as follows. Let $U \subset Q$ be open and consider the group $\mathfrak{F}(U)$ of G-diffeomorphisms $\Phi(t,x)$ of X_U, $t \in C$, such that:

(1) $\Phi(t,x)$ is a continuous family of G-diffeomorphisms of X_U of type $\mathcal{F}$.
(2) For $t \in N$, $\Phi(t,x)$ is the identity, i.e., $\Phi(t,x) = x$ for all $x \in X_U$.
(3) For $t \in H$, $\Phi(t,x)$ is holomorphic in x.

Note that condition (1) is the same as saying that the partial derivatives of Φ in x are continuous in (t,x). The topology on $\mathfrak{F}(U)$ is uniform convergence of partial derivatives on compact sets. Similarly we define the sheaf $\mathfrak{L}\mathfrak{F}$ of continuous families of G-invariant vector fields of type $\mathcal{LF}$ on open subsets U of Q. The vector fields are zero for $t \in N$, holomorphic for $t \in H$ and of type $\mathcal{LF}$ for all $t \in C$. The topology on $\mathfrak{L}\mathfrak{F}(U)$ is again uniform convergence of derivatives on compact sets and it is a Fréchet space. We can also view $\mathfrak{L}\mathfrak{F}(U)$ as a closed subspace of $\mathcal{C}^0(C) \widehat{\otimes} \mathcal{LF}(U)$. Since $\mathcal{LF}(U)$ is a vector subspace of a nuclear space, hence nuclear, the topology on the tensor product and the completion are unique (take the π or ε topology). Since $\mathcal{LF}(U)$ is Fréchet (Theorem 7.1), so is $\mathfrak{L}\mathfrak{F}(U)$.

We say that a continuous function $f(t,x)$ on $C \times X_U$ is an *NHC-function* if it is G-invariant, zero for $t \in N$, holomorphic for $t \in H$ and smooth for all $t \in C$. The NHC-functions form a Fréchet space with the topology of uniform convergence of partial derivatives on compact sets. It is a closed subspace of $\mathcal{C}^0(C) \widehat{\otimes} \mathcal{C}^\infty(X)^G$. We may consider an NHC-function $f(t,x)$ as a function $\tilde{f}(t,q)$ for $q \in Q$. But then $\tilde{f}$ may not be smooth in q (see Example 7.8).

We now quote a lemma about surjections of Fréchet spaces from [Car58, Appendix].

Lemma 9.2. *Let $\pi\colon E \to E'$ be a continuous linear surjection of Fréchet spaces. Let B be a closed subset of the compact Hausdorff space A. Suppose that we have continuous mappings $f'\colon A \to E'$ and $h\colon B \to E$ such that f' agrees with $\pi \circ h$ on B. Then there is a continuous map $f\colon A \to E$ which extends h such that $\pi \circ f = f'$.*

Lemma 9.3. *Let $A(t,x)$ be in $\mathfrak{L}\mathfrak{F}(U)$ where U is a Stein open subset of Q. Suppose that $A_1, \dots, A_k$ generate $\operatorname{Der}_U(X_U)^G$ over $\mathscr{O}(U)$. Then there are NHC-functions $a_i(t,x)$, $x \in X_U$, such that*

$$A(t,x) = \sum_i a_i(t,x) A_i(x) \text{ for } x \in X_U.$$

Proof. For notational convenience we may suppose that $U = Q$. Let $E = \mathscr{O}(Q)^k$ and $E' = \operatorname{Der}_Q(X)^G$. Then we have the surjection π which sends $(h_1, \dots, h_k) \in E$ to $\sum h_i A_i \in E'$. Now $A(t,x)$ is the zero mapping from N to E', and it lifts to the zero mapping h of N to E. By Lemma 9.2, h extends to a continuous mapping $(h_i(t,x))$ of H to E which covers $A(t,x)$. Now consider the surjection of $E = (\mathcal{C}^\infty(X)^G)^k$ onto E', the space of smooth vector fields of type $\mathcal{LF}$, sending $(a_1, \dots, a_k)$ to $\sum a_i A_i$. The $h_i(t,x)$, considered as smooth functions, cover $A(t,x)\colon H \to \operatorname{Der}_Q(X)^G \subset E'$. By Lemma 9.2 we can find extensions of the $h_i(t,x)$ to NHC-functions $a_i(t,x)$ such that $A = \sum_i a_i(t,x) A_i$. ∎

Using the open mapping theorem one obtains:

Corollary 9.4. *Let Ω' be a neighbourhood of zero in the space of NHC-functions over U. Then there is a neighbourhood Ω of zero in $\mathfrak{LF}(U)$ such that any $A(t,x) \in \Omega$ is $\sum_i a_i(t,x)A_i(x)$ where $a_i(t,x) \in \Omega'$, $i = 1,\dots,k$.*

Here is a basic result about sections of $\mathfrak{F}$ and $\mathfrak{LF}$.

