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Homotopy spherical space forms—a numerical bound for homotopy types

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ABSTRACT. Let G be a finite group. We show that for a fixed $n \geq 1$ the set of homotopy types of orbit spaces of all free G-actions on homotopy $(2n-1)$-spheres is finite and bounded by the order of some quotient group associated with G. In particular, we deduce that there are at most two homotopy types of lens spaces determined by all free $\mathbf{Z}/p^m$-actions on homotopy 3-spheres when p is an odd prime, and only one homotopy type of those spaces provided that $4 \nmid p-1$. There is also only one homotopy type of lens spaces of dimension $2n-1$ determined by all $\mathbf{Z}/2^m$-free actions provided that n is odd.

Introduction

A finite group has the periodic homology if it acts freely on a finite homology sphere [1, Chap. III] and [3, 4]. On the other hand, Swan [10] showed that any finite group with periodic homology of period l acts freely on a finite CW complex of the homotopy type of an $(nl-1)$-sphere for some positive integer n. This CW complex may not have the homotopy type of a closed manifold. The symmetric group S_3 has a periodic homology but does not act freely on any manifold which has the homotopy type of a sphere due to the result of Milnor [8]: *Every element of order two must be in the center*.

The study of free actions of a group on homotopy spheres is related to the study of their orbit spaces. In the case of the cyclic group $\mathbf{Z}/2$ of order two, the orbit space has the homotopy type of the real projective space and the problem of classifying manifolds of this homotopy type has been studied extensively in [7, 11]. An old result of homotopy type theory (see e.g. [9]) says that, up to homotopy, the set of lens spaces exhausts all the homotopy types of orbit spaces of free actions of the cyclic group $\mathbf{Z}/m$ of order m on homotopy $(2n-1)$-spheres. Thus the classification of all the $\mathbf{Z}/m$-actions on those homotopy spheres is equivalent to the classification of manifolds of the homotopy type of lens spaces studied by Browder in [2]. The set of homotopy

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types of lens spaces is evidently finite and their homotopy classification has been presented in [5, 9]. The main purpose of this paper is to present in Corollary 1.3 a proof of the following result to estimate the number of homotopy types of the orbit spaces.

Let G be a finite group with $H^{2n}(G;\mathbf{Z}) = \mathbf{Z}/|G|$ for a fixed $n \geq 1$, where $|G|$ is the order of the group G. Then the set of homotopy types of orbit spaces of all free G-actions on all homotopy $(2n-1)$-spheres is finite and bounded by the order of the quotient group $\mathrm{Aut}(\mathbf{Z}/|G|)/\{\pm\varphi^*; \varphi \in \mathrm{Aut}\, G\}$ of the automorphism group $\mathrm{Aut}(\mathbf{Z}/|G|)$, where φ^* is the induced automorphism on the cohomology $H^{2n}(G;\mathbf{Z}) = \mathbf{Z}/|G|$ by φ in the automorphism group $\mathrm{Aut}\, G$.

In particular, under some assumptions on n and m, we calculate in Corollary 2.3 the number of homotopy types of lens spaces given by free $\mathbf{Z}/p^m$-actions on homotopy $(2n-1)$-spheres and we estimate in Corollary 2.5 the number of homotopy types of orbit spaces given by free actions of the generalized quaternion group $Q_{2^{m+2}}$ on homotopy $(4n-1)$-spheres.

The paper is divided into two sections. In §1 we use tom Dieck's theorem [6, p. 126] to prove our main result. Then in Proposition 1.5 we deduce that the group of homotopy classes of homotopy equivalences of the orbit space of a free G-action on a homotopy sphere is isomorphic to the subgroup of $\mathrm{Aut}\, G$ which consists of all the automorphisms φ with $\varphi^* \equiv \pm 1 \pmod{|G|}$ provided that $|G| > 2$.

In §2 we examine the quotient group $\mathrm{Aut}(\mathbf{Z}/|G|)/\{\pm\varphi^*; \varphi \in \mathrm{Aut}\, G\}$ for a cyclic group G of finite order. In particular, there are at most two homotopy types of lens spaces determined by all the free $\mathbf{Z}/p^m$-actions on homotopy 3-spheres when p is an odd prime, and only one homotopy type of those spaces provided that $4 \nmid p-1$. There is also only one homotopy type of lens spaces of dimension $2n-1$, determined by all the $\mathbf{Z}/2^m$-free actions provided that n is odd. We prove that the number of homotopy types of the orbit spaces of free actions of the quaternion group Q_8 on homotopy 3-spheres is bounded by 2.

