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HOMOTOPY THEORY OF MIXED HODGE COMPLEXES

In memoriam Alexander Grothendieck

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Abstract. We show that the category of mixed Hodge complexes admits a Cartan-Eilenberg structure, a notion introduced by Guillén-Navarro-Pascual-Roig leading to a good calculation of the homotopy category in terms of (co)fibrant objects. Using Deligne's décalage, we show that the homotopy categories associated with the two notions of mixed Hodge complex introduced by Deligne and Beilinson respectively, are equivalent. The results provide a conceptual framework from which Beilinson's and Carlson's results on mixed Hodge complexes and extensions of mixed Hodge structures follow easily.

1. Introduction.

Mixed Hodge complexes were introduced by Deligne [9] in order to extend his theory of mixed Hodge structures on the cohomology of algebraic varieties to the singular case, via simplicial resolutions. Since their appearance, these objects and their variants (see for example [28]) have become a fruitful source of interest. In particular, they have proved crucial in the theory of Hodge invariants for the homotopy of complex algebraic varieties (see for example [22], [10], [14], [23]). A natural question arising is to ask for a homotopical structure in the category of such objects. Unfortunately, the derived categories of Verdier [31] and the model categories of Quillen [27], that are nowadays considered the standard basis of homological and homotopical algebra respectively, do not adequately fulfill the needs to express the properties of diagram categories of complexes with filtrations.

In this paper we study the homotopy theory of mixed Hodge complexes within the homotopical framework of Cartan-Eilenberg categories of [13], a weaker framework than the one provided by Quillen model categories, but sufficient to study homotopy categories and to extend the classical theory of derived additive functors, to non-additive settings. To achieve this, we overcome two problems of distinct nature.

The first of these problems is to understand the different homotopical structures carried by filtered and bifiltered complexes. Filtered derived categories were first studied by Illusie (see Chapter V of [18]) following the classical theory for abelian categories. An alternative

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approach in the context of exact categories was developed by Laumon [20]. In certain situations, the filtrations under study are not well defined, and become a proper invariant only in higher stages of the associated spectral sequences. This is the case of the mixed Hodge theory of Deligne, in which the weight filtration of a variety depends on the choice of a hyperresolution, and is only well defined at the second stage. This circumstance is somewhat hidden by the degeneration of the spectral sequences, but it already highlights the interest of studying more flexible structures. In this paper we generalize the results of Illusie by considering the class of weak equivalences given by morphisms of filtered complexes inducing an isomorphism at a fixed stage of the associated spectral sequence (see also [24], [15] and [6]).

The second problem is to obtain a theory of rectification of morphisms of diagrams up to homotopy, allowing the construction of level-wise fibrant models for diagram categories. This problem is of great interest in the field of abstract homotopical algebra, and has only been solved for some specific situations by means of Quillen-type theories (see for example [16], [2], [7]). In this paper we address the problem for diagrams of complexes over additive categories satisfying certain compatibility conditions. This sets-up the basis for a more general abstract homotopy theory for diagram categories, as done in [5]. While mixed Hodge complexes fit into this more general context, their homotopy theory carries stronger properties. Thanks to the additive properties of complexes here we obtain a much simpler and manageable homotopy theory. We next explain the main results of this paper.

The category $\mathbf{F}\mathcal{A}$ of filtered objects (with finite filtrations) of an abelian category $\mathcal{A}$ is additive, but not abelian in general. Consider the category $\mathbf{C}^{\#}(\mathbf{F}\mathcal{A})$ of complexes over $\mathbf{F}\mathcal{A}$, where # denotes the boundedness condition. For $r \geq 0$, denote by $\mathcal{E}_r$ the class of *E_r-quasi-isomorphisms*: these are morphisms of filtered complexes inducing a quasi-isomorphism at the r-stage of the associated spectral sequence. The *r-derived category* is defined by $\mathbf{D}^{\#}_r(\mathbf{F}\mathcal{A}) := \mathbf{C}^{\#}(\mathbf{F}\mathcal{A})[\mathcal{E}_r^{-1}]$. The case $r = 0$ corresponds to the original filtered derived category, studied by Illusie. To study the weight filtration of complex algebraic varieties, Deligne [8] introduced the décalage of a filtered complex, which shifts the associated spectral sequence by one stage. This defines a functor $\mathrm{Dec} : \mathbf{C}^{\#}(\mathbf{F}\mathcal{A}) \to \mathbf{C}^{\#}(\mathbf{F}\mathcal{A})$ which is the identity on morphisms and satisfies $\mathrm{Dec}(\mathcal{E}_{r+1}) \subset \mathcal{E}_r$. The following result exhibits how Deligne's décalage is already a key tool in the study of filtered derived categories.

THEOREM 2.19. *For all $r \geq 0$, Deligne's décalage induces an equivalence of categories*

$$\mathrm{Dec} : \mathbf{D}^{\#}_{r+1}(\mathbf{F}\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^{\#}_r(\mathbf{F}\mathcal{A})\,.$$

The notion of homotopy between morphisms of complexes over an additive category is defined via a translation functor. In the filtered setting, we find that an r-shift on the filtration of the translation functor leads to different notions of *r-homotopy*, suitable to the study of the r-derived category. The associated class $\mathcal{S}_r$ of *r-homotopy equivalences* satisfies $\mathcal{S}_r \subset \mathcal{E}_r$.

As in the classical case, we study the bounded below r-derived category of filtered objects $\mathbf{F}\mathcal{A}$ under the assumption that $\mathcal{A}$ has enough injectives. Denote by $\mathbf{C}^{+}_r(\mathbf{F}\mathrm{Inj}\mathcal{A})$ the full subcategory of those bounded below filtered complexes (K, F) over injective objects of $\mathcal{A}$

whose differential satisfies $dF^pK \subset F^{p+r}K$, for all $p \in \mathbb{Z}$. Objects in this category are called *r-injective*. We prove:

THEOREM 2.26. *Let $\mathcal{A}$ be an abelian category with enough injectives. For all $r \geq 0$ the triple $(\mathbf{C}^+(\mathbf{F}\mathcal{A}), \mathcal{S}_r, \mathcal{E}_r)$ is a Cartan-Eilenberg category with fibrant models in $\mathbf{C}^+_r(\mathbf{F}\mathrm{Inj}\mathcal{A})$. The inclusion induces an equivalence of categories $\mathbf{K}^+_r(\mathbf{F}\mathrm{Inj}\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^+_r(\mathbf{F}\mathcal{A})$ between the category of r-injective complexes modulo r-homotopy and the r-derived category.*

Denote by **MHC** the category of mixed Hodge complexes (see Definition 8.1.5 of [9]). The spectral sequences associated with the weight and the Hodge filtrations of a mixed Hodge complex degenerate at the stages E_2 and E_1 respectively. It proves to be convenient to consider the category **AHC** of absolute Hodge complexes as introduced by Beilinson [1], in which all spectral sequences degenerate at the first stage. Deligne's décalage with respect to the weight filtration induces a functor $\mathrm{Dec}_W : \mathbf{MHC} \to \mathbf{AHC}$. We study the homotopy categories $\mathrm{Ho}(\mathbf{MHC}) := \mathbf{MHC}[\mathcal{Q}^{-1}]$ and $\mathrm{Ho}(\mathbf{AHC}) := \mathbf{AHC}[\mathcal{Q}^{-1}]$ defined by inverting the class $\mathcal{Q}$ of level-wise quasi-isomorphisms compatible with filtrations.

The category of mixed (resp. absolute) Hodge complexes is a diagram category whose vertices are categories of filtered and bifiltered complexes. We develop a homotopy theory for such diagram categories and show that, under certain hypothesis, one can define a level-wise Cartan-Eilenberg structure on the diagram category (see Theorem 3.23). Its application to mixed Hodge theory gives the following result.

Denote by $\pi\left(\mathbf{G}^+(\mathsf{MHS})^h\right)$ the category whose objects are non-negatively graded mixed Hodge structures and whose morphisms are given by homotopy classes of level-wise morphisms compatible up to a filtered homotopy (ho-morphisms for short). Denote by $\mathcal{H}$ the class of morphisms of absolute Hodge complexes that are homotopy equivalences as ho-morphisms. We prove:

THEOREM 4.9. *The triple $(\mathbf{AHC}, \mathcal{H}, \mathcal{Q})$ is a Cartan-Eilenberg category and $\mathbf{G}^+(\mathsf{MHS})$ is a full subcategory of fibrant minimal models. The inclusion induces an equivalence of categories $\pi\left(\mathbf{G}^+(\mathsf{MHS})^h\right) \xrightarrow{\sim} \mathrm{Ho}(\mathbf{AHC})$.*

We prove an analogous result for mixed Hodge complexes (see Theorem 4.10). Note that although every mixed (resp. absolute) Hodge complex is quasi-isomorphic to its cohomology (which has trivial differentials), the full subcategory of fibrant minimal models has non-trivial homotopies. This reflects the fact that mixed Hodge structures have non-trivial extensions.

The following result rests on Theorems 4.9 and 4.10, and relates the different definitions of mixed Hodge complex due to Deligne and Beilinson respectively.

THEOREM 4.11. *Deligne's décalage induces an equivalence of categories*

$$\mathrm{Dec}_W : \mathrm{Ho}(\mathbf{MHC}) \xrightarrow{\sim} \mathrm{Ho}(\mathbf{AHC})\,.$$

Beilinson established an equivalence of categories between the homotopy category of absolute Hodge complexes and the derived category of mixed Hodge structures (see [1], Theorem 3.4). As an application of Theorem 4.9 we provide a more standard and modern proof of Beilinson's result (see Theorem 4.12). We also compute the extensions of mixed Hodge structures (Proposition 4.13) and describe morphisms in Ho(**MHC**) in terms of morphisms and extensions of mixed Hodge structures (Corollary 4.14).

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2. Décalage and filtered derived categories.

Deligne [8] introduced the shift and décalage of filtered complexes and proved that their associated spectral sequences are related by a shift of indexing. We collect some important properties of shift and décalage which are probably known to experts, but which do not seem to have appeared in the literature. We introduce the r-derived category of filtered complexes as the localization of (bounded below) filtered complexes with respect to E_r-quasi-isomorphisms and, using Deligne's décalage functor, we generalize results of Illusie for $r = 0$, to an arbitrary $r \geq 0$, within the framework of Cartan-Eilenberg categories.

2.1. Homotopy in additive categories. Let $\mathcal{A}$ be an additive category. Denote by $\mathbf{C}^{\#}(\mathcal{A})$ the category of cochain complexes of $\mathcal{A}$, where # denotes the boundedness condition ($+$ and $-$ for bounded below and above respectively, b for bounded and $\emptyset$ for unbounded).

Recall that the classical translation functor $T : \mathbf{C}^{\#}(\mathcal{A}) \to \mathbf{C}^{\#}(\mathcal{A})$ is defined on objects by $T(K)^n = K^{n+1}$ and $d^n_{T(K)} = -d_{K^{n+1}}$ and on morphisms by $T(f)^n = f^{n+1}$. We next define the notion of a translation functor in the category of complexes which is induced by an additive automorphism of $\mathcal{A}$.

DEFINITION 2.1. Let $\alpha : \mathcal{A} \to \mathcal{A}$ be an additive automorphism of $\mathcal{A}$ with a natural transformation $\eta : \alpha \to 1$. The *translation functor induced by* α is the automorphism $T_\alpha : \mathbf{C}^{\#}(\mathcal{A}) \to \mathbf{C}^{\#}(\mathcal{A})$ given by the composition $T_\alpha := T \circ \alpha = \alpha \circ T$.

Such translation functor will prove to be useful in the context of complexes over filtered abelian categories, in which a shift by r on the filtration of the classical translation leads to the different notions of r-homotopy, as we shall see in the following section.

For the rest of this section we fix a translation functor T_α induced by an automorphism α of $\mathcal{A}$ with a natural transformation $\eta : \alpha \to 1$.

DEFINITION 2.2. Let $f, g : K \to L$ be morphisms of complexes. An *α-homotopy* from f to g is a degree preserving map $h : T_\alpha(K) \to L$ such that $dh + hd = (g - f) \circ \eta_K$. We denote $h : f \underset{\alpha}{\simeq} g$.

The additive operation between maps makes the homotopy relation into an equivalence relation compatible with the composition.

Denote by $[K, L]_\alpha$ the set of morphisms of complexes from K to L modulo α-homotopy, and by $\mathbf{K}^{\#}_\alpha(\mathcal{A}) := \mathbf{C}^{\#}(\mathcal{A})/\underset{\alpha}{\simeq}$ the corresponding quotient category.

DEFINITION 2.3. A morphism $f : K \to L$ is said to be an *α-homotopy equivalence* if there exists a morphism $g : L \to K$ together with α-homotopies $fg \underset{\alpha}{\simeq} 1_L$ and $gf \underset{\alpha}{\simeq} 1_K$. Denote by $\mathcal{S}_\alpha$ the class of α-homotopy equivalences.

The following are standard constructions useful in the study of the homotopy theory of complexes over $\mathcal{A}$ (see for example Section III.3.2 of [11]). We will later generalize these constructions in Section 2, for diagrams of complexes.