Theorem 9.5. *Let $K \subset Q$ be compact and U a neighbourhood of K. Let U' be a neighbourhood of K in U with compact closure in U. Then there is a neighbourhood Ω of the identity family in $\mathfrak{F}(U)$ and a continuous mapping $\log\colon \Omega \to \mathfrak{LF}(U')$ such that $\exp\log\Phi = \Phi|_{C\times U'}$ for $\Phi \in \Omega$.*

Proof. We may assume that we have a standard generating set $\{f_i\}$ for $\mathscr{O}_{\mathrm{fin}}(X)$. By Theorem 6.36 there is a neighbourhood Ω' of the identity in $\mathcal{F}(U)$ such that any $\Psi \in \Omega'$ admits a logarithm $\log\Psi \in \mathcal{LF}(U')$. The mapping $\Omega' \ni \Psi \mapsto \log\Psi$ is continuous. Let $\mathcal{F}(C\times U)$ denote continuous families of elements of $\mathcal{F}(U)$. Let Ω be the open subset of $\mathcal{F}(C\times U)$ of families $\Phi(t,x)$ such that $\Phi(t,x) \in \Omega'$ for all t. Then Ω is open in $\mathcal{F}(C\times U)$ and the family $t \mapsto \log\Phi(t,x)$ is a continuous family in $\mathcal{LF}(U')$. Now the intersection of Ω with $\mathfrak{F}(U)$ is open, and if $\Phi(t,x) \in \Omega\cap\mathfrak{F}(U)$, then $\log\Phi(t,x)$ is zero if $t \in N$ and is holomorphic if $t \in H$. Hence $\log\Phi(t,x) \in \mathfrak{LF}(U')$. Clearly $\log\colon \mathfrak{F}(U) \to \mathfrak{LF}(U')$ is continuous. □

If U, U' and Ω are as above, we say that every $\Phi \in \Omega$ *admits a logarithm in* $\mathfrak{LF}(U')$.

Corollary 9.6. *Suppose that $A_1,\dots,A_k$ generate $\mathrm{Der}_{U'}(X_{U'})^G$ over $\mathscr{O}(U')$. Let Ω' be a neighbourhood of zero in the space of NHC-functions over U'. Then there is a neighbourhood Ω of the identity family in $\mathfrak{F}(U)$ such that for any $\Phi \in \Omega$, $\log\Phi = \sum a_i(t,x)A_i(x)$ where the $a_i \in \Omega'$.*

Using topological tensor products one establishes the following variant of Lemma 6.27.

Lemma 9.7. *Let $S \times T_B$ be a standard neighbourhood in X and let K be a compact subset of $S \times Q_B$. Let $\{f_i\}$ be a standard generating set for $S \times T_B$. Then there is a neighbourhood Ω of the identity in $\mathfrak{F}(S\times Q_B)$ such that any $\Phi \in \Omega$ corresponds to a matrix $(b_{ij}(t,s,q))$, $t \in C$, $s \in S$, $q \in Q_B$, where $||(b^u_{ij}) - I||_{C\times K} < 1/2$, $0 \le u \le 1$.*

Proposition 9.8. *Let $U \subset Q$ be open, let $q \in U$ and $F = X_q$. Suppose that $\Phi \in \mathfrak{F}(U)$ is the identity when restricted to F. Then there is a neighbourhood U' of q such that Φ admits a logarithm in $\mathfrak{LF}(U')$.*

Proof. We may assume that we have a standard generating set for $\mathscr{O}_{\mathrm{fin}}(X_U)$. Shrinking U we are in the situation where $X_U = S\times T_B$ is a standard neighbourhood and $q = (s_0, q_0)$ where $s_0 \in S$ and q_0 is the image of $[e,0] \in T_B$. We have the action of $[0,1]$ on X_U sending (s,x) to $(s, u\cdot x)$, $s \in S$, $x \in T_B$, $u \in [0,1]$. Let Φ_u be the corresponding deformation of Φ. Then Φ_u is a continuous family of elements of $\mathfrak{F}(S\times Q_B)$, $u \in [0,1]$. Since Φ is the identity on F, $\Phi_0(t,s_0,x)$ is the identity, $t \in C$, $x \in T_B$. By Theorem 9.5 and Lemma 9.7 there is a neighbourhood Ω of the identity section in $\mathfrak{F}(S\times Q_B)$ and a neighbourhood U_0 of (s_0,q_0) such that every $\Psi \in \Omega$ admits a logarithm in $\mathfrak{LF}(U_0)$ and has a matrix (b_{ij}) such that $||(b^u_{ij}) - I||_{C\times\overline{U_0}} < 1/2$, $u \in [0,1]$. For u in a neighbourhood of 0, Φ_u lies in Ω, hence admits a logarithm $D_u \in \mathfrak{LF}(U_0)$. As in the proof of Lemma 6.28 this implies that Φ admits a logarithm over $U' = u \cdot U_0$. □

Remark 9.9. Let $K \subset Q$ be compact. Let U be a neighbourhood of K and let U' be a relatively compact neighbourhood of K in U. Let $\Phi \in \mathfrak{F}(U)$ and suppose that Φ is represented by the matrix $(a_{ij}(t,q))$ and admits a logarithm $D \in \mathfrak{LF}(U')$ with matrix $(d_{ij}(t,q))$ where the d_{ij} are close to zero. There is no a priori guarantee that the $d_{ij}(t,q)$ are holomorphic for $t \in H$ or zero for $t \in N$. However, by Lemma 9.3 and Corollary 9.4, we can arrange this to be true. It follows that we can arrange that, over U', $(a_{ij}(t,q))$ is holomorphic for $t \in H$ and the identity matrix for $t \in N$.

