# Homotopy Types of Subspace Arrangements via Diagrams of Spaces

Günter M. Ziegler & Rade T. Živaljević

Institut Mittag-Leffler
Auravägen 17
S-18262 Djursholm, Sweden
ziegler@ml.kva.se

Matematički Institut
Knez Mihailova 35/1, p.p. 367
11001 Beograd, Yugoslavia
ezivalje@yubgef51.bitnet

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**Abstract.** We prove combinatorial formulas for the homotopy type of the union of the subspaces in an (affine, compactified affine, spherical or projective) subspace arrangement. From these formulas we derive results of Goresky & MacPherson on the homology of the arrangement and the cohomology of its complement.

The union of an arrangement can be interpreted as the direct limit of a diagram of spaces over the intersection poset. A closely related space is obtained by taking the homotopy direct limit of this diagram. Our method consists in constructing a combinatorial model diagram over the same poset, whose homotopy limit can be compared to the original one by usual homotopy comparison results for diagrams of spaces.

## 0. Introduction.

In this paper we describe a general method to construct the homotopy type of an arrangement in terms of its combinatorial data. We demonstrate its use in the cases of linear, affine, spherical and projective subspace arrangements.

The key lemmas for our approach are standard tools in algebraic topology, used for example to define localizations of spaces in the setting of semisimplicial theory [BK]. It seems, however, that they have not previously been applied in a combinatorial setting.

To make them more easily accessible for other possible applications (e.g., arrangements of quadrics in projective spaces, of trains in Grassmannians, etc.), we give a summary of the set-up in Section 1. Simple proofs are given in the appendix to this paper. The following is an outline of our approach.

**0. Arrangement**

An *arrangement* is a finite set of subspaces $\mathcal{A} = \{A_1, \ldots, A_m\}$ in a topological *ambient space* $U$. We assume that the arrangement is closed under intersection. By the *union* of the arrangement we mean the space $D := \bigcup \mathcal{A} = \bigcup_{i=1}^{m} A_i$, by the *complement* we mean $M := U \backslash D$.

**1. Combinatorial data**

The intersection poset $P$ is a partially ordered set that is isomorphic to the set of all subspaces in $\mathcal{A}$, ordered by reversed inclusion. Thus for every element $p \in P$ there is a corresponding subspace $A_p \in \mathcal{A}$, and $q \leq p$ means $A_q \supseteq A_p$. The *combinatorial data* specify for every $p \in P$ the homotopy type of $A_p$, and for every $q \leq p$ the homotopy class of the inclusion map $A_p \hookrightarrow A_q$.

**2. Diagram of spaces**

A *diagram of spaces* $\mathcal{D} = \mathcal{D}(\mathcal{A})$ is a functor $\mathcal{D} : P \longrightarrow$ CW-Top, which associates a topological space $A_p$ to every $p \in P$ and the inclusion map $A_p \hookrightarrow A_q$ to every order relation $q \leq p$ in $P$. The union of an arrangement is the (direct) limit of its diagram of spaces. Assuming that the inclusion maps are cofibrations, this limit is homotopy equivalent to the *homotopy (direct) limit* $\|\mathcal{D}\|$ [Projection Lemma 1.6].

**3. Normalization**

From the combinatorial data, one can construct a diagram $\mathcal{D}'$ of spaces over the same poset $P$. The homotopy limit of this *model diagram* serves as a combinatorial model for the union $D(\mathcal{A})$. If there is a homomorphism of diagrams $\mathcal{D} \longrightarrow \mathcal{D}'$ which induces a homotopy equivalence $D_i \longrightarrow D'_i$ for all $i$, then the homotopy limits are homotopy equivalent [Homotopy Lemma 1.7], which yields a combinatorial formula $D(\mathcal{A}) \simeq \|\mathcal{D}\| \simeq \|\mathcal{D}'\|$ for the homotopy type of the union.

**4. Simplification**

Often the homotopy type of the model diagram can be further simplified. For example, if all the maps are homotopically trivial, the homotopy limit has a wedge decomposition over the poset $P$ [Wedge Lemma 1.8]. If the homotopy types of the spaces and the embedding maps are (up to homotopy) determined by dimensions, then one can sometimes get a wedge decomposition over the rank levels of the poset $P$.

**5. Duality**

If the arrangement determines a relative (homology) manifold $(U, D)$, then Lefschetz duality yields information about the compactified complement. If the ambient space is a (homology) sphere, then Alexander duality can be used to deduce the cohomology of the complement.

Let us give one example for the formulas derived by this approach.

For this, denote by $\mathcal{A} = \{A_1, \ldots, A_m\}$ a set of affine subspaces (not necessarily containing the origin) in $U = \mathbb{R}^n$. Assume (without affecting the topology) that the arrangement $\mathcal{A}$ is closed under intersection, let $P$ be an abstract poset that is isomorphic to the set of all non-empty intersections of subspaces in $\mathcal{A}$, ordered by reversed inclusion, and let $d : P \longrightarrow \mathbb{N}_0$ be the dimension function. Then $D := \bigcup \mathcal{A}$ is homotopy equivalent to the order complex $\Delta(P)$ – this follows from the nerve theorem.

**Theorem (2.2).** *The one-point compactification $\widehat{D} = D \uplus \{\infty\}$ is homotopy equivalent to*

$$\widehat{D} \simeq \bigvee_{p \in P} \Delta(P_{<p}) * S^{d(p)},$$

From this, it is easy to see that

$$\widetilde{\mathrm{H}}_i(\widehat{D}; \mathbb{Z}) \cong \bigoplus_{p \in P} \widetilde{\mathrm{H}}_{i-1-d(p)}(\Delta(P_{<p}); \mathbb{Z})$$

and thus by Alexander duality in $\widehat{\mathbb{R}^n} \cong S^n$ one gets Corollary 2.3 [GM, III.1.5 "Theorem A"]:

$$\widetilde{\mathrm{H}}^i(\mathbb{R}^n \backslash D; \mathbb{Z}) \cong \bigoplus_{p \in P} \widetilde{\mathrm{H}}_{n-2-i-d(p)}(\Delta(P_{<p}); \mathbb{Z})$$

for all $i \in \mathbb{Z}$.

This result is best possible in several respects. First, neither the homeomorphism type of the union nor the homotopy type of the complement are determined by the combinatorial data. In fact, the algebra structure of the cohomology of the complement is not determined by the combinatorial data. These observations even apply in the special case where $\mathcal{A}$ is an "even" arrangement (as considered by Goresky & McPherson [GM, p. 257]), that is, if the subspaces in $\mathcal{A}$ have codimension 2 and all intersections have even codimensions, see [Z1].

Our approach does not utilize any differentiable structure. Thus it carries over verbatim to arrangements of flats in an oriented matroid [BLSWZ] and their affine and projective versions.

## 1. Diagrams of Spaces.

A functor $\mathcal{D} : S \longrightarrow A$ from a small category $S$ to an arbitrary category $A$ will be called an *$S$-diagram of objects in $A$.* All $S$-diagrams of objects in $A$ form a category, where morphisms are natural transformations of functors.

A finite partially ordered set $(P, \leq)$ is here seen as a small category with arrows pointing downwards, i.e., $p \to q$ is equivalent to $p \geq q$. We will be solely interested in

$(P,\leq)$-diagrams of topological spaces, usually spheres and projective spaces and almost always CW-complexes, so for us a $P$-diagram of spaces is a functor

$$\mathcal{D}:\ (P,\leq)\ \longrightarrow\ \text{CW-Top}$$

to the category of spaces having the homotopy type of a finite CW-complex. If $\mathcal{D}$ is a $P$-diagram of spaces then the space associated with the element $p\in P$ is denoted by $D_p$, and the morphism corresponding to $p\geq q$ is denoted by $d_{pq}:=\mathcal{D}(p\to q)$.

**Example 1.1.** Let $\mathcal{A}$ be a finite collection of subspaces of a topological space $M$. Then $P:=(\mathcal{A},\supseteq)$ is a finite partially ordered set and the identity map $\mathcal{I}:P\longrightarrow\mathcal{A}$ defines a $P$-diagram of spaces with inclusion maps as morphisms. Diagrams of this type are very frequent and will be referred to as *subset diagrams*.

**Definition 1.2.** An *arrangement* is a finite collection $\mathcal{A}=\{A_1,\ldots,A_m\}$ of closed subspaces of a topological space $U$ such that

(i) $\mathcal{A}$ is closed under intersection, that is, $A,B\in\mathcal{A}$ implies $A\cap B\in\mathcal{A}$, and

(ii) for $A,B\in\mathcal{A}$ and $A\subseteq B$ the inclusion map $A\hookrightarrow B$ is a cofibration.

The *union* of the arrangement $\mathcal{A}$ in $U$ is $D:=\bigcup\mathcal{A}$, the *complement* is $M:=U\backslash D$.

Thus every arrangement gives rise to an associated $P$-diagram $\mathcal{D}=\mathcal{D}(\mathcal{A})$, where $P$ is the *intersection poset* of $\mathcal{A}$: a finite join semi-lattice whose maximal element $\hat{1}$ corresponds to $A_{\hat{1}}=\bigcap\mathcal{A}$, which may be empty.

The key to our treatment of arrangements is the replacement of the limit of its subset diagram, which is $\bigcup\mathcal{A}$, by the homotopy direct limit [a. k. a. "the classifying space of the corresponding small category of spaces"], which we will now define. This construction belongs to the general class of "geometric realization of semisimplicial sets" constructions, essentially started by Milnor [Mi] and widely used in topology, see for example [Se], [BK], [GZ], [Vo] etc.

The primitives for the construction are the *order complex* $\Delta(P)$, i.e., the geometric realization of the simplicial complex of chains in $P$, and the *mapping cylinder* $Z(f)$ of a map $f:A\longrightarrow B$, obtained from $A\times[0,1]\cup B$ by identification of $(a,1)$ with $f(a)$ for all $a\in A$. The subspace $A\times\{0\}$ of $Z(f)$ is referred to as the *top* of the cylinder, the image of $B$ as its *base*.