DEFINITION 2.4. Let $f : K \to L$ and $g : K \to L'$ be two morphisms of complexes. The *α-double mapping cylinder of f and g* is the complex $\mathcal{C}yl_\alpha(f, g) = T_\alpha(K) \oplus L \oplus L'$ with differential

$$D = \begin{pmatrix} -d & 0 & 0 \\ -\eta_L \circ \alpha(f) & d & 0 \\ \eta_{L'} \circ \alpha(g) & 0 & d \end{pmatrix}.$$

Let $i : L' \to \mathcal{C}yl_\alpha(f, g)$, $j : L \to \mathcal{C}yl_\alpha(f, g)$ and $k : T_\alpha(K) \to \mathcal{C}yl_\alpha(f, g)$ denote the inclusions into the corresponding direct summands. Then i and j are morphisms of complexes and k is an α-homotopy from jf to ig. With these notations:

LEMMA 2.5. *For any complex X, the map*

$$\mathrm{Hom}(\mathcal{C}yl_\alpha(f, g), X) \to \left\{ (h, u, v)\,;\ h : uf \underset{\alpha}{\simeq} vg,\ \begin{matrix} u \in \mathrm{Hom}(L, X) \\ v \in \mathrm{Hom}(L', X) \end{matrix} \right\}$$

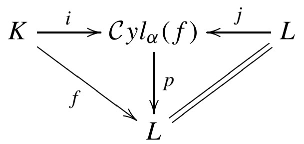
defined by $t \mapsto (tk, tj, ti)$ is a bijection.

PROOF. An inverse is given by $t(x, y, z) = h(x) + u(y) + v(z)$. □

DEFINITION 2.6. Let $f : K \to L$ be a morphism of complexes.

(1) The *α-mapping cylinder of f* is the complex $\mathcal{C}yl(f) := \mathcal{C}yl(f, 1_K) = T_\alpha(K) \oplus L \oplus K$.

There is a commutative diagram of morphisms of complexes



where as before i and j denote the inclusions and $p(x, y, z) = y + f(z)$.

(2) The *α-mapping cone of f* is the complex $C_\alpha(f) := \mathcal{C}yl_\alpha(0, f) = T_\alpha(K) \oplus L$.

For every complex X, the map

$$\mathrm{Hom}(C_\alpha(f), X) \to \left\{ (h, v)\,;\ h : vf \underset{\alpha}{\simeq} 0,\ v \in \mathrm{Hom}(L, X) \right\}$$

defined by $t \mapsto (tk, tj)$ is a bijection.

DEFINITION 2.7. The *α-cylinder* of a complex K is the complex $\mathrm{Cyl}_\alpha(K) := \mathcal{C}yl_\alpha(1_K)$.

By Lemma 2.5 an α-homotopy $h : T_\alpha(K) \to L$ between morphisms $f, g : K \to L$ is equivalent to a morphism of complexes $H : \mathrm{Cyl}_\alpha(K) \to L$ satisfying $Hj = f$ and $Hi = g$.

An important property of the cylinder is the following.

PROPOSITION 2.8. *For every complex K, the map $p : \mathrm{Cyl}_\alpha(K) \longrightarrow K$ defined by $p(x, y, z) = y + z$ is an α-homotopy equivalence, with homotopy inverse j.*

PROOF. Define an α-homotopy $h : T_\alpha(\mathrm{Cyl}_\alpha(K)) \to \mathrm{Cyl}_\alpha(K)$ from jp to the identity by letting $h(x, y, z) = (z, 0, 0)$. □

COROLLARY 2.9. *The localization functor $\delta : \mathbf{C}^\#(\mathcal{A}) \to \mathbf{C}^\#(\mathcal{A})[\mathcal{S}_\alpha^{-1}]$ induces a canonical isomorphism $\mathbf{K}^\#_\alpha(\mathcal{A}) \to \mathbf{C}^\#(\mathcal{A})[\mathcal{S}_\alpha^{-1}]$.*

PROOF. It follows from Proposition 2.8 and Proposition 1.3.3 of [13]. □

2.2. Filtered complexes. Let $\mathbf{F}\mathcal{A}$ denote the additive category of filtered objects of an abelian category $\mathcal{A}$. Throughout this paper we will consider filtered complexes $(K, F) \in \mathbf{C}^\#(\mathbf{F}\mathcal{A})$ whose filtration is regular and exhaustive: for each $n \geq 0$ there exists $q \in \mathbb{Z}$ such that $F^q K^n = 0$ and $K = \bigcup_p F^p K$. We will denote by $E_r(K)$ the spectral sequence associated with a filtered complex (K, F), omitting the filtration F whenever there is no danger of confusion. For the rest of this section we fix an integer $r \geq 0$.

DEFINITION 2.10. A morphism of filtered complexes $f : K \to L$ is called *E_r-quasi-isomorphism* if the map $E_r(f) : E_r(K) \to E_r(L)$ is a quasi-isomorphism of complexes (the map $E_{r+1}(f)$ is an isomorphism).

Denote by $\mathcal{E}_r$ the class of E_r-quasi-isomorphisms. The *r-derived category* is the localized category $\mathbf{D}^\#_r(\mathbf{F}\mathcal{A}) := \mathbf{C}^\#(\mathbf{F}\mathcal{A})[\mathcal{E}_r^{-1}]$. For $r = 0$ we recover the notions of filtered quasi-isomorphism and filtered derived category studied by Illusie in [18] (see also [19] and [25] for an account in the frameworks of exact categories and Cartan-Eilenberg categories respectively). There is a chain of functors

$$\mathbf{D}^\#_0(\mathbf{F}\mathcal{A}) \to \mathbf{D}^\#_1(\mathbf{F}\mathcal{A}) \to \cdots \to \mathbf{D}^\#_r(\mathbf{F}\mathcal{A}) \to \cdots \to \mathbf{D}^\#(\mathbf{F}\mathcal{A}),$$

where the rightmost category denotes the localization with respect to quasi-isomorphisms. Each of these categories keeps less and less information of the original filtered homotopy type.

NOTATION 2.11. Given $(A, F) \in \mathbf{F}\mathcal{A}$ denote by $(A, F(r))$ the filtered object given by $F(r)^p A := F^{p+r} A$. This defines an automorphism α_r of $\mathbf{F}\mathcal{A}$, and the identity defines a natural transformation $\alpha_r \to 1$.

Denote by $T_r : \mathbf{C}^\#(\mathbf{F}\mathcal{A}) \to \mathbf{C}^\#(\mathbf{F}\mathcal{A})$ the translation functor induced by the automorphism α_r. For every filtered complex (K, F) we have

$$F^p T_r(K)^n = F(r)^p K^{n+1} = F^{p+r} K^{n+1}.$$

Given morphisms $f, g : K \to L$, an *r-homotopy* from f to g is given by a degree preserving filtered map $h : T_r(K) \to L$ such that $dh + hd = g - f$. The condition that h is compatible with the filtrations is equivalent to $h(F^p K^{n+1}) \subset F^{p-r} L^n$ for all $n \geq 0$ and all $p \in \mathbb{Z}$. Therefore our notion of r-homotopy coincides with the notion of r-homotopy of [4, p. 321].

Denote by $\mathcal{S}_r$ the associated class of *r-homotopy equivalences*. By Proposition 3.1 of [4] we have $\mathcal{S}_r \subset \mathcal{E}_r$. Hence the triple $(\mathbf{C}^+(\mathbf{F}\mathcal{A}), \mathcal{S}_r, \mathcal{E}_r)$ is a category with strong and weak equivalences in the sense of [13]. This is a category $\mathcal{C}$ together with two classes of morphisms $\mathcal{S}$ and $\mathcal{W}$ of $\mathcal{C}$ containing all isomorphisms, closed by composition and satisfying $\mathcal{S} \subset \mathcal{W}$.

2.3. Deligne's Décalage functor. We next recall the definition of the shift, the décalage and its dual construction, and prove some properties.

DEFINITION 2.12. The *shift* of a filtered complex (K, F) is the filtered complex (K, SF) defined by $SF^pK^n = F^{p-n}K^n$. This defines a functor $S : \mathbf{C}^{\#}(\mathbf{F}\mathcal{A}) \to \mathbf{C}^{\#}(\mathbf{F}\mathcal{A})$ which is the identity on morphisms.

The shift functor does not admit an inverse, since the differentials would not necessarily be compatible with filtrations. However, it has both a right and a left adjoint: these are the décalage and its dual construction.

DEFINITION 2.13 ([8]). The *décalage* of a filtered complex (K, F) is the filtered complex $(K, \mathrm{Dec}F)$ given by

$$\mathrm{Dec}F^pK^n = F^{p+n}K^n \cap d^{-1}(F^{p+n+1}K^{n+1})\,.$$

The *dual décalage* is the filtered complex (K, Dec^*F) given by

$$\mathrm{Dec}^*F^pK^n = d(F^{p+n-1}K^{n-1}) + F^{p+n}K^n\,.$$

These give functors $\mathrm{Dec}, \mathrm{Dec}^* : \mathbf{C}^{\#}(\mathbf{F}\mathcal{A}) \to \mathbf{C}^{\#}(\mathbf{F}\mathcal{A})$ which are the identity on morphisms.

EXAMPLE 2.14. Consider the trivial filtration $0 = G^1K \subset G^0K = K$ of a complex K. Then $SG = \sigma$ is the bête filtration and $\mathrm{Dec}G = \mathrm{Dec}^*G = \tau$ is the canonical filtration.

The next result is a matter of verification.

LEMMA 2.15. *The following identities are satisfied:*

(1) $\mathrm{Dec} \circ S = 1$, *and* $(S\mathrm{Dec}F)^p = F^p \cap d^{-1}(F^{p+1})$,

(2) $\mathrm{Dec}^* \circ S = 1$, *and* $(S\mathrm{Dec}^*F)^p = F^p + d(F^{p-1})$.

In particular, there are natural transformations $S \circ \mathrm{Dec} \to 1$ *and* $1 \to S \circ \mathrm{Dec}^*$.

As a consequence of the above lemma we obtain:

PROPOSITION 2.16. *The functor S is left adjoint to* Dec *and right adjoint to* Dec*. *In particular* $\mathrm{Hom}(SK, L) = \mathrm{Hom}(K, \mathrm{Dec}L)$ *and* $\mathrm{Hom}(\mathrm{Dec}^*K, L) = \mathrm{Hom}(K, SL)$.

We next show that for a particular type of complexes, the décalage and its dual construction coincide, and define an inverse functor to the shift.

NOTATION 2.17. Denote by $\mathbf{C}^{\#}_r(\mathbf{F}\mathcal{A})$ the full subcategory of $\mathbf{C}^{\#}(\mathbf{F}\mathcal{A})$ of those filtered complexes (K, F) satisfying $d(F^pK) \subset F^{p+r}K$. In particular, the induced differential at the s-stage of their associated spectral sequence is trivial for all $s < r$.

LEMMA 2.18. *For all $r \geq 0$, the functors* $\mathrm{Dec} = \mathrm{Dec}^* : \mathbf{C}^{\#}_{r+1}(\mathbf{F}\mathcal{A}) \rightleftarrows \mathbf{C}^{\#}_r(\mathbf{F}\mathcal{A}) : S$ *are inverse to each other.*

PROOF. If $K \in \mathbf{C}^{\#}_{r+1}(\mathbf{F}\mathcal{A})$ then $d(F^pK) \subset F^{p+1}K$. Hence $\mathrm{Dec}F^pK^n = \mathrm{Dec}^*F^pK^n = F^{p+n}K^n$. Therefore $S \circ \mathrm{Dec}(K) = S \circ \mathrm{Dec}^*(K) = K$. A simple verification shows that $\mathrm{Dec}K \in \mathbf{C}^{\#}_r(\mathbf{F}\mathcal{A})$. Conversely, if $K \in \mathbf{C}^{\#}_r(\mathbf{F}\mathcal{A})$ it is straightforward that $SK \in \mathbf{C}^{\#}_{r+1}(\mathbf{F}\mathcal{A})$ and $\mathrm{Dec} \circ S = 1$. □

From the definition of the shift it follows that $E_{r+1}^{p+n,-p}(SK) \cong E_r^{p,n-p}(K)$ for all $r \geq 0$. Therefore we have $S(\mathcal{E}_r) \subset \mathcal{E}_{r+1}$. Moreover, by Proposition 1.3.4 of [8] the canonical maps

$$E_{r+1}^{p,n-p}(\mathrm{Dec}K) \longrightarrow E_{r+2}^{p+n,-p}(K) \longrightarrow E_{r+1}^{p,n-p}(\mathrm{Dec}^*K)$$

are isomorphisms for all $r \geq 0$. We have $\mathcal{E}_{r+1} = \mathrm{Dec}^{-1}(\mathcal{E}_r) = (\mathrm{Dec}^*)^{-1}(\mathcal{E}_r)$.

