HOOK REPRESENTATIONS OF THE SYMMETRIC GROUPS

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1. Introduction. In this paper we are concerned with the representation theory of the symmetric groups over a field K of characteristic p. Every field is a splitting field for the symmetric groups. Consequently, in order to study the modular representation theory of these groups, it is sufficient to work over the prime fields. However, we take K to be an arbitrary field of characteristic p, since the presentation of the results is not affected by this choice. S_n denotes the group of permutations of $\{x_1, \ldots, x_n\}$, where $x_1, \ldots, x_n$ are independent indeterminates over K. The group algebra of S_n with coefficients in K is denoted by Φ_n.

Let (λ) be a partition of n. We can define the Specht module $S^{(\lambda)}$ corresponding to (λ) (over the field K). For the construction, the reader is referred to [**3**]. $S^{(\lambda)}$ consists of K-linear combinations of the polynomials $f^{(\lambda)}(y)$ defined for any tableau y of (λ) as in [**3**, Introduction]. These modules are studied in the author's Ph.D. thesis [**4**]. If K is a field of characteristic zero, the Specht modules constitute a full set of irreducible, non-isomorphic left Φ_n-modules.

It turns out that the problem of determining the p-decomposition numbers of the symmetric groups is equivalent to the problem of constructing the composition factors of the Specht modules over a field of characteristic p. We attempt to analyse the Specht modules corresponding to partitions of the form $(n-r, 1^r)$, $0 \leqq r \leqq n-1$. The diagram of such a partition is a hook, and accordingly, these Specht modules are referred to as *Hook Representations*. The Specht module corresponding to the partition $(n-r, 1^r)$ is denoted by $S(r, n)$.

Throughout most of this paper, K is a field of characteristic p not equal to 2. We prove that $S(r, n)$ is irreducible if p does not divide n. In the case in which p divides n, we find a composition series of the form $0 \subset M \subset S(r, n)$, $0 < r < n-1$. The proof is by induction on n. In the first few sections we lay the foundations of the inductive argument which is carried out in Section 5.

The paper is concluded in Section 6 with some remarks concerning the hook representations over a field of characteristic 2. We establish a connection between these modules and the *natural representation modules* to be defined. If $2r \leqq n$, $S(r, n)$ contains a submodule isomorphic to the Specht module corresponding to the partition $(n-r, r)$. The problem of analysing the hook representations is equivalent to that of analysing the natural representation modules.

For a general reference on the representations of the symmetric groups, see [**5**].

2. General remarks. We begin with a few general remarks concerning the hook representations.

PROPOSITION 1. *If K is a field of characteristic not equal to* 2, *then $S(r, n)$ is an indecomposable Φ_n-module.*

Indeed the Specht modules $S^{(\lambda)}$ are all indecomposable over a field of characteristic not equal to 2. One proves that $\mathrm{Hom}_{\Phi_n}(S^{(\lambda)}, S^{(\lambda)}) \cong K$.

For the rest of this section, K is an arbitrary field. We write $x_i \leqq x_j$ if $i \leqq j$, and $\{y_1, \ldots, y_r\}$ will denote a subset of the set $\{x_1, \ldots, x_n\}$.

PROPOSITION 2. *A K-basis for $S(r, n)$ is given by*

$$X = \{\Delta(x_1, y_1, \ldots, y_r): x_1 < y_1 < \ldots < y_r \leqq x_n\}.$$

Thus

$$\dim_K S(r, n) = \binom{n-1}{r}.$$

This is a special case of the result that a K-basis for $S^{(\lambda)}$ is given by $\{f^{(\lambda)}(y): y$ is a standard tableau of $(\lambda)\}$ (standard tableaux are defined in [5], to which the reader is referred for the meaning of any undefined terms). The proof that the set X is linearly independent over K is by induction on n. $S(r, n)$ is generated over K by the polynomials of the form $\Delta(y_1, \ldots, y_{r+1})$, $x_1 \leqq y_1 < \ldots < y_{r+1} \leqq x_n$. The formula

$$\Delta(y_1, y_2, \ldots, y_{r+1}) = \sum_{s=1}^{r+1} (-1)^{s-1} \Delta(x_1, y_1, \ldots, \hat{y}_s, \ldots, y_{r+1}), \tag{1}$$

where $\hat{y}_s$ is meant to indicate that y_s is missing from the sequence $(x_1, y_1, \ldots, y_s, \ldots, y_{r+1})$, shows that X generates $S(r, n)$ over K. This formula can be proved by expressing the difference products as Vandermonde determinants, and noting that the determinant

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ x_1 & y_1 & \ldots & y_{r+1} \\ \vdots & \vdots & & \vdots \\ x_1^r & y_1^r & \ldots & y_{r+1}^r \end{vmatrix}$$

is zero.

The partitions $(n-r, 1^r)$ and $(r+1, 1^{n-r-1})$, $0 \leqq r \leqq n-1$, are conjugate in the sense of [5, p. 36]. Write $\Delta = \Delta(x_1, \ldots, x_n)$. Then $K\Delta = S(n-1, n)$.

PROPOSITION 3. *$S(r, n)$ and $K\Delta \otimes_K S(n-r-1, n)$ have the same composition factors, counted according to multiplicity.*

This again is a special case of a general result. Let (λ) and (λ') be conjugate partitions with corresponding ordinary irreducible characters $\zeta^{(\lambda)}$ and $\zeta^{(\lambda')}$ respectively. Then $\zeta^{(\lambda)} = \zeta^{(1^n)}\zeta^{(\lambda')}$, where $\zeta^{(1^n)}$ is the alternating character. This is the basis of the result that $S^{(\lambda)}$ and $K\Delta \otimes_K S^{(\lambda')}$ have the same composition factors.