10. Grauert's proof

We now show how one can modify Cartan's version [Car58] of the proof of Grauert's Oka principle [Gra57a, Gra57b, Gra58] to obtain Theorem 9.1. Let $N \subset H \subset C$ be compact Hausdorff spaces as before, and we consider the corresponding sheaves $\mathfrak{F}$ and $\mathfrak{LF}$ defined in the previous section, together with the topologies on their sections. We recall that an open subset $U \subset Q$ is a *Runge domain* if it is Stein and $\mathscr{O}(Q)$ is dense in $\mathscr{O}(U)$. Here is the main theorem.

Theorem 10.1. *Suppose that N is a deformation retract of C. Then the following hold.*

(1) *The topological group $H^0(Q,\mathfrak{F})$ is pathwise connected.*
(2) *If $U \subset Q$ is a Runge domain, then $H^0(Q,\mathfrak{F})$ is dense in $H^0(U,\mathfrak{F})$.*
(3) $H^1(Q,\mathfrak{F}) = 0$.

We actually only need (3), but the proof of the theorem is by an induction involving all three statements.

Proof of Theorem 9.1. Let $\Phi\colon X \to Y$ be a G-diffeomorphism of type $\mathcal{F}$. We have an open cover $\{U_i\}$ of Q and G-biholomorphisms $\Gamma_i\colon X_i = X_{U_i} \to Y_i = Y_{U_i}$ covering id_{U_i}. We may assume that $X_i \simeq S_i \times T_{B_i} \subset S_i \times T_{W_i}$ is a standard neighbourhood in X where S_i is smoothly contractible, say a ball. Let $\Psi = {\Gamma_i}^{-1} \circ \Phi$. By Lemma 8.2 we have a homotopy Ψ_t of G-diffeomorphisms of type $\mathcal{F}$ where Ψ_0 is special and corresponds to a smooth mapping $\sigma\colon S_i \to L_{\mathrm{vb}}$. Here L_{vb} is the subgroup of $\mathrm{Aut}_{\mathrm{vb}}(T_{W_i})^G$ fixing the invariants. Since S_i is smoothly contractible, we have a smooth homotopy of σ to a constant mapping to L_{vb}, hence we have a homotopy of Ψ_0 to a G-biholomorphism inducing the identity on $S_i \times Q_{B_i}$. Combining the two homotopies we obtain a homotopy $\Psi_i(t,x)$ of ${\Gamma_i}^{-1} \circ \Phi$ which equals ${\Gamma_i}^{-1} \circ \Phi$ when $t = 1$ and is holomorphic when $t = 0$. Moreover, all the $\Psi_i(t,x)$ are of type $\mathcal{F}$. Let $\Phi_i(t,x) = \Gamma_i \circ \Psi_i(t,x)$. Then $\Phi_i(1,x) = \Phi$ on X_i and $\Phi_i(0,x)\colon X_i \to Y_i$ is a G-biholomorphism. Now $\Phi_i(t,x)^{-1} \circ \Phi_j(t,x)$ gives an element of $H^1(Q,\mathfrak{F})$ where $C = [0,1]$, $N = \{1\}$ and $H = \{0,1\}$. By (3) of Theorem 10.1 there are sections $c_i(t,x) \in H^0(U_i,\mathfrak{F})$ such that $\Phi_i(t,x)^{-1} \circ \Phi_j(t,x) = c_i(t,x)^{-1} \circ c_j(t,x)$. Then the $\Phi_i(t,x) \circ c_i(t,x)^{-1}$ combine to give a homotopy $\Phi(t,x)$ with $\Phi(1,x) = \Phi(x)$ and $\Phi(0,x)\colon X \to Y$ a G-biholomorphism inducing id_Q. □

We recall some definitions from [Car58]. Let $a_j \le b_j$ and $c_j \le d_j$ be real numbers, $j = 1,\ldots,m$. Let $z_j = x_j + iy_j$ be the usual coordinate functions on $\mathbb{C}^m$. The corresponding *cube* in $\mathbb{C}^m$ is the subset defined by the inequalities $a_j \le x_j \le b_j$ and $c_j \le y_j \le d_j$, $j = 1, \dots, m$. Let $K \subset Q$ be compact and let U be a neighbourhood of K. Suppose that we have a holomorphic mapping $f\colon Q \to \mathbb{C}^m$ which restricts to a biholomorphism of U onto an analytic subset of a neighbourhood of a cube Γ where Γ has real dimension k. We say that K is *k-special* or *special of dimension k* if $K = U \cap f^{-1}(\Gamma)$. Special compact sets K have nice properties. For example, K is holomorphically convex, so every holomorphic function on a neighbourhood of K can be uniformly approximated (on a neighbourhood of K) by functions holomorphic on Q. Also, Q is the union of special compact sets K_n where K_n lies in the interior of K_{n+1} for all n.

