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Hop Dominating Sets in Graphs Under Binary Operations

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Abstract. Let G be a (simple) connected graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. A set $S \subseteq V(G)$ is a hop dominating set of G if for each $v \in V(G) \setminus S$, there exists $w \in S$ such that $d_G(v, w) = 2$. The minimum cardinality of a hop dominating set of G, denoted by $\gamma_h(G)$, is called the hop domination number of G. In this paper we revisit the concept of hop domination, relate it with other domination concepts, and investigate it in graphs resulting from some binary operations.

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1. Introduction

Domination in graph and several variations of the concept have been widely studied by many researchers. The two books by Haynes et al. [3, 4] give an excellent treatment of the standard domination concept and some of its variants.

Recently, Natarajan and Ayyaswamy [6] introduced and studied the concept of hop domination in a graph. In another study, Ayyaswamy et al. [2] investigated the same concept and gave bounds of the hop domination number of some graphs. Henning and Rad [5] also studied the concept and answered a question posed by Ayyaswamy and Natarajan in [6]. They presented probabilistic upper bounds for the hop domination number and showed that the decision problems for the 2-step dominating set and hop dominating set problems are NP-complete for planar bipartite graphs and planar chordal graphs. Pabilona and Rara [7] considered the variant called connected hop domination and studied it in graphs under some binary operations.

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Let $G = (V(G), E(G))$ be a simple graph. The *open neighbourhood* of a vertex v of G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and its *closed neighbourhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. The degree of v, denoted by $deg_G(v)$, is equal to $|N_G(v)|$ and the maximum degree of G, denoted by $\Delta(G)$, is equal to $\max\{deg_G(v) : v \in V(G)\}$. The *open hop neighbourhood* of vertex v is the set $N_G(v, 2) = \{w \in V(G) : d_G(v, w) = 2\}$, where $d_G(v, w)$ denotes the distance between v and w (the length of a shortest path joining v and w). The *open neighbourhood* of a subset S of $V(G)$ is the set $N_G(S) = \cup_{v\in S} N_G(v)$ and its *closed neighbourhood* is the set $N_G[S] = N_G(S) \cup S$.

A set $S \subseteq V(G)$ is a *dominating set* (resp. *total dominating set*) of G if $N_G[S] = V(G)$ (resp. $N_G(S) = V(G)$). The smallest cardinality of a dominating (resp. total dominating) set of G, denoted by $\gamma(G)$ (resp. $\gamma_t(G)$), is called the *domination number* (resp. *total domination number*) of G. A dominating (resp. total dominating) set S of G with $|S| = \gamma(G)$ (resp. $|S| = \gamma_t(G)$), is called a *γ-set* (resp. γ_t-set) of G. It should be noted that only graphs without isolated vertices admit total dominating sets.

A set $S \subseteq V(G)$ is a *hop dominating set* (*total hop dominating set*) of G if for each $x \in V(G) \setminus S$ (resp. $x \in V(G)$), there exists $z \in S$ such that $d_G(x, z) = 2$. The smallest cardinality of a hop dominating (total hop dominating) set of G, denoted by $\gamma_h(G)$ (resp. $\gamma_{th}(G)$), is called the *hop domination number* (*total hop domination number*) of G. A hop dominating (total hop dominating) set S of G with $|S| = \gamma_h(G)$ (resp. $|S| = \gamma_{th}(G)$) is called a *γ_h-set* (resp. *γ_{th}-set*) of G.

A set $S \subseteq V(G)$ is a *$(1,2)^*$-dominating set* (resp. *$(1,2)^*$-total dominating set*) of G if it is a dominating (resp. total dominating) set of G and for each $x \in V(G) \setminus S$, there exists $z \in S$ such that $d_G(x, z) = 2$. The smallest cardinality of a $(1,2)^*$-dominating (resp. $(1,2)^*$-total dominating) set of G, denoted by $\gamma^*_{1,2}(G)$ (resp. $\gamma^{*t}_{1,2}(G)$), is called the *$(1,2)^*$-domination number* (resp. *$(1,2)^*$-total domination number*) of G. A $(1,2)^*$-dominating (resp. $(1,2)^*$- total dominating) set S with $|S| = \gamma^*_{1,2}(G)$ (resp. $|S| = \gamma^{*t}_{1,2}(G)$) is called a $\gamma^*_{1,2}$-set (resp. $\gamma^{*t}_{1,2}$-set) of G. Clearly, $S \subseteq V(G)$ is a $(1,2)^*$-dominating (resp. $(1,2)^*$-total dominating) set if and only if it is both a dominating (resp. total dominating) and a hop dominating set. The concept of $(1,2)^*$-domination (a variation of $(1,2)$-domination) is introduced and investigated in [1].