**Definition 1.3.** Let $\mathcal{D}$ be a diagram over the finite poset $P$. The *homotopy direct limit* $\|\mathcal{D}\|:=\underrightarrow{\text{holim}}\,\mathcal{D}$ is obtained from the disjoint union $X:=\coprod_{p\in P}\ \Delta(P_{\leq p})\times D_p$ by "making the obvious identifications", as follows.

Let $Y$ be defined by $Y:=\coprod_{p>q}\ \Delta(P_{\leq q})\times D_p$. There exist two obvious maps $\alpha,\beta:Y\longrightarrow X$, where $\alpha$ consists of the maps $\Delta(P_{\leq q})\times D_p\longrightarrow\Delta(P_{\leq p})\times D_p$, induced by inclusions $\Delta(P_{\leq q})\longrightarrow\Delta(P_{\leq p})$ for $p>q$, and $\beta$ consists of the maps $\Delta(P_{\leq q})\times D_p\longrightarrow\Delta(P_{\leq q})\times D_q$, induced by the maps $d_{pq}:D_p\longrightarrow D_q$. Now $\|\mathcal{D}\|$ is the difference cokernel of these two maps, i.e., $\|\mathcal{D}\|$ is the space obtained from $X$ by identifying $(x,u)$ and $(x',u')$ whenever $\alpha(x,u)=\beta(x',u')$.

A less formal and more geometric description of this construction is the following. One starts with the disjoint union of all spaces $D_p$, $p\in P$. Then one attaches to this, for every map $d_{pq}: D_p \longrightarrow D_q$, a copy of the mapping cylinder $Z(d_{pq})$ by identifying the top with $D_p$ and the base with $D_q$. The process is continued by attaching, for every $p>q>r$, a copy of $X_p \times \Delta(\{p,q,r\})$, where $\Delta(\{p,q,r\})$ is a two dimensional simplex spanned by $\{p,q,r\}$, for example the order complex of the poset $p>q>r$. This space is attached along the sides of the triangle $\Delta(\{p,q,r\})$ to the mapping cylinders of maps $d_{pq}, d_{qr}$ and $d_{pr}$. This construction is continued inductively for the chains in $P$ of all lengths.

Let $\alpha: \mathcal{D} \longrightarrow \mathcal{E}$ be a morphism of $P$-diagrams, i.e., a collection $\alpha=(\alpha_p)_{p\in P}$ of continuous maps $\alpha_p: D_p \longrightarrow E_p$ that satisfy the usual commutation relations $e_{pq}\circ\alpha_p=\alpha_q\circ d_{pq}$. Then $\alpha$ uniquely determines a continuous map $\overline{\alpha}: \|\mathcal{D}\| \longrightarrow \|\mathcal{E}\|$ of the corresponding homotopy limits.

There are other approaches to the definition of $\|\mathcal{D}\|$, and some of them, including the two mentioned above, can be found in [BK, Sect. XII.2]. There exists an even more general approach which deals with the case of diagrams of spaces which commute only up to coherent homotopies, see [Vo].

**Examples 1.4.**

(a) A map $f: A \longrightarrow B$ can be seen as a diagram $\mathcal{D}$ over a poset of two elements $\{p>q\}$. In this case $\|\mathcal{D}\|$ is the mapping cylinder of $f$.

(b) The mapping cone of $f: A \longrightarrow B$ is obtained as $\|\mathcal{D}\|$ for the diagram over the poset $q<p>r$ where $D_p=A$, $D_q=B$, $D_r$ is a one point space, and $d_{pq}=f$.

(c) The poset $P$ can be seen as a diagram $\mathcal{P}$ over itself having a one point space $\{p\}$ associated to each $p\in P$. In this case $\|\mathcal{P}\|$ is the order complex $\Delta(P)$.

(d) In case of the diagram $\mathcal{I}: P \longrightarrow \mathcal{A}$ of subspaces of a given space $M$, see Example 1.1, there exists a natural "collapsing" map $\xi: \|\mathcal{I}\| \longrightarrow M$. If $\mathcal{A}$ is a covering of $M$ that is closed under intersections, then $\xi$ is a continuous map with contractible fibers, so it is "usually" a homotopy equivalence (see Projection Lemma 1.6 and Example 4.3).

The map $\xi$ defined in Example 1.4(d) can be used for comparison of the homotopy limit of a subspace diagram with the underlying space. It can be seen as a special instance of the map arising in the following construction.

There is one more category naturally associated with diagrams, the category of all diagrams over all finite posets. A morphism $\alpha: \mathcal{D} \longrightarrow \mathcal{E}$ between a $P$-diagram $\mathcal{D}$ and a $Q$-diagram $\mathcal{E}$ is a pair $(\nu,(\alpha_p)_{p\in P})$, where $\nu: P \longrightarrow Q$ is an order preserving map and $\alpha_p: D_p \longrightarrow E_{\nu(p)}$, $p\in P$, is a family of continuous maps satisfying the usual commutativity relations. A map $\overline{\alpha}: \|\mathcal{D}\| \longrightarrow \|\mathcal{E}\|$ arises naturally and the map $\xi$ from Example 1.4(d) is seen as a special case of $\overline{\alpha}$.

**Examples 1.5.**

(a) If $\mathcal{D}$ is a $P$-diagram and $P' \subseteq P$ is a subposet, then there is a natural restriction of $\mathcal{D}$ to a $P'$-diagram $\mathcal{D}' = \mathcal{D}|P'$, the inclusion map $P' \subseteq P$ induces a morphism of diagrams $\mathcal{D}' \longrightarrow \mathcal{D}$. The corresponding map of homotopy limits embeds $\|\mathcal{D}'\|$ as a subspace of $\|\mathcal{D}\|$.

(b) Assume that $P$ contains a maximal element $\hat{1}$ and that $\mathcal{D}$ is a $P$-diagram for which $D_{\hat{1}} = \emptyset$. Then for $P' := P\backslash\hat{1}$ and $\mathcal{D}' := \mathcal{D}|P'$ we get that the map $\|\mathcal{D}'\| \hookrightarrow \|\mathcal{D}'\|$ is a homeomorphism.

(c) Let $\mathcal{D}$ be a $P$-diagram, and let $p_0$ be a minimal element of $P$. Then each of the posets $P^1 := P\backslash p_0$, $P^2 := P_{\geq p_0}$ and $P^{12} := P_{>p_0} = P^1 \cap P^2$ inherits a diagram structure from $\mathcal{D}$, where we write $\mathcal{D}^1 = \mathcal{D}|P^1$, $\mathcal{D}^2 = \mathcal{D}|P^2$ and $\mathcal{D}^{12} = \mathcal{D}|P^{12}$. In this situation we have
$$\|\mathcal{D}^1\| \cap \|\mathcal{D}^2\| = \|\mathcal{D}^{12}\| \qquad \text{and} \qquad \|\mathcal{D}^1\| \cup \|\mathcal{D}^2\| = \|\mathcal{D}\|.$$
This leads to the "deletion and contraction" approach to the construction of $\mathcal{D}$. For example, a Mayer-Vietoris sequence can be applied to compute the homology of $\mathcal{D}$ by induction on the size of $P$.

(d) Let $Q = \{\hat{0}\}$ be a one element poset, so $Q$-diagrams can be identified with spaces. Then a morphism from a $P$-diagram $\mathcal{D}$ to a space $E$, seen as a $Q$-diagram $\mathcal{E}$, is just a collection $\alpha$ of maps $\alpha_p : D_p \longrightarrow E$ satisfying the condition $\alpha_q \circ d_{pq} = \alpha_p$ for all $p \geq q$. In this case $\|\mathcal{E}\| = E$ and $\overline{\alpha}$ will be as in Example 1.4(d) denoted by $\xi$. Note that a collection $\alpha = (\alpha_p)_{p\in P}$ of maps which define a morphism from $\mathcal{D}$ to $\mathcal{E}$, naturally define a diagram $\overline{\mathcal{D}}$ over $\overline{P} = \{\hat{0}\} \cup P$ where $\hat{0} < p$ for all $p \in P$. For this extended diagram we get $\|\overline{\mathcal{D}}\| \cong Z_\xi$, where $Z_\xi$ is the mapping cylinder of $\xi$.
Conversely, whenever $\overline{\mathcal{D}}$ is a diagram over a poset $\overline{P}$ with a unique element $\hat{0}$, then $\|\overline{\mathcal{D}}\|$ is the mapping cylinder of the map $\|\mathcal{D}\| \longrightarrow D_{\hat{0}}$, where $\mathcal{D}$ is the restriction of $\overline{\mathcal{D}}$ to $P := \overline{P}\backslash\hat{0}$.

The following two propositions will serve as our primary tools for finding homotopy models of arrangements. The first of them, referred to as the Projection Lemma, was proved in [Se]. The second, called the Homotopy Lemma, is essentially proved in [tD]. Special cases of this result were of course known before; for this see the references of [tD]. A very general treatment, where it is shown that these lemmas hold under mild restrictions for diagrams over arbitrary small categories, can be found in [BK] and [Vo]. In this paper we are interested in the special case of diagrams over finite posets. So, we refer the reader to the appendix, where direct and elementary proofs of these statements are outlined.

**Projection Lemma 1.6.** [Se] [BK, XII.3.1(iv)]
*Let $\mathcal{A}$ be an arrangement (Definition 1.2) in $U$ with intersection poset $P$, let $\mathcal{D}$ be the corresponding $P$-diagram of spaces, and $\|\mathcal{D}\|$ its homotopy limit.*

*Then the natural collapsing map $\xi : \|\mathcal{D}\| \longrightarrow D$ (see Example 1.4(d)) is a homotopy equivalence.*

**Homotopy Lemma 1.7.** [tD] [BK, XII.4.2] [Vo]
*Let $\alpha = (\alpha_p)_{p\in P}$ be a morphism of two $P$-diagrams $\mathcal{D}$ and $\mathcal{E}$.*

*If $\alpha_p : D_p \longrightarrow E_p$ is a homotopy equivalence for all $p \in P$, then the associated map $\overline{\alpha} : \|\mathcal{D}\| \longrightarrow \|\mathcal{E}\|$ is also a homotopy equivalence.*

We note that this homotopy lemma becomes trivial if we assume the existence of a homotopy equivalence between the diagrams $\mathcal{D}$ and $\mathcal{E}$, that is, if we assume that the homotopy equivalences between the "stalks" $D_p$ and $E_p$ can be chosen compatibly. However, there is no compatibility assumption in Lemma 1.7, which makes it quite powerful.