Let $K \subset Q$ be a special compact set, so that we can think of it as a subset of a cube Γ corresponding to real numbers $a_j \le b_j$ and $c_j \le d_j$, $j = 1, \dots, m$. Let $c \in [a_1, b_1]$. Let Γ' be the points of Γ where $x_1 \le c$ and let Γ'' be the points where $x_1 \ge c$. Let K' denote $K \cap \Gamma'$ and let K'' denote $K \cap \Gamma''$. Then K', K'' and $K' \cap K''$ are special. We say that the triple (K, K', K'') is a *special configuration*.

Let K be a compact set in Q. Define $H^0(K, \mathfrak{F})$ to be the direct limit (with the direct limit topology) of the groups $H^0(U, \mathfrak{F})$ for U an open set containing K. Our proof of Theorem 10.1 uses the following two key results.

Proposition 10.2. *Let K be a special compact subset of Q. Then the image of $H^0(Q, \mathfrak{F})$ in $H^0(K, \mathfrak{F})$ is dense in a neighbourhood of the identity element of $H^0(K, \mathfrak{F})$.*

Proposition 10.3. *Let (K, K', K'') be a special configuration in Q. Then any element $\Phi \in H^0(K' \cap K'', \mathfrak{F})$, sufficiently close to the identity, can be written in the form*

$$\Phi = \Phi' \circ (\Phi'')^{-1}$$

where $\Phi' \in H^0(K', \mathfrak{F})$ and $\Phi'' \in H^0(K'', \mathfrak{F})$.

We assume the propositions for now. By induction on $k \ge 0$ we prove the following statements.

- (i)$_k$ If K is a k-special compact set, then $H^0(K, \mathfrak{F})$ is pathwise connected.
- (ii)$_k$ If K is a k-special compact set, then $H^0(Q, \mathfrak{F}) \to H^0(K, \mathfrak{F})$ has dense image.
- (iii)$_k$ If (K, K', K'') is a special configuration where $K' \cap K''$ is k-special, then every $\Phi \in H^0(K' \cap K'', \mathfrak{F})$ can be written in the form $\Phi' \circ (\Phi'')^{-1}$ where $\Phi' \in H^0(K', \mathfrak{F})$ and $\Phi'' \in H^0(K'', \mathfrak{F})$.

Proof. Consider (i)$_0$. We have that $K = \{q_0\} \subset Q$ and we need to show that $H^0(\{q_0\}, \mathfrak{F})$ is pathwise connected. Let $\Phi(t, x)$ be a section of $\mathfrak{F}$ defined in a neighbourhood of $p^{-1}(q_0) = F$ and let $\lambda(t)$ denote the restriction of $\Phi(t, x)$ to F. Then for every $t_0 \in C$, $\lambda(t_0) \in L_{\mathrm{hr}}$, the algebraic subgroup of $\operatorname{Aut}(F)^G$ of holomorphically reachable points (see Theorem 6.32 and discussion preceding it). Since N is a deformation retract of C, λ has values in the identity component of L_{hr} and we have a homotopy of $\lambda(t)$ to the identity section on F. By Remark 6.34 we have a homomorphism $\nu\colon L_{\mathrm{hr}} \to \operatorname{Aut}_U(X_U)^G$ for a neighbourhood U of q_0 such that $\nu(\ell)$ restricted to F is ℓ for $\ell \in L_{\mathrm{hr}}$. This allows us to reduce to the case that $\Phi(t, x)$ restricts to the identity section on F. Then it follows from Proposition 9.8 that the logarithm $D(t, x)$ of $\Phi(t, x)$ exists in a neighbourhood of F, and $D(t, x)$ is a section of $\mathfrak{L}\mathfrak{F}$. Then we have the homotopy $\Phi(t, u, x) = \exp(uD(t, x))$,

$0 \leq u \leq 1$. This is a homotopy of $\Phi(t,x)$ to the identity map on a neighbourhood of F. Hence we have (i)$_0$.

Suppose that we have (i)$_k$. Let K be compact and k-special. Then (i)$_k$ implies that every element of $H^0(K,\mathfrak{F})$ is a product of finitely many elements of $H^0(K,\mathfrak{F})$ which are arbitrarily close to the identity. Apply Proposition 10.2 to the elements close to the identity. Then one obtains (ii)$_k$.

Suppose that we have (ii)$_k$. Let (K,K',K'') be a special configuration such that $K' \cap K''$ is special of dimension k. Let $\Phi \in H^0(K' \cap K'', \mathfrak{F})$. By (ii)$_k$ one can write $\Phi = \Psi \cdot \Phi_1$ where $\Phi_1 \in H^0(K' \cap K'', \mathfrak{F})$ is close to the identity section and $\Psi \in H^0(K',\mathfrak{F})$. By Proposition 10.3, we can write $\Phi_1 = \Phi' \cdot (\Phi'')^{-1}$ where $\Phi' \in H^0(K',\mathfrak{F})$ and $\Phi'' \in H^0(K'',\mathfrak{F})$. Then $\Phi = (\Psi \cdot \Phi') \cdot (\Phi'')^{-1}$ and we have (iii)$_k$.