A set $D \subseteq V(G)$ is a *point-wise non-dominating set* of G if for each $v \in V(G) \setminus S$, there exists $u \in S$ such that $v \notin N_G(u)$. The smallest cardinality of a point-wise non-dominating set of G, denoted by $pnd(G)$, is called the *point-wise non-domination number* of G. A dominating set S which is also a point-wise non-dominating set of G is called a *dominating point-wise non-dominating set* of G. The smallest cardinality of a dominating point-wise non-dominating set of G will be denoted by $\gamma_{pnd}(G)$. Any point-wise non-dominating (resp. dominating point-wise non-dominating) set S of G with $|S| = pnd(G)$ (resp. $|S| = \gamma_{pnd}(G)$), is called a *pnd-set* (resp. *γ_{pnd}-set*) of G.

2. Results

The first result, which will be needed later, is found in [1].

Proposition 1. [1] *Let G be a graph. Then $1 \leq pnd(G) \leq |V(G)|$. Moreover,*

(i) *$pnd(G) = |V(G)|$ if and only if G is a complete graph;*

(ii) *$pnd(G) = 1$ if and only if G has an isolated vertex; and*

(iii) *$pnd(G) = 2$ if and only if G has no isolated vertex and there exist distinct vertices a and b of G such that $N_G(a) \cap N_G(b) = \varnothing$.*

The *join* of graphs G and H is the graph $G + H$ with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$.

Theorem 1. *Let G and H be any two graphs. A set $S \subseteq V(G + H)$ is hop dominating set of $G+H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are point-wise non-dominating sets of G and H, respectively.*

Proof. Suppose that S is a hop dominating set of $G + H$. Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. If S_G were empty, then $S = S_H$. Since $V(G) \subseteq N_G(S)$, it follows that S is not a hop dominating set, a contradiction. Thus, $S_G \neq \varnothing$. Similarly, $S_H \neq \varnothing$. Now let $v \in V(G) \setminus S_G$. Since S is hop dominating set, there exists $z \in S$ such that $d_{G+H}(v, z) = 2$. Hence, $z \in S_G$ and $v \notin N_G(z)$. This shows that S_G is a point-wise non-dominating set of G. Similarly, S_H is a point-wise non-dominating set of H.

For the converse, suppose that $S = S_G \cup S_H$, where S_G and S_H are point-wise non-dominating sets of G and H, respectively. Let $v \in V(G + H) \setminus S$. If $v \in V(G)$, then $v \in N_{G+H}(S_H)$. Since S_G is a point-wise non-dominating set of G, there is a vertex $y \in S_G \setminus N_G(v)$. It follows that $d_{G+H}(v, y) = 2$. The same argument can be used if $v \in V(H)$. Therefore S is a hop dominating set of $G + H$. □

The next result is a consequence of Theorem 1 and Proposition 1

Corollary 1. *Let G and H be any two graphs of orders m and n, respectively. Then*

$$\gamma_h(G + H) = pnd(G) + pnd(H).$$

In particular,

(i) *$\gamma_h(G + H) = m + n$ if G and H are complete;*

(ii) *$\gamma_h(G + H) = 2$ if G and H have isolated vertices;*

(iii) *$\gamma_h(G + H) = 1 + pnd(H)$ if $G = K_1$;*

(iv) *$\gamma_h(G + H) = 4$ if $G = P_m$ and $H = P_n$ $(m, n \geq 2)$; and*

(v) *$\gamma_h(G + H) = 4$ if $G = C_m$ and $H = C_n$ $(m, n \geq 4)$.*

The *corona* of graphs G and H, denoted by $G \circ H$, is the graph obtained from G by taking a copy H^v of H and forming the join $\langle v\rangle + H^v = v + H^v$ for each $v \in V(G)$.

Theorem 2. *Let G and H be any two graphs. A set $C \subseteq V(G \circ H)$ is a hop dominating set of $G \circ H$ if and only if*

$$C = A \cup (\cup_{v \in V(G) \cap N_G(A)} S_v) \cup (\cup_{w \in V(G) \setminus N_G(A)} E_w),$$

where

(*i*) *$A \subseteq V(G)$ such that for each $w \in V(G) \setminus A$, there exists $x \in A$ with $d_G(w, x) = 2$ or there exists $y \in V(G) \cap N_G(w)$ with $V(H^y) \cap C \neq \varnothing$,*