THEOREM 2.19. *For all $r \geq 0$, Deligne's décalage induces an equivalence of categories*

$$\mathrm{Dec} : \mathbf{D}^{\#}_{r+1}(\mathbf{F}\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^{\#}_r(\mathbf{F}\mathcal{A})\,.$$

PROOF. Consider the functor $\mathcal{J}_r := (S^r \circ \mathrm{Dec}^r) : \mathbf{C}^{\#}(\mathbf{F}\mathcal{A}) \to \mathbf{C}^{\#}_r(\mathbf{F}\mathcal{A})$ and denote by $i_r : \mathbf{C}^{\#}_r(\mathbf{F}\mathcal{A}) \hookrightarrow \mathbf{C}^{\#}(\mathbf{F}\mathcal{A})$ the inclusion. The functors $\mathcal{J}_r$ and i_r induce inverse equivalences $\mathcal{J}_r : \mathbf{D}^{\#}_r(\mathbf{F}\mathcal{A}) \rightleftarrows \mathbf{C}^{\#}_r(\mathbf{F}\mathcal{A})[\mathcal{E}_r^{-1}] : i_r$ at the localizations with respect to $\mathcal{E}_r$. Indeed, since $1 = \mathcal{J}_r \circ i_r$ it suffices to show that the map $\varepsilon_r : i_r \circ \mathcal{J}_r \to 1$ induced by the counit of the adjunction $S \dashv \mathrm{Dec}$ is an E_r-quasi-isomorphism. By (1) of Lemma 2.15 one has $\mathrm{Dec}^r \circ (i_r \circ \mathcal{J}_r) = \mathrm{Dec}^r$ and $\mathrm{Dec}^r(\varepsilon_r)$ is the identity morphism. Since $\mathcal{E}_{r+1} = \mathrm{Dec}^{-1}(\mathcal{E}_r)$, it follows that the map ε_r is an E_r-quasi-isomorphism, and the equivalence follows.

Since $\mathrm{Dec}(\mathcal{E}_{r+1}) \subset \mathcal{E}_r$ and $S(\mathcal{E}_r) \subset \mathcal{E}_{r+1}$, the inverse functors of Lemma 2.18 induce an equivalence $\mathrm{Dec} : \mathbf{C}^{\#}_{r+1}(\mathbf{F}\mathcal{A})[\mathcal{E}_{r+1}^{-1}] \xrightarrow{\sim} \mathbf{C}^{\#}_r(\mathbf{F}\mathcal{A})[\mathcal{E}_r^{-1}]$. We have a diagram of functors

$$\begin{array}{ccc}
\mathbf{D}^{\#}_{r+1}(\mathbf{F}\mathcal{A}) & \xrightarrow{\mathrm{Dec}} & \mathbf{D}^{\#}_r(\mathbf{F}\mathcal{A}) \\
\wr\downarrow \mathcal{J}_{r+1} & & \wr\downarrow \mathcal{J}_r \\
\mathbf{C}^{\#}_{r+1}(\mathbf{F}\mathcal{A})[\mathcal{E}_{r+1}^{-1}] & \xrightarrow[\sim]{\mathrm{Dec}} & \mathbf{C}^{\#}_r(\mathbf{F}\mathcal{A})[\mathcal{E}_r^{-1}]
\end{array}$$

where the bottom and vertical arrows are equivalences. Since $\mathrm{Dec} \circ S = 1$ this diagram commutes. □

The following result is a matter of verification and establishes the behavior of r-homotopies and r-homotopy equivalences by shift and décalage.

LEMMA 2.20. *The following diagram of functors commutes*

$$\begin{array}{ccccccc}
\mathbf{C}^{\#}(\mathbf{F}\mathcal{A}) & \xrightarrow{\mathrm{Dec}} & \mathbf{C}^{\#}(\mathbf{F}\mathcal{A}) & \xrightarrow{S} & \mathbf{C}^{\#}(\mathbf{F}\mathcal{A}) & \xrightarrow{\mathrm{Dec}^*} & \mathbf{C}^{\#}(\mathbf{F}\mathcal{A}) \\
\downarrow T_{r+1} & & \downarrow T_r & & \downarrow T_{r+1} & & \downarrow T_r \\
\mathbf{C}^{\#}(\mathbf{F}\mathcal{A}) & \xrightarrow{\mathrm{Dec}} & \mathbf{C}^{\#}(\mathbf{F}\mathcal{A}) & \xrightarrow{S} & \mathbf{C}^{\#}(\mathbf{F}\mathcal{A}) & \xrightarrow{\mathrm{Dec}^*} & \mathbf{C}^{\#}(\mathbf{F}\mathcal{A})\,.
\end{array}$$

In particular:

(1) Dec *sends $(r+1)$-homotopies from f to g to r-homotopies from* $\mathrm{Dec}f$ *to* $\mathrm{Dec}g$.

(2) *S sends r-homotopies from f to g to $(r+1)$-homotopies from Sf to Sg.*

COROLLARY 2.21. *We have inclusions* $\mathrm{Dec}(\mathcal{S}_{r+1})$, $\mathrm{Dec}^*(\mathcal{S}_{r+1}) \subset \mathcal{S}_r$ *and* $S(\mathcal{S}_r) \subset \mathcal{S}_{r+1}$.

2.4. Injective models. Generalizing the notion of filtered injective complex of Illusie, we introduce r-injective complexes and show that these are fibrant objects (in the sense of [13], Definition 2.2.1), with respect to the classes of r-homotopy equivalences and E_r-quasi-isomorphisms. We then prove the existence of r-injective models for bounded below filtered complexes (a similar result is proved by Paranjape in [24]), giving rise to a Cartan-Eilenberg structure. Our results are based on the original results of Illusie, using induction over $r \geq 0$ via Deligne's décalage functor.

DEFINITION 2.22. A filtered complex (K, F) is called *r-injective* if for all $p \in \mathbb{Z}$:

(i) the differential satisfies $d(F^pK) \subset F^{p+r}K$, that is $K \in \mathbf{C}_r^{\#}(\mathbf{F}\mathcal{A})$.
(ii) the graded object $Gr_F^pK \in \mathbf{C}^+(\mathrm{Inj}\mathcal{A})$ is a complex of injective objects of $\mathcal{A}$.

For $r = 0$ the first condition becomes trivial and we recover the original notion of filtered complex of injective type introduced by Illusie.

Denote by $\mathbf{C}_r^+(\mathbf{F}\mathrm{Inj}\mathcal{A})$ the full subcategory of $\mathbf{C}^+(\mathbf{F}\mathcal{A})$ of r-injective complexes.

LEMMA 2.23. *The functors* $\mathrm{Dec} = \mathrm{Dec}^* : \mathbf{C}_{r+1}^+(\mathbf{F}\mathrm{Inj}\mathcal{A}) \rightleftarrows \mathbf{C}_r^+(\mathbf{F}\mathrm{Inj}\mathcal{A}) : S$ *are inverse to each other.*

PROOF. It follows from Lemma 2.18 and the identity $Gr_{SF}^pK^n = Gr_F^{p-n}K^n$. □

PROPOSITION 2.24. *Let I be an r-injective complex. Every E_r-quasi-isomorphism $w : K \to L$ induces a bijection $w^* : [L, I]_r \longrightarrow [K, I]_r$ between r-homotopy classes of morphisms.*

PROOF. The case $r = 0$ follows from Lemma V.1.4.3 of [18]. We proceed by induction. Assume that I is an $(r+1)$-injective complex. By Lemma 2.20 we have a diagram

$$\begin{array}{ccc} [L, I]_{r+1} & \xrightarrow{\mathrm{Dec}^*} & [\mathrm{Dec}^*L, \mathrm{Dec}^*I]_r \\ \downarrow{\scriptstyle w^*} & & \downarrow{\scriptstyle w^*} \\ [K, I]_{r+1} & \xrightarrow{\mathrm{Dec}^*} & [\mathrm{Dec}^*K, \mathrm{Dec}^*I]_r \end{array}$$

where Dec^*I is r-injective by Lemma 2.23. By induction hypothesis, the vertical arrow on the right is a bijection. Hence to conclude the proof it suffices to show that the horizontal arrows are bijections. By Lemma 2.23 we have $S \circ \mathrm{Dec}^*I = I$. Together with the adjunction $\mathrm{Dec}^* \dashv S$ of Proposition 2.16 we obtain

$$\mathrm{Hom}(K, I) = \mathrm{Hom}(K, S \circ \mathrm{Dec}^*I) = \mathrm{Hom}(\mathrm{Dec}^*K, \mathrm{Dec}^*I)\,.$$

This identity is valid in the more general setting of degree preserving filtered maps. Together with Lemma 2.20 this gives a bijection between the set of $(r+1)$-homotopies $h : T_{r+1}(K) \to$

I from f to g and the set of r-homotopies $h : T_r(\mathrm{Dec}^*K) \to \mathrm{Dec}^*I$ from Dec^*f to Dec^*g. Therefore the horizontal arrows are bijections. □

PROPOSITION 2.25. *Let $\mathcal{A}$ be an abelian category with enough injectives. For every filtered complex K of $\mathbf{C}^+(\mathbf{F}\mathcal{A})$ there is an r-injective complex I and an E_r-quasi-isomorphism $\rho : K \to I$.*

PROOF. The case $r = 0$ follows from Lemma V.1.4.4 of [18]. We proceed by induction. Let $\rho : \mathrm{Dec}^*K \to I$ be an E_r-quasi-isomorphism, where I is r-injective. The adjunction $\mathrm{Dec}^* \dashv S$ of Proposition 2.16 gives an E_{r+1}-quasi-isomorphism $\rho : K \to SI$. By Lemma 2.23, the filtered complex SI is $(r + 1)$-injective. □

THEOREM 2.26. *Let $\mathcal{A}$ be an abelian category with enough injectives. For all $r \geq 0$ the triple $(\mathbf{C}^+(\mathbf{F}\mathcal{A}), \mathcal{S}_r, \mathcal{E}_r)$ is a right Cartan-Eilenberg category with fibrant models in $\mathbf{C}^+_r(\mathbf{F}\mathrm{Inj}\mathcal{A})$. The inclusion induces an equivalence of categories*

$$\mathbf{K}^+_r(\mathbf{F}\mathrm{Inj}\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^+_r(\mathbf{F}\mathcal{A})$$

between the category of r-injective complexes modulo r-homotopy, and the r-derived category.

PROOF. By Proposition 2.24 every r-injective complex I is a fibrant object with respect to the classes $\mathcal{S}_r \subset \mathcal{E}_r$, that is, every morphism $w : K \to L$ of $\mathcal{E}_r$ induces a bijection

$$\mathbf{C}^+(\mathbf{F}\mathcal{A})[\mathcal{S}_r^{-1}](L, I) \to \mathbf{C}^+(\mathbf{F}\mathcal{A})[\mathcal{S}_r^{-1}](K, I)\,.$$

By Proposition 2.25 every filtered complex K has an r-injective model: this is an r-injective complex I together with an E_r-quasi-isomorphism $w : K \to I$. Hence the triple $(\mathbf{C}^+(\mathbf{F}\mathcal{A}), \mathcal{S}_r, \mathcal{E}_r)$ is a Cartan-Eilenberg category with fibrant models in $\mathbf{C}^+_r(\mathbf{F}\mathrm{Inj}\mathcal{A})$. The equivalence of categories follows from Theorem 2.3.4 of [13]. □

2.5. Bifiltered complexes. We will consider bifiltered complexes $(K, W, F) \in \mathbf{C}^+(\mathbf{F}^2\mathcal{A})$ with W an increasing filtration and F a decreasing filtration. For the sake of simplicity, and given our interest in Hodge theory, we shall only describe the homotopy theory of bifiltered complexes with respect to the class of $E_{r,0}$-quasi-isomorphisms. Denote by Dec_W the functor defined by taking the décalage with respect to the filtration W and leaving F intact. The functor Dec^*_W is defined analogously via the dual décalage.

DEFINITION 2.27. Denote by $\mathcal{E}_{0,0}$ the class of maps $f : K \to L$ of $\mathbf{C}^+(\mathbf{F}^2\mathcal{A})$ inducing isomorphisms $H(Gr^W_p Gr^q_F f) : H(Gr^W_p Gr^q_F K) \to H(Gr^W_p Gr^q_F L)$ for all $p, q \in \mathbb{Z}$. Inductively over $r \geq 0$, define a class $\mathcal{E}_{r,0}$ of weak equivalences by letting

$$\mathcal{E}_{r+1,0} := \mathrm{Dec}_W^{-1}(\mathcal{E}_{r,0}) = (\mathrm{Dec}^*_W)^{-1}(\mathcal{E}_{r,0})\,.$$

Morphisms of $\mathcal{E}_{r,0}$ are called *$E_{r,0}$-quasi-isomorphisms.*

The $(r, 0)$-*derived category* is defined by $\mathbf{D}^+_{r,0}(\mathbf{F}^2\mathcal{A}) := \mathbf{C}^+(\mathbf{F}^2\mathcal{A})[\mathcal{E}_{r,0}^{-1}]$. There is an obvious notion of $(r, 0)$-translation functor induced by the automorphism of $\mathbf{F}^2\mathcal{A}$ that sends every bifiltered object (A, W, F) to the bifiltered object $(A, W(r), F)$. This defines a notion of $(r,0)$-homotopy. The associated class $\mathcal{S}_{r,0}$ of $(r,0)$-homotopy equivalences satisfies $\mathcal{S}_{r,0} \subset \mathcal{E}_{r,0}$. Hence the triple $(\mathbf{C}^+(\mathbf{F}^2\mathcal{A}), \mathcal{S}_{r,0}, \mathcal{E}_{r,0})$ is a category with strong and weak equivalences.