In the following we analyze the situation when $\mathcal{D} : P \longrightarrow$ CW-Top is a *diagram with trivial maps.* It turns out that in this case the homotopy type of $\|\mathcal{D}\|$ has a simple description in terms of the subcomplexes $\Delta(P_{<p})$ of the order complex $\Delta(P)$, and the spaces $D_p$. As a consequence one obtains a direct sum decomposition of the homology $\widetilde{\mathrm{H}}_*(\|\mathcal{D}\|;\mathbb{Z})$. We refer to the Appendix (Section 4) for a list of basic constructions and for proofs.

**Wedge Lemma 1.8.** *Let $P$ be a poset with a unique maximal element $\hat{1}$, and let $\mathcal{D}$ be a $P$-diagram so that there exist points $c_p \in D_p$ for all $p < \hat{1}$ such that $d_{pq}$ is the constant map $d_{pq} : x \mapsto c_q \in D_q$ for all $p > q$. Then*

$$\|\mathcal{D}\| \;=\; \bigvee_{p\in P} (\Delta(P_{<p}) * D_p),$$

*where the wedge is formed by identifying $c_p \in \Delta(P_{<p}) * D_p$ with $p \in \Delta(P_{<\hat{1}}) * D_{\hat{1}}$ for every $p < \hat{1}$.*

**Corollary 1.9.** *In the situation of the Wedge Lemma 1.8,*

$$\widetilde{\mathrm{H}}_*(\|\mathcal{D}\|;\mathbb{Z}) \;\cong\; \bigoplus_{p\in P} \widetilde{\mathrm{H}}_*(\Delta(P_{<p}) * D_p).$$

## 2. Homotopy Types of Arrangements.

We will now demonstrate the power of the diagram technique by computing combinatorial formulas for the homotopy types of four types of subspace arrangements:

(a) affine arrangements,

(b) compactified affine arrangements,

(c) spherical arrangements (links of central arrangements),

(d) projective arrangements.

The computation of the cohomology of a complement of a hyperplane arrangement (subspaces of codimension 1) is a classical problem [OS] [Or] [BZ]. The cohomology of the complement $M$ of an arbitrary subspace arrangement was first computed by Goresky & MacPherson [GM, Part III], who used this example to demonstrate the power of their newly developed "Stratified Morse Theory". They show that the cohomology groups can be constructed from the data given by the intersection poset of the arrangement, together with its dimension function. (An alternative approach is due independently to Jewell, Orlik & Shapiro [JOS] and to Vassiliev [Va, 4.2.A], see also Section 3(e).)

We obtain the cohomology of the complement by Alexander duality from case (c) for central arrangements and from case (b) for affine subspace arrangements. The computation in the projective case needs extra arguments, see Theorem 2.8.

### 2(a) Affine Arrangements.

Our first application of the diagram technique will be a description of the homotopy type of an affine real subspace arrangement.

For this, let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a finite set of affine subspaces in $\mathbb{R}^n$, closed under intersection. Let $P$ be the intersection poset of $\mathcal{A}$, where $p \in P$ corresponds to $A_p \in \mathcal{A}$. We order by reversed inclusion: $p \geq q$ means $A_p \subseteq A_q$. The poset $P$ includes a maximal element $\hat{1}$, corresponding to $A_{\hat{1}} = \bigcap \mathcal{A}$, which is either empty or contractible. Since all spaces $A_p$ for $p < \hat{1}$ are contractible there is a unique homotopy class of maps $A_p \longrightarrow A_q$, and a complete set of combinatorial invariants (in the sense of the introduction) is given by the poset $P$ together with the information whether $A_{\hat{1}}$ is empty.

**Theorem 2.1.** *Let $\mathcal{A}$ be an affine arrangement with intersection poset $P$. Then*

$$\bigcup \mathcal{A} \;\simeq\; \begin{cases} \Delta(P \backslash \hat{1}), & \text{if } A_{\hat{1}} = \emptyset, \\ \{\hat{1}\}, & \text{otherwise.} \end{cases}$$

**Proof.** The $P$-diagram $\mathcal{D}$ of $\mathcal{A}$ is a subspace diagram that satisfies the conditions of the Projection Lemma 1.6, so we conclude that $\xi : \|\mathcal{D}\| \longrightarrow \bigcup \mathcal{A}$ is a homotopy equivalence.

Now let $\mathcal{D}'$ be the trivial $P$-diagram, with $D'_{\hat{1}} = \emptyset$ if $A_{\hat{0}} = \emptyset$, and $A_p = \{c_p\}$ otherwise. Then there is an obvious map of diagrams $\mathcal{D} \longrightarrow \mathcal{D}'$, to which the Homotopy Lemma 1.7 applies. Finally $\|\mathcal{D}\| = \Delta(P \backslash \hat{1})$ if $A_{\hat{1}}$ is empty, and $\|\mathcal{D}\| = \Delta(P) \simeq \hat{1}$ otherwise. □

This theorem is also proved in [GM, Chapt. III.2]. It can alternatively be derived by two applications of the nerve theorem [Bj], as in [BLY, Prop. 4.1].

### 2(b) Compactified Affine Arrangements.

If $\mathcal{A}$ is an arrangement of affine subspaces of $\mathbb{R}^n$, then another natural invariant is $\widehat{D} = \widehat{\bigcup \mathcal{A}} = \bigcup \mathcal{A} \cup \{\infty\} \subset \mathbb{R}^n \cup \{\infty\}$, the one-point compactification of $D = \bigcup \mathcal{A}$ seen as a subspace of the one-point compactification $\widehat{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\} \cong S^n$ of the ambient space. We interpret $\widehat{D}$ as the union of an arrangement $\widehat{\mathcal{A}}$ of compactified affine subspaces $\widehat{A}_p$ with intersection poset $P$. We assume that the arrangement includes $\widehat{A}_{\hat{1}} = \{\infty\}$ as the compactification of the empty subspace. Define a *dimension function* $d : P_{<\hat{1}} \longrightarrow \mathbb{N}_0$ by $d(p) := \dim_{\mathbb{R}} A_p$. Then we get that the pair $(P, d)$ determines the homotopy type of $\widehat{A}_p \cong S^{d(p)}$, while the maps $S_p \longrightarrow S_q$ for $p > q$ are all pointed embedding maps of a sphere (or a point) into a higher-dimensional sphere and thus null-homotopic. Hence we consider $(P, d)$ as a complete set of combinatorial data.

**Theorem 2.2.** *The one-point compactification $\widehat{D}$ of the union of all flats in an affine arrangement $\mathcal{A}$ has the homotopy type of the wedge*

$$\widehat{D} \;\simeq\; \{\infty\} \vee \bigvee_{p<\hat{1}} (\Delta(P_{<p}) * S^{d(p)}).$$

**Proof.** For simplicity, let $\widehat{\mathcal{A}}$ also denote the $P$-diagram of the arrangement $\widehat{\mathcal{A}}$. From the Projection Lemma we know $\widehat{D} \simeq \|\widehat{\mathcal{A}}\|$. Let us show now that $\widehat{\mathcal{A}}$ can be replaced by a $P$-diagram $\mathcal{A}'$, defined by $A'_{\hat{1}} = \{\infty\}$, $A'_p := S^{d(p)}$ for $p < \hat{1}$, and trivial maps $s'_{pq} : A_p \mapsto \mathbf{e}_1 \in S^{d(q)} = A_q$ for $p > q$.

Indeed, for every $p < \hat{1}$ choose $c_p \in A_p \backslash \bigcup_{q>p} A_q$. Then choose homotopy equivalences $\alpha_p : \widehat{A}_p \longrightarrow S^{d(p)}$ for all $p < \hat{1}$, in such a way that the map $\alpha_p$ contracts the complement of a small, open disc around $c_p \in S_p$ to a point and maps it to $\mathbf{e}_1 \in S^{d(p)-1}$. From this construction we get that the following diagram is commutative.

$$\begin{array}{ccccc}
\widehat{A}_p & \xrightarrow{\alpha_p} & A'_p & = & S^{d(p)} \text{ resp. } \{\infty\} \\
{\scriptstyle s_{pq}}\downarrow & & \downarrow{\scriptstyle s'_{pq}} & & \\
\widehat{A}_q & \xrightarrow[\alpha_q]{} & A'_q & = & S^{d(q)}
\end{array}$$

By the Homotopy Lemma $\|\widehat{\mathcal{A}}\|$ and $\|\mathcal{A}'\|$ have the same homotopy type and by the Wedge Lemma $\|\mathcal{A}'\|$ has the homotopy type of the wedge of the associated spaces. The space $\Delta(P_{<1}) * A_{\hat{1}}$, being contractible, can be omitted, so

$$\widehat{D} \simeq \|\widehat{\mathcal{A}}\| \simeq \|\mathcal{A}'\| \simeq \bigvee_{p<\hat{1}} (\Delta(P) * S^{d(p)}). \qquad \square$$

**Corollary 2.3.** [GM, III.1.5 "Theorem A"] *Let $\mathcal{A}$ be an affine subspace arrangement in $\mathbb{R}^n$ with combinatorial data $(P, d)$. Then the homology of the one-point compactification $\widehat{D}$ of $D = \bigcup \mathcal{A}$ and the cohomology of the complement $M := \mathbb{R}^n \backslash D$ are given by*

$$\widetilde{\mathrm{H}}_i(\widehat{D}; \mathbb{Z}) \;\cong\; \bigoplus_{p\in P} \widetilde{\mathrm{H}}_{i-d(p)-1}(\Delta(P_{<p}); \mathbb{Z}),$$

$$\widetilde{\mathrm{H}}^i(M; \mathbb{Z}) \;\cong\; \bigoplus_{p\in P} \widetilde{\mathrm{H}}_{n-i-d(p)-2}(\Delta(P_{<p}); \mathbb{Z}).$$

**2(c) Spherical Arrangements.**

A subspace arrangement $\mathcal{A}$ is *central* if each of its subspaces contains the origin $0 \in \mathbb{R}^n$. In this case $D = \bigcup \mathcal{A}$ is contractible, but the *link* $L := D \cap S^{n-1}$, corresponding to the intersection of the arrangement with the unit sphere, is interesting.