The final proposition of this section will be of use in Section 5.

PROPOSITION 4. *Let M be an indecomposable Φ_n-module with a composition series of the form $0 \subset M_1 \subset M$. Then M has exactly one proper submodule.*

Proof. Let $0 \subset M_1 \subset M$ be a composition series for M, and suppose that N is a proper Φ_n-submodule of M. The chain $0 \subset N \subset M$ can be refined to a composition series, and by the

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Jordan–Hölder Theorem, it must be a composition series. If $M_1 \cap N = 0$, then $M_1 + N$ (direct sum) would be a submodule of M properly containing M_1. It would follow that $M = M_1 + N$ (direct sum), contradicting the assumption that M is indecomposable. Thus $M_1 \cap N \neq 0$. Since M_1 and N are irreducible, $M_1 = M_1 \cap N = N$, proving that M_1 is the unique proper submodule of M.

3. The case p divides n. The results of this section form part of the author's Ph.D Thesis [**4**]. Throughout this section we assume that the characteristic p of the field K divides n. We set

$$s_i = \sum_{k=1}^{n} x_k^i, \qquad i = 1, 2, \ldots .$$

Suppose that $r < n-1$. Proposition 1 gives a K-basis for $S(r, n)$. Using this basis, we define a linear transformation $\theta^r : S(r, n) \to S(r+1, n)$ as follows:

$$\theta^r(\Delta(x_1, y_1, \ldots, y_r)) = \sum_{k=1}^{n} \Delta(x_1, y_1, \ldots, y_r, x_k)$$

$$= \begin{vmatrix} 1 & 1 & \ldots & 1 & n \\ x_1 & y_1 & \ldots & y_r & s_1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_1^{r+1} & y_1^{r+1} & \ldots & y_r^{r+1} & s_{r+1} \end{vmatrix}, \tag{2}$$

extending to the whole of $S(r, n)$ by K-linearity. We claim that θ^r is a Φ_n-homomorphism. To see this, let $\{y_1, \ldots, y_{r+1}\} \subseteq \{x_1, \ldots, x_n\}$ with $y_1 < y_2 < \ldots < y_{r+1}$. From (1), we have

$$\theta^r(\Delta(y_1, \ldots, y_{r+1})) = -\begin{vmatrix} 1 & 1 & \ldots & 1 & 0 \\ 1 & 1 & \ldots & 1 & n \\ x_1 & y_1 & \ldots & y_{r+1} & s_1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_1^{r+1} & y_1^{r+1} & \ldots & y_{r+1}^{r+1} & s_{r+1} \end{vmatrix} + \begin{vmatrix} 1 & \ldots & 1 & n \\ y_1 & \ldots & y_{r+1} & s_1 \\ \vdots & & \vdots & \vdots \\ y_1^{r+1} & \ldots & y_{r+1}^{r+1} & s_{r+1} \end{vmatrix}.$$

Since p divides n,

$$\theta^r(\Delta(y_1, \ldots, y_{r+1})) = \begin{vmatrix} 1 & \ldots & 1 & n \\ y_1 & \ldots & y_{r+1} & s_1 \\ \vdots & & \vdots & \vdots \\ y_1^{r+1} & \ldots & y_{r+1}^{r+1} & s_{r+1} \end{vmatrix}.$$

From this it follows that θ^r is a Φ_n-homomorphism.

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Thus we have a chain of Φ_n-modules and Φ_n-homomorphisms:

$$0 \to S(0, n) \xrightarrow{\theta^0} S(1, n) \xrightarrow{\theta^1} \dots \to S(n-2, n) \xrightarrow{\theta^{n-2}} S(n-1, n) \to 0. \tag{3}$$

We proceed to investigate the properties of this sequence.

Certainly $S(r, n-1)$ is a Φ_{n-1}-submodule of $S(r, n)$. Let ϕ^r denote the restriction of θ^r to this submodule. Then ϕ^r is a Φ_{n-1}-homomorphism.

LEMMA 1. $\operatorname{Im}\theta^r = \operatorname{Im}\phi^r$.

Proof. All we need to show is that $\operatorname{Im}\theta^r \subseteq \operatorname{Im}\phi^r$. We do this by checking that $\theta^r(\Delta(x_1, y_1, \dots, y_r)) \in \operatorname{Im}\phi^r$ for all $y_1, \dots, y_r$ with $x_1 < y_1 < \dots < y_r \leqq x_n$. Clearly it is sufficient to take $y_r = x_n$.

$$\sum_{k=1}^{n-1} \theta^r(\Delta(x_1, y_1, \dots, y_{r-1}, x_k)) = \sum_{k,l=1}^{n} \Delta(x_1, y_1, \dots, y_{r-1}, x_k, x_l) - \sum_{l=1}^{n} \Delta(x_1, y_1, \dots, y_{r-1}, x_n, x_l)$$
$$= -\theta^r(\Delta(x_1, y_1, \dots, y_{r-1}, x_n)).$$

Lemma 1 follows from this.

For the rest of this section, we assume that $0 < r < n-1$.