DEFINITION 2.28. A bifiltered complex (K, W, F) is $(r,0)$-*injective* if for all $p, q \in \mathbb{Z}$,

(i) the differential satisfies $d(W_p F^q K) \subset W_{p-r} F^q K$, and
(ii) the complex $Gr_p^W Gr_F^q K$ is an object of $\mathbf{C}^+(\mathrm{Inj}\mathcal{A})$.

THEOREM 2.29. *Let $\mathcal{A}$ be an abelian category with enough injectives. For all $r \geq 0$ the triple $(\mathbf{C}^+(\mathbf{F}^2\mathcal{A}), \mathcal{S}_{r,0}, \mathcal{E}_{r,0})$ is a right Cartan-Eilenberg category with fibrant models in $\mathbf{C}^+_{r,0}(\mathbf{F}^2\mathrm{Inj}\mathcal{A})$. The inclusion induces an equivalence of categories*

$$\mathbf{K}^+_{r,0}(\mathbf{F}^2\mathrm{Inj}\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^+_{r,0}(\mathbf{F}^2\mathcal{A})$$

between the category of $(r,0)$-injective complexes modulo $(r,0)$-homotopy, and the $(r,0)$-derived category.

PROOF. The proof is analogous to that of Theorem 2.26. □

3. Homotopy theory of diagrams of complexes.

In this section we study the homotopy theory of diagram categories whose vertices are categories of complexes over additive categories. The homotopical structure of the vertex categories is transferred to the diagram category with level-wise weak equivalences and level-wise fibrant models, via a rectification of homotopy commutative morphisms. Our approach to the problem fits in the context of the ∞-categories, but due to the simplicity of the situation (we consider diagrams of complexes of zig-zag type), only homotopies up to second order appear. The results will be applied to mixed Hodge complexes in the following section.

3.1. Diagrams of complexes.

DEFINITION 3.1. Let $\mathcal{C} : I \to \mathsf{Cat}$ be a functor from a small category I, to the category of categories Cat. Denote $\mathcal{C}_i := \mathcal{C}(i) \in \mathsf{Cat}$ for all $i \in I$ and $u_* := \mathcal{C}(u) \in \mathsf{Cat}(\mathcal{C}_i, \mathcal{C}_j)$ for all $u : i \to j$. The *category of diagrams* $\Gamma\mathcal{C}$ *associated with the functor* $\mathcal{C}$ is defined as follows:

- An object X of $\Gamma\mathcal{C}$ is given by a family of objects $\{X_i \in \mathcal{C}_i\}$, for all $i \in I$, together with a family of morphisms $\varphi_u : u_*(X_i) \to X_j$, called *comparison morphisms*, for every map $u : i \to j$. Such an object is denoted as $X = \big(X_i \overset{\varphi_u}{\dashrightarrow} X_j\big)$.
- A morphism $f : X \to Y$ of $\Gamma\mathcal{C}$ is given by a family of morphisms $\{f_i : X_i \to Y_i\}$ of $\mathcal{C}_i$ for all $i \in I$, such that for every map $u : i \to j$ of I, the diagram

$$\begin{array}{ccc} u_*(X_i) & \xrightarrow{\varphi_u} & X_j \\ \downarrow{\scriptstyle u_*(f_i)} & & \downarrow{\scriptstyle f_j} \\ u_*(Y_i) & \xrightarrow{\varphi_u} & Y_j \end{array}$$

commutes in $\mathcal{C}_j$. Such a morphism is denoted $f = (f_i) : X \to Y$.

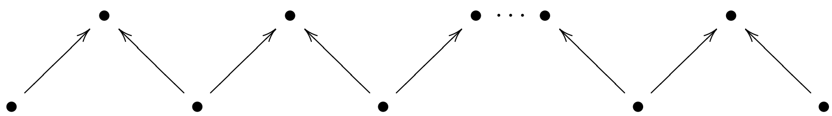
We will often write X_i for $u_*(X_i)$ and f_i for $u_*(f_i)$, whenever there is no danger of confusion.

REMARK 3.2. The category of diagrams $\Gamma\mathcal{C}$ associated with $\mathcal{C}$ is the category of sections of the projection functor $\pi : \int_I \mathcal{C} \to I$, where $\int_I \mathcal{C}$ is the *Grothendieck construction* of $\mathcal{C}$ (see [30]). If $\mathcal{C} : I \to \mathsf{Cat}$ is the constant functor $i \mapsto \mathcal{C}$ then $\Gamma\mathcal{C} = \mathcal{C}^I$ is the diagram category of $\mathcal{C}$ under I.

NOTATION 3.3. We will restrict our study to diagram categories indexed by a finite directed category I whose degree function takes values in $\{0, 1\}$. That is:

(D_0) There exists a *degree function* $|\cdot| : \mathrm{Ob}(I) \longrightarrow \{0, 1\}$ such that $|i| < |j|$ for every non-identity morphism $u : i \to j$ of I.

A finite category I satisfying (D_0) is a particular case of a Reedy category for which $I^+ = I$. The main examples of such categories are given by finite zig-zags.



We will also assume that the functor $\mathcal{C} : I \to \mathsf{Cat}$ satisfies the following axioms:

(D_1) For all $i \in I$, $\mathcal{C}_i = \mathbf{C}^+(\mathcal{A}_i)$ is a category of bounded below complexes, where $\mathcal{A}_i$ is an additive category with an additive automorphism $\alpha_i : \mathcal{A}_i \to \mathcal{A}_i$ and a natural transformation $\eta_i : \alpha_i \to 1$. In particular, there is a class $\mathcal{S}_i$ of homotopy equivalences associated with the translation T_{α_i} induced by α_i (see Definition 2.1).
(D_2) There is a class of weak equivalences $\mathcal{W}_i$ of $\mathcal{C}_i$ such that $\mathcal{S}_i \subset \mathcal{W}_i$. In particular, the triple $(\mathcal{C}_i, \mathcal{S}_i, \mathcal{W}_i)$ is a category with strong and weak equivalences.
(D_3) For all $u : i \to j$, the functor u_* is induced by an additive functor $A_i \to A_j$ compatible with α_i. Furthermore, u_* preserves strong and weak equivalences and fibrant objects (these are the objects Q of $\mathcal{C}_i$ such that every morphism $w : X \to Y$ of $\mathcal{W}_i$ induces a bijection $\mathcal{C}_i[\mathcal{S}_i^{-1}](Y, Q) \to \mathcal{C}_i[\mathcal{S}_i^{-1}](X, Q)$).

With the above hypotheses, the category $\Gamma\mathcal{C}$ is a category of complexes of diagrams of additive categories. The level-wise translation functors define a translation functor on $\Gamma\mathcal{C}$. This gives a notion of homotopy between morphisms of $\Gamma\mathcal{C}$. Denote by $\mathcal{S}$ the associated class of homotopy equivalences. Note that if $f = (f_i) \in \mathcal{S}$, then $f_i \in \mathcal{S}_i$ for all $i \in I$, but the converse is not true in general. Denote by $\mathcal{W}$ the class of level-wise weak equivalences of $\Gamma\mathcal{C}$. Since $\mathcal{S}_i \subset \mathcal{W}_i$ for all $i \in I$, it follows that $\mathcal{S} \subset \mathcal{W}$. Therefore the triple $(\Gamma\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a category with strong and weak equivalences.

3.2. Morphisms up to homotopy. We next introduce a new category $\Gamma\mathcal{C}^h$ whose objects are those of $\Gamma\mathcal{C}$ but whose morphisms are defined by level-wise morphisms compatible up to fixed homotopies.

For the sake of simplicity, from now on we will omit the notations T_i, α_i and $\eta_i : \alpha_i \to 1$ of the translation functors of $\mathcal{C}_i$, and use the standard notation $X[n]$ instead of $T_i^n(X)$, for $X \in \mathcal{C}_i$.

DEFINITION 3.4. Let X and Y be two objects of $\Gamma\mathcal{C}$ and let $n \in \mathbb{Z}$. A *pre-morphism of degree n from X to Y* is given by a pair of families $f = (f_i, F_u)$ indexed by $i \in I$ and $u \in \mathrm{Hom}_I(i, j)$, where

(i) $f_i : X_i[-n] \to Y_i$ is a degree preserving map in $\mathcal{C}_i$.
(ii) $F_u : X_i[-n+1] \to Y_j$ is a degree preserving map in $\mathcal{C}_j$.

Given a pre-morphism $f = (f_i, F_u)$ of degree n from X to Y, define its differential by

$$Df = \left(df_i - (-1)^n f_i d,\ F_u d + (-1)^n dF_u + (-1)^n (f_j \varphi_u - \varphi_u f_i)\right) .$$

DEFINITION 3.5. A *ho-morphism* $f : X \rightsquigarrow Y$ is a pre-morphism f of degree 0 from X to Y such that $Df = 0$. It is given by a pair of families $f = (f_i, F_u)$ such that for all $i \in I$ and $u : i \to j$,

(i) $f_i : X_i \to Y_i$ is a morphism of complexes in $\mathcal{C}_i$.
(ii) $F_u : X_i[1] \to Y_j$ satisfies $dF_u + F_u d = \varphi_u f_i - f_j \varphi_u$, that is, F_u is a homotopy from $f_j \varphi_u$ to $\varphi_u f_i$.

Given ho-morphisms $f : X \rightsquigarrow Y$ and $g : Y \rightsquigarrow Z$ let $gf : X \rightsquigarrow Z$ be the ho-morphism given by $gf = (g_i f_i, G_u f_i + g_j F_u)$. This defines an associative composition between ho-morphisms. The identity ho-morphism of X is $1 = (1_{X_i}, 0)$. A ho-morphism $f : X \rightsquigarrow Y$ is invertible if and only if the morphism f_i is invertible for all $i \in I$. Then $f^{-1} = (f_i^{-1}, -f_j^{-1} F_u f_i^{-1})$.

Denote by $\Gamma\mathcal{C}^h$ the category whose objects are those of $\Gamma\mathcal{C}$ and whose morphisms are ho-morphisms. Every morphism $f = (f_i)$ can be made into a ho-morphism by letting $F_u = 0$. Hence $\Gamma\mathcal{C}$ is a subcategory of $\Gamma\mathcal{C}^h$.

DEFINITION 3.6. A ho-morphism $f = (f_i, F_u)$ is said to be a *weak equivalence* if f_i is a weak equivalence in $\mathcal{C}_i$ for all $i \in I$.

DEFINITION 3.7. Let $f, g : X \rightsquigarrow Y$ be ho-morphisms. A *homotopy from f to g* is a pre-morphism h of degree -1 from X to Y such that $Dh = g - f$. Hence $h = (h_i, H_u)$ is such that, for all $i \in I$ and $u : i \to j$,

(i) $h_i : X_i[1] \to Y_i$ satisfies $dh_i + h_i d = g_i - f_i$, that is, h_i is a homotopy from f_i to g_i.
(ii) $H_u : X_i[2] \to Y_j$ satisfies $H_u d - dH_u = G_u - F_u + h_j \varphi_u - \varphi_u h_i$.

We denote such a homotopy by $h : f \simeq g$.

LEMMA 3.8. *The homotopy relation between ho-morphisms is an equivalence relation compatible with composition.*

PROOF. Symmetry and reflexivity are trivial. To prove transitivity consider three ho-morphisms $f, f', f'' : X \rightsquigarrow Y$ such that $h : f \simeq f'$, and $h' : f' \simeq f''$. A homotopy from f to f'' is given by $h'' = h + h' = (h_i + h'_i, H_u + H'_u)$. This proves that $\simeq$ is an equivalence relation. Let $h : f \simeq f'$ be a homotopy from f to f'. Given a ho-morphism $g : Y \rightsquigarrow Z$, a homotopy from gf to gf' is given by $gh = (g_i h_i, G_u h_i + g_j H_u)$. Given a ho-morphism $g' : W \rightsquigarrow X$, a homotopy from fg' to $f'g'$ is given by $hg' = (h_i g'_i, H_u g'_i + h_j G'_u)$. □

We will denote by $[X, Y]^h$ the class of ho-morphisms from X to Y modulo homotopy and by $\pi(\Gamma\mathcal{C}^h) := \Gamma\mathcal{C}^h/\simeq$ the corresponding quotient category $\pi(\Gamma\mathcal{C}^h)(X, Y) := [X, Y]^h$.

The following generalizes the constructions given in Section 2.1.