Suppose that we have (i)$_k$, hence (iii)$_k$. Let K be compact and $(k+1)$-special. Let $\Phi \in H^0(U,\mathfrak{F})$ for U a neighbourhood of K. Let the cube in $\mathbb{C}^m$ corresponding to K be Γ where the condition on x_1 is $a_1 \leq x_1 \leq b_1$. Let $\lambda \in [a_1, b_1]$ and let K_λ denote the special compact subset of K defined by $x_1 = \lambda$. We may assume that $a_1 < b_1$ so that each K_λ has dimension k. By (i)$_k$, K_λ has a neighbourhood V_λ in K (we may choose closed neighbourhoods) such that the restriction of Φ to V_λ is homotopic to the identity section. There are a finite number of K_{λ_i} (with $\lambda_1 < \lambda_2 < \cdots$) such that the corresponding V_{λ_i} cover K. We can assume that the V_{λ_i} are $(k+1)$-special such that $V_{\lambda_i} \cap V_{\lambda_{i+1}}$ is k-special for each i. Let K_i denote V_{λ_i} and let $\Phi_i(u) \in H^0(K_i,\mathfrak{F})$ be a section in a neighbourhood of K_i, depending continuously on the parameter $u \in [0,1]$, such that $\Phi_i(0)$ is the section induced by Φ and $\Phi_i(1)$ is the identity section e.

The homotopies $\Phi_i(u)$ do not have to agree on the intersections $K_i \cap K_{i+1}$. We now modify them to agree. One is reduced to what Cartan calls an *elementary problem*. Let (K,K_1,K_2) be a special configuration where $K_1 \cap K_2$ is special of dimension k. Let $\Phi \in H^0(K,\mathfrak{F})$ and let $\Phi_i(u) \in H^0(K_i,\mathfrak{F})$ be homotopies such that $\Phi_i(0)$ is the restriction of Φ and $\Phi_i(1)$ is e. Then one wants to find $\Psi(u) \in H^0(K,\mathfrak{F})$ such that $\Psi(0) = \Phi$ and $\Psi(1) = e$.

Now $\Phi_1(u)^{-1}\Phi_2(u) \in H^0(K_1 \cap K_2, \mathfrak{F})$ is a homotopy from the identity section of $H^0(K_1 \cap K_2, \mathfrak{F})$ to itself. It is an element of $H^0(K_1 \cap K_2, \mathfrak{F}')$ where $\mathfrak{F}'$ is the sheaf of groups relative to the compact sets

$$C' = C \times [0,1],\ N' = (N \times [0,1]) \cup (C \times \{0\}) \cup (C \times \{1\}),\ H' = (H \times [0,1]) \cup N'$$

where N' is still a deformation retract of C'. Thus applying (iii)$_k$ to $\mathfrak{F}'$ we have that

$$\Phi_1(u)^{-1}\Phi_2(u) = \Phi'(u)\Phi''(u)^{-1}$$

where $\Phi'(u) \in H^0(K_1,\mathfrak{F})$ and $\Phi''(u) \in H^0(K_2,\mathfrak{F})$ depend continuously upon the parameter $u \in [0,1]$. They are the identity element for $u = 0$ and $u = 1$. Let $\Psi(u) = \Phi_1(u)\Phi'(u)$ in a neighbourhood of K_1 and let it equal $\Phi_2(u)\Phi''(u)$ in a neighbourhood of K_2. Then the definitions agree on the overlaps and we have $\Psi(u) \in H^0(K,\mathfrak{F})$ giving the desired deformation. This solves the elementary problem and shows that we have (i)$_{k+1}$. □

We have established (i), (ii) and (iii) which are the statements (i)$_k$, etc. with k omitted.

Proof of Theorem 10.1. Note that (2) follows from (ii). Now we consider (1). We know that there are special compact sets $K_1 \subset K_2 \subset \cdots$ where K_i is contained in the interior V_{i+1} of K_{i+1} for all i and where $Q = \cup K_i$. Let $\Phi \in H^0(Q, \mathfrak{F})$. By (i), the image of Φ in $H^0(K_n, \mathfrak{F})$ is homotopic to the identity section. Thus the image Φ_n of Φ in $H^0(V_n, \mathfrak{F})$ is homotopic to the identity section. Let $\Phi_n(u)$ be such a homotopy with $\Phi_n(0) = \Phi_n$ and $\Phi_n(1) = e$. Then $\Phi_n(u)^{-1}\Phi_{n+1}(u)$, over V_n, is an element of $H^0(V_n, \mathfrak{F}')$ where $\mathfrak{F}'$ is the sheaf of groups associated to the sets $N' \subset H' \subset C'$ given above. Applying (ii) to $\mathfrak{F}'$ we see that $\Phi_n(u)^{-1}\Phi_{n+1}(u)$ may be approximated over $K_{n-1} \subset V_n$ by elements of $H^0(V_{n+1}, \mathfrak{F}')$. Hence, without changing $\Phi_n(u)$, we can modify $\Phi_{n+1}(u)$ such that $\Phi_n(u)^{-1}\Phi_{n+1}(u)$ is arbitrarily close to the identity section over K_{n-1}. Thus one can arrange that the sequence $\Phi_n(u)$ converges over every compact subset of Q. Let $\Phi(u)$ denote the limit. Then $\Phi(u)$ is a homotopy of elements of $H^0(Q, \mathfrak{F})$ with $\Phi(0) = \Phi$ and $\Phi(1)$ the identity section. Here we have used Corollary 6.37. This proves (1).