We interpret the situation as an arrangement of subspheres $S_p := A_p \cap S^{n-1}$ (with the same intersection poset $P$ as $\mathcal{A}$) in the ambient space $U = S^{n-1}$, whose diagram we denote by $\mathcal{S}$. Define a *dimension function* $d : P \longrightarrow \mathbb{N}_0$ by $d(p) := \dim_{\mathbb{R}} A_p$. Again the pair $(P, d)$ amounts to a complete set of combinatorial data.

**Theorem 2.4.** *Let $\mathcal{A}$ be a linear subspace arrangement. The homotopy type of its link $\bigcup \mathcal{A} \cap S^{n-1}$ is completely determined by the intersection poset $P$ together with the dimension function $d : P \longrightarrow \mathbb{N}_0$, $p \mapsto \dim(p)$, as*

$$L \;\simeq\; \bigvee_{p \in P} \left(\Delta(P_{<p}) * S^{d(p)-1}\right),$$

*where the wedge is formed by identifying $\mathbf{e}_1 \in \Delta(P_{<p}) * S^{d(p)-1}$ with $p \in \Delta(P_{<\hat{1}}) * S^{d(p)-1}$, for every $p < \hat{1}$.*

Note that in the case where the arrangement $\mathcal{A}$ is *essential*, with $\bigcap \mathcal{A} = A_{\hat{1}} = \{0\}$, we get $d(\hat{1}) = 0$, $S^{d(\hat{1})-1} = S^{-1} = \emptyset$, so that the formula can be rewritten as

$$\bigcup \mathcal{A} \cap S^{n-1} \;\simeq\; \Delta(P \backslash \hat{1}) \vee \bigvee_{p < \hat{1}} \left(\Delta(P_{<p}) * S^{d(p)-1}\right).$$

**Proof.** The subspace diagram $\mathcal{S}$ satisfies $\|\mathcal{S}\| \simeq L$, by Projection Lemma 1.6. As in the proof for Theorem 2.2, $\mathcal{S}$ can be replaced by a diagram $\mathcal{S}'$, defined by $S'_p := S^{d(p)-1}$ and trivial maps $s'_{pq} : S^{d(p)-1} \mapsto \mathbf{e}_1 \in S^{d(q)-1}$ for $p > q$. With this, the Homotopy Lemma and the Wedge Lemma finish the proof. □

**Corollary 2.5.** *The homology of the link, and the cohomology of the complement, of a linear subspace arrangement in $\mathbb{R}^n$ can be computed from the data $(P, d)$ and $n$ as*

$$\widetilde{\mathrm{H}}_i(L; \mathbb{Z}) \;\cong\; \bigoplus_{p \in P} \widetilde{\mathrm{H}}_{i-d(p)}(\Delta(P_{<p}); \mathbb{Z}),$$

$$\widetilde{\mathrm{H}}^i(M; \mathbb{Z}) \;\cong\; \bigoplus_{p \in P} \widetilde{\mathrm{H}}_{n-i-d(p)-2}(\Delta(P_{<p}); \mathbb{Z}).$$

Note that every linear subspace arrangement is affine, so Corollary 2.5 also follows from Corollary 2.3, where $\widehat{D} \cong \Sigma L$.

**2(d) Real Projective Arrangements.**

Let $\mathcal{A}$ be, as before, a central arrangement in $\mathbb{R}^n$ with intersection poset $P$ and dimension function $d : P \longrightarrow \mathbb{N}_0$, $d(p) := \dim_{\mathbb{R}} A_p$. Let $\mathcal{PA}$ be the projective arrangement associated to $A$, i.e., $\mathcal{PA} = \{\mathrm{Proj}(A_p) : p \in P\}$, where $\mathrm{Proj}(V) = \{l \subseteq V : \dim_{\mathbb{R}} l = 1\}$ denotes the projective space associated to $V$. Let $PL := \bigcup \mathcal{PA}$ be the "projective" link of $\mathcal{A}$. Then $PL$ is the direct limit of a diagram $\mathcal{R} = \mathcal{R}(\mathcal{A})$ over $P$ defined by $R_p := \mathrm{Proj}(A_p)$.

The problem of describing the homotopy type of $PL$ seems to be more difficult than in previous examples. Nevertheless, the method of simplifying a diagram and relating it to a combinatorially defined model diagram still applies. Let us describe a combinatorial model diagram suitable for this purpose, which depends only on $P$ and on $d : P \longrightarrow \mathbb{N}_0$.

**Definition 2.6.** Let $\mathcal{A}$ be a central arrangement in $\mathbb{R}^n$ with intersection poset $P$ and dimension function $d : P \longrightarrow \mathbb{N}_0$. Let $\mathcal{D} = \mathcal{D}(\mathcal{A}) = \{A_p\}_{p \in P}$ be the corresponding $P$-diagram of linear spaces. Similarly, for the projective arrangement $\mathcal{PA}$, let $\mathcal{R} = \mathcal{R}(\mathcal{A}) = \{R_p\}_{p\in P}$, $R_p = \mathrm{Proj}(A_p)$, be the corresponding $P$-diagram of projective spaces. Choose a flag $F = \{F_i\}_{i=0}^n$, $\{0\} = F_0 \subset F_1 \subset \ldots \subset F_n = \mathbb{R}^n$. Then the *projective flag diagram* $\mathcal{R}[F]$ associated with $\mathcal{PA}$ is defined by $R_p[F] := \mathrm{Proj}(F_{d(p)})$, where the morphisms $R_p[F] \longrightarrow R_q[F]$ are the obvious inclusion maps. Every two flag diagrams $\mathcal{R}[F]$ and $\mathcal{R}[F']$ are naturally isomorphic, thus the isomorphism type of $\mathcal{R}[F]$ depends only on $P$ and $d$. Therefore, the associated projective flag diagram $\mathcal{R}[F]$ will be simply denoted by $\mathcal{R}'$, $R'_p := R_p[F]$, $p \in P$.

For technical reasons, it is convenient to also introduce a linear flag diagram, denoted by $\mathcal{E} = \mathcal{E}[F]$, where $E_p := F_{d(p)}$, $p \in P$, so the diagram $\mathcal{R}[F]$ can be seen as the projectivization of $\mathcal{E}[F]$.

**Remark 2.7.** It will be convenient to choose the flag $F$ in sufficiently general position with respect to the arrangement $\mathcal{A}$. This means that, given a spanning basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of the flag $F$, we assume that all corresponding Plücker coordinates of all flats in $\mathcal{A}$ are nonzero.

Unlike in the previous examples, there does not seem to exist an easy way of comparing diagrams $\mathcal{R} = \mathcal{R}(\mathcal{A})$ and $\mathcal{R}' = \mathcal{R}[F]$, i.e., of constructing a morphism $\alpha : \mathcal{R} \longrightarrow \mathcal{R}'$ such that $\alpha_p : R_p \longrightarrow R'_p$ is a homotopy equivalence. One way of getting around this difficulty is to apply a stronger version of Homotopy Lemma 1.7, which says that a family $\alpha = \{\alpha_p\}_{p\in P}$ of homotopy equivalences, $\alpha_p : R_p \longrightarrow R'_p$, in some cases still gives rise to a homotopy equivalence between $\|\mathcal{R}\|$ and $\|\mathcal{R}'\|$, even if the diagrams

$$\begin{array}{ccc} R_p & \xrightarrow{\alpha_p} & R'_p \\ {\scriptstyle r_{pq}}\downarrow & & \downarrow{\scriptstyle r'_{pq}} \\ R_q & \xrightarrow[\alpha_q]{} & R'_q \end{array}$$

are not commutative. Instead, it is assumed that these diagrams are commutative up to homotopy and that the homotopies can be chosen in a coherent way. A detailed treatment and a very general form of this result can be found in [Vo, Theorem 1.4].

**Theorem 2.8.** *Let $\mathcal{A} = \{A_p\}_{p\in P}$ be a linear subspace arrangement and $\mathcal{PA}$ the associated projective arrangement. Then the projective link $PL = \bigcup \mathcal{PA}$ has the homotopy type*

$$PL \;\simeq\; \|\mathcal{R}'\|,$$

*where $\mathcal{R}'$ is the combinatorially defined projective flag diagram associated with the intersection poset $P$ and the dimension function $d : P \longrightarrow \mathbb{N}_0$, $d(p) = \dim(A_p)$, see Definition 2.6.*

**Proof.** Let us assume that the flag $F$ is chosen in general position with respect to the arrangement $\mathcal{A}$, see Remark 2.7. This assumption permits us to deform our arrangement "in the direction" of the flag $F$. Namely, let $C_\epsilon : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the linear map defined by $C_\epsilon(\mathbf{e}_i) = \epsilon^i \mathbf{e}_i$ for $\epsilon > 0$ and $i = 1, \ldots, n$, and let $\mathcal{A}_\epsilon = \{A_p^\epsilon : p \in P\}$, where $A_p^\epsilon := C_\epsilon(A_p)$. The arrangement $\mathcal{A}_\epsilon$ is linearly isomorphic to $\mathcal{A}$ and the corresponding projective links $PL(\mathcal{A})$ and $PL(\mathcal{A}_\epsilon)$ are homeomorphic. By assumption, all Plücker coordinates $\mu_J(A_p)$, are nonzero, where $J = \{i_1 < \ldots < i_k\}$, $k = \dim(A_p)$. As a consequence, if $I = \{1, \ldots, k\}$, one has $\mu_I(A_p^\epsilon) = \epsilon^{n(n-1)/2} \cdot \mu_I(A_p)$ and for $J \neq I$, $\mu_J(A_p^\epsilon) = o(\epsilon^{n(n-1)/2})$, so we observe that if $\epsilon$ is small enough, all $k$-dimensional flats $A_p^\epsilon$ in $\mathcal{A}_\epsilon$ will be in a small neighborhood of $F_k$.