DEFINITION 3.9. Let $f : X \rightsquigarrow Y$ and $g : X \rightsquigarrow Y'$ be two ho-morphisms. The *double mapping cylinder of f and g* is the object of $\Gamma\mathcal{C}$ defined by

$$\mathcal{C}yl(f, g) = \left(\mathcal{C}yl(f_i, g_i) \overset{\psi_u}{\dashrightarrow} \mathcal{C}yl(f_j, g_j)\right),$$

where $\mathcal{C}yl(f_i, g_i)$ is the double mapping cylinder of f_i and g_i, and for all $u : i \to j$ the comparison map $\psi_u : \mathcal{C}yl(f_i, g_i) \to \mathcal{C}yl(f_j, g_j)$ is given by

$$\psi_u = \begin{pmatrix} \varphi_u & 0 & 0 \\ \text{-}F_u & \varphi_u & 0 \\ G_u & 0 & \varphi_u \end{pmatrix}.$$

Let $i : Y' \to \mathcal{C}yl(f, g)$, $j : Y \to \mathcal{C}yl(f, g)$ and $k : X[1] \to \mathcal{C}yl(f, g)$ be defined level-wise by the inclusions into the corresponding direct summands. Then i and j are morphisms of diagrams and k is a homotopy of ho-morphisms from jf to ig. With these notations:

LEMMA 3.10. *For any object Z of $\Gamma\mathcal{C}$, the map*

$$\Gamma\mathcal{C}^h(\mathcal{C}yl(f, g), Z) \to \left\{(h, u, v)\,;\ h : uf \simeq vg,\ \begin{matrix} u \in \Gamma\mathcal{C}^h(Y, Z) \\ v \in \Gamma\mathcal{C}^h(Y', Z) \end{matrix}\right\}$$

defined by $t \mapsto (tk, tj, ti)$ is a bijection.

PROOF. Define an inverse $(h, u, v) \mapsto t = (t_i, T_u)$ by letting

$$t_i(x, y, z) = h_i(x) + u_i(y) + v_i(z) \text{ and } T_u(x, y, z) = H_u(x) + U_u(y) + V_u(z).$$

□

DEFINITION 3.11. Let $f : X \rightsquigarrow Y$ be a ho-morphism.

(1) The *mapping cylinder* of f is the diagram

$$\mathcal{C}yl(f) := \mathcal{C}yl(f, 1_X) = \left(\mathcal{C}yl(f_i) \overset{\psi_u}{\dashrightarrow} \mathcal{C}yl(f_j)\right).$$

(2) The *mapping cone* of f is the diagram

$$C(f) := \mathcal{C}yl(0, f) = \left(C(f_i) \overset{\psi_u}{\dashrightarrow} C(f_j)\right).$$

COROLLARY 3.12. *For any object Z of $\Gamma\mathcal{C}$ the map*

$$\Gamma\mathcal{C}^h(C(f), Z) \longrightarrow \{(h, v)\,;\ h : 0 \simeq vf, v \in \Gamma\mathcal{C}^h(Y, Z)\}$$

defined by $t \mapsto (tk, tj)$ is a bijection.

DEFINITION 3.13. The *cylinder* of a diagram X is the diagram

$$\mathrm{Cyl}(X) := \mathcal{C}yl(1_X) = \left(\mathrm{Cyl}(X_i) \overset{\psi_u}{\dashrightarrow} \mathrm{Cyl}(X_j)\right).$$

By Lemma 3.10, a homotopy $h : f \simeq g$ between two ho-morphisms $f, g : X \rightsquigarrow Y$ is equivalent to a ho-morphism $H : \mathrm{Cyl}(X) \rightsquigarrow Y$ satisfying $Hj = f$ and $Hi = g$.

3.3. Rectification of ho-morphisms. The notion of homotopy between ho-morphisms allows to define a new class of strong equivalences of $\Gamma\mathcal{C}$ as follows.

DEFINITION 3.14. A morphism $f : X \to Y$ of $\Gamma\mathcal{C}$ is said to be a *ho-equivalence* if there exists a ho-morphism $g : Y \rightsquigarrow X$, together with homotopies $gf \simeq 1_X$ and $fg \simeq 1_Y$. We say that the ho-morphism g is a *homotopy inverse* of f.

Denote by $\mathcal{H}$ the class of ho-equivalences of $\Gamma\mathcal{C}$. This class is closed by composition, and satisfies $\mathcal{S} \subset \mathcal{H} \subset \mathcal{W}$, where $\mathcal{S}$ and $\mathcal{W}$ denote the classes of homotopy and weak equivalences of $\Gamma\mathcal{C}$.

Using the approach of Cartan-Eilenberg categories of [13] we study the localized category $\mathrm{Ho}(\Gamma\mathcal{C}) := \Gamma\mathcal{C}[\mathcal{W}^{-1}]$ by means of the localization $\Gamma\mathcal{C}[\mathcal{H}^{-1}]$.

We first describe $\Gamma\mathcal{C}[\mathcal{H}^{-1}]$ in terms of homotopy classes of ho-morphisms. Consider the solid diagram of functors

$$\begin{array}{ccccc} \Gamma\mathcal{C} & \xrightarrow{i} & \Gamma\mathcal{C}^h & \xrightarrow{\pi} & \pi(\Gamma\mathcal{C}^h) \\ \big\downarrow{\scriptstyle\delta} & & & \nearrow{\scriptstyle\Psi} & \\ \Gamma\mathcal{C}[\mathcal{H}^{-1}] & & & & \end{array}.$$

Since every morphism of $\mathcal{H}$ is an isomorphism in $\pi(\Gamma\mathcal{C}^h)$, there exists a unique dotted functor Ψ making the diagram commute. Our next objective is to show that Ψ admits an inverse functor.

Let $f : X \rightsquigarrow Y$ be a ho-morphism. Define a ho-morphism $p : \mathcal{C}yl(f) \rightsquigarrow Y$ by the level-wise morphisms $p_i(x, y, z) = y + f_i(z)$, together with the homotopies $P_u(x, y, z) = F_u(z)$.

With these notations we have the following Brown-type factorization lemma for ho-morphisms.

PROPOSITION 3.15. *Let $f : X \rightsquigarrow Y$ be a ho-morphism. Then the diagram*

$$\begin{array}{ccccc} X & \xrightarrow{i} & \mathcal{C}yl(f) & \xleftarrow{j} & Y \\ & {\scriptstyle f}\searrow & \big\downarrow{\scriptstyle p} & \swarrow\!\!= & \\ & & Y & & \end{array}$$

commutes in $\Gamma\mathcal{C}^h$. In addition,

(1) *The maps j and p are weak equivalences.*
(2) *The morphism j is a ho-equivalence with homotopy inverse p.*

(3) *If f is a weak equivalence, then i is a weak equivalence.*

PROOF. It is a matter of verification that the diagram commutes. Since weak equivalences are defined level-wise, (1) and (3) are straightforward. We prove (2). Let $h_i : \mathcal{C}yl(f_i)[1] \to \mathcal{C}yl(f_i)$ be defined by $h_i(x, y, z) = (z, 0, 0)$. Then the pair $h = (h_i, H_u)$ with $H_u = 0$, is a homotopy from jp to the identity. Indeed, $(dh_i + h_i d) = 1 - j_i p_i$ and $0 = -j_j P_u + h_j \psi_u - \psi_u h_i$. □

We will use the following result on ho-morphisms between mapping cylinders.

LEMMA 3.16. *Given a commutative diagram in $\Gamma\mathcal{C}^h$*

$$\begin{array}{ccc} X & \overset{a}{\rightsquigarrow} & Z \\ {\scriptstyle f}\wr\downarrow & & \wr\downarrow{\scriptstyle g} \\ Y & \underset{b}{\rightsquigarrow} & W \end{array}$$

there is an induced ho-morphism $(a, b)_* : \mathcal{C}yl(f) \rightsquigarrow \mathcal{C}yl(g)$ which is compatible with i, j and p. In addition, if a, b are morphisms of $\Gamma\mathcal{C}$ then $(a, b)_*$ is also a morphism of $\Gamma\mathcal{C}$.

PROOF. Since $g_i a_i = f_i b_i$, the assignation $(x, y, z) \mapsto (a_i(x), b_i(y), a_i(z))$ gives a well-defined morphism of complexes $(a_i, b_i)_* : \mathcal{C}yl(f_i) \to \mathcal{C}yl(g_i)$. Since $G_u a_i + g_j A_u = b_j F_u + B_u f_i$, the assignation $(x, y, z) \mapsto (-A_u(x), B_u(y), -A_u(z))$ defines a homotopy from $\psi_u(a_i, b_i)_*$ to $(a_j, b_j)_* \psi_u$. We get a ho-morphism $(a, b)_*$ making the following diagram commute in $\Gamma\mathcal{C}^h$

$$\begin{array}{ccccccc} X & \xrightarrow{i} & \mathcal{C}yl(f) & \overset{p}{\rightsquigarrow} & Y & \xrightarrow{j} & \mathcal{C}yl(f) \\ {\scriptstyle a}\wr\downarrow & & {\scriptstyle (a,b)_*}\wr\downarrow & & {\scriptstyle b}\wr\downarrow & & {\scriptstyle (a,b)_*}\wr\downarrow \\ Z & \xrightarrow{i} & \mathcal{C}yl(g) & \overset{p}{\rightsquigarrow} & W & \xrightarrow{j} & \mathcal{C}yl(g) \end{array} .$$

If a and b are in $\Gamma\mathcal{C}$ then $A_u = 0$ and $B_u = 0$, and $(a, b)_*$ is also a morphism of $\Gamma\mathcal{C}$. □

NOTATION 3.17. Given two objects X and Y of $\Gamma\mathcal{C}$, define a map

$$\Phi_{X,Y} : \Gamma\mathcal{C}^h(X, Y) \longrightarrow \Gamma\mathcal{C}[\mathcal{H}^{-1}](X, Y)$$

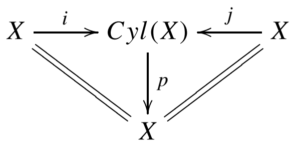
as follows. Let $f : X \rightsquigarrow Y$ be a ho-morphism. By Proposition 3.15 we have $f = p_f i_f$ in $\Gamma\mathcal{C}^h$, where i_f is a morphism of $\Gamma\mathcal{C}$ and p_f is a homotopy inverse for the ho-equivalence j_f. We then let

$$\Phi_{X,Y}(f) := j_f^{-1} i_f \in \Gamma\mathcal{C}[\mathcal{H}^{-1}] .$$

THEOREM 3.18. *The above map induces a well-defined functor $\Phi : \pi(\Gamma\mathcal{C}^h) \longrightarrow \Gamma\mathcal{C}[\mathcal{H}^{-1}]$ which is an inverse of Ψ.*

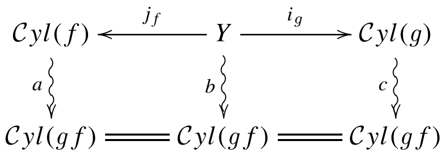
PROOF. Since $\mathcal{S} \subset \mathcal{H}$, the functor $\delta : \Gamma\mathcal{C} \to \Gamma\mathcal{C}[\mathcal{H}^{-1}]$ factors through $\gamma : \Gamma\mathcal{C} \to \Gamma\mathcal{C}[\mathcal{S}^{-1}]$. Hence to prove that two morphisms in $\Gamma\mathcal{C}[\mathcal{H}^{-1}]$ coincide, it suffices to prove that they coincide in $\Gamma\mathcal{C}[\mathcal{S}^{-1}]$.

We first show that the assignation $(X, Y) \mapsto \Phi_{X,Y}$ defines a functor $\Phi : \Gamma\mathcal{C}^h \to \Gamma\mathcal{C}[\mathcal{H}^{-1}]$. Given an object $X \in \Gamma\mathcal{C}$, the diagram

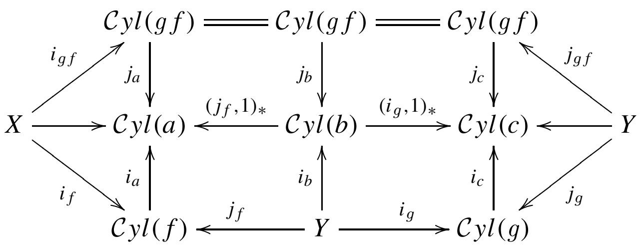


commutes in $\Gamma\mathcal{C}$. Hence $j^{-1}i = 1_X$ in $\Gamma\mathcal{C}[\mathcal{H}^{-1}]$, and $\Phi(1_X) = 1_X$.

Let $f : X \rightsquigarrow Y$ and $g : Y \rightsquigarrow Z$ be two ho-morphisms. To show that $\Phi(g) \circ \Phi(f) = \Phi(g \circ f)$ consider the following diagram

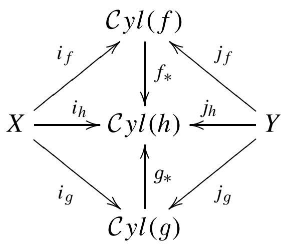


where $a := (1_X, g)_*$, $b := j_{gf} \circ g$ and $c := j_{gf} \circ p_g$. By Lemma 3.16 the above diagram commutes in $\Gamma\mathcal{C}^h$. Taking factorizations we obtain a diagram of morphisms in $\Gamma\mathcal{C}$



which commutes in $\Gamma\mathcal{C}[\mathcal{S}^{-1}]$. Therefore $j_{gf}^{-1} i_{gf} = j_g^{-1} i_g j_f^{-1} i_f$ in $\Gamma\mathcal{C}[\mathcal{H}^{-1}]$. This proves that $\Phi : \Gamma\mathcal{C}^h \to \Gamma\mathcal{C}[\mathcal{H}^{-1}]$ is a functor.