LEMMA 2. *Suppose that $S(r, n-1)$ and $S(r-1, n-1)$ are irreducible Φ_{n-1}-modules. Then*

$$\operatorname{Im}\theta^{r-1} = \ker\theta^r.$$

Proof. $\theta^r\theta^{r-1}(\Delta(y_1, \dots, y_r)) = \sum_{k,l=1}^{n} \Delta(y_1, \dots, y_r, x_k, x_l) = 0.$ Hence $\operatorname{Im}\theta^{r-1} \subseteq \ker\theta^r$. We prove the reverse inclusion by counting dimensions.

By hypothesis, $S(r, n-1)$ is an irreducible Φ_{n-1}-module, and certainly $\phi^r \neq 0$. Hence $\ker\phi^r = 0$. Consequently, by Lemma 1 and Proposition 1,

$$\dim_K \operatorname{Im}\theta^r = \dim_K \operatorname{Im}\phi^r = \dim_K S(r, n-1) = \binom{n-2}{r}.$$

Therefore

$$\dim_K \ker\theta^r = \dim_K S(r, n) - \dim_K \operatorname{Im}\theta^r = \binom{n-1}{r} - \binom{n-2}{r} = \binom{n-2}{r-1}.$$

By hypothesis, $S(r-1, n-1)$ is irreducible; whence $\ker\phi^{r-1} = 0$. Hence by Lemma 1 and Proposition 1,

$$\dim_K \operatorname{Im}\theta^{r-1} = \dim_K \operatorname{Im}\phi^{r-1} = \dim_K S(r-1, n-1) = \binom{n-2}{r-1}.$$

Thus $\dim_K \operatorname{Im}\theta^{r-1} = \dim_K \ker\theta^r$, and this completes the proof of Lemma 2.

THEOREM 1. *Assuming that p divides n, let* $0 < r < n-1$, *and suppose that* $S(r, n-1)$ *and* $S(r-1, n-1)$ *are irreducible* Φ_{n-1}*-modules. Then a composition series for* $S(r, n)$ *is given by*

$$0 \subset \ker\theta^r \subset S(r, n),$$

where θ^r *is the* Φ_n*-homomorphism* (2). *Further,* $\ker\theta^r \cong S(r-1, n-1)$ *over* Φ_{n-1}.

Proof. By Lemmas 1 and 2, there are Φ_{n-1}-isomorphisms

$$\ker\theta^r = \operatorname{Im}\theta^{r-1} = \operatorname{Im}\phi^{r-1} \cong S(r-1, n-1)$$

because $\ker\phi^{r-1} = 0$. It follows that $\ker\theta^r$ is irreducible as a Φ_{n-1}-module, and hence as a Φ_n-module. Also there are Φ_{n-1}-isomorphisms

$$\frac{S(r, n)}{\ker\theta^r} \cong \operatorname{Im}\theta^r = \operatorname{Im}\phi^r \cong S(r, n-1).$$

The irreducibility of $S(r, n-1)$ over Φ_{n-1} implies that of $S(r, n)/\ker\theta^r$ over Φ_n.

COROLLARY. *Take* $n = p$. *Let* θ^r *be the homomorphism* (2). *Then if* $0 < r < p-1$, *a composition series for* $S(r, p)$ *is given by*

$$0 \subset \ker\theta^r \subset S(r, p).$$

Proof. This follows immediately because we know that $S(r, p-1)$ is irreducible for all r.

REMARKS. By Lemma 2, the sequence (3) is exact in the case in which $n = p$; thus if $r < p-2$,

$$\frac{S(r, p)}{\ker\theta^r} \cong \operatorname{Im}\theta^r = \ker\theta^{r+1}.$$

Hence if $0 \leqq r \leqq p-2$, $S(r, p)$ and $S(r+1, p)$ have a common composition factor. Further, Proposition 3 says that $S(r-1, p)$ and $K\Delta\otimes_K S(p-r, p)$ have the same composition factors. A composition series for the latter is given by

$$0 \subset K\Delta\otimes_K \ker\theta^{p-r} \subset K\Delta\otimes_K S(p-r, p).$$

Counting dimensions we see that

$$K\Delta\otimes_K \ker\theta^{p-r} \cong \frac{S(r-1, p)}{\ker\theta^{r-1}} \cong \ker\theta^r.$$

We cannot have $\ker\theta^r \cong \ker\theta^s$ with $r \neq s$, since

$$\binom{p-2}{r-1} = \dim_K \ker\theta^r = \dim_K \ker\theta^s = \binom{p-2}{s-1}$$

can only occur when $r = p-s$, and when this is satisfied, the relationship between $\ker\theta^r$ and $\ker\theta^s$ is as described.

These results give the modular representation theory of S_p over the field of p elements as obtained by T. Nakayama [**2**, Theorems 2 and 4], and R. M. Thrall and C. Nesbitt [**6**]. The

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Specht modules $S^{(\lambda)}$, where (λ) is a partition of p whose diagram is not a hook, together with the $\ker\theta^r$, $r = 0, 1, \ldots, p-2$, and $K\Delta$ constitute a full set of irreducible, non-isomorphic left Φ_p-modules.