At the end, we deal with the group of homotopy types of self-homotopy equivalences of a lens space given by a free $\mathbf{Z}/p$-action on a homotopy $(2n-1)$-sphere, provided that p is a prime.

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1. Main result

An n-dimensional CW complex X with the homotopy type of an n-sphere S^n is called a *homotopy n-sphere*. An orientation for X consists of a choice of a generator $z(X)$ in the cohomology group $H^n(X;\mathbf{Z}) = \mathbf{Z}$. Suppose that a finite group G of order $|G|$ acts freely on X. Then by [4], the group G has periodic cohomology groups of period $n+1$ with $H^{n+1}(G;\mathbf{Z}) = \mathbf{Z}/|G|$. Given two oriented homotopy n-spheres X^1 and X^2 with free G-actions, we can associate to each G-map $f : X^1 \to X^2$ a degree $d(f)$, defined by $f^*z(X^2) = d(f)z(X^1)$, where f^* is the induced map on cohomology groups. Throughout this section G will be a finite group acting freely and cellularly on an odd dimensional homotopy sphere X. Then all the hypotheses of Theorem 4.11 in [6, p. 126] are satisfied for homotopy spheres and it can be formulated as follows.

Theorem 1.1. *Suppose that X^1 and X^2 are oriented homotopy n-spheres with free cellular actions of a finite group G. Then the following holds:*

(1) *The set $[X^1, X^2]_G$ of G-homotopy types of all G-maps $X^1 \to X^2$ is not empty.*
(2) *Let $f : X^1 \to X^2$ be a G-map. Suppose that $d \equiv d(f) \pmod{|G|}$. Then there exists a G-map $h : X^1 \to X^2$ such that $d(h) = d$.*
(3) *Suppose that $f_0, f_1 : X^1 \to X^2$ are G-maps. Then $d(f_0) \equiv d(f_1) \pmod{|G|}$.*
(4) *Two G-maps $f_0, f_1 : X^1 \to X^2$ are G-homotopic if and only if $d(f_0) = d(f_1)$.*

From Theorem 1.1 it follows that for a G-map $f : X^1 \to X^2$ of homotopy $(2n-1)$-spheres with free G-actions there is a G-map $g : X^2 \to X^1$ with $d(gf) = d(g)d(f) \equiv 1 \pmod{|G|}$. Hence the degree $d(f)$ determines a unit in $\mathbf{Z}/|G|$ and then an automorphism of the cyclic group $\mathbf{Z}/|G|$. Since $H^{2n}(G;\mathbf{Z}) = \mathbf{Z}/|G|$, we have the induced map of automorphism groups $\operatorname{Aut} G \to \operatorname{Aut}(H^{2n}(G;\mathbf{Z})) = \operatorname{Aut}(\mathbf{Z}/|G|)$. For an automorphism φ of the group G, let φ^* be its image in $\operatorname{Aut}(\mathbf{Z}/|G|)$. Observe that φ^* can be identified with a unit of the ring $\mathbf{Z}/|G|$ so in the sequel we may also identify φ^* with some positive integer less than the order $|G|$ of the group G, which is invertible in this ring.

Write X_γ for a homotopy $(2n-1)$-sphere X with a free cellular G-action $\gamma : G \times X \to X$ and let X/γ be the associated orbit space. By the Serre spectral sequence argument $H^k(X/\gamma;\mathbf{Z}) = H^k(G;\mathbf{Z})$ for $0 \le k < 2n-1$, $H^{2n-1}(X/\gamma;\mathbf{Z}) = \mathbf{Z}$ and $H^k(X/\gamma;\mathbf{Z}) = 0$ for $k > 2n-1$. Consequently, with any map between G-orbit spaces we can also associate its degree. Fix an arbitrary free cellular G-action γ_0 on a homotopy $(2n-1)$-sphere X^0. Then by Theorem 1.1, there is a G-map $f_\gamma : X_\gamma \to X^0_{\gamma_0}$.

Theorem 1.2. *Two orbit spaces X^1/γ_1 and X^2/γ_2 of homotopy $(2n-1)$-spheres X^1 and X^2 have the same homotopy type if and only if $d(f_{\gamma_1}) \equiv \pm\varphi^* d(f_{\gamma_2}) \pmod{|G|}$ for some φ in the group* Aut G *and G-maps $f_{\gamma_1} : X^1_{\gamma_1} \to X^0_{\gamma_0}$ and $f_{\gamma_2} : X^2_{\gamma_2} \to X^0_{\gamma_0}$.*

Proof. Suppose that for two orbit spaces X^1/γ_1 and X^2/γ_2 there is an automorphism φ of the group G such that $d(f_{\gamma_1}) \equiv \pm\varphi^* d(f_{\gamma_2}) \pmod{|G|}$. Then, in view of Theorem 1.1, there is a G-map $f'_{\gamma_2} : X^0_{\gamma_0} \to X^2_{\gamma_2}$ such that $d(f'_{\gamma_2} f_{\gamma_1}) = d(f'_{\gamma_2}) d(f_{\gamma_1}) \equiv \pm\varphi^* \pmod{|G|}$.