Finally, we want to prove (3), that $H^1(Q, \mathfrak{F}) = 0$. Let K be a special compact subset of Q. As in [Car58, Section 5], using (iii) and an induction over a decomposition of K into a union of smaller special compact sets, one shows that $H^1(K, \mathfrak{F}) = 0$. Let K_n be a sequence of special compact sets as above, with V_n denoting the interior of K_n. Let $\{U_i\}$ be an open cover of Q and let $\{\Phi_{ij}\}$ be a cocycle with values in $\mathfrak{F}$. Then for each n we have sections $c_i^n \in H^0(U_i \cap V_n, \mathfrak{F})$ such that $\Phi_{ij} = (c_i^n)^{-1}c_j^n$ on $U_i \cap U_j \cap V_n$. Hence we have

$$c_i^{n+1}(c_i^n)^{-1} = c_j^{n+1}(c_j^n)^{-1} \text{ on } U_i \cap U_j \cap V_n.$$

Thus the $c_i^{n+1}(c_i^n)^{-1}$ define a section $\Phi_n \in H^0(V_n, \mathfrak{F})$. Using (ii) as above, we can arrange that the c_i^n converge on compact sets as $n \to \infty$. The limiting sections $c_i \in H^0(U_i, \mathfrak{F})$ have the property that

$$\Phi_{ij} = c_i^{-1}c_j \text{ on } U_i \cap U_j.$$

Hence $H^1(Q, \mathfrak{F}) = 0$. □

We are left with the proofs of the two propositions.

Proof of Proposition 10.2. Let $\Phi \in H^0(K, \mathfrak{F})$ where K is special and Φ is near to the identity section. By Theorem 9.5, there is a relatively compact neighbourhood U of K such that $D = \log \Phi$ exists and is in $\mathfrak{L}\mathfrak{F}(U)$. We may suppose that U is a Runge domain. By Lemma 9.3 we can write $D = \sum a_i(t, x)A_i$ where the a_i are NHC and the $A_i \in \operatorname{Der}_Q(X)^G$ generate $\operatorname{Der}_U(X_U)^G$. Thus it suffices to approximate NHC-functions on U by global ones. Using a partition of unity on C one is reduced to the problem of approximating holomorphic (resp. smooth G-invariant) functions on U (resp. X_U) by functions holomorphic on Q (resp. smooth and G-invariant on X). There is no problem for smooth functions, and for holomorphic functions approximation is possible because U is a Runge domain. □

Before giving the proof of Proposition 10.3 we need some preliminary results. First we consider a result on differential equations on Lie groups from [Car58, p. 115]. Let L be a Lie group and let $f(u)$, $f'(u)$ and $f''(u)$ be elements of L which are $\mathcal{C}^1$ in $u \in [0, 1]$ such that $f'(u) = f(u)f''(u)$ and $f(0) = f'(0) = f''(0) = e$, the identity of L. Then df/du is a tangent vector at $f(u)$, hence it is of the form $d\rho(f(u)) \cdot a(u)$ where $a(u)$ is a continuous family of elements of $T_e(L)$ and ρ denotes right multiplication. We write this as

$$\frac{df}{du} = a(u) \cdot f(u).$$

Similarly we have continuous $a'(u)$ and $a''(u)$ with

$$\frac{df'}{du} = a'(u) \cdot f'(u) \text{ and } \frac{df''}{du} = a''(u) \cdot f''(u) \tag{10.1}$$

Consider $a(u)$, $a'(u)$ and $a''(u)$ as elements of $\mathfrak{l}$, the Lie algebra of L.

Lemma 10.4. *Let $f(u)$, $a(u)$ etc. be as above. Then we have*

$$a'(u) = a(u) + \operatorname{Ad}(f(u)) \cdot a''(u). \tag{10.2}$$

Conversely, given $f(u)$ which is $\mathcal{C}^1$ in u with $f(0) = e$, define $a(u)$ as above and suppose that there are continuous $a'(u)$ and $a''(u)$ with values in $\mathfrak{l}$ satisfying (10.2)*. Then integrating the equations* (10.1) *with the conditions that $f'(0) = f''(0) = e$ we obtain $f'(u)$ and $f''(u)$ such that $f'(u) = f(u)f''(u)$.*

Let $U \subset Q$ be open. Let $\Phi(t, x) \in \mathfrak{F}(U)$ and let $A(t, x)$, $D(t, x) \in \mathfrak{LF}(U)$. Let F be a fibre of $p\colon X_U \to U$. Then the restriction of $\Phi(t, x)$ to F is a family $\Phi(t)|_F$ in L_{hr}, the reachable automorphisms in $\operatorname{Aut}(F)^G$. We have the adjoint action $\operatorname{Ad}\Phi(t)|_F$ on $A(t)|_F \in \mathfrak{l}_{\mathrm{hr}}$. If $\Phi = \exp D$, then $\operatorname{Ad}\Phi(t)|_F$ is the exponential of the action of $D(t)|_F$ by ad. It is not hard to see that the $\operatorname{Ad}\Phi(t)|_F$ combine to give an action $\operatorname{Ad}\Phi$ of Φ on $\mathfrak{LF}(U)$ and, of course, the $\operatorname{ad}D(t)|_F$ give us an action $\operatorname{ad}D$ of D on $\mathfrak{LF}(U)$ which is just bracket with D.