H. K. Farahat [**1**, Theorem 5.2] proves that $S(1, n)$ is irreducible if p does not divide n. Further, taking p to be a factor of n, he obtains the same composition series for $S(1, n)$ as is given by applying Theorem 1 with $r = 1$. In [**4**], we prove that $S(2, n)$ is irreducible if p does not divide n and $p \neq 2$. In Section 5 we prove the conjecture made in [**4**] that $S(r, n)$ is irreducible if p does not divide n and $p \neq 2$. The composition factors of $S(r, n)$ in the case when p divides n are then as in Theorem 1. We shall see in Section 6 that if the ground field has characteristic 2, the hook representations are reducible, except for $r = 0$, $n-1$, and possibly $r = 1$, $n-2$. In this situation, Theorem 1 is not applicable.

3. Restriction to S_{n-1}. In preparation for the analysis of the hook representations by induction on n, we examine the structure of $S(r, n)$ regarded as a Φ_{n-1}-module. In this section, K is an arbitrary field and $0 < r < n-1$.

Using the basis of $S(r, n)$ given by Proposition 2, we define a linear transformation $\psi^r: S(r, n) \to S(r-1, n-1)$ as follows:

$$\psi^r(\Delta(x_1, y_1, \ldots, y_r)) = \begin{cases} \Delta(x_1, y_1, \ldots, y_{r-1}), & x_1 < y_1 < \ldots < y_{r-1} < y_r = x_n \\ 0, & x_1 < y_1 < \ldots < y_{r-1} < y_r < x_n, \end{cases}$$

extending to the whole of $S(r, n)$ by K-linearity. Using (1), it is clear that ψ^r is a Φ_{n-1}-homomorphism. Obviously ψ^r is an epimorphism. Note also that $S(r, n-1)$ is a Φ_{n-1}-submodule of $S(r, n)$ contained in $\ker\psi^r$.

LEMMA 3. *The following sequence of Φ_{n-1}-modules is exact:*

$$0 \to S(r, n-1) \xrightarrow{\text{incl.}} S(r, n) \xrightarrow{\psi^r} S(r-1, n-1) \to 0. \tag{4}$$

Proof. We have $S(r, n-1) \subseteq \ker\psi^r$. It follows by counting dimensions that

$$S(r, n-1) = \ker\psi^r.$$

For the moment, let S_{n-2} be the group of permutations of the set $\{x_2, \ldots, x_{n-1}\}$. The map $\omega_0^r: S(r-1, n-1) \to S(r, n)$ defined by

$$\omega_0^r(\Delta(x_1, y_1, \ldots, y_{r-1})) = \Delta(x_1, y_1, \ldots, y_{r-1}, x_n) \qquad x_1 < y_1 < \ldots < y_{r-1} < x_n,$$

is a Φ_{n-2}-homomorphism, where Φ_{n-2} is the group algebra of S_{n-2} with coefficients in K. Clearly $\psi^r\omega_0^r$ is the identity map on $S(r-1, n-1)$. Suppose that p does not divide $n-1$, and define $\omega^r: S(r-1, n-1) \to S(r, n)$ by

$$\omega^r(z) = (n-1)^{-1} \sum_{k=1}^{n-1} (x_1\, x_k)\omega_0^r((x_1\, x_k)z) \qquad (z \in S(r-1, n-1)).$$

Then ω^r is a Φ_{n-1}-homomorphism, and $\psi^r\omega^r$ is the identity map on $S(r-1, n-1)$. This means that the exact sequence (4) splits; thus $\operatorname{Im}\omega^r \cong S(r-1, n-1)$ and $S(r, n) = S(r, n-1) + \operatorname{Im}\omega^r$, (direct sum). This proves

LEMMA 4. *If the characteristic of the field K does not divide n−1, there is a Φ_{n-1}-isomorphism*

$$S(r, n) \cong S(r-1, n-1)+S(r, n-1) \qquad (\textit{direct sum}).$$

5. Analysis of hook representations. The one theorem of this section gives the composition factors of the hook representations when the ground field has characteristic p not equal to 2.

THEOREM 2. *Let K be a field of characteristic p not equal to 2. If p does not divide n, $S(r, n)$ is irreducible for each r, $0 \leqq r \leqq n-1$. If p divides n, a composition series for $S(r, n)$, $0 < r < n-1$, is given by*

$$0 \subset \ker\theta^r \subset S(r, n),$$

where $\theta^r: S(r, n) \to S(r+1, n)$ is the Φ_n-homomorphism (2). $S(0, n)$ and $S(n-1, n)$ are of course still irreducible when p divides n.

Proof. Throughout the proof we assume that $0 < r < n-1$ and $n \geqq 3$. We proceed by induction on n.

Induction hypothesis: $p \nmid n$ implies that $S(r, n)$ is irreducible for each r, $0 < r < n-1$; $p \mid n$ implies that each $S(r, n)$ has a composition series of the form $0 \subset M \subset S(r, n)$.

The remarks at the end of Section 3, together with Proposition 3, are sufficient to cover the cases when $n \leqq 6$. Suppose that $n > 3$, and that the induction hypothesis holds for smaller values of the inductive argument. The proof is divided into three parts.

(a) $p \mid n$. In this case, p does not divide $n-1$, and so $S(r, n-1)$ and $S(r-1, n-1)$ are irreducible Φ_{n-1}-modules by the induction hypothesis. By Theorem 1, $S(r,n)$, has a composition series

$$0 \subset \ker\theta^r \subset S(r, n),$$

where θ^r is the Φ_n-homomorphism (2).