Let $(X^1/\gamma_1)_{2n-1}$ be the $(2n-1)$-stage of the Postnikov tower of X^1/γ_1 and consider the canonical maps $(X^1/\gamma_1)_{2n-1} \to K(G,1)$ to the Eilenberg-MacLane space $K(G,1)$ and $X^1/\gamma_1 \to (X^1/\gamma_1)_{2n-1}$. The map $K(\varphi^{-1},1) : K(G,1) \to K(G,1)$ determined by the automorphism φ^{-1} yields a map $\bar\varphi : (X^1/\gamma_1)_{2n-1} \to (X^1/\gamma_1)_{2n-1}$ with a homotopy commutative square

$$\begin{array}{ccc} (X^1/\gamma_1)_{2n-1} & \xrightarrow{\bar\varphi} & (X^1/\gamma_1)_{2n-1} \\ \downarrow & & \downarrow \\ K(G,1) & \xrightarrow{K(\varphi^{-1},1)} & K(G,1). \end{array}$$

Since the homotopy fiber of the map $X^1/\gamma_1 \to (X^1/\gamma_1)_{2n-1}$ is $(2n-2)$-connected, by obstruction theory the map $\bar\varphi : (X^1/\gamma_1)_{2n-1} \to (X^1/\gamma_1)_{2n-1}$ admits a map $\Phi : X^1/\gamma_1 \to X^1/\gamma_1$ such that the diagram

$$\begin{array}{ccc} X^1/\gamma_1 & \xrightarrow{\Phi} & X^1/\gamma_1 \\ \downarrow & & \downarrow \\ (X^1/\gamma_1)_{2n-1} & \xrightarrow{\bar\varphi} & (X^1/\gamma_1)_{2n-1} \end{array}$$

commutes up to homotopy. If $\bar x_0$ is a base point in X^1/γ_1 and $\tilde\Phi : X^1 \to X^1$ is a cover of the map Φ, then $\tilde\Phi(gx) = \varphi^{-1}(g)\tilde\Phi(x)$ for all g in the group G and x in the space X^1. This means that $\tilde\Phi : X^1_{\gamma_1'} \to X^1_{\gamma_1}$ is a G-map with $d(\tilde\Phi) \equiv (\varphi^*)^{-1} \pmod{|G|}$, where $\gamma_1' : G \times X^1 \to X^1$ is the G-action such that $\gamma_1'(g,x) = \varphi(g)x$ for all $g \in G$ and $x \in X^1$. For the composite G-map $f'_{\gamma_2} f_{\gamma_1} \tilde\Phi : X^1_{\gamma_1'} \to X^2_{\gamma_2}$, we get that $d(f'_{\gamma_2} f_{\gamma_1} \tilde\Phi) \equiv \pm 1 \pmod{|G|}$. Then Theorem 1.1 yields a G-map $h : X^1_{\gamma_1'} \to X^2_{\gamma_2}$ with $d(h) = \pm 1$ which in view of [9] induces a homotopy equivalence $X^1/\gamma_1 \to X^2/\gamma_2$ of the orbit spaces, since the spaces X^1/γ_1 and X^1/γ_1' are homeomorphic.

Suppose now that $f : X^1/\gamma_1 \to X^2/\gamma_2$ is a homotopy equivalence and fix a base point x_0 in the orbit space X^1/γ_1. Then we get an automorphism $\varphi = \pi_1(f)$ of the group G, determined by the fundamental group functor π_1. Let $\tilde f : X^1 \to X^2$ be a cover of the map f and $\tilde\Phi : X^1 \to X^1$ the map associated to the automorphism φ^{-1} in the light of the first part of this proof. Then $\tilde f(gx) = \varphi(g)\tilde f(x)$ and $\tilde\Phi(gx) = \varphi^{-1}(g)\tilde\Phi(x)$ for all $g \in G$ and $x \in X^1$, so the composite