We next show that if $f \simeq g$ then $\Phi(f) = \Phi(g)$. A homotopy from f to g defines a ho-morphism $h : \mathrm{Cyl}(X) \rightsquigarrow Y$ such that $hi_X = f$ and $hj_X = g$. Consider the morphisms $f_* := (i_X, 1_Y)_*$ and $g_* := (j_X, 1_Y)_*$ defined by Lemma 3.16. Then the following diagram commutes in $\Gamma\mathcal{C}[\mathcal{S}^{-1}]$.



Hence $\Phi(f) = \Phi(g)$. This proves that Φ induces a functor $\Phi : \pi(\Gamma\mathcal{C}^h) \to \Gamma\mathcal{C}[\mathcal{H}^{-1}]$.

Lastly, we show that Φ is an inverse functor of Ψ. Let $f : X \rightsquigarrow Y$ be a ho-morphism. Then $\Psi(\Phi([f])) = \Psi(j_f^{-1} i_f) = [p_f i_f] = [f]$. To check the other composition it suffices to show that if $g : X \to Y$ is a ho-equivalence, then $\Phi(\Psi(g^{-1})) = g^{-1}$. Let $h : Y \rightsquigarrow X$ be a homotopy inverse of g. By definition we have $\Psi(g^{-1}) = [h]$. Then $g \circ \Phi(\Psi(g^{-1})) = [g] \circ \Phi([h]) = \Phi([g \circ h]) = 1$. Therefore $\Phi(\Psi(g^{-1})) = g^{-1}$. □

3.4. Fibrant models of diagrams.

NOTATION 3.19. Denote by $\Gamma\mathcal{C}_f$ the full subcategory of $\Gamma\mathcal{C}$ of objects $Q = (Q_i \overset{\varphi_u}{\dashrightarrow} Q_j)$ of $\Gamma\mathcal{C}$ such that for all $i \in I$, Q_i is fibrant in $(\mathcal{C}_i, \mathcal{S}_i, \mathcal{W}_i)$ in the sense of [13], that is: every weak equivalence $f : X \to Y$ in $\mathcal{C}_i$ induces a bijection $w^* : [Y, Q_i] \to [X, Q_i]$. Condition ($D_3$) of 3.3 ensures that for all $u : i \to j$, the object $u_*(Q_i)$ is fibrant in $(\mathcal{C}_j, \mathcal{S}_j, \mathcal{W}_j)$.

PROPOSITION 3.20. *Let $Q \in \Gamma\mathcal{C}_f$ and let $w : X \rightsquigarrow Y$ be a ho-morphism. If w is a weak equivalence then it induces a bijection $w^* : [Y, Q]^h \longrightarrow [X, Q]^h$ between homotopy classes of ho-morphisms.*

PROOF. We first prove surjectivity. Let $f : X \rightsquigarrow Q$ be a ho-morphism. Since Q_i is fibrant in $\mathcal{C}_i$, there exists a morphism $g_i : Y_i \to Q_i$ together with a homotopy $h_i : g_i w_i \simeq f_i$, for all $i \in I$. The chain of homotopies

$$g_j \varphi_u w_i \overset{-g_j W_u}{\simeq} g_j w_j \varphi_u \overset{h_j \varphi_u}{\simeq} f_j \varphi_u \overset{F_u}{\simeq} \varphi_u f_i \overset{-\varphi_u h_i}{\simeq} \varphi_u g_i w_i$$

gives a homotopy $G'_u := h_j \varphi_u - \varphi_u h_i + F_u - g_j W_u$ from $g_j \varphi_u w_i$ to $\varphi_u g_i w_i$.

By Lemma 2.5 the triple $(G'_u, g_j \varphi_u, \varphi_u g_i)$ defines a map $t_u : \mathcal{C}yl(w_i, w_i) \to Q_j$. We have a solid diagram

$$\begin{array}{ccc} \mathcal{C}yl(w_i, w_i) & \xrightarrow{t_u} & Q_j \\ {\scriptstyle w_i \oplus 1 \oplus 1}\downarrow & \overset{t'_u}{\nearrow} & \\ \mathrm{Cyl}(Y_i) & & \end{array}$$

where the vertical arrow is a weak equivalence. Since $u_*(Q_i)$ is fibrant in $\mathcal{C}_j$, there exists a dotted morphism t'_u making the diagram commute up to homotopy. By Lemma 2.5 t'_u defines a triple (G''_u, g'_u, g''_u) such that $(H_u, h'_u, h''_u) : (G''_u, g'_u, g''_u) \circ (w_i \oplus 1 \oplus 1) \simeq (G'_u, g_j \varphi_u, \varphi_u g_i)$. Let $G_u := -h'_u + G''_u + h''_u$. Then $G_u : g_j \varphi_u \simeq \varphi_u g_i$, and $H_u : G_u w_i \simeq G'_u$. The pair $g = (g_i, G_u)$ is a ho-morphism, and $H = (h_i, H_u)$ is a homotopy from gw to f.

To prove injectivity it suffices to show that if $f : Y \rightsquigarrow Q$ is such that $h : 0 \simeq fw$, then $0 \simeq f$. By Lemma 3.12 we have a solid diagram

$$\begin{array}{ccc} C(w) & \overset{(h,f)}{\rightsquigarrow} & Q \\ {\scriptstyle w \oplus 1}\wr\downarrow & \overset{(h', f')}{\nearrow} & \\ C(1_Y) & & \end{array} .$$

Since w is a weak equivalence, the map $(w \oplus 1)^* : [C(1_Y), Q]^h \longrightarrow [C(w), Q]^h$ is surjective. In particular, there exists a ho-morphism $f' : Y \rightsquigarrow Q$ such that $f' \simeq 0$, together with a homotopy $f \simeq f'$. Using transitivity we have $f \simeq 0$. Therefore the map w^* is injective. □

COROLLARY 3.21. *Let $Q \in \Gamma\mathcal{C}_f$. Every weak equivalence $w : X \to Y$ in $\Gamma\mathcal{C}$ induces a bijection $w^* : \Gamma\mathcal{C}[\mathcal{H}^{-1}](Y, Q) \longrightarrow \Gamma\mathcal{C}[\mathcal{H}^{-1}](X, Q)$.*

PROOF. This follows from Proposition 3.20 and Theorem 3.18. □

We next prove the existence of level-wise fibrant models in $\Gamma\mathcal{C}$.

PROPOSITION 3.22. *Let $\Gamma\mathcal{C}$ be a category of diagrams satisfying the hypotheses* (D_0)–(D_3) *of* 3.3. *Assume that every object of $\mathcal{C}_i$ has a fibrant model in $(\mathcal{C}_i, \mathcal{S}_i, \mathcal{W}_i)$. Then for every object X of $\Gamma\mathcal{C}$ there is an object $Q \in \Gamma\mathcal{C}_f$, together with a ho-morphism $X \rightsquigarrow Q$, which is a weak equivalence.*

PROOF. Let $f_i : X_i \to Q_i$ be fibrant models in $(\mathcal{C}_i, \mathcal{S}_i, \mathcal{W}_i)$. By ($D_3$) of 3.3, $u_*(Q_i)$ is fibrant in $\mathcal{C}_j$. Therefore given the solid diagram

$$\begin{array}{ccc} X_i & \xrightarrow{\varphi_u} & X_j \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_j} \\ Q_i & \overset{\varphi'_u}{\dashrightarrow} & Q_j \end{array}$$

there exists a dotted arrow φ'_u together with a homotopy $F_u : f_j\varphi_u \simeq \varphi'_u f_i$. This defines an object Q of $\Gamma\mathcal{C}_f$ and a ho-morphism $f = (f_i, F_u) : X \rightsquigarrow Q$ which is a weak equivalence. □

In the following we consider a full subcategory $\mathcal{D} \subset \Gamma\mathcal{C}$ of a category of diagrams. This will be useful for the application to the category of mixed Hodge complexes.

THEOREM 3.23. *Let $\Gamma\mathcal{C}$ be a category of diagrams satisfying the hypotheses* (D_0)–(D_3) *of* 3.3. *Let $\mathcal{D}$ be a full subcategory of $\Gamma\mathcal{C}$ such that:*

(i) *If $f : X \rightsquigarrow Y$ is a ho-morphism between objects of $\mathcal{D}$, the mapping cylinder $Cyl(f)$ also an object of $\mathcal{D}$.*
(ii) *There is a full subcategory $\mathcal{M} \subset \mathcal{D} \cap \Gamma\mathcal{C}_f$ such that for every object D of $\mathcal{D}$, there exists an object $Q \in \mathcal{M}$ together with a ho-morphism $D \rightsquigarrow Q$ which is a weak equivalence.*

Then the triple $(\mathcal{D}, \mathcal{H}, \mathcal{W})$ is a right Cartan-Eilenberg category with fibrant models in $\mathcal{M}$. The inclusion induces an equivalence of categories $\pi(\mathcal{M}^h) \xrightarrow{\sim} \mathcal{D}[\mathcal{W}^{-1}]$, where $\pi(\mathcal{M}^h)$ is the category whose objects are those of $\mathcal{M}$ and whose morphisms are homotopy classes of ho-morphisms.

PROOF. Consider the solid diagram of functors

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\ \pi\ } & \pi(\mathcal{D}^h) \\ {\scriptstyle\gamma}\big\downarrow & \nearrow{\scriptstyle\Psi} & \\ \mathcal{D}[\mathcal{H}^{-1}] & & \end{array}.$$

Since every morphism of $\mathcal{H}$ is an isomorphism in $\pi(\mathcal{D}^h)$, by the universal property of the localizing functor γ, there is a unique dotted functor Ψ making the diagram commute. By (i), the mapping cylinder $\mathcal{C}yl(f)$ of a ho-morphism $f : X \rightsquigarrow Y$ is in $\mathcal{D}$. Hence the map $\Phi_{X,Y} : \mathcal{D}^h(X, Y) \to \mathcal{D}[\mathcal{H}^{-1}](X, Y)$ given by $f \mapsto j_f^{-1} i_f$ is well-defined. By Theorem 3.18 it induces a functor $\Phi : \pi(\mathcal{D}^h) \to \mathcal{D}[\mathcal{H}^{-1}]$ which is an inverse of Ψ. In particular, by restriction to $\mathcal{M}$ we obtain an equivalence of categories $\Phi : \pi(\mathcal{M}^h) \xrightarrow{\sim} \mathcal{M}[\mathcal{H}^{-1}; \mathcal{D}]$, where $\mathcal{M}[\mathcal{H}^{-1}; \mathcal{D}]$ denotes the full subcategory of $\mathcal{D}[\mathcal{H}^{-1}]$ whose objects are in $\mathcal{M}$.

By (ii) and Proposition 3.21, for every object D of $\mathcal{D}$ there exists a fibrant object $Q \in \mathcal{M}$ and a ho-morphism $\rho : D \rightsquigarrow Q$ which is a weak equivalence. Then the morphism $\Phi_{D,Q}(\rho) : D \to Q$ of $\mathcal{D}[\mathcal{H}^{-1}]$ is a fibrant model of D. This proves that the triple $(\mathcal{D}, \mathcal{H}, \mathcal{W})$ is a Cartan-Eilenberg category with fibrant models in $\mathcal{M}$. By Theorem 2.3.4 of [13] the inclusion induces an equivalence of categories $\mathcal{M}[\mathcal{H}^{-1}; \mathcal{D}] \xrightarrow{\sim} \mathcal{D}[\mathcal{W}^{-1}]$. □

4. Mixed and absolute Hodge complexes.

4.1. Diagrams of filtered complexes. Let $I = \{0 \to 1 \leftarrow 2 \to \cdots \leftarrow s\}$ be an index category of zig-zag type and fixed length s, and let $\mathbf{k} \subset \mathbb{R}$ be a field. We next define the category of diagrams of filtered complexes of type I over $\mathbf{k}$. This is a diagram category whose vertices are categories of filtered and bifiltered complexes defined over $\mathbf{k}$ and $\mathbb{C}$ respectively. Additional assumptions on the filtrations will lead to the notions of mixed and absolute Hodge complexes.

DEFINITION 4.1. Let $\mathbf{F} : I \to \mathsf{Cat}$ be the functor defined by

$$\begin{array}{ccccccccc} 0 & \xrightarrow{\ u\ } & 1 & \longleftarrow & \cdots & \longrightarrow & s-1 & \xleftarrow{\ v\ } & s \\ \big\downarrow & & \big\downarrow & & & & \big\downarrow & & \big\downarrow \\ \mathbf{C}^+(\mathbf{Fk}) & \xrightarrow{\ u_*\ } & \mathbf{C}^+(\mathbf{F}\mathbb{C}) & \xleftarrow{\ Id\ } & \cdots & \xrightarrow{\ Id\ } & \mathbf{C}^+(\mathbf{F}\mathbb{C}) & \xleftarrow{\ v_*\ } & \mathbf{C}^+(\mathbf{F}^2\mathbb{C}) \end{array}$$

where $u_*(K_{\mathbf{k}}, W) = (K_{\mathbf{k}}, W) \otimes \mathbb{C}$ is given by extension of scalars and $v_*(K_{\mathbb{C}}, W, F) = (K_{\mathbb{C}}, W)$ is the forgetful functor for the second filtration. All intermediate functors are identities. The *category of diagrams of filtered complexes* is the category of diagrams $\Gamma\mathbf{F}$ associated with $\mathbf{F}$. Objects of $\Gamma\mathbf{F}$ are denoted by $K = ((K_{\mathbf{k}}, W) \overset{\varphi}{\leftarrow\text{-}\text{-}\rightarrow} (K_{\mathbb{C}}, W, F))$.