Our goal is to relate the $P$-diagram $\mathcal{D}(\mathcal{A}_\epsilon)$, associated to $\mathcal{A}_\epsilon$, to the linear flag diagram $\mathcal{E} = \mathcal{E}[F]$, $E_p = F_{d(p)}$ for $p \in P$, by linear isomorphisms $\alpha_p : A_p^\epsilon \longrightarrow E_p$ which commute up to "coherently chosen homotopies", see [Vo, Definitions 2.3 and 2.7]. Actually, $\alpha_p$ will be defined as a restriction of a linear map $\beta_p : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $\|\mathbb{1} - \beta_p\| < 1$. For a fixed $p \in P$, let $a_p, b_p \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ be defined as the orthogonal projection operators on $A_p^\epsilon$ and $E_p = F_{d(p)}$ respectively. Let $\beta_p := \mathbb{1} - a_p - b_p + 2\, a_p b_p$. Then it is easily checked that $a_p\beta_p = \beta_p b_p$, in particular $\beta_p(A_p^\epsilon) \subseteq E_p$ and $\alpha_p := \beta_p | A_p^\epsilon$ is well defined. Since $\|\mathbb{1} - \beta_p\| = \|(a_p - b_p)(\mathbb{1} - 2b_p)\| < 1$ if $\|a_p - b_p\| < 1/3$, which is certainly true if $\epsilon$ is small enough, we observe that $\alpha_p$ and $\beta_p$ are linear isomorphisms. So, $\alpha = (\alpha_p)_{p\in P}$ defines a *homotopy homomorphism* in the sense of [Vo, Definition 2.7]. Indeed, all maps $\beta_p$, $p \in P$, are contained in the unit open ball $B := \{x \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n) : \|\mathbb{1} - x\| < 1\}$, so the linear homotopies between them give rise to the required family of coherently chosen homotopies. This construction shows that there exists a homotopy homomorphism $\gamma = (\gamma_p)_{p\in P}$, $\gamma_p : R_p(\mathcal{A}_\epsilon) \longrightarrow R'_p$, between the $P$-diagram $\mathcal{R}(\mathcal{A}^\epsilon)$ of projective spaces associated to $\mathcal{A}_\epsilon$ and the projective flag diagram $\mathcal{R}'$ such that $\gamma_p$ is a homotopy equivalence for all $p \in P$. By [Vo, Theorem 1.4], $\|\mathcal{R}(\mathcal{A}^\epsilon)\| \simeq \|\mathcal{R}'\|$, hence $PL \simeq \|\mathcal{R}'\|$, which proves the theorem. □

The problem of finding a purely combinatorial description of the projective link of an arrangement is solved in principle by Theorem 2.8. So, the problem of studying projective links is reduced to the problem of understanding topological properties of homotopy limits of projective flag diagrams $\mathcal{R}'$. As an example of this, we prove a direct sum decomposition for the $\mathbb{Z}/2$-homology of the projective link $PL$ which is "dual" to the corresponding decomposition of the homology of the complement of $PL$, proved in [GM, III.1.7 "Theorem C"]. The resemblance of formulas indicates that there ought to be a natural "duality" type map between the cohomology of the complement and homology of the projective link.

**Proposition 2.9.** *Let $(P, d)$ be a finite poset with a dimension function $d : P \longrightarrow \mathbb{N}_0$. Let $\mathcal{R} = \mathcal{R}[F]$ be the $P$-diagram of projective spaces associated with the standard flag $F : \{0\} \subset \mathbb{R}^1 \subset \ldots \subset \mathbb{R}^n$, so that $R_p = \mathbb{RP}^{d(p)-1}$ and $r_{pq} : R_p \longrightarrow R_q$, $p > q$, is the canonical embedding. The dimension function $d$ determines subposets $P^{(u)} := \{p \in P : d(p) \geq u\}$, where $k \leq u \leq m$ and $k := \min\{\dim(R_p) : p \in P\}$, $m := \max\{\dim(R_p) : p \in P\}$.*

*Then the homology of $\|\mathcal{R}\|$ with $\mathbb{Z}/2$-coefficients has the following direct sum decomposition:*

$$\widetilde{\mathrm{H}}_*(\|\mathcal{R}\|; \mathbb{Z}/2) \;\cong\; \widetilde{\mathrm{H}}_*(\mathbb{RP}^k \times \Delta(P); \mathbb{Z}/2) \;\oplus\; \bigoplus_{u=k+1}^{m} \widetilde{\mathrm{H}}_{*-u}(\Delta(P^{(u)}); \mathbb{Z}/2).$$

**Proof.** The proof is by induction on $m - k$, the case $m = k$ being trivial. We will prove a slightly stronger statement which includes the description of naturally defined subdiagrams of $\mathcal{R}$ which induce the desired decomposition. We assume $\mathbb{Z}/2$-coefficients for all of the following.

Let $\mathcal{T} = \{T_p\}_{p \in P}$ be the "maximal" constant subdiagram of $\mathcal{R}$, i.e., $T_p = \mathbb{RP}^k$ and $t_{pq} : T_p \longrightarrow T_q$ is the identity map. Let $\mathcal{T}^{(k+1)}$ be the restriction of $\mathcal{T}$ to $P^{(k+1)}$. Clearly,

$$\widetilde{\mathrm{H}}_*(\|\mathcal{T}\|) \;\cong\; \widetilde{\mathrm{H}}_*(\mathbb{RP}^k \times \Delta(P)) \;\cong\; \widetilde{\mathrm{H}}_*(\mathbb{RP}^k) \otimes \widetilde{\mathrm{H}}_*(\Delta(P)).$$

We will prove that $\widetilde{\mathrm{H}}_*(\|\mathcal{R}\|)$ has the desired decomposition, where $\widetilde{\mathrm{H}}_*(\mathbb{RP}^k \times \Delta(P))$ comes from the subdiagram $\mathcal{T}$ and the rest of decomposition is determined by subdiagrams of $\mathcal{R}^{(k+1)}$. By excision, $\mathrm{H}_*(\|\mathcal{R}\|, \|\mathcal{T}\|) \cong \mathrm{H}_*(\|\mathcal{R}^{(k+1)}\|, \|\mathcal{T}^{(k+1)}\|)$, so the long exact sequence of the pair $(\|\mathcal{R}\|, \|\mathcal{T}\|)$ has the form

$$\ldots \longrightarrow \widetilde{\mathrm{H}}_*(\|\mathcal{T}\|) \longrightarrow \widetilde{\mathrm{H}}_*(\|\mathcal{R}\|) \xrightarrow{\pi} \mathrm{H}_*(\|\mathcal{R}^{(k+1)}\|, \|\mathcal{T}^{(k+1)}\|) \longrightarrow \ldots .$$

One observes that $\widetilde{\mathrm{H}}_*(\|\mathcal{T}^{(k+1)}\|)$ is a direct summand of $\widetilde{\mathrm{H}}_*(\mathbb{RP}^k \times \Delta(P^{(k+1)}))$ and that the last group, by the inductive assumption, can be seen as a part of the first summand in the direct decomposition of $\widetilde{\mathrm{H}}_*(\|\mathcal{R}^{(k+1)}\|)$. This implies

$$\widetilde{\mathrm{H}}_*(\|\mathcal{R}^{(k+1)}\|, \|\mathcal{T}^{(k+1)}\|) \cong \bigoplus_{u=k+1}^{m} \widetilde{\mathrm{H}}_{*-u}(\Delta(P^{(u)})).$$

Also, since $\mathrm{H}_*(\|\mathcal{R}^{(k+1)}\|, \|\mathcal{T}^{(k+1)}\|)$ is a direct summand of $\widetilde{\mathrm{H}}_*(\mathcal{R}^{(k+1)})$ and the last group can be naturally embedded in $\widetilde{\mathrm{H}}_*(\|\mathcal{R}\|)$, one observes that $\pi$ is an epimorphism which has a right inverse. So, the long exact sequence consists of short, splitting exact sequences which immediately leads to the desired decomposition for $\widetilde{\mathrm{H}}_*(\|\mathcal{R}\|)$. □

The following proposition shows that the homology of the projective link $PL$ with rational coefficients, has a different direct sum decomposition at least if all projective spaces are odd dimensional.

**Proposition 2.10.** *Let us suppose that $\mathcal{R}$ is a $P$-diagram of projective spaces associated with the standard flag. Let us assume that all projective spaces in the diagram are odd dimensional. Then the homology of $\|\mathcal{R}\|$ with rational coefficients has the following direct sum decomposition.*

$$\widetilde{\mathrm{H}}_i(\|\mathcal{R}\|;\mathbb{Q}) \;\cong\; \bigoplus_{p\in P} \widetilde{\mathrm{H}}_{i-d(p)}(\Delta(P_{<p});\mathbb{Q}).$$

**Proof.** Let $\mathcal{D}$ be the diagram of spheres associated with the standard flag and let $\alpha : \mathcal{D} \longrightarrow \mathcal{R}$ be the morphism of diagrams such that $\alpha_p : R_p \longrightarrow D_p$ is the double covering. The morphism $\alpha$ induces a map of corresponding spectral sequences, see Section 3(e), which turns out to be an isomorphism at the $E^2$-term. Knowing that the spectral sequence of $\mathcal{D}$ collapses at the $E^2$-term, one easily deduces that the same holds for the diagram $\mathcal{R}$. This, together with the fact that the decomposition of the homology of $\|\mathcal{D}\|$ given in Corollary 2.5, coincides with the decomposition coming from the $E^2$-term of the spectral sequencs, leads to the desired observation. □

We note that if $\mathcal{A}$ is a linear even subspace arrangement in $\mathbb{R}^{2n}$, then we get the rational homology of the projective complement $\mathbb{RP}^{2n-1}\backslash\mathcal{PA}$ from Alexander duality in $\mathbb{RP}^{2n-1}$, which is a rational homology sphere.

It would be interesting to work out complete formulas for the $\mathbb{Z}$-homology of $\bigcup\mathcal{PA}$ (which exist by Theorem 2.8) and for the $\mathbb{Z}$-homology of the complements (where they are known for $\mathbb{Z}/2$-coefficients [GM, III.1.7 "Theorem C"], the dual result to our Proposition 2.9).

### 2(e) Complex Projective Arrangements.

In the problem of describing the homotopy types of linear, affine or spherical arrangements, one can forget the complex or quaternionic structure and deal only with the real case. In the case of projective arrangements the distinction is necessary, although the proofs are similar and without new ideas. So we will omit the details and restrict ourselves to formulation of results in the case of complex projective arrangements. Let us note that Definition 2.6 can be extended by allowing scalars to be either complex numbers or quaternions.