$\tilde{f}\tilde{\Phi} : X^1_{\gamma_1} \to X^2_{\gamma_2}$ is a G-map with $d(\tilde{f}\tilde{\Phi}) \equiv \pm(\varphi^{-1})^*$ (mod $|G|$). Finally, in the light of Theorem 1.1 we get that $d(f_{\gamma_1}) \equiv \pm(\varphi^{-1})^* d(f_{\gamma_2})$ (mod $|G|$) and the proof is complete. □

In particular, if two orbit spaces X^1/γ_1 and X^2/γ_2 of homotopy $(2n-1)$-spheres X^1 and X^2 are homeomorphic, then $d(f_{\gamma_1}) \equiv \pm\varphi^* d(f_{\gamma_2})$ (mod $|G|$) for some $\varphi \in \operatorname{Aut} G$ and G-maps $f_{\gamma_1} : X^1_{\gamma_1} \to X^0_{\gamma_0}$ and $f_{\gamma_2} : X^2_{\gamma_2} \to X^0_{\gamma_0}$.

Write $\mathscr{K}^{2n-1}_G$ for the set of all homeomorphism classes $[X/\gamma]$ of orbit spaces X/γ of all homotopy $(2n-1)$-spheres X with respect to all free cellular G-actions. Then the map

$$\mathscr{D}^{2n-1}_G : \mathscr{K}^{2n-1}_G \to \operatorname{Aut}(\mathbf{Z}/|G|)/\{\pm\varphi^*; \varphi \in \operatorname{Aut} G\}$$

given by $\mathscr{D}^{2n-1}_G([X/\gamma]) = [d(f_\gamma)]$, where $[d(f_\gamma)]$ is the class of the automorphism in $\operatorname{Aut}(\mathbf{Z}/|G|)$ determined by the integer $d(f_\gamma)$, is well-defined. Here we note that the equality $d(f_\gamma) = -1$ holds if f_γ is the identity map of a homotopy $(2n-1)$-sphere to itself with the opposite orientation. Let $\mathscr{K}^{2n-1}_G/_{\simeq}$ be the quotient set of $\mathscr{K}^{2n-1}_G$ by the homotopy relation $\simeq$. Then we can state

Corollary 1.3. *The map* $\mathscr{D}^{2n-1}_G : \mathscr{K}^{2n-1}_G \to \operatorname{Aut}(\mathbf{Z}/|G|)/\{\pm\varphi^*;\ \varphi \in \operatorname{Aut} G\}$ *induces an injection* $\widetilde{\mathscr{D}^{2n-1}_G} : \mathscr{K}^{2n-1}_G/_{\simeq} \to \operatorname{Aut}(\mathbf{Z}/|G|)/\{\pm\varphi^*;\ \varphi \in \operatorname{Aut} G\}$. *Consequently, the set* $\mathscr{K}^{2n-1}_G/_{\simeq}$ *is finite and bounded by the order of the quotient group* $\operatorname{Aut}(\mathbf{Z}/|G|)/\{\pm\varphi^*; \varphi \in \operatorname{Aut} G\}$.

If G is the cyclic group $\mathbf{Z}/m$ of order m, then by [5, 9] the map $\widetilde{\mathscr{D}^{2n-1}_G}$ is surjective on the homotopy types of the lens spaces. Thus we deduce from Theorem 1.2 one of the classical results of homotopy theory.

Corollary 1.4. *Given a free cellular action γ of the cyclic group $\mathbf{Z}/m$ on a homotopy $(2n-1)$-sphere X, the orbit space X/γ is homotopy equivalent to a lens space.*

Let $\mathscr{E}[X^1/\gamma_1, X^2/\gamma_2]$ denote the set of homotopy classes of homotopy equivalences $f : X^1/\gamma_1 \to X^2/\gamma_2$. Then from the proof of Theorem 1.2, we may deduce the following result which is a generalization of the result in [5, p. 96] stated for cyclic groups only.

Proposition 1.5. *If $|G| > 2$ and orbit spaces X^1/γ_1, X^2/γ_2 of homotopy spheres are homotopy equivalent, then there is a bijection from the set $\mathscr{E}[X^1/\gamma_1, X^2/\gamma_2]$ to the subgroup of the group* $\operatorname{Aut} G$ *consisting of all automorphisms φ such that $\varphi^* \equiv \pm 1$ (mod $|G|$). In particular, the group $\mathscr{E}[X/\gamma, X/\gamma]$ is isomorphic to that subgroup of the group* $\operatorname{Aut} G$, *for the orbit space X/γ of a homotopy sphere X.*

Proof. Let $f : X^1/\gamma_1 \to X^2/\gamma_2$ be a homotopy equivalence, fix a base point in X^1/γ_1 and let $\varphi = \pi_1(f)$ be the automorphism of the group G induced by the fundamental group functor π_1. Then $d(f) = \pm 1$ and by the Serre spectral sequence argument $\varphi^* \equiv d(f) = \pm 1 \pmod{|G|}$. Define $\lambda([f]) = \pi_1(f)$ for $[f] \in \mathscr{E}[X^1/\gamma_1, X^2/\gamma_2]$. We show that λ gives the required bijection from the set $\mathscr{E}[X^1/\gamma_1, X^2/\gamma_2]$.