(b) $p \nmid n, p \nmid n-1$. By Lemma 4, there is a Φ_{n-1}-isomorphism

$$S(r, n) \cong S(r, n-1)+S(r-1, n-1) \qquad (\text{direct sum}).$$

Thus, by the induction hypothesis $S(r, n)$ is completely reducible as a Φ_{n-1}-module. Let X be a Φ_n-submodule of $S(r, n)$. Then X is a Φ_{n-1}-submodule of $S(r, n)$, and as such is a direct summand. Hence there exists a Φ_{n-1}-homomorphism $\eta_0: S(r, n) \to X$ such that the restriction of η_0 to X is the identity map on X, namely the projection onto X. We are supposing that p does not divide n, so that n^{-1} exists in K. We can therefore define $\eta: S(r, n) \to X$ by

$$\eta(z) = n^{-1} \sum_{i=1}^{n} (x_i x_n)\eta_0((x_i x_n)z) \qquad (z \in S(r, n)).$$

Then η is a Φ_n-homomorphism, and the restriction of η to X is the identity map on X. It follows that $S(r, n) = X+\operatorname{Im}\eta$, (direct sum), where both X and $\operatorname{Im}\eta$ are Φ_n-submodules of $S(r, n)$. But $S(r, n)$ is indecomposable according to Proposition 1; whence $X = 0$ or $X = S(r, n)$. Thus $S(r, n)$ is irreducible.

(c) $p \nmid n, p \mid n-1$. In Lemma 3 we found the following exact sequence of Φ_{n-1}-modules:

$$0 \to S(r, n-1) \to S(r, n) \xrightarrow{\psi^r} S(r-1, n-1) \to 0.$$

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Since p divides $n-1$, there exist Φ_{n-1}-homomorphisms $\theta^t: S(t, n-1) \to S(t+1, n-1)$, $0 \leqq t \leqq n-2$; $\operatorname{Im}\theta^{t-1} = \ker\theta^t$ for $0 < t \leqq n-2$, as in Section 3. By the induction hypothesis, each of $\ker\theta^r$, $S(r, n-1)/\ker\theta^r$, $\ker\theta^{r-1}$ and $S(r-1, n-1)/\ker\theta^{r-1}$ is an irreducible Φ_{n-1}-module.

From the above exact sequence of Φ_{n-1}-modules, we can construct a chain of Φ_{n-1}-submodules of $S(r, n)$ of the form

$$0 \subset X_1 \subseteq X_2 \subseteq X_3 \subset S(r, n), \tag{5}$$

where $X_2 = S(r, n-1)$ and $X_1 = \ker\theta^r$: in the case in which $r = n-2$, we have $X_2 \cong K\Delta(x_1, \ldots, x_{n-1})$, and $X_1 = X_2$; otherwise X_1 is a proper submodule of X_2. Also the map ψ^r induces a Φ_{n-1}-isomorphism

$$\bar{\psi}^r: \frac{S(r, n)}{X_2} \to S(r-1, n-1),$$

and X_3 is the Φ_{n-1}-submodule of $S(r, n)$ containing X_2 and corresponding to $\ker\theta^{r-1}$; thus

$$\bar{\psi}^r: \frac{X_3}{X_2} \to \ker\theta^{r-1} = \operatorname{Im}\theta^{r-2}$$

is a Φ_{n-1}-isomorphism: in the case in which $r = 1$, we have $S(r, n)/X_2 \cong K$, and $X_3 = X_2$; in all other cases $X_2 \subset X_3$.

At this point we construct a specific submodule X_3 which has the properties just described. We take $r > 1$. Propositions 1 and 4 show that there is exactly one proper Φ_{n-1}-submodule of $S(r, n)$ containing X_2 properly.

Let X_3 be the smallest Φ_{n-1}-submodule of $S(r, n)$ containing X_2 and all polynomials of the form

$$\sum_{k=1}^{n-1} \Delta(y_1, \ldots, y_{r-1}, x_k, x_n), \qquad x_1 \leqq y_1 < \ldots < y_{r-1} < x_n.$$

By (1), and using the assumption that p divides $n-1$, we have

$$\sum_{k=1}^{n-1} \Delta(y_1, \ldots, y_{r-1}, x_k, x_n) = \sum_{k=1}^{n-1}\sum_{l=1}^{r-1} (-1)^{l-1}\Delta(x_1, y_1, \ldots, \hat{y}_l, \ldots, y_{r-1}, x_k, x_n)$$
$$+(-1)^r \sum_{k=1}^{n-1} \Delta(x_1, y_1, \ldots, y_{r-1}, x_k).$$

Further (if $r > 2$),

$$\sum_{l=1}^{n-1}\sum_{k=1}^{n-1} \Delta(x_1, y_1, \ldots, y_{r-3}, x_l, x_k, x_n) = 0.$$

It follows that X_3 is generated over K by the set

$$\{\Delta(x_1, y_1, \ldots, y_r): x_1 < y_1 < \ldots < y_r < x_n\} \cup \left\{\sum_{k=1}^{n-1} \Delta(x_1, y_1, \ldots, y_{r-2}, x_k, x_n):\right.$$
$$\left. x_1 < y_1 < \ldots < y_{r-2} < x_{n-1}\right\}.$$

(In the case in which $r=2$, the second set in this union consists of the element $\sum_{k=1}^{n-1}\Delta(x_1, x_k, x_n)$ alone.) It is easy to check that this set is linearly independent over K because

$$\{\Delta(x_1, y_1, \ldots, y_r) : x_1 < y_1 < \ldots < y_r \leqq x_n\}$$

is linearly independent over K. We therefore have a K-basis for X_3 which yields a K-basis for X_3/X_2. We have now constructed a Φ_{n-1}-composition series (5) for $S(r, n)$.