**Theorem 2.11.** *Let $\mathcal{A} = \{A_p\}_{p\in P}$ be a complex linear subspace arrangement and $\mathcal{PA}$ the corresponding complex projective arrangement. Then the projective link $PL = \bigcup\mathcal{PA}$ has the homotopy type*

$$PL \;\simeq\; \|\mathcal{R}'\|,$$

*where $\mathcal{R}'$ is the projective flag diagram associated with the intersection poset $P$ and the dimension function $d : P \longrightarrow \mathbb{N}_0$, $d(p) = \dim_{\mathbb{C}}(A_p)$, see Definition 2.6.*

**Proof.** The proof of this theorem follows closely the proof of Theorem 2.8 and we omit the details. □

**Proposition 2.12.** *Let $(P,d)$ be a finite poset with a dimension function $d : P \longrightarrow \mathbb{N}_0$. Let $\mathcal{C}$ be the $P$-diagram of complex projective spaces associated with the standard flag $F : \{0\} \subset \mathbb{C}^1 \subset \ldots \subset \mathbb{C}^n$, so that $C_p = \mathbb{C}\mathrm{P}^{d(p)-1}$ and $r_{pq} : C_p \longrightarrow C_q$, $p > q$, is the canonical embedding. The dimension function $d$ determines, as before, subposets $P^{(u)} := \{p \in P : d(p) \geq u\}$, where $k \leq u \leq m$ and $k := \min\{\dim_{\mathbb{C}}(C_p) : p \in P\}$, $m := \max\{\dim_{\mathbb{C}}(C_p) : p \in P\}$. Then the homology of $\|\mathcal{C}\|$ with $\mathbb{Z}$ coefficients has the following direct sum decomposition:*

$$\widetilde{\mathrm{H}}_*(\|\mathcal{C}\|;\mathbb{Z}) \;\cong\; \widetilde{\mathrm{H}}_*(\mathbb{C}\mathrm{P}^k \times \Delta(P);\mathbb{Z}) \;\oplus\; \bigoplus_{u=k+1}^{m} \widetilde{\mathrm{H}}_{*-u}(\Delta(P^{(u)});\mathbb{Z}).$$

**Proof.** The proof is very similar to the proof of Proposition 2.9. □

## 3. Remarks.

In this section we have collected some additional remarks about the homotopy theory of arrangements and some corollaries that follow from the results of Section 2. We concentrate on the case of spherical arrangements, only indicating the changes that occur in other cases.

### (a) Explicit Maps.

Let $\mathcal{A}$ denote a spherical arrangement in $S^{n-1} \subset \mathbb{R}^n$ (the affine case can be treated analogously). In this case, Theorem 2.4 asserts a homotopy equivalence between the geometrically given space $D = \bigcup \mathcal{A}$ and its *combinatorial model* $\Gamma(P,d) = \bigvee_{p\in P} \Delta(P_{<p}) * S^{d(p)}$. Here we observe that one can explicitly construct maps $\Phi : \Gamma(P,d) \longrightarrow D$ and $\Psi : D \longrightarrow \Gamma(P,d)$ that induce isomorphisms in homology.

The construction of $\Phi$ is actually easy. For every $p \in P$, choose a point $\phi(p) \in S_p \backslash \bigcup_{q>p} S_q =: S_p^o$, which can be interpreted as a "generic" point on $S_p$. To construct $\Phi$, we start with a homeomorphism $\Phi_p : S^{d(p)-1} \longrightarrow S_p$ for each $p \in P$, which maps $\mathbf{e}_1$ to $\phi(p) \in D$. For $q < p$, we define $\Phi_p(q) := \phi(q)$. For every chain $q_1 < \ldots < q_k < p$ we have a chain of spheres $S_{q_1} \supset \ldots \supset S_{q_k} \supset S_p$. The choice of $\phi(q_i)$ guarantees that the set $\{\phi(q_1),\ldots,\phi(q_k),p_0\}$ is linearly independent for all $p_0 \in S_p$. Thus $\Phi_p$ and $\phi$ together determine a geodesic embedding of the closed simplex $[q_1,\ldots,q_k,p_0]$ into $S(q_1) \subseteq D$, which depends continuously on $p_0$ and is natural with respect to the selection of subchains of $(q_1 < \ldots < q_k)$. Hence we get a continuous map $\Phi_p : \Delta(P_{<p}) * S^{d(p)-1} \longrightarrow D$. Furthermore, the condition $\Phi_p(\mathbf{e}_1) = \phi(p)$ ensures that the maps $\Phi_p$ fit together to give a continuous map $\Phi : \Gamma(P,d) \longrightarrow D$.

The map $\Phi$ is surjective since $S_p$ is in the image of $\Phi_p$. To see that it induces an isomorphism in homology, one can proceed by induction on $m := |P| = |\mathcal{A}|$, together with a Mayer-Vietoris argument (using Example 1.5(c) for an argument parallel to that used for proofs in the appendix), and the Five-Lemma – see [Z2] for details. This yields the simplest proof we know for the Goresky-MacPherson homology formulas (Corollaries 2.3 and 2.5).

The construction of the map $\Psi$ is more involved, relying on a partition of unity that defines a system of tubular neighborhoods for the sets $S_p^o$. The basic ideas can be derived from [GM, p. 243]. Again we refer to [Z2] for further details.

There are three main disadvantages to this "explicit approach":

– it relies more than necessary on metric/differentiable structure,

– it is easy to show that the maps are isomorphisms in homology, but it is not immediate they are homotopy equivalences. In fact, in order to get that $\Phi$ and $\Psi$ are homotopy inverses, one would have to be very careful in the choice of orientations etc.,

– such explicit maps are not easily available for other cases, such as the projective ones.

All three problems do not occur in the diagram approach.

### (b) Universality

For every finitely presented group $G$ there is a connected finite simplicial complex $\Delta$ with fundamental group $G$ and such that every star of a vertex is non-empty and connected. (For this, one can take the CW complex with only one vertex associated with a presentation of $G$, triangulate it, and take the product with a unit interval.)

Furthermore, for every finite simplicial complex $\Delta \subseteq 2^{[n]}$ there is an arrangement of coordinate subspaces $\mathcal{A}_\Delta := \{\text{span}_{\mathbb{R}}\{e_i : i \in A\} : A \in \Delta\}$ in $\mathbb{R}^n$ whose poset of intersections is the order dual of the face poset $P_\Delta$. (See [Z2] for more details.) With these observations, Theorem 2.4 implies the following universality results.

**Corollary 3.1.** *For every finite simplicial complex $\Delta$ there is a connected spherical arrangement $\mathcal{A}$ for which the cohomology algebra $\mathrm{H}^*(\Delta; \mathbb{Z})$ is a direct summand of the algebra $\mathrm{H}^*(D; \mathbb{Z})$ of the link of $\mathcal{A}$. In particular, $\mathrm{H}^*(D; \mathbb{Z})$ can contain arbitrary amounts of torsion.*

**Corollary 3.2.** *For every finitely presented group $G$ there is a connected spherical arrangement $\mathcal{A}$ with fundamental group $\pi_1(D) \cong G$.*

We get analogous results for affine arrangements and for compactified affine arrangements, the first case being trivial. The situation for projective arrangements is more subtle.

### (c) Wedges of Spheres

In contrast to the universality results that we have just derived, one finds that many arrangements that "occur in nature" have the homotopy type of a wedge of spheres.

From Theorem 2.4 we see that the crucial condition for this is that the posets $P_{<p}$ all have the homotopy type of a wedge of spheres. (Note that this condition is completely independent of the dimension function). This is satisfied in particular when $P$ is a shellable

poset (see [Bj]). In this case we can read off the number of spheres of various dimensions from the Möbius function on $\widehat{P} := \hat{0} \uplus P$. This covers in particular the cases of

- all *c-arrangements*: arrangements of subspaces of codimension $c$ in $\mathbb{R}^n$ for which the codimension of any intersection is a multiple of $c$; this includes the cases of real hyperplane arrangements for $c = 1$, *even subspace arrangements* (including all complex hyperplane arrangements) for $c = 2$, and quaternionic arrangements with $c = 4$,
- the subspace arrangements that correspond to shellable simplicial complexes (via the construction mentioned in Section 3(b)), and
- the arrangements of the $k$-equal problem, that is, the subspace arrangement given by all points in $\mathbb{R}^n$ with at least $k$ identical coordinates [BLY]. The poset of this arrangement is quite trivial if $2k > n$; for $2k \leq n$ one can show that all lower intervals of the poset are wedges of spheres [We].

In the following corollary we consider the case where the intersection poset $P$ of a linear arrangement $\mathcal{A}$ is shellable. Let $r : \hat{P} \longrightarrow \mathbb{N}_0$ be the rank function on $\widehat{P} := \hat{0} \uplus P$, and let $\mu$ be the Möbius function on $\widehat{P}$. The case when $r(\hat{1}) \leq 1$ is trivial and can be excluded. We use $\bigvee^{\mu} S^k$ to denote the wedge of $\mu$ copies of the $k$-sphere.

**Corollary 3.3.** *Let $\mathcal{A}$ be a linear arrangement in $\mathbb{R}^n$ whose intersection poset is shellable. Then the link $L = S^{n-1} \cap \bigcup \mathcal{A}$ has the homotopy type of a wedge of spheres,*

$$L \simeq \bigvee_{p \in P} \bigvee^{|\mu(\hat{0},p)|} S^{d(p)+r(p)-2}.$$

*if $r(\hat{1}) > 2$ or $d(\hat{1}) > 0$, otherwise $L$ is a disjoint union of spheres $L \simeq \biguplus_{p<\hat{1}} S^{d(p)-1}$.*

*In particular, homology and cohomology of the link are free, and the cohomology algebra has trivial multiplication.*

**Proof.** We compute

$$L \simeq \bigvee_{p \in P} \Sigma^{d(p)} \Delta(P_{<p}) \simeq \bigvee_{p \in P} \Sigma^{d(p)} \bigvee^{|\mu(\hat{0},p)|} S^{r(p)-2} \simeq \bigvee_{p \in P} \bigvee^{|\mu(\hat{0},p)|} S^{d(p)+r(p)-2},$$

where the first homotopy equivalence is from Theorem 2.4, the second one is from shellability, and the third one follows since suspension and wedge commute. □

For complex hyperplane arrangements in $\mathbb{C}^d = \mathbb{R}^{2d}$, Corollary 3.3 was proved in [BZ, Theorem 6.6] for the case $d \geq 4$, where $L$ is simply connected, and independently by Orlik & Terao. The proof given in [BZ] is equally valid for the more general situation of even subspace arrangements ($c = 2$).