Consider two homotopy equivalences $f, g : X^1/\gamma_1 \to X^2/\gamma_2$ with $\pi_1(f) = \pi_1(g) = \varphi$. If $\tilde{f}, \tilde{g} : X^1 \to X^2$ are covers of f and g, respectively then $\tilde{f}\tilde{\Phi}, \tilde{g}\tilde{\Phi} : X^1_{\gamma_1} \to X^2_{\gamma_2}$ are G-maps, where $\tilde{\Phi} : X^1 \to X^1$ is a map associated with the automorphism φ^{-1} as in the proof of Theorem 1.2. In view of Theorem 1.1, $d(f)(\varphi^*)^{-1} \equiv d(\tilde{f}\tilde{\Phi}) \equiv d(\tilde{g}\tilde{\Phi}) \equiv d(g)(\varphi^*)^{-1} \pmod{|G|}$. Consequently, $d(f) \equiv d(g) \pmod{|G|}$, hence $d(f) = d(g)$ since $d(f), d(g) = \pm 1$ and $|G| > 2$. Finally, in the light of [9], the maps $f, g : X^1/\gamma_1 \to X^2/\gamma_2$ are homotopic.

Let now ψ be an automorphism of the group G such that $\psi^* \equiv \pm 1 \pmod{|G|}$ and $f : X^1/\gamma_1 \to X^2/\gamma_2$ be a fixed homotopy equivalence with $\varphi = \pi_1(f)$. Then, by the proof of Theorem 1.2, there is a map $\Omega : X^1/\gamma_1 \to X^1/\gamma_1$ associated with the automorphism $\omega = \varphi^{-1}\psi$ and $d(\Omega) = \pm 1$, so it is a homotopy equivalence by [9]. But $\pi_1(f\Omega) = \psi$ and the proof is complete. □

Remark 1.6. It is well-known that the group $\mathscr{E}[X/\gamma, X/\gamma]$ is isomorphic to the cyclic group $\mathbf{Z}/2$ provided that $|G| \leq 2$ for any homotopy sphere X.

In the next section we calculate the group $\mathscr{E}[X/\gamma, X/\gamma]$, when G is a cyclic group $\mathbf{Z}/p$ with p an odd prime and X a homotopy $(2n-1)$-sphere.

2. Estimation of the number of orbit spaces

Let g_2 be a generator of the cohomology group $H^2(\mathbf{Z}/m; \mathbf{Z}) = \mathbf{Z}/m$. The results of [4, Chap. XII, § 11] show that $(g_2)^n$ generates $H^{2n}(\mathbf{Z}/m; \mathbf{Z}) = \mathbf{Z}/m$. Then, for any $\varphi \in \mathrm{Aut}(\mathbf{Z}/m)$, the induced automorphism φ^* of the group $H^{2n}(\mathbf{Z}/m; \mathbf{Z}) = \mathbf{Z}/m$ is determined by the power k^n, for some k with $(k, m) = 1$, that is $k \in (\mathbf{Z}/m)^*$.

Let now $Q_{4m} = \{x, y; x^m = y^2, xyx = y\}$ be the generalized quaternion group of order $4m$ and g_4 a generator of $H^4(Q_{4m}; \mathbf{Z}) = \mathbf{Z}/4m$. Then by [10], the element $(g_4)^n$ generates $H^{4n}(Q_{4m}; \mathbf{Z}) = \mathbf{Z}/4m$, and for any φ in the group $\mathrm{Aut}\, Q_{4m}$ the induced automorphism φ^* of the group $H^{4n}(Q_{4m}; \mathbf{Z}) = \mathbf{Z}/4m$ is determined by the power k^{2n}, for some k with $(k, 4m) = 1$.