Let M be a non-zero Φ_n-submodule of $S(r, n)$, and consider the Φ_{n-1}-submodule $Y = M \cap X_2$. There are three possibilities: $Y=0$, $0 \subset Y \subset X_2$, or $Y = X_2$. In the third case, $X_2 \subseteq M$; whence $\Delta(x_1, \ldots, x_{r+1}) \in M$. Since M is a Φ_n-submodule of $S(r, n)$, this implies that $M = S(r, n)$. We shall prove that the other conditions are impossible by deducing a contradiction from each. It will follow from this that $S(r, n)$ is an irreducible Φ_n-module.

First suppose that $0 \subset Y \subset X_2$ (this does not arise if $r = n-2$). By Propositions 1 and 4, $Y = X_1$. It follows that $X_1 \subseteq M$. Now $X_1 = \ker\theta^r = \operatorname{Im}\theta^{r-1}$, and hence M contains

$$\theta^{r-1}(\Delta(x_1, x_2, \ldots, x_r)) = \sum_{k=1}^{n-1}\Delta(x_1, x_2, \ldots, x_r, x_k).$$

Applying the permutation $(x_n x_{n-1})$ to this element gives

$$\sum_{\substack{k=1 \\ k\neq n-1}}^{n} \Delta(x_1, x_2, \ldots, x_r, x_{n-1}).$$

The difference of these two, namely,

$$\Delta(x_1, \ldots, x_r, x_n) - \Delta(x_1, \ldots, x_r, x_k) \tag{6}$$

belongs to M. Applying $(x_r x_n)$ to (6) and adding the result to (6), we find that

$$-\Delta(x_1, \ldots, x_{r-1}, x_n, x_{n-1}) - \Delta(x_1, \ldots, x_{r-1}, x_r, x_{n-1})$$

belongs to M. Apply $(x_r x_{n-1})$ to this element and add the result to (6); this yields $2\Delta(x_1, \ldots, x_{r-1}, x_r, x_n) \in M$. Since $p \neq 2$, $\Delta(x_1, \ldots, x_r, x_n) \in M$, whence $M = S(r, n)$. This contradicts $Y = M \cap X_2 \subset X_2$.

For the final part of the proof, suppose that $Y = 0$. Then $M + X_2$ (direct sum) is a Φ_{n-1}-submodule of $S(r, n)$ containing X_2 properly. Suppose first that $M + X_2$ is a proper submodule of $S(r, n)$ (if $r = 1$, this situation does not arise). Then, by Propositions 1 and 4,

$$\frac{M+X_2}{X_2} = \frac{X_3}{X_2},$$

i.e. $M + X_2 = X_3$. Let $q: X_3 \to X_2$ be the projection onto $X_2 = S(r, n-1)$, and write $\sum_{k=1}^{n-1}\Delta(x_1, \ldots, x_{r-1}, x_k, x_n) = \xi \in X_3$. Since $q(\xi) \in X_2$, we have

$$q(\xi) = \sum_{1 < i_1 < \ldots < i_r < n} \lambda_{i_1 \ldots i_r}\Delta(x_1, x_{i_1}, \ldots, x_{i_r}),$$

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where each $\lambda_{i_1 \ldots i_r} \in K$. In each sequence $(i_1, \ldots, i_r)$ with $1 < i_1 < \ldots < i_r < n$, both $i_r > r-1$ and $i_{r-1} > r-1$. Since $(x_{i_r} x_{i_{r-1}})\xi = \xi$, and because q is a Φ_{n-1}-homomorphism, we have $(x_{i_r} x_{i_{r-1}})q(\xi) = q(\xi)$. Equating coefficients of $\Delta(x_1, x_{i_1}, \ldots, x_{i_r})$, we have $\lambda_{i_1 \ldots i_r} = 0$. Thus $q(\xi) = 0$, and this means that $\xi \in M$, i.e.

$$\sum_{k=1}^{n-1} \Delta(x_1, \ldots, x_{r-1}, x_k, x_n) \in M.$$

Apply the permutation (x_1, x_l) to this element, $l \neq 1, 2, \ldots, r-1, n$. We have by (1)

$$\begin{aligned}\sum_{k=1}^{n-1} \Delta(x_l, x_2, \ldots, x_{r-1}, x_k, x_n) = {} & \sum_{k=1}^{n-1} \Delta(x_1, x_2, \ldots, x_{r-1}, x_k, x_n) \\ & + \sum_{k=1}^{n-1} \sum_{s=2}^{r-1} (-1)^{s-1} \Delta(x_1, x_l, x_2, \ldots, \hat{x}_s, \ldots, x_{r-1}, x_k, x_n) \\ & + (n-1)(-1)^{r-1} \Delta(x_1, x_l, x_2, \ldots, x_{r-1}, x_n) \\ & + (-1)^r \sum_{k=1}^{n-1} \Delta(x_1, x_l, x_2, \ldots, x_{r-1}, x_k).\end{aligned}$$

Since p divides $n-1$, $\sum_{k=1}^{n-1} \Delta(x_1, x_2, \ldots, x_{r-1}, x_k) \in M$. This element also belongs to $S(r, n-1)$, contradicting $M \cap S(r, n-1) = 0$.