(*ii*) *$S_v \subseteq V(H^v)$ for each $v \in V(G) \cap N_G(A)$, and*

(*iii*) *$E_w \subseteq V(H^w)$ is a point-wise non-dominating set of H^w for each $w \in V(G) \setminus N_G(A)$.*

Proof. Suppose C is a hop dominating set of $G \circ H$ and set $A = C \cap V(G)$. Let $w \in V(G) \setminus A$. Then there exists $x \in C$ such that $d_{G \circ H}(w, x) = 2$. If $x \in A$, then $d_G(w, x) = 2$. Suppose that $x \notin A$. Then there exists $y \in V(G)$ such that $x \in V(H^y)$. Since $d_{G \circ H}(w, x) = 2$, it follows that $y \in N_G(w)$. Thus, (i) holds. Let $v \in V(G)$. Set $S_v = C \cap V(H^v)$ if $v \in V(G) \cap N_G(A)$ and $E_w = C \cap V(H^w)$ if $v \in V(G) \setminus N_G(A)$. Then, clearly, $S_v \subseteq V(H^v)$ and $E_w \subseteq V(H^w)$. Suppose that $w \in V(G) \setminus N_G(A)$ and let $q \in V(H^w) \setminus E_w$. Since C is a hop dominating set of $G \circ H$, there exists $u \in C$ such that $d_{G \circ H}(q, u) = 2$. By assumption, $u \notin A$. Thus, $u \in E_w$ and $qu \notin E(H^w)$. Therefore E_w is a point-wise non-dominating set of H^w, showing that (iii) holds.

For the converse, suppose that C has the given form and satisfies properties (i), (ii), and (iii). Let $z \in V(G \circ H) \setminus C$ and let $v \in V(G)$ such that $z \in V(v + H^v)$. Consider the following cases:

Case 1. $z = v$

Then $z \notin A$. From the assumption that (i) holds, it follows that there exists $y \in C$ such that $d_{G \circ H}(z, y) = 2$.

Case 2. $z \neq v$

Then $z \in V(H^v)$. If $v \in N_G(A)$, say $vw \in E(G)$ for some $w \in A$, then $d_{G \circ H}(z, w) = 2$. Suppose that $v \notin N_G(A)$. Then $z \in V(H^v) \setminus E_v$ where E_v is a point-wise non-dominating set of H^v by property (iii). Thus, there exists $p \in E_v \subset C$ such that $d_{G \circ H}(x, p) = 2$.

Accordingly, C is a hop dominating set of $G \circ H$. □

Corollary 2. *Let G be a connected non-trivial graph and let H be any graph. Then:*

(*i*) $\gamma_h(G \circ H) \leq \min\{\gamma^{*t}_{1,2}(G), [1 + pnd(H)]\gamma(G)\}$.

(*ii*) $\gamma_h(G \circ H) = 2$ *if* $\gamma^{*t}_{1,2}(G) = 2$.

(*iii*) $\gamma_h(G \circ H) = 2$ *if* $\gamma(G) = 1$ *and H has an isolated vertex.*

Let A be a $\gamma^{*t}_{1,2}$-set of G. Since A is a total dominating set of G, $V(G) \setminus N_G(A) = \varnothing$. Let $w \in V(G) \setminus A$. Since A is a hop dominating set of G, there exists $x \in A$ such that $d_G(x, w) = 2$. Setting $S_v = \varnothing$ for each $v \in A \cap N_G(A) = A$, we find that $C = A$ satisfies conditions (i), (ii), and (iii) of Theorem 2. Thus, $C = A$ is a hop dominating set of $G \circ H$ and $\gamma_h(G \circ H) \leq |C| = |A| = \gamma^{*t}_{1,2}(G)$.

Next, let A_0 be a γ-set of G and let D_0 be a *pnd*-set of H. Set $S_v = D_v$, where $D_v \subseteq V(H^v)$ and $\langle D_v \rangle \cong \langle D \rangle$, for each $v \in A_0$. Since A_0 is a dominating set of G, $w \in N_G(A_0)$ for each $w \in V(G) \setminus A_0$ (hence, $[V(G) \setminus A_0] \setminus N_G(A_0) = \varnothing$). Thus, by Theorem 2, $C_0 = A_0 \cup (\cup_{u \in A_0} S_v)$ is a hop dominating set of $G \circ H$, and $\gamma_h(G \circ H) \leq |C_0| = |A_0| + |A_0|.pnd(H) = [1 + pnd(H)]\gamma(G)$. Therefore,

$$\gamma_h(G \circ H) \leq \min\{\gamma^{*t}_{1,2}(G), [1 + pnd(H)]\gamma(G)\},$$

showing that (i) holds. Statements (ii) and (iii) are immediate from (i) and the fact that $\gamma_h(G \circ H) \geq 2$. □