The automorphism group $\mathrm{Aut}(\mathbf{Z}/m)$ is isomorphic to the unit group $(\mathbf{Z}/m)^*$ of the ring $\mathbf{Z}/m$. Therefore, we are led to compute the quotient group $(\mathbf{Z}/m)^*/\{\pm x^n; x \in (\mathbf{Z}/m)^*\}$. If $m = p_1^{k_1} \cdots p_s^{k_s}$ is the prime factorization of an integer $m \geq 1$ with $k_i \geq 1$ for $i = 1, \dots, s$, then it is well-known (see e.g. [12, Chapter IV]) that $(\mathbf{Z}/m)^* = (\mathbf{Z}/p_1^{k_1})^* \times \cdots \times (\mathbf{Z}/p_s^{k_s})^*$. Moreover, $(\mathbf{Z}/p^l)^* =$

$\mathbf{Z}/(p-1) \times \mathbf{Z}/p^{l-1}$ for p an odd prime or $l < 3$ and $(\mathbf{Z}/2^l)^* = \mathbf{Z}/2 \times \mathbf{Z}/2^{l-2}$ for $l \geq 3$ with generators, multiplication by -1 giving the element of order two, and multiplication by 5 or -3 giving the element of order 2^{l-2}.

In the group $\mathbf{Z}/(p-1) \times \mathbf{Z}/p^{m-1}$, if p is an odd prime, $(p-1)/2$ is the only element of order 2, which corresponds to -1 in the group $(\mathbf{Z}/p^m)^*$ by the isomorphism above. Therefore, the equation $-1 = x^n$ has a solution in the multiplicative group $(\mathbf{Z}/p^m)^*$ if and only if $(p-1)/2$ is divisible by the integer n in the additive group $\mathbf{Z}/(p-1)$. Note that $(2k+1)(p-1)/2 \equiv (p-1)/2$ $(\mathrm{mod}(p-1))$. On the other hand $-1 \neq x^n$ for any element x in the group $(\mathbf{Z}/2^m)^*$ if n is an even integer. Thus, we will get the proposition below and we may concentrate in the sequel on the quotient group $(\mathbf{Z}/p^m)^*/\{\pm x^n; x \in (\mathbf{Z}/p^m)^*\}$, for positive integers m, n and a prime p.

Proposition 2.1. *If $m = p_1^{k_1} \cdots p_s^{k_s}$ is the prime factorization of an integer $m \geq 1$, then for any integer $n \geq 1$ there is an extension*

$$0 \to (\mathbf{Z}/2)^t \to (\mathbf{Z}/m)^*/\{\pm x^n; x \in (\mathbf{Z}/m)^*\}$$
$$\to (\mathbf{Z}/p_1^{k_1})^*/\{\pm x^n; x \in (\mathbf{Z}/p_1^{k_1})^*\} \times \cdots \times (\mathbf{Z}/p_s^{p_s})^*/\{\pm x^n; x \in (\mathbf{Z}/p_s^{k_s})^*\}$$
$$\to 0,$$

where t is determined as follows:

(1) *If $-1 \in \{x^n; x \in \mathbf{Z}/p_i^{k^i}\}$ for any $i = 1, \dots, s$, then $t = 0$;*

(2) *Otherwise $t = s - 1 - \#\{i; -1 \in \{x^n; x \in \mathbf{Z}/p_i^{k^i}\}\}$. In particular, when n is an odd integer, $-1 \in \{x^n; x \in \mathbf{Z}/p_i^{k^i}\}$ for any $i = 1, \dots, s$ and hence $t = 0$; when n is an even integer, $t+1$ is the number of odd primes p_i which appears in the prime factorization of m and $(p_i - 1)/2$ is not divisible by n in the group $\mathbf{Z}/(p_i - 1)$.*

Proof. Of course, we have the extension

$$0 \to \{\pm x^n; x \in (\mathbf{Z}/p_1^{k_1})^*\} \times \cdots \times \{\pm x^n; x \in (\mathbf{Z}/p_s^{k_s})^*\}/\{\pm x^n; x \in (\mathbf{Z}/m)^*\}$$
$$\to (\mathbf{Z}/m)^*/\{\pm x^n; x \in (\mathbf{Z}/m)^*\} \to (\mathbf{Z}/p_1^{k_1})^*/\{\pm x^n; x \in (\mathbf{Z}/p_1^{k_1})^*\}$$
$$\times \cdots \times (\mathbf{Z}/p_s^{k_s})^*/\{\pm x^n; x \in (\mathbf{Z}/p_s^{k_s})^*\} \to 0.$$

The square of every element in $\{\pm x^n; x \in (\mathbf{Z}/p_1^{k_1})^*\} \times \cdots \times \{\pm x^n; x \in (\mathbf{Z}/p_s^{k_s})^*\}$ lies in $\{x^n; x \in (\mathbf{Z}/m)^*\}$, because $(\pm x^n)^2 = (x^2)^n$. So, it follows that the group

$$\{\pm x^n; x \in (\mathbf{Z}/p_1^{k_1})^*\} \times \cdots \times \{\pm x^n; x \in (\mathbf{Z}/p_s^{k_s})^*\}/\{\pm x^n; x \in (\mathbf{Z}/m)^*\}$$

is isomorphic to a direct sum of $\mathbf{Z}/2$'s. Certainly, $t = s - 1$ in the case (1).