This rules out the possibility that $M + X_2$ is a proper submodule of $S(r, n)$ i.e. $M + X_2 = S(r, n)$. Let $q_1 : S(r, n) \to M$ be the projection onto M. Then q_1 is a Φ_{n-1}-homomorphism. Define $q : S(r, n) \to M$ by

$$q(z) = n^{-1} \sum_{k=1}^{n} (x_n x_k) q_1((x_n x_k)z) \qquad (z \in S(r, n)).$$

Then q is a Φ_n-homomorphism, and $S(r, n) = M + \operatorname{Im} q$ (direct sum). This contradicts Proposition 1, and completes the proof of Theorem 2.

6. Hook representations and natural representations at characteristic 2. We begin this section by defining the natural representation modules. These are studied with a view to constructing irreducible and indecomposable representation modules of the symmetric groups.

Let K be an arbitrary field. The Φ_n-module $M^r(n)$ defined by

$$\begin{aligned} M^r(n) &= \Phi_n x_1 \ldots x_r \qquad (0 < r \leqq n), \\ M^0(n) &= K, \end{aligned}$$

is called the rth natural representation module of S_n (over K). A K-basis for $M^r(n)$ is given by

$$\{y_1 \ldots y_r : x_1 \leqq y_1 < \ldots < y_r \leqq x_n\}.$$

Thus $\dim_K M^r(n) = \binom{n}{r}$. For example, $M^n(n) \cong K = M^0(n)$.

Define the map $\omega_r: M^r(n) \to M^{n-r}(n)$ by

$$\omega_r(\sigma x_1 \dots x_r) = \sigma x_{r+1} \dots x_n \qquad (\sigma \in \Phi_n).$$

It is easy to check that ω_r is a well defined Φ_n-isomorphism, so that $M^r(n) \cong M^{n-r}(n)$. Thus, in studying the natural representation modules, it is sufficient to treat only the cases when $2r \leqq n$.

If $2r \leqq n$, $(n-r, r)$ is a partition of n. We denote by $S^r(n)$ the corresponding Specht module. Thus $S^r(n)$ consists of all K-linear combinations of polynomials of the form

$$(x_{a_1} - x_{b_1})(x_{a_2} - x_{b_2}) \dots (x_{a_r} - x_{b_r}),$$

where $a_1, \dots, a_r, b_1, \dots, b_r$ are all distinct. It is clear that $S^r(n) \subseteq M^r(n)$. A K-basis for $S^r(n)$ can be found in terms of the standard tableaux of the partition $(n-r, r)$.

Let $d: K[x_1, \dots, x_n] \to K[x_1, \dots, x_n]$ be the map defined in terms of the partial differential operators by

$$d = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}.$$

Restricting to $M^{r+1}(n)$, $r < n$, we obtain a Φ_n-homomorphism

$$d_r: M^{r+1}(n) \to M^r(n) \quad (r = 0, 1, \dots, n-1).$$

We clearly have

$$d_r(y_1 \dots y_{r+1}) = \sum_{j=1}^{r+1} y_1 \dots \hat{y}_j \dots y_{r+1},$$

where $\hat{y}_j$ is meant to indicate that y_j is missing from the product $y_1 \dots y_{r+1}$.

For the rest of this section we assume that K has characteristic 2. We intend to study the following sequence of Φ_n-modules and Φ_n-homomorphisms:

$$0 \to M^n(n) \xrightarrow{d_{n-1}} M^{n-1}(n) \to \dots \to M^2(n) \xrightarrow{d_1} M^1(n) \xrightarrow{d_0} M^0(n) \to 0. \tag{7}$$

Clearly $\ker d_{n-1} = 0$ and $\operatorname{Im} d_0 = M^0(n)$. We prove that (7) is exact. Obviously, because K has characteristic 2,

$$d_r d_{r+1} = 0 \qquad (r = 0, 1, \dots, n-2).$$

Thus $\operatorname{Im} d_{r+1} \subseteq \ker d_r$.

A K-basis for $S(r, n)$ is given by Proposition 2. Using this basis, we define a linear transformation $\eta_r: S(r, n) \to M^r(n)$ by

$$\eta_r(\Delta(x_1, y_1, \dots, y_r)) = d_r(x_1 y_1 \dots y_r),$$

extending to the whole of $S(r, n)$ by K-linearity. By (1)

$$\begin{aligned}
\eta_r(\Delta(y_1, \dots, y_{r+1})) &= \sum_{s=1}^{r+1} d(x_1 y_1 \dots \hat{y}_s \dots y_{r+1}) \\
&= x_1 d\left(\sum_{s=1}^{r+1} y_1 \dots \hat{y}_s \dots y_{r+1}\right) + \sum_{s=1}^{r+1} y_1 \dots \hat{y}_s \dots y_{r+1} \\
&= x_1 d^2(y_1 \dots y_{r+1}) + d(y_1 \dots y_{r+1}) \\
&= d(y_1 \dots y_{r+1}).
\end{aligned}$$

It follows that η_r is a Φ_n-homomorphism.