For $r \in \{0, 1\}$ consider the classes of weak equivalences of $\mathbf{C}^+(\mathbf{Fk})$ and $\mathbf{C}^+(\mathbf{F}\mathbb{C})$ given by E_r-quasi-isomorphisms, and the notion of r-homotopy defined by the corresponding r-translation functors. Likewise, in $\mathbf{C}^+(\mathbf{F}^2\mathbb{C})$, consider the class of weak equivalences given by $E_{r,0}$-quasi-isomorphisms, and the notion of $(r, 0)$-homotopy defined by the $(r, 0)$-translation functor. With these choices, the category of diagrams $\Gamma\mathbf{F}$ satisfies conditions (D_0)–(D_3) of 3.3.

Denote by $\mathcal{E}_{r,0}$ the class of level-wise weak equivalences of $\Gamma\mathbf{F}$. The level-wise homotopies define the notions of *ho$_r$-morphism* of diagrams and *$(r,0)$-homotopy* between ho$_r$-morphisms. Denote by $\mathcal{H}_{r,0}$ the corresponding class of ho-equivalences: these are morphisms of $\Gamma\mathbf{F}$ that are $(r,0)$-homotopy equivalences as ho$_r$-morphisms. It satisfies $\mathcal{H}_{r,0} \subset \mathcal{E}_{r,0}$.

Denote by $\mathcal{Q}$ the class of weak equivalences of $\Gamma\mathbf{F}$ given by level-wise quasi-isomorphisms compatible with filtrations. Since filtrations are regular and exhaustive, we have $\mathcal{E}_{0,0} \subset \mathcal{E}_{1,0} \subset \mathcal{Q}$. Hence we have localization functors

$$\mathrm{Ho}_{0,0}(\Gamma\mathbf{F}) \to \mathrm{Ho}_{1,0}(\Gamma\mathbf{F}) \to \mathrm{Ho}(\Gamma\mathbf{F}) .$$

Deligne's décalage with respect to the filtration W defines a functor $\mathrm{Dec}_W : \Gamma\mathbf{F} \to \Gamma\mathbf{F}$ which sends a diagram K to the diagram

$$\mathrm{Dec}_W K := ((K_{\mathbf{k}}, \mathrm{Dec}W) \overset{\varphi}{\dashleftarrow\dashrightarrow} (K_{\mathbb{C}}, \mathrm{Dec}W, F)) .$$

It has a left adjoint $S_W : \Gamma\mathbf{F} \to \Gamma\mathbf{F}$ defined by shifting the filtration W. We have:

THEOREM 4.2. *Deligne's décalage induces an equivalence of categories*

$$\mathrm{Dec}_W : \mathrm{Ho}_{1,0}(\Gamma\mathbf{F}) \longrightarrow \mathrm{Ho}_{0,0}(\Gamma\mathbf{F}) .$$

PROOF. It follows from Theorems 2.19 and 2.29. □

4.2. Hodge complexes.

DEFINITION 4.3 ([9], 8.1.5). A *mixed Hodge complex* is a diagram of filtered complexes

$$K = \Big((K_{\mathbf{k}}, W) \overset{\varphi}{\dashleftarrow\dashrightarrow} (K_{\mathbb{C}}, W, F)\Big) ,$$

satisfying the following conditions:

(MH_0) The comparison map φ is a string of E_1^W-quasi-isomorphisms. The cohomology $H^*(K_{\mathbf{k}})$ has finite type.

(MH_1) For all $p \in \mathbb{Z}$, the differential of $Gr_p^W K_{\mathbb{C}}$ is strictly compatible with F.

(MH_2) For all $n \geq 0$ and all $p \in \mathbb{Z}$, the filtration F induced on $H^n(Gr_p^W K_{\mathbb{C}})$ defines a pure Hodge structure of weight $p+n$ on $H^n(Gr_p^W K_{\mathbf{k}})$.

Denote by **MHC** the category of mixed Hodge complexes of a fixed type I, omitting the index category in the notation. Note that axiom (MH_2) implies that for all $n \geq 0$ the triple $(H^n(K_{\mathbf{k}}), \mathrm{Dec}W, F)$ is a mixed Hodge structure.

Deligne showed that, given a mixed Hodge complex K, the spectral sequences associated with $(K_{\mathbb{C}}, F)$ and $(Gr_p^W K_{\mathbb{C}}, F)$ degenerate at E_1, while the spectral sequences associated with $(K_{\mathbf{k}}, W)$ and $(Gr_F^q K_{\mathbb{C}}, W)$ degenerate at E_2 (see Scholie 8.1.9 of [9]).

For convenience, we consider a shifted version of mixed Hodge complex in which all spectral sequences degenerate at the first stage. This corresponds to the notion of mixed Hodge complex given by Beilinson in [1] (see also Chapter 8 of [17] or Section 2.3 of [21] and [29] where *standard* and *absolute* weight filtrations are compared).

DEFINITION 4.4. An *absolute Hodge complex* is a diagram of filtered complexes

$$K = \Big((K_{\mathbf{k}}, W) \overset{\varphi}{\dashleftarrow\dashrightarrow} (K_{\mathbb{C}}, W, F) \Big) ,$$

satisfying the following conditions:

(AH_0) The comparison map φ is a string of E_0^W-quasi-isomorphisms. The cohomology $H^*(K_{\mathbf{k}})$ has finite type.

(AH_1) The four spectral sequences associated with K degenerate at E_1.

(AH_2) For all $n \geq 0$ and all $p \in \mathbb{Z}$, the filtration F induced on $H^n(Gr_p^W K_{\mathbb{C}})$ defines a pure Hodge structure of weight p on $H^n(Gr_p^W K_{\mathbf{k}})$.

Denote by **AHC** the category of absolute Hodge complexes. By Scholie 8.1.9 of [9] Deligne's décalage with respect to the weight filtration sends every mixed Hodge complex to an absolute Hodge complex. Hence we have a functor $\mathrm{Dec}_W : \mathbf{MHC} \to \mathbf{AHC}$. Note however that the shift $S_W K$ of an absolute Hodge complex K is not in general a mixed Hodge complex.

Since the category of mixed Hodge structures is abelian ([8], Theorem 2.3.5), every complex of mixed Hodge structures is an absolute Hodge complex. Denote by $\mathbf{G}^+(\mathsf{MHS})$ the category of non-negatively graded mixed Hodge structures. We have full subcategories

$$\mathbf{G}^+(\mathsf{MHS}) \longrightarrow \mathbf{C}^+(\mathsf{MHS}) \longrightarrow \mathbf{AHC} .$$

Our objective is to study the homotopy categories

$$\mathrm{Ho}(\mathbf{MHC}) := \mathbf{MHC}[\mathcal{Q}^{-1}] \text{ and } \mathrm{Ho}(\mathbf{AHC}) := \mathbf{AHC}[\mathcal{Q}^{-1}]$$

defined by inverting level-wise quasi-isomorphisms.

The following result follows easily from Theorem 2.3.5 of [8], stating that morphisms of mixed Hodge structures are strictly compatible with filtrations.

LEMMA 4.5. *We have* $\mathcal{Q} \cap \mathbf{AHC} = \mathcal{E}_{0,0} \cap \mathbf{AHC}$ *and* $\mathcal{Q} \cap \mathbf{MHC} = \mathcal{E}_{1,0} \cap \mathbf{MHC}$. *In particular*

$$\mathrm{Ho}_{0,0}(\mathbf{AHC}) = \mathrm{Ho}(\mathbf{AHC}) \textit{ and } \mathrm{Ho}_{1,0}(\mathbf{MHC}) = \mathrm{Ho}(\mathbf{MHC}) .$$

We will also use the following result (c.f. [26], Theorem 3.22).

LEMMA 4.6. *Let* $f : K \rightsquigarrow L$ *be a* ho_0*-morphism (resp.* ho_1*-morphism) of absolute (resp. mixed) Hodge complexes. Then* $\mathcal{C}yl(f)$ *is an absolute (resp. mixed) Hodge complex.*

PROOF. By Lemma 3.15, given a ho_r-morphism $f : K \to L$ of diagrams of filtered complexes, the level-wise inclusion $j : L \to \mathcal{C}yl(f)$ is in $\mathcal{E}_{r,0}$. Therefore condition (AH_0) (resp. (MH_0)) follows from the two-out-of-three property of E_r-quasi-isomorphisms. Assume that $f : K \rightsquigarrow L$ is a ho_0-morphism of absolute Hodge complexes. To show that $\mathcal{C}yl(f)$ is an absolute Hodge complex it suffices to note that condition (AH_1) is preserved by $E_{0,0}$-quasi-isomorphisms, while condition (AH_2) is a consequence of the following isomorphisms:

$$H^n(Gr_p^W \mathcal{C}yl_{\mathbf{k}}) \cong H^n(Gr_p^W L_{\mathbf{k}}),\ H^n(Gr_p^W Gr_F^q \mathcal{C}yl_{\mathbb{C}}) \cong H^n(Gr_p^W Gr_F^q L_{\mathbb{C}}) .$$

Assume that $f : K \rightsquigarrow L$ is a ho_1-morphism of mixed Hodge complexes. For all $p \in \mathbb{Z}$,

$$Gr_p^W \mathcal{C}yl(f_{\mathbb{C}}) = Gr_{p-1}^W K_{\mathbb{C}}[1] \oplus Gr_p^W L_{\mathbb{C}} \oplus Gr_p^W K_{\mathbb{C}}\,.$$

Hence at the graded level, the differential of $\mathcal{C}yl(f)$ is given by the differential at each component, and we have a direct sum decomposition of complexes compatible with the Hodge filtration. Hence (MH_1) and (MH_2) follow. □

4.3. Minimal models. The following technical lemma will be of use for the construction of minimal models.

LEMMA 4.7. *Let K be an absolute Hodge complex. Let $Z^n(K_i) := \mathrm{Ker}(d : K_i^n \to K_i^{n+1})$.*

(1) *There are sections $\sigma_{\mathbf{k}}^n : H^n(K_{\mathbf{k}}) \to Z^n(K_{\mathbf{k}})$ and $\sigma_i^n : H^n(K_i) \to Z^n(K_i)$ of the projection, which are compatible with W.*
(2) *There exists a section $\sigma_{\mathbb{C}}^n : H^n(K_{\mathbb{C}}) \to Z^n(K_{\mathbb{C}})$ of the projection, which is compatible with both filtrations W and F.*

PROOF. The first assertion follows from the degeneration of the spectral sequence associated with $(K_{\mathbf{k}}, W)$ at the first stage. To prove the second assertion we use Deligne's splitting of mixed Hodge structures. Since the cohomology $H^n(K_{\mathbf{k}})$ is a mixed Hodge structure, by Lemma 1.2.11 of [8] (see also Lemma 1.12 of [12]) there is a direct sum decomposition $H^n(K_{\mathbb{C}}) = \bigoplus I^{p,q}$ with $I^{p,q} \subset W_{p+q}F^pH^n(K_{\mathbb{C}})$ and such that

$$W_mH^n(K_{\mathbb{C}}) = \bigoplus_{p+q\le m+n} I^{p,q} \text{ and } F^lH^n(K_{\mathbb{C}}) = \bigoplus_{p\ge l} I^{p,q}\,.$$

Therefore it suffices to define sections $\sigma^{p,q} : I^{p,q} \to Z^n(K_{\mathbb{C}})$. By ($\mathrm{AH}_1$) the four spectral sequences associated with K degenerate at E_1. It follows that the induced filtrations in cohomology are given by:

$$W_pF^qH^n(A_{\mathbb{C}}) = \mathrm{Im}\{H^n(W_pF^qK_{\mathbb{C}}) \to H^n(K_{\mathbb{C}})\}\,.$$

Since $I^{p,q} \subset W_{p+q}F^qH^n(K_{\mathbb{C}})$ we have $\sigma^{p,q}(I^{p,q}) \subset W_{p+q}F^pK_{\mathbb{C}}$. □

THEOREM 4.8. *Let K be an absolute Hodge complex. There exists a ho_0-morphism $\rho : K \rightsquigarrow H(K)$ which is a quasi-isomorphism.*

PROOF. By Lemma 4.7 we can find sections $\sigma_{\mathbf{k}} : H^*(K_{\mathbf{k}}) \to K_{\mathbf{k}}$ and $\sigma_i : H^*(K_i) \to K_i$ compatible with the filtration W, together with a section $\sigma_{\mathbb{C}} : H^*(K_{\mathbb{C}}) \to K_{\mathbb{C}}$ compatible with W and F. By definition, all maps are quasi-isomorphisms. Let $\varphi_u : K_i \to K_j$ be a component of the quasi-equivalence φ of K. The diagram

$$\begin{array}{ccc} H^*(K_i) & \xrightarrow{\varphi_u^*} & H^*(K_j) \\ {\scriptstyle\sigma_i}\downarrow & & \downarrow{\scriptstyle\sigma_j} \\ K_i & \xrightarrow{\varphi_u} & K_j \end{array}$$

is not necessarily commutative, but for any element $x \in H^*(K_i)$, the difference $(\sigma_j \varphi_u^* - \varphi_u \sigma_j)(x)$ is a coboundary. Since the differentials are strictly compatible with the weight filtration there exists a linear map $\Sigma_u : H^*(K_i)[1] \to K_j$ compatible with the weight filtration W and such that $\sigma_j \varphi_u^* - \varphi_u \sigma_i = d\Sigma_u$. The morphisms $\sigma_{\mathbf{k}}$, σ_i and $\sigma_{\mathbb{C}}$ together with the homotopies Σ_u define a ho$_0$-morphism $\sigma : H(K) \rightsquigarrow K$ which by construction is a quasi-isomorphism. Since every object in **AHC** is fibrant, by Lemma 3.20 this lifts to a ho$_0$-morphism $\rho : K \rightsquigarrow H(K)$ which is a quasi-isomorphism. □

Denote by $\pi\left(\mathbf{G}^+(\mathsf{MHS})^h\right)$ the category whose objects are non-negatively graded mixed Hodge structures and whose morphisms are $(0, 0)$-homotopy classes of ho$_0$-morphisms.