In the case (2), $-1 \in \{x^n; x \in (\mathbf{Z}/m)^*\}$; we can replace $\pm x^n$ with x^n for $(\mathbf{Z}/m)^*$ and for $(\mathbf{Z}/p_i^{k_i})^*$ with $-1 \in \{x^n; x \in (\mathbf{Z}/p_i^{k_i})^*\}$. Then it is not hard to see the result. $\square$

Of course, we can assume that $n = q_1^{t_1} \cdots q_l^{t_l}$ and $p - 1 = q_1^{u_1} \cdots q_l^{u_l}$, where $q_1, \dots, q_l$ are different primes, $t_1, \dots, t_l, u_1, \dots, u_l$ are non-negative integers and t_i or u_i is positive for all $i = 1, \dots, l$.

Proposition 2.2. *Let m, n be positive integers, p a prime and $q_1, \dots, q_l$ primes from the factorization above. Then the quotient group $(\mathbf{Z}/p^m)^*/\{x^n;\ x \in (\mathbf{Z}/p^m)^*\}$ is isomorphic to one of the following groups:*

(1) $\mathbf{Z}/q_1^{\min(t_1,u_1)} \times \cdots \times \mathbf{Z}/q_l^{\min(t_l,u_l)}$, *if $p \neq 2$ and $p \neq q_1, \dots, q_l$;*

(2) $\mathbf{Z}/q_1^{\min(t_1,u_1)} \times \cdots \times \mathbf{Z}/q_{i_0-1}^{\min(t_{i_0-1},u_{i_0-1})} \times \mathbf{Z}/p^{\min(m-1,t_{i_0})} \times \mathbf{Z}/q_{i_0+1}^{\min(t_{i_0+1},u_{i_0+1})} \times \cdots \times \mathbf{Z}/q_l^{\min(t_l,u_l)}$, *if $p \neq 2$ and $p = q_{i_0}$ for some $1 \leq i_0 \leq l$;*

(3) *the trivial group E, if $p = 2$ and n is odd;*

(4) $\mathbf{Z}/2 \times \mathbf{Z}/2^{\min(t_{i_0},m-2)}$, *if $p = 2$ and $q_{i_0} = 2$ for some $1 \leq i_0 \leq l$.*

Proof. Let A be a finite additive abelian group and n a positive integer. Then, for the endomorphism $\bar{n} : A \to A$ given by the multiplication by n, the quotient group $A/\operatorname{Im}\bar{n}$ is isomorphic to the kernel $\operatorname{Ker}\bar{n}$, where $\operatorname{Im}\bar{n}$ is the image of $\bar{n}$. We consider the endomorphism of the multiplicative group $(\mathbf{Z}/p^m)^*$ given by taking the n-th power. But the group $(\mathbf{Z}/p^m)^*$ is isomorphic to the additive group $\mathbf{Z}/(p-1) \times \mathbf{Z}/p^{m-1}$, for p an odd prime or $m < 3$ and to the group $\mathbf{Z}/2 \times \mathbf{Z}/2^{m-2}$ for $p = 2$ and $m \geq 3$. Thus the result follows. Remark that $u_{i_0} = 0$ in (ii). $\square$

Consequently, the following holds.

Corollary 2.3. *Let m, n and p be as in Proposition 2.2 and let $T_p(m,n)$ denote the number of homotopy types of lens spaces given by free $\mathbf{Z}/p^m$-actions on homotopy $(2n-1)$-spheres.*

(1) *If p is an odd prime and $(p-1)/2$ is divisible by n in the group $\mathbf{Z}/(p-1)$, or $p = 2$ and n is an odd integer, then $T_p(m,n)$ is equal to:*
  - (i) $q_1^{\min(t_1,u_1)} \cdots q_l^{\min(t_l,u_l)}$, *if $p \neq 2$ and $p \neq q_1, \dots, q_l$;*
  - (ii) $q_1^{\min(t_1,u_1)} \cdots q_{i_0-1}^{\min(t_{i_0-1},u_{i_0-1})} p^{\min(m-1,t_{i_0})} q_{i_0+1}^{\min(t_{i_0+1},u_{i_0+1})} \cdots q_l^{\min(t_l,u_l)}$, *if $p \neq 2$ and $p = q_{i_0}$ for some $1 \leq i_0 \leq l$;*
  - (iii) 1, *if $p = 2$.*