Proposition 10.5. *Let $K \subset Q$ be compact and let U be a relatively compact neighbourhood of K. If $\Phi \in \mathfrak{F}(U)$ is sufficiently close to the identity, then for any $A \in \mathfrak{LF}(U)$, $\operatorname{Ad}\Phi(A)$ is close to A over a neighbourhood of K. In particular, if A is close to zero, so is $\operatorname{Ad}\Phi(A)$.*

Proof. Let $D_1, \dots, D_k \in \operatorname{Der}_U(X_U)^G$ generate $\operatorname{Der}_U(X_U)^G$ as an $\mathscr{O}(U)$-module. Let U' be a neighbourhood of K relatively compact in U. By Theorem 9.5 and Corollary 9.6, by choosing $\Phi \in \mathfrak{F}(U)$ close to the identity, we have $\Phi = \exp D$, $D \in \mathfrak{LF}(U')$, where $D = \sum a_i D_i$ and the NHC functions a_i can be chosen close to zero. Then $\operatorname{Ad}\Phi(A) = (\exp \sum a_i \operatorname{ad}D_i)(A)$ is close to A in a neighbourhood of K. □

The following is essentially Hilfssatz 1 in [Gra57b].

Lemma 10.6. *Let (K, K', K'') be a special configuration. For every $t \in H$ suppose that we have a square matrix $M(t, q)$ which is holomorphic and invertible for q in a neighbourhood U of $K' \cap K''$, and suppose that $M(t, q)$ is continuous in (t, q). If $M(t, q)$ is sufficiently close to the identity on $H \times (K' \cap K'')$, then*

$$M(t, q) = M'(t, q) \cdot M''(t, q)^{-1}$$

where $M'(t, q)$ (resp. $M''(t, q)$) is continuous in (t, q), and for fixed $t \in H$ is holomorphic and invertible for q in a neighbourhood of K' (resp. K''). Moreover, $M'(t, q)$ and $M''(t, q)$ can be chosen in any given neighbourhood of the identity if $M(t, q)$ is sufficiently close to the identity.

Proof of Proposition 10.3. Let $\Phi \in H^0(K' \cap K'', \mathfrak{F})$ be close to the identity. Then we want $\Phi' \in H^0(K', \mathfrak{F})$ and $\Phi'' \in H^0(K'', \mathfrak{F})$ so that

$$\Phi(t,x) = \Phi'(t,x) \circ \Phi''(t,x)^{-1} \text{ for } p(x) \text{ close to } K' \cap K''.$$

We first do this for $t \in H$, where the sections have to be the identity on N. Cartan calls this the *fundamental problem.* If Φ is sufficiently close to the identity, then Theorem 9.5 shows that $\Phi(t,x) = \exp(D(t,x))$ for $t \in H$ where D vanishes on N (since Φ is the identity on N) and is holomorphic on a relatively compact neighbourhood U of $K' \cap K''$. Set $\Phi(t,u,x) = \exp(u \cdot D(t,x))$, $0 \le u \le 1$. Then $\partial\Phi/\partial u = D(t,x) \cdot \Phi(t,u,x)$. Suppose that we have parameterised G-invariant holomorphic vector fields $D'(t,u,x)$ and $D''(t,u,x)$ over the inverse image of neighbourhoods of K' and K'', respectively, zero on N, annihilating the invariants, such that

$$D'(t,u,x) = D(t,x) + \mathrm{Ad}\Phi(t,u,x) \cdot D''(t,u,x) \text{ for } p(x) \text{ close to } K' \cap K''.$$

By Lemma 10.4, integrating in u we obtain holomorphic parameterised sections $\Phi'(t,u,x)$ and $\Phi''(t,u,x)$ such that

$$\Phi(t,u,x) = \Phi'(t,u,x) \circ \Phi''(t,u,x)^{-1}$$

For $u = 1$ we get a solution of the fundamental problem. We will construct D' and D'' close to zero so that Φ' and Φ'' are close to the identity.