THEOREM 4.9. *The triple* $(\mathbf{AHC}, \mathcal{H}_{0,0}, \mathcal{Q})$ *is a right Cartan-Eilenberg category and* $\mathbf{G}^+(\mathsf{MHS})$ *is a full subcategory of fibrant minimal models. The inclusion induces an equivalence of categories*

$$\pi\left(\mathbf{G}^+(\mathsf{MHS})^h\right) \xrightarrow{\sim} \mathrm{Ho}\,(\mathbf{AHC})\ .$$

PROOF. We show that the conditions of Theorem 3.23 are satisfied, with $\Gamma\mathcal{C} = \Gamma\mathbf{F}$, $\mathcal{D} = \mathbf{AHC}$ and $\mathcal{M} = \mathbf{G}^+(\mathsf{MHS})$. By Lemma 4.5, the class $\mathcal{Q}$ of quasi-isomorphisms coincides with the class $\mathcal{E}_{0,0}$ of level-wise E_0- (resp. $E_{0,0}$-) quasi-isomorphisms. Therefore conditions (D_0)–(D_3) of 3.3 are trivially satisfied. Condition (i) of Theorem 3.23 follows from Lemma 4.6. By Proposition 2.24 and Corollary 3.21 every object of **AHC** is fibrant. Hence $\mathbf{AHC} \subset \Gamma\mathbf{F}_f$. Then condition (ii) follows from Theorem 4.8. □

Let H be a graded object in the category of mixed Hodge structures. Then the shift $S_W H$ is a mixed Hodge complex with trivial differentials, since $Gr_p^{SW} H_{\mathbf{k}}^n = Gr_{p+n}^W H_{\mathbf{k}}^n$. This gives a functor $S_W : \mathbf{G}^+(\mathsf{MHS}) \to \mathbf{MHC}$. Denote by $\pi(S_W(\mathbf{G}^+(\mathsf{MHS}))^h)$ the category whose objects are those of $S_W(\mathbf{G}^+(\mathsf{MHS}))$ and whose morphisms are $(1, 0)$-homotopy classes of ho$_1$-morphisms.

THEOREM 4.10. *The triple* $(\mathbf{MHC}, \mathcal{H}, \mathcal{Q})$ *is a right Cartan-Eilenberg category, and* $S_W(\mathbf{G}^+(\mathsf{MHS}))$ *is a full subcategory of fibrant minimal models. The inclusion induces an equivalence of categories*

$$\pi(S_W(\mathbf{G}^+(\mathsf{MHS}))^h) \xrightarrow{\sim} \mathrm{Ho}\,(\mathbf{MHC})\ .$$

PROOF. We show that the conditions of Theorem 3.23 are satisfied, with $\Gamma\mathcal{C} = \Gamma\mathbf{F}$, $\mathcal{D} = \mathbf{MHC}$ and $\mathcal{M} = S_W(\mathbf{G}^+(\mathsf{MHS}))$. By Lemma 4.5, the class $\mathcal{Q}$ of quasi-isomorphisms coincides with the class $\mathcal{E}_{1,0}$ of level-wise E_1- (resp. $E_{1,0}$-) quasi-isomorphisms. Therefore conditions (D_0)–(D_3) of 3.3 are trivially satisfied. Condition (i) of Theorem 3.23 follows from Lemma 4.6. We verify condition (ii). Let K be a mixed Hodge complex. Then $\mathrm{Dec}_W K$ is an absolute Hodge complex and by Theorem 4.8 there exists a ho$_0$-morphism $\sigma : H(\mathrm{Dec}_W K) \rightsquigarrow \mathrm{Dec}_W K$. The adjunction $S_W \dashv \mathrm{Dec}_W$ defined at the level of diagrams of filtered complexes together with Lemma 2.20.(ii) gives a ho$_1$-morphism $S_W H(\mathrm{Dec}_W K) \rightsquigarrow K$ which is a quasi-isomorphism. From Proposition 2.24 and Corollary 3.21 every object in $S_W(\mathbf{G}^+(\mathsf{MHS}))$ is fibrant. □

THEOREM 4.11. *Deligne's décalage induces an equivalence of categories*

$$\mathrm{Dec}_W : \mathrm{Ho}(\mathbf{MHC}) \xrightarrow{\sim} \mathrm{Ho}(\mathbf{AHC})\,.$$

PROOF. It suffices to note that when restricted to complexes with trivial differentials, the functors Dec_W and S_W are inverse to each other. The result follows from Theorems 4.9 and 4.10. □

4.4. Applications. We give an alternative proof of Beilinson's Theorem on absolute Hodge complexes and study further properties of morphisms of absolute Hodge complexes in the homotopy category.

THEOREM 4.12 ([1], Theorem 3.4). *The inclusion induces an equivalence of categories*

$$\mathbf{D}^+\left(\mathsf{MHS}\right) \xrightarrow{\sim} \mathrm{Ho}\,(\mathbf{AHC})\ .$$

PROOF. It suffices to verify the hypotheses of Theorem 3.23 for the subcategory of complexes of mixed Hodge structures $\mathbf{C}^+\left(\mathsf{MHS}\right) \subset \Gamma\mathbf{F}$. Conditions ($D_0$)–($D_3$) of 3.3 and condition (ii) of Theorem 3.23 follow analogously to Theorem 4.9. We verify condition (i), that is, the mapping cylinder $\mathcal{C}yl(g)$ of a ho-morphism $g = (g_{\mathbf{k}}, g_{\mathbb{C}}, G) : K \rightsquigarrow L$ of complexes of mixed Hodge structures is a complex of mixed Hodge structures. The morphism $\psi : \mathcal{C}yl(g_{\mathbf{k}}) \otimes \mathbb{C} \to \mathcal{C}yl(g_{\mathbb{C}})$ defined by

$$\psi = \begin{pmatrix} 1 & 0 & 0 \\ -G & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is invertible. Define a filtration F on $\mathcal{C}yl(f_{\mathbf{k}}) \otimes \mathbb{C}$ by letting

$$F^p(\mathcal{C}yl(f_{\mathbf{k}}) \otimes \mathbb{C}) := \psi^{-1}(F^p\mathcal{C}yl(f_{\mathbb{C}}))\,.$$

Since the category of mixed Hodge structures is abelian, this endows $\mathcal{C}yl(f_{\mathbf{k}})$ with mixed Hodge structures. □

Every mixed Hodge structure can be identified with a complex of mixed Hodge structures concentrated in degree 0. With this identification, and since the category MHS is abelian, given mixed Hodge structures H and H' over a field $\mathbf{k}$, one can compute their extensions as

$$\mathrm{Ext}^n(H, H') = \mathbf{D}^+(\mathsf{MHS})(H, H'[n])\,.$$

Given filtered (resp. bifiltered) vector spaces X and Y over $\mathbf{k}$, denote by $\mathrm{Hom}^W(X, Y)$ (resp. $\mathrm{Hom}^W_F(X, Y)$) the set of morphisms from X to Y that are compatible with the filtration W (resp. the filtrations W and F).

We next recover a result of Carlson [3] regarding extensions of mixed Hodge structures, by studying the morphisms in the homotopy category of absolute Hodge complexes (see also Section I.3 of [26] and Proposition 8.1 of [22]).

PROPOSITION 4.13. *Let H and H' be mixed Hodge structures. Then*

$$\mathrm{Ext}^1(H, H') = \frac{\mathrm{Hom}^W(H_{\mathbb{C}}, H'_{\mathbb{C}})}{\mathrm{Hom}^W(H_{\mathbf{k}}, H'_{\mathbf{k}}) + \mathrm{Hom}^W_F(H_{\mathbb{C}}, H'_{\mathbb{C}})}\,,$$

and $\mathrm{Ext}^n(H, H') = 0$ *for all* $n > 1$.

PROOF. By Theorems 4.9 and 4.12 we have $\mathrm{Ext}^n(H, H') = [H, H'[n]]^h$ for all $n \geq 0$.

A pre-morphism g from H to $H'[n]$ of degree 0 is given by a triple $g = (g_{\mathbf{k}}, g_{\mathbb{C}}, G)$ where $g_{\mathbf{k}} \in \mathrm{Hom}^W(H_{\mathbf{k}}, H'_{\mathbf{k}}[n])$, $g_{\mathbb{C}} \in \mathrm{Hom}_F^W(H_{\mathbb{C}}, H'_{\mathbb{C}}[n])$ and $G \in \mathrm{Hom}^W(H_{\mathbb{C}}, H'_{\mathbb{C}}[n-1])$. Its differential is given by $Dg = (0, 0, (-1)^m(g_{\mathbb{C}} - g_{\mathbf{k}} \otimes \mathbb{C}))$.

If $n = 1$ it follows that $g_{\mathbf{k}} = 0$, and $g_{\mathbb{C}} = 0$. In particular, it satisfies $Dg = 0$. Such a cocycle $g = (0, 0, G)$ is a coborder if and only if $G = h_{\mathbf{k}} \otimes \mathbb{C} - h_{\mathbb{C}}$, where $h_{\mathbf{k}} \in \mathrm{Hom}^W(H_{\mathbf{k}}, H'_{\mathbf{k}})$ and $h_{\mathbb{C}} \in \mathrm{Hom}_F^W(H_{\mathbb{C}}, H'_{\mathbb{C}})$. This proves the formula for $\mathrm{Ext}^1(H, H')$.

If $n > 1$ then $g_{\mathbf{k}} = 0$, $g_{\mathbb{C}} = 0$ and $G = 0$. Hence $\mathrm{Ext}^n(H, H') = 0$. □

Morphisms in the homotopy category of **AHC** are characterized as follows.

COROLLARY 4.14. *Let K and L be absolute Hodge complexes. Then*

$$\mathrm{Ho}(\mathbf{AHC})(K, L) = \bigoplus_n \left(\mathrm{Hom}_{\mathsf{MHS}}(H^n K, H^n L) \oplus \mathrm{Ext}^1_{\mathsf{MHS}}(H^n K, H^{n-1} L)\right).$$

PROOF. By Theorem 4.9 there is a bijection $\mathrm{Ho}(\mathbf{AHC})(K, L) \cong [H(K), H(L)]^h$. A ho-morphism $g : H(K) \rightsquigarrow H(L)$ is given by a morphism $g^*_{\mathbf{k}} : H^*(K_{\mathbf{k}}) \to H^*(L_{\mathbf{k}})$ compatible with W, a morphism $g^*_{\mathbb{C}} : H^*(K_{\mathbb{C}}) \to H^*(L_{\mathbb{C}})$ compatible with W and F, such that $g_{\mathbf{k}} \otimes \mathbb{C} \cong g_{\mathbb{C}}$, together with a morphism $G^* : H^*(K_{\mathbb{C}})[1] \to H^*(L_{\mathbb{C}})$ compatible with W.

Such a ho-morphism is a coboundary if $g = Dh$, for some pre-morphism h of degree -1. This implies $g_{\mathbf{k}} = 0$ and $g_{\mathbb{C}} = 0$, and there is a morphism $h^*_{\mathbf{k}} : H^*(K_{\mathbf{k}})[1] \to H^*(L_{\mathbf{k}})$ compatible with W, and a morphism $h^*_{\mathbb{C}} : H^*(K_{\mathbb{C}})[1] \to H^*(L_{\mathbb{C}})$ compatible with W and F, such that $G \cong h_{\mathbf{k}} \otimes \mathbb{C} - h_{\mathbb{C}}$. The result now follows from Proposition 4.13. □

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