(2) *Otherwise $T_p(m,n)$ is equal to:*
  - (i) $2^{-1} q_1^{\min(t_1,u_1)} \cdots q_l^{\min(t_l,u_l)}$, *if $p \neq 2$ and $p \neq q_1, \dots, q_l$;*
  - (ii) $2^{-1} q_1^{\min(t_1,u_1)} \cdots q_{i_0-1}^{\min(t_{i_0-1},u_{i_0-1})} p^{\min(m-1,t_{i_0})} q_{i_0+1}^{\min(t_{i_0+1},u_{i_0+1})} \cdots q_l^{\min(t_l,u_l)}$, *if $p \neq 2$ and $p = q_{i_0}$ for some $1 \leq i_0 \leq l$;*
  - (iii) $2^{\min(t_{i_0},m-2)}$, *if $p = 2$, $m \geq 3$ and $q_{i_0} = 2$;*
  - (iv) 1, *if $p = 2$, $m \leq 2$ and $q_{i_0} = 2$.*

In particular, we get

Corollary 2.4. *Let p be a prime and m a positive integer. Then the number of homotopy types of lens spaces given by free* $\mathbf{Z}/p^m$*-actions on a homotopy* 3*-sphere is equal to:*
(1) 2*, if either* $p-1$ *is divisible by* 4 *or* $p=2$ *and* $m \geq 2$*;*
(2) 1*, if either* $p-1$ *is divisible by* 2 *but not divisible by* 4 *or* $p=2$ *and* $m=1$.

Moreover, for the generalized quaternion group $Q_{2^{m+2}}$ one can deduce

Corollary 2.5. *The number of homotopy types of orbit spaces given by free* $Q_{2^{m+2}}$*-actions on homotopy* $(4n-1)$*-spheres is bounded by* 2 *for any* $m > 0$*, provided that n is an odd positive integer.*

Proof. In the light of [10, Proposition 8.1], for any φ in the group $\operatorname{Aut} Q_{2^{m+2}}$, the induced automorphism φ^* of $H^{4n}(Q_{2^{m+2}};\mathbf{Z}) = \mathbf{Z}/2^{m+2}$ is determined by the power k^{2n} for some k with $(k, 2^{m+2}) = 1$. The result follows from the same argument as for Corollary 2.3. □

Finally, let L be a lens space determined by a $\mathbf{Z}/p$-free action on a homotopy $(2n-1)$-sphere with p an odd prime. Then, by Proposition 1.5, the group $\mathscr{E}[L,L]$ of homotopy types of all its self-homotopy equivalences is isomorphic to the subgroup of $(\mathbf{Z}/p)^*$ consisting of all elements k with $k^n \equiv \pm 1 \pmod{p}$. But $(\mathbf{Z}/p)^* = \mathbf{Z}/(p-1)$, so the corresponding subgroup of $\mathbf{Z}/(p-1)$ consisting of all elements l with $nl = 0$ or $nl = (p-1)/2$. If $d = (n, p-1)$ is the greatest common divisor of the integers n and $p-1$, then the subgroup G_d of $\mathbf{Z}/(p-1)$ generated by $(p-1)/d$ consists of all elements l in the group $\mathbf{Z}/(p-1)$ with $nl = 0$. Moreover, if there is a solution l_0 in the group $\mathbf{Z}/(p-1)$ of the equation $nl = (p-1)/2$, then the set of all solution of that equation in the group $\mathbf{Z}/(p-1)$ is equal to the coset $l_0 + G_d$. We can conclude the paper with the following result deduced from the consideration above.

Proposition 2.6. *Let L be a lens space determined by a free* $\mathbf{Z}/p$*-action on a homotopy* $(2n-1)$*-sphere with an odd prime p. Then the group* $\mathscr{E}[L,L]$ *of homotopy classes of all its self-homotopy equivalences is isomorphic either to the subgroup* G_d *of the cyclic group* $\mathbf{Z}/(p-1)$ *or to the subgroup* $G_d \cup (l_0 + G_d)$*, provided that there is a solution* l_0 *of the equation* $nl = (p-1)/d$ *in the group* $\mathbf{Z}/(p-1)$.

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