Let $\{A_i\}$ be $\mathscr{O}(U)$-module generators of $\mathrm{Der}_U(X_U)^G$. Then

$$\mathrm{Ad}\Phi(t,u,x) \cdot A_i = \sum m_{ij} A_j$$

where the coefficients $m_{ij}(t,u,q)$ are holomorphic for $q \in U$, and continuous for $t \in H$ and $u \in [0,1]$. This uses Lemma 9.3. If $\Phi(t,x)$ is sufficiently close to the identity section, then $\Phi(t,u,x) \cdot A_i$ is close to A_i and we may choose (m_{ij}) close to the identity. By Lemma 10.6 we have

$$\sum_i m_{ji} m'_{ik} = m''_{jk}$$

where the matrices $m'_{ij}(t,u,q)$ and $m''_{ij}(t,u,q)$ are invertible and depend continuously on $t \in H$ and $u \in [0,1]$ and are close to the identity. The first is holomorphic near K' and the second near K''. We have

$$D(t,x) = \sum a_i(t,q) A_i$$

where the $a_i(t,q)$ are holomorphic near $K' \cap K''$. By Corollary 9.4 we may assume that the $a_i(t,q)$ are near zero. It suffices to find $a'_i(t,u,q)$ and $a''_i(t,u,q)$ holomorphic near K' and K'', respectively, near zero, such that

$$a'_i(t,u,q) = a_i(t,q) + \sum_j m_{ji} a''_j(t,u,q)$$

for q near $K' \cap K''$. For this it suffices that

$$\sum_i m'_{ik} a'_i = \sum_i m'_{ik} a_i + \sum_i m''_{ik} a''_i.$$

Since our matrices are invertible, in place of the a'_i and a''_i one can take as unknowns the terms

$$b'_k = \sum_i m'_{ik} a'_i \text{ and } b''_k = \sum_i m''_{ik} a''_i.$$

Set

$$b_k = \sum m'_{ik}(t,u,q)a_i(t,q).$$

Then our equation becomes

$$b'_k(t,u,q) - b''_k(t,u,q) = b_k(t,u,q)$$

which we can always solve with $b'_k(t,u,q)$ and $b''(t,u,q)$ small if $b_k(t,u,q)$ is small [Car58, footnote p. 117].

So we have solved the fundamental problem. Given $\Phi(t,x)$ near the identity we find $\Phi'(t,x)$ and $\Phi''(t,x)$, near the identity, such that $\Phi = \Phi'(\Phi'')^{-1}$ for $t \in H$. Since Φ' and Φ'' are near the identity we may write them as $\exp D'$ and $\exp D''$, respectively, where D' and D'' are near zero. We have continuous maps $a'_i(t,q)$ and $a''_i(t,q)$ from H to holomorphic functions which are small on neighbourhoods U' of K' and U'' of K'', respectively, zero on N, such that

$$D' = \sum_i a'_i(t,q)A_i \text{ and } D'' = \sum a''_i(t,q)A_i.$$

Now the space of holomorphic functions on U' is nuclear. Since $\mathcal{C}^0(C) \to \mathcal{C}^0(H)$ is surjective, we find that

$$\mathcal{C}^0(C, \mathscr{O}(U')) \simeq \mathcal{C}^0(C) \mathbin{\widehat{\otimes}} \mathscr{O}(U') \to \mathcal{C}^0(H) \mathbin{\widehat{\otimes}} \mathscr{O}(U') \simeq \mathcal{C}^0(H, \mathscr{O}(U'))$$

is surjective. See [Trè67, Proposition 43.9]. Thus we have an extension of D' to an invariant holomorphic vector field annihilating the invariants for all $t \in C$, and similarly for D''. The extensions may not have coefficients close to zero for $t \in C$, but we can arrange this by multiplying by continuous functions $g'(t,x)$ and $g''(t,x)$ which are 1 for $t \in H$ and smooth and invariant in x. Thus we have smooth extensions $D' \in \mathfrak{LF}(U')$ and $D'' \in \mathfrak{LF}(U'')$ which have exponentials $\widetilde{\Phi}'(t,x)$ and $\widetilde{\Phi}''(t,x)$, respectively, near the identity. Consider the product

$$\Psi(t,x) = \widetilde{\Phi}'(\widetilde{\Phi}'')^{-1}\Phi^{-1}.$$

Then Ψ is a section of $\mathfrak{F}$ on a neighbourhood of $K' \cap K''$ which is the identity for $t \in H$. Since Ψ is near to the identity, $\log \Psi$ exists for $p(x)$ near $K' \cap K''$. Multiply $\log \Psi(t,x)$ by a smooth invariant cutoff function which is 1 on a neighbourhood of $K' \cap K''$ such that the closure of the support is compact in our original neighbourhood of $K' \cap K''$. Then the corresponding automorphism $\widetilde{\Psi}(t,x)$ is defined everywhere, in particular, on a neighbourhood of K'. Now we set

$$\Phi'(t,x) = \widetilde{\Psi}(t,x)^{-1}\widetilde{\Phi}'(t,x) \text{ for } p(x) \text{ in a neighbourhood of } K',$$

$$\Phi''(t,x) = \widetilde{\Phi}''(t,x) \text{ for } p(x) \text{ in a neighbourhood of } K''.$$

Then on a neighbourhood of $K' \cap K''$ we have

$$\Phi(t,x) = \Phi'(t,x) \cdot \Phi''(t,x)^{-1} \text{ for all } t \in C.$$

This completes the proof of the proposition. □

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