In the case of a $c$-arrangement in $\mathbb{R}^n$ we have $d(p) = n - c{\cdot}r(p)$, which can be used to simplify the formula to $L \simeq \bigvee_{p \in P} \bigvee^{|\mu(\hat{0},p)|} S^{n-(c-1)r(p)-2}$. In particular, if $c = 1$ we get that all spheres in the wedge have dimension $n-2$. Passing to homology and applying Alexander duality we get the cohomology of the complement of a $c$-arrangements, which is due to Goresky & MacPherson [GM, III.1.6 “Theorem B”].

We note that analogously we get a wedge of spheres in the affine and the compactified affine cases. To see that this covers the case of affine $c$-arrangements we need the fact that their face posets are inverted geometric semilattices and thus shellable [WW].

### (d) On the Complements

The homotopy type of the *complement* $M = \mathbb{R}^n \backslash \bigcup \mathcal{A}$ of an affine arrangement is not determined by the combinatorial data $(P, d)$. For example, let $\mathcal{A}$ be an arrangement of a line and two points in the plane $\mathbb{R}^2$. If the points are on different sides of the line, then the complement has the homotopy type of two disjoint circles, $M \simeq S^1 \uplus S^1$; if the points are on the same side of the line, then $M$ is homotopy equivalent to the wedge of two circles plus an extra point, $M \simeq S^1 \vee S^1 \vee S^0$.

In fact, although the integral cohomology of $M$ is determined by $(P, d)$, this is not true for the cohomology algebra, even in the special case of even subspace arrangements [Z1] [Z2]. (In the case of complex hyperplane arrangements the cohomology algebra is determined by the combinatorial data by a result of Orlik & Solomon [OS]; that the homotopy type is determined by $(P, d)$ in this case is a notorious conjecture [Or].)

Here we note that there are combinatorial formulas available for the stable homotopy type of the complement, as a direct consequence of Theorem 2.4 together with Spanier-Whitehead duality [SW] [Ad, p. 9]. For simplicity, we state only the linear/spherical case.

**Theorem 3.4.** *The stable homotopy type of the complement $M = S^{n-1} \backslash D$ of a spherical arrangement is determined by its combinatorial data $(P, d, n)$, as*

$$M \;\sim\; \bigvee_{p \in P} \Sigma^{n-1-d(p)}\, S(\Delta(P_{<p})),$$

*where $S(\Delta)$ is the Spanier-Whitehead dual [SW] [Sw, p. 321] of the simplicial complex $\Delta$. Here “$\sim$” denotes stable homotopy equivalence, that is,*

$$\Sigma^N M \;\simeq\; \bigvee_{p \in P} \Sigma^{N+n-1-d(p)}\, S(\Delta(P_{<p}))$$

*for large enough $N$.*

*In particular, if $L$ is a homotopy wedge of spheres, then $M$ has the stable homotopy type of a wedge of spheres.*

**Proof.** This follows by taking Spanier-Whitehead duals (“$S$-duals”) with respect to $S^{n-1}$ of the formula $L \simeq \bigvee_{p \in P} \Sigma^{d(p)} \Delta(P_{<p})$ of Theorem 2.5, using that $S$-duality commutes with wedge products [SW, 4.13]. □

We are grateful to Boris Shapiro for the observations of this section.

### (e) Spectral Sequences

There is a natural spectral sequence "built in" for every diagram of spaces.

**Proposition 3.5.** *Let $\mathcal{D}$ be a $P$-diagram of spaces. Then there exists a spectral sequence abutting to $\widetilde{\mathrm{H}}_*(\|\mathcal{D}\|)$ with the $E^2$-term described by*

$$E^2_{m,n} \cong \widetilde{\mathrm{H}}_m(\mathcal{H}_n(\mathcal{D})).$$

*Here, $\mathcal{H}_n(\mathcal{D})$ is the $P$-diagram [a.k.a. local system, or presheaf] of groups obtained by applying the functor $\mathcal{H}_n : \mathrm{Top} \longrightarrow \mathrm{Ab}$, $X \mapsto H_n(X)$, and $E^2_{m,n}$ is the $m^{th}$ homology with the coefficients in this diagram.*

This spectral sequence is a consequence of a very general construction given in [Se, Proposition 5.1], which applies to any semi-simplicial space. A closely related spectral sequence entered combinatorics with the paper of Quillen [Qu], see also [Ba]. It turns out that, if applied to diagrams of spaces associated with arrangements, this spectral sequence has often a form which is simple enough to assure that it collapses at the $E^2$-term. For example, in the case of an arrangement of spheres, Section 2(c), the $E^2$-term has the following simple form: $E^2_{m,n} \cong \widetilde{\mathrm{H}}_m(\mathcal{H}_n)$ where

$$\mathcal{H}_n(p) = \begin{cases} \mathbb{Z}, & d(p) = 0 \text{ or } n, \\ 0, & \text{otherwise.} \end{cases}$$

The proof of Theorem 2.4 shows, since both diagrams $\mathcal{S}$ and $\mathcal{S}'$ defined there induce the same spectral sequence, that this sequence collapses at the $E^2$-term. From here one can easily observe that

$$E^\infty_{m,n} \;\cong\; E^2_{m,n} \;\cong\; \bigoplus_{d(p)-1=n} \widetilde{\mathrm{H}}_{m-1}(\Delta(P_{<p}); \mathbb{Z}).$$

Vassiliev [Va] outlined a spectral sequence approach to the proof of the Goresky-MacPherson formula (Corollary 2.3). Independently, Jewell, Orlik & Shapiro [JOS] gave a complete solution of this problem using the Mayer-Vietoris spectral sequence. Both of these approaches are direct and do not rely on the technique of diagrams of spaces.

## 4. Appendix: Basic Constructions.

We refer to [Mu] for definition and basic properties of the constructions of cell complexes, to [Wh] for homotopy theory, and to [Bj] for combinatorial notions. In this appendix, we will start with a review of some essential properties of spheres, wedges, joins, suspensions and their homology. All spaces we construct are CW-complexes, and can easily be triangulated. Thus path-connectivity is equivalent to connectivity, and cellular chains can be used for homology [Mu, §39].

The $k$-dimensional *sphere* is denoted by $S^k$, for $k \geq -1$. This includes the cases of $k = -1, 0$, where $S^{-1}$ is empty and $S^0$ consists of two points. We use the first coordinate vector $\mathbf{e}_1 = (1, 0, \ldots, 0)$ as a standard basepoint in $S^k$, for $k \geq 0$.

We usually compute reduced (co)homology. This implies that contractible spaces are zero homology in all dimensions, and for $k \geq -1$:

$$\widetilde{\mathrm{H}}_i(S^k; \mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

The *wedge* $X \vee Y$ of two spaces $X, Y$ is obtained by identification of one point from each complex. The homology is given by a natural isomorphism

$$\widetilde{\mathrm{H}}_*(X \vee Y, \mathbb{Z}) \;\cong\; \widetilde{\mathrm{H}}_*(X; \mathbb{Z}) \oplus \widetilde{\mathrm{H}}_*(Y, \mathbb{Z}),$$

which follows from the Mayer-Vietoris sequence of $(X, Y)$.

The *join* $X * Y$ is obtained as a quotient of the set $X \uplus (X \times [0,1] \times Y) \uplus Y$, where $x \in X$ is identified with $(x, 0, y)$ for all $y \in Y$, and
$y \in Y$ is identified with $(x, 1, y)$ for all $x \in X$.

Thus $X*Y$ can be written as a disjoint union of $X, Y$ and a segment $\{(x, y, t) : 0<t<1\}$ for every $x \in X$ and $y \in Y$. This differs from the usual definition of a join [Mu, p. 378] in the case when $X = \emptyset$ or $Y = \emptyset$, where we insist that $X * \emptyset = X$ and $\emptyset * Y = Y$. The join operation is well-defined, commutative and associative up to homeomorphism. The homology of a join can again be computed from a Mayer-Vietoris sequence [Mu, §25].

The *suspension* $\Sigma X$ can be defined as the join with $S^0$, that is, $\Sigma X := X * S^0$. Furthermore, we have $\Sigma S^k \cong S^{k+1}$ for $k \geq -1$, so that (using associativity of the join operation) the $k$-fold suspension is given by the join with the $(k-1)$-sphere,

$$\Sigma^k(X) \;\cong\; X * S^{k-1} \qquad \text{for } k \geq -1.$$

Since the join $X * S^0$ can be written as a union of two cones that intersect in $X$, the homology of the suspension is given by a natural isomorphism

$$\widetilde{\mathrm{H}}_i(X * S^k; \mathbb{Z}) \;\cong\; \widetilde{\mathrm{H}}_{i-k-1}(X; \mathbb{Z}) \qquad \text{for } k \geq -1.$$

Note that this reduces to a trivial statement for $k = -1$, with $X * S^{-1} = X * \emptyset = X$.

We will now prove the basic lemmas about the homotopy types of diagrams. The assumption that the underlying small category is a finite poset permits us to prove the Projection Lemma and the Homotopy Lemma by elementary inductive arguments. A silent assumption throughout this is that all inclusion maps $i : A \longrightarrow X$ of all pairs of spaces which appear in subspace diagrams are closed cofibrations, i.e., these maps possess the homotopy lifting property. Equivalently, this property can be reformulated as the statement that $X \times \{0\} \cup A \times I$ is a retract of $X \times I$, or that $(X, A)$ is an NDR-pair, see [Wh, I.5].

The following lemma and its corollary are used in [tD] for a similar purpose, namely for proving that a map $f : X \to Y$ is a homotopy equivalence if it is "locally" a homotopy equivalence.