Suppose that

$$\eta_r\left(\sum_{1<i_1<\ldots<i_r\leqq n} \lambda_{i_1,\ldots,i_r}\Delta(x_1, x_{i_1}, \ldots, x_{i_r})\right) = 0,$$

where each $\lambda_{i_1,\ldots,i_r} \in K$. Then

$$\sum_{1<i_1<\ldots<i_r\leqq n} \lambda_{i_1,\ldots,i_r} d(x_1 x_{i_1} \ldots x_{i_r}) = 0,$$

i.e.

$$\sum_{1<i_1<\ldots<i_r\leqq n} \lambda_{i_1,\ldots,i_r} x_{i_1} \ldots x_{i_r} + \sum_{1<i_1<\ldots<i_r\leqq n} \lambda_{i_1,\ldots,i_r} x_1 d(x_{i_1} \ldots x_{i_r}) = 0.$$

Equating coefficients of $x_{i_1} \ldots x_{i_r}$, $1 < i_1 < \ldots < i_r \leqq n$, we find that $\lambda_{i_1,\ldots,i_r} = 0$. This proves that

$$\ker \eta_r = 0,$$

i.e. η_r is a monomorphism.

Now define $\alpha_r: M^{r+1}(n) \to S(r, n)$, $r = 0, 1, \ldots, n-1$, by

$$\alpha_r(y_1 \ldots y_{r+1}) = \Delta(y_1, \ldots, y_{r+1}) \qquad (1 \leqq y_1 < \ldots < y_{r+1} \leqq n),$$

extending to the whole of $M^{r+1}(n)$ by K-linearity. It is clear that α_r is a Φ_n-homomorphism, and $\operatorname{Im}\alpha_r = S(r, n)$, i.e. α_r is an epimorphism.

Obviously

$$d_r = \eta_r \alpha_r.$$

We have thus factorised d_r into the composition of an epimorphism and a monomorphism. It follows that $\operatorname{Im}\eta_r = \operatorname{Im} d_r$ and $\ker \alpha_r = \ker d_r$. We have already noted that $\operatorname{Im} d_{r+1} \subseteq \ker d_r$. We prove the reverse inclusion by counting dimensions. Thus

$$\dim_K \operatorname{Im} d_{r+1} = \dim_K \operatorname{Im} \eta_{r+1} = \dim_K S(r+1, n) = \binom{n-1}{r+1},$$

and

$$\dim_K \ker d_r = \dim_K M^{r+1}(n) - \dim_K \operatorname{Im} d_r = \binom{n}{r+1} - \binom{n-1}{r} = \binom{n-1}{r+1}.$$

Thus $\dim_K \operatorname{Im} d_{r+1} = \dim_K \ker d_r$, and it follows that $\operatorname{Im} d_{r+1} = \ker d_r$. This completes the proof of

THEOREM 3. *The sequence* (7) *of* Φ_n*-modules is exact.*

Suppose that $0 < 2r \leqq n$. It is clear that $S^r(n) \subseteq \ker d_{r-1} = \operatorname{Im} d_r$, so that $S^r(n) \subseteq \operatorname{Im}\eta_r$. We determine $\eta_r^{-1}(S^r(n))$.

Choose integers $a_1, \ldots, a_r, b_1, \ldots, b_r$ all different and satisfying $1 < a_i, b_j \leqq n$. Define the element $\Delta^{a_1,b_1,a_2,b_2,\ldots,a_r,b_r} \in S(r, n)$ by

$$\Delta^{a_1,b_1,\ldots,a_r,b_r} = \sum_c \Delta(x_1, x_{c_1}, \ldots, x_{c_r}),$$

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summing over all sequences $c = (c_1, \ldots, c_r)$ in which, for each i, either $c_i = a_i$ or $c_i = b_i$. By the definition of η_r,

$$\eta_r(\Delta^{a_1,b_1,\ldots,a_r,b_r}) = \sum_c \eta_r(\Delta(x_1, x_{c_1}, \ldots, x_{c_r}))$$
$$= \sum_c d(x_1 x_{c_1} \ldots x_{c_r})$$
$$= x_1 \sum_c d(x_{c_1} \ldots x_{c_r}) + \sum_c x_{c_1} \ldots x_{c_r}.$$

But

$$\sum_c x_{c_1} \ldots x_{c_r} = (x_{a_1}+x_{b_1})(x_{a_2}+x_{b_2}) \ldots (x_{a_r}+x_{b_r}).$$

Consequently

$$\eta_r(\Delta^{a_1,b_1,\ldots,a_r,b_r}) = (x_{a_1}+x_{b_1})(x_{a_2}+x_{b_2}) \ldots (x_{a_r}+x_{b_r}).$$

Note also that of k is different from $1, a_1, \ldots, a_r, b_1, \ldots, b_r$, then

$$\eta_r\left(\sum_c \Delta(x_k, x_{c_1}, \ldots, x_{c_r})\right) = (x_{a_1}+x_{b_1}) \ldots (x_{a_r}+x_{b_r}).$$

Thus, since η_r is a monomorphism,

$$\Delta^{a_1,b_1,\ldots,a_r,b_r} = \sum_c \Delta(x_k, x_{c_1}, \ldots, x_{c_r}).$$

We have now proved

THEOREM 4. *Let K be a field of characteristic 2, and choose r such that $0 < 2r \leqq n$. Then $S(r, n)$ contains a submodule isomorphic to $S^r(n)$ over Φ_n, namely the Φ_n-submodule generated by the elements $\Delta^{a_1,b_1,\ldots,a_r,b_r}$.*

Take $r = 1$. In this case $\Delta^{a_1,b_1} = (x_{a_1}+x_{b_1})$, and the submodule is $S(1, n)$. Note that $S^1(n) = S(1, n)$. If $r > 1$, the submodule is a proper submodule. The results of [3] yield the composition factors of $S(2, n)$ when the ground field has characteristic 2. The problem of analysing the hook representations is equivalent to the problem of analysing the natural representation modules.

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