**Lemma 4.1.** *Let $f : (X, A) \longrightarrow (Y, B)$ be a map of pairs of spaces such that both $f : X \longrightarrow Y$ and $f' = f|A : A \longrightarrow B$ are homotopy equivalences. Let $g' : B \longrightarrow A$ be a homotopy equivalence and $\alpha : A \times I \longrightarrow A$ a homotopy with $\alpha(a, 0) = g' \circ f'(a)$ and $\alpha(a, 1) = a$ for $a \in A$. Then there exist a homotopy equivalence $g : Y \longrightarrow X$ and a homotopy $\beta : X \times I \longrightarrow X$ such that $g|B = g'$, $\beta(x, 0) = g \circ f(x)$, $\beta(x, 1) = x$ for $x \in X$, and for all $a \in A$,*

$$\beta(a,t) \;=\; \begin{cases} \alpha(a,2t), & \text{if } t \leq 1/2 \\ a\,, & \text{if } t \geq 1/2\,. \end{cases}$$

**Corollary 4.2.** *Let $X = A \cup B$ and $Y = C \cup D$ be spaces and $f : X \longrightarrow Y$ a continuous map with the property $f(A) \subset C$ and $f(B) \subset D$.*

*If $f|A : A \longrightarrow C$, $f|B : B \longrightarrow D$ and $f|(A \cap B) : A \cap B \longrightarrow C \cap D$ are all homotopy equivalences, then the map $f : X \longrightarrow Y$ is a homotopy equivalence.*

**Proof of the Projection Lemma.** The proof will be carried out by induction on the size of the poset $(P, \supseteq)$. For $|P| \leq 1$ there is nothing to show.

Choose a minimal element $p_0$ in $P$. Let $\mathcal{A}^1 = \mathcal{A} \backslash A_{p_0}$ be the arrangement obtained by deleting $A_{p_0}$ from $\mathcal{A}$, and $\mathcal{A}^2 := \{A_p \cap A_{p_0} : A_p \in \mathcal{A}\}$ the arrangement of all intersections with $A_{p_0}$. By $\mathcal{A}^{12} := \mathcal{A}^2 \backslash A_{p_0} = \mathcal{A}^1 \cap \mathcal{A}^2$ we denote the arrangement of all proper intersections with $A_{p_0}$.

The unions of the arrangements are $D = \bigcup \mathcal{A}$, $D^1 := \bigcup \mathcal{A}^1$, $D^2 := \bigcup \mathcal{A}^2 = A_{p_0}$ and $D^{12} := \bigcup \mathcal{A}^{12} = D^1 \cap A_{p_0}$. The corresponding posets are $P^1 := P \backslash p_0$, $P^2 := P_{\geq p_0}$ and $P^{12} := P_{>p_0} = P^1 \cap P^2$. Each of them inherits a diagram structure from $\mathcal{D} = \mathcal{D}[\mathcal{A}]$, as in Example 1.5(c). One has the following diagram of inclusions:

$$\begin{array}{ccc} \|\mathcal{D}^{12}\| & \longrightarrow & \|\mathcal{D}^1\| \\ \Big\downarrow & & \Big\downarrow \\ \|\mathcal{D}^2\| & \longrightarrow & \|\mathcal{D}\| \end{array}$$

The collapsing map $\xi : \|\mathcal{D}\| \longrightarrow M$, described in Example 1.4(d), is an extension of the collapsing maps $\xi^1 : \|\mathcal{D}^1\| \longrightarrow D^1$, $\xi^2 : \|\mathcal{D}^2\| \longrightarrow A_{p_0}$ and $\xi^{12} : \|\mathcal{D}^{12}\| \longrightarrow D^1 \cap A_{p_0}$.

The map $\xi^2$ is a homotopy equivalence because it coincides with the collapsing of a mapping cylinder to its base (Example 1.5(d)). The maps $\xi^1$ and $\xi^{12}$ are homotopy equivalences by the inductive assumption. So, by Corollary 4.2, $\xi$ is also a homotopy equivalence if we can prove that all pairs of spaces involved are NDR-pairs. In other words, we need a statement which tells us that $(\|\mathcal{D}\|, \|\mathcal{D}'\|)$ is a NDR-pair for every subdiagram $\mathcal{D}'$ of $\mathcal{D}$ obtained by restricting $\mathcal{D}'$ to a filter $P' \subseteq P$. This statement can be proved by a parallel inductive argument, similar to the argument used above. To this end, one can use well known properties of NDR-pairs. [Wh, I.5]. □

The collapsing map $\xi$ in the Projection Lemma 1.6 is always a map with contractible fibers. It is worth reminding ourselves that, as the following example shows, this condition does not guarantee that $\xi$ is a homotopy equivalence.

**Example 4.3.** Let $X = [0,1]\times\{0,1\} \cup \bigcup_{n=0}^{\infty} A_n \subset [0,1]^2$ , where $A_0 = \{0\} \times [0,1]$ and $A_n = \{1/n\} \times [0,1]$ for $n \geq 1$. Let $Y$ be the quotient space of $X$ obtained by contracting $\{1/n\} \times [0,1]$ to the point $\{(1/n, 0)\}$ for $n \geq 1$, and $A_0$ to the point $(0,0)$. One can easily check that the projection map $p : X \longrightarrow Y$ is not a homotopy equivalence in spite of the fact that $p$ has contractible fibers.

**Proof of the Homotopy Lemma 1.7.** Again we use induction on the size of $P$.

Let $\alpha = (\alpha_p)_{p\in P}$ be a morphism of two $P$-diagrams $\mathcal{D}$ and $\mathcal{E}$ such that $\alpha_p : D_p \longrightarrow E_p$ is a homotopy equivalence for all $p \in P$. Let $p_0$ be a minimal element in $P$, $P^1 = P\backslash p_0$, $P^2 = P_{\geq p_0}$ and $P^{12} = P^1 \cap P^2$. The restrictions of diagrams $\mathcal{D}$ and $\mathcal{E}$ to these posets will be denoted by $\mathcal{D}^1$, $\mathcal{E}^1$, etc.. The restriction of the morphism $\alpha$ to $\mathcal{D}^1$, $\mathcal{D}^2$ and $\mathcal{D}^{12}$ will be denoted by $\alpha^1$, $\alpha^2$ and $\alpha^{12}$, so the corresponding maps at the level of homotopy limits are $\overline{\alpha^1}$, $\overline{\alpha^2}$ and $\overline{\alpha^{12}}$. By Example 1.5(d), there is a commutative diagram

$$\begin{array}{ccc} \|\mathcal{D}^2\| & \xrightarrow{\overline{\alpha^2}} & \|\mathcal{E}^2\| \\ \xi \big\downarrow & & \big\downarrow \xi' \\ D_{p_0} & \xrightarrow[\alpha_{p_0}]{} & E_{p_0} \end{array}$$

where the naturally defined maps $\xi$ and $\xi'$ are homotopy equivalences, since they are just collapsing maps associated with the corresponding mapping cylinders. Since $\alpha_{p_0}$ is by assumption a homotopy equivalence, so is the map $\overline{\alpha^2}$. By the inductive assumption $\overline{\alpha^1} : \|\mathcal{D}^1\| \longrightarrow \|\mathcal{E}^1\|$ and $\overline{\alpha^{12}} : \|\mathcal{D}^{12}\| \longrightarrow \|\mathcal{E}^{12}\|$ are homotopy equivalences. By the remark at the end of proof of the Projection Lemma all spaces involved are NDR-pairs, so by Corollary 4.2 the map $\overline{\alpha}$ is also a homotopy equivalence. □

**Proof of the Wedge Lemma 1.8.** For every $p \in P$, let $\mathcal{C}[p]$ be a diagram over $P$ defined as follows:

$$C[p]_q \;=\; \begin{cases} D_p, & \text{if } q = p, \\ \{c_q\}, & \text{if } q < p, \\ \emptyset, & \text{otherwise.} \end{cases}$$

with the connecting functions inherited from $\mathcal{D}$. Then again one has an obvious map $\gamma[p] : \mathcal{C}[p] \longrightarrow \mathcal{D}$ which defines an embedding $\overline{\gamma[p]} : \|\mathcal{C}[p]\| \longrightarrow \|\mathcal{D}\|$.

Now observe that $\|\mathcal{C}[p]\|$ is homeomorphic to the join $\Delta(P_{<p}) * D_p$ and that $\Delta(P_{\leq p}) = \Delta(P_{<p}) * \{c_p\} \subseteq \|\mathcal{C}[p]\|$.

The structure of $\|\mathcal{D}\|$ is easily described in terms of the "building blocks" $\|\mathcal{C}[p]\| \cong \Delta(P_{<p}) * D_p$. One observes that $\|\mathcal{C}[p]\| \cap \Delta(P\backslash\hat{1}) = \Delta(P_{\leq p})$, which is a contractible set, and that $\|\mathcal{C}[p]\| \cap \|\mathcal{C}[q]\| = \Delta(P_{\leq p}) \cap \Delta(P_{\leq q}) \subseteq \Delta(P\backslash\hat{1})$ for $q \neq p$ and $q, p < \hat{1}$. Hence,

$\|\mathcal{D}\|$ is the space obtained from $\Delta(P\backslash\hat{1}) * D_{\hat{1}}$ by attaching, for every $p < \hat{1}$, the space $\Delta(P_{<p}) * D_p$ to $\Delta(P\backslash\hat{1}) * D_{\hat{1}}$, along the common copy of $\Delta(P_{\leq p})$.

The homotopy type of this space is not changed if one replaces the attaching maps of $\|\mathcal{C}[p]\|$ to $\Delta(P_{\hat{1}}) * D_{\hat{1}}$ by wedge operations, to get

$$\|\mathcal{D}\| \;=\; \Delta(P_{\hat{1}}) * D_{\hat{1}} \;\vee\; \bigvee_{p<\hat{1}} (\Delta(P_{<p}) * D_p)$$

as claimed. □

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**G.M. Ziegler's address after April 1, 1992:**
Konrad-Zuse-Zentrum für Informationstechnik Berlin,
Heilbronner Str. 10, W-1000 Berlin 31, Germany
`ziegler@zib-berlin.de`