Observation: The bound given in Corollary 2(i) is attainable (as given in (ii) and (iii)). It can also be verified easily that $\gamma_h(C_5 \circ P_3) = \gamma^{*t}_{1,2}(C_5) = 3 < 6 = [1 + pnd(P_3)]\gamma(C_5)$ and $\gamma_h(K_4 \circ P_3) = [1 + pnd(P_3)]\gamma(K_4) = 3 < 4 = \gamma^{*t}_{1,2}(K_4)$. It is worth noting that the inequality is also attainable. As a matter of fact, it can be shown that $\gamma_h(K_5 \circ K_4) = 3 < 5 = \min\{[1 + pnd(K_4)]\gamma(K_5), \gamma^{*t}_{1,2}(K_5)\}$.

The *lexicographic product* of graphs G and H, denoted by $G[H]$, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ such that $(v, a)(u, b) \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $ab \in E(H)$. Note that every non-empty subset C of $V(G) \times V(H)$ can be expressed as $C = \cup_{x \in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$.

Theorem 3. *Let G and H be connected non-trivial graphs. A subset $C = \cup_{x \in S}[\{x\} \times T_x]$ of $V(G[H])$ is a hop dominating set of $G[H]$ if and only if the following conditions hold:*

(i) S is a hop dominating set of G;

(ii) T_x is a point-wise non-dominating set of H for each $x \in S$ with $|N_G(x, 2) \cap S| = 0$.

Proof. Suppose C is a hop dominating set of $G[H]$. Let $u \in V(G) \setminus S$ and pick any $a \in V(H)$. Since C is a hop dominating set and $(u, a) \notin C$, there exists $(y, b) \in C$ such that $d_{G[H]}((u, a)(y, b)) = 2$. This implies that $y \in S$ and $d_G(u, y) = 2$. Since u was arbitrarily chosen, it follows that S is a hop dominating set of G. Thus, (i) holds.

Now let $x \in S^*$ and let $p \in V(H) \setminus T_x$. Then $(x, p) \notin C$. Again, noting that C is a hop dominating set of $G[H]$, there exists $(z, q) \in C$ such that $d_{G[H]}((x, p)(z, q)) = 2$. By the assumption that $x \in S^*$, we find that $x = z$. Hence, $q \in T_x$ and $q \notin N_H(p)$. Thus, T_x is a point-wise non-dominating set of H, showing that (ii) holds.

For the converse, suppose that C satisfies properties (i) and (ii). Let $(v, t) \in V(G[H]) \setminus C$ and consider the following cases:

Case 1. $v \notin S$

Since S is a hop dominating set of G, there exists $w \in S$ such that $d_G(v, w) = 2$. Pick any $d \in T_w$. Then $(w, d) \in C$ and $d_{G[H]}((v, t)(w, d)) = 2$.

Case 2. $v \in S$

If $v \notin S^*$, then there exists $z \in S$ such that $d_G(v, z) = 2$. It follows that $d_{G[H]}((v, t)(z, a)) = 2$ for any $a \in T_z$. Suppose that $v \in S^*$. Then, by property (ii), there exists $c \in T_v$ such that $tc \notin E(H)$. Since G is non-trivial and connected, $d_{G[H]}((v, t)(v, c)) = 2$.

Accordingly, C is a hop dominating set of $G[H]$. □

Lemma 1. *A non-trivial graph G admits a total hop dominating set if and only if $\gamma(C) \neq 1$ for every component C of G.*

Proof. Suppose G admits a total hop dominating set, say S. Suppose further that there exists a component C of G such that $\gamma(C) = 1$. Let $v \in V(C)$ be such that $\{v\}$ is a dominating set of C. Since S is a hop dominating set of G, $v \in S$. This, however, contradicts the fact that S is a total hop dominating set. Thus, $\gamma(C) \neq 1$ for every component C of G.

For the converse, suppose that $\gamma(C) \neq 1$ for every component C of G. Clearly, $S = V(G)$ is a hop dominating set of G. Let $w \in V(G)$ and C_w be the component of G with $w \in V(C_w)$. Since $\{w\}$ is not a dominating set of C_w, there exists $u \in V(C) \setminus \{w\}$ such that $d_C(u, w) = d_G(u, w) = 2$. This shows that $S = V(G)$ is a total hop dominating set of G. □

Theorem 4. *Let G be a connected graph with $\gamma(G) \neq 1$. If S is a hop dominating set of G, then $\gamma_{th}(G) \leq |S \cap N_G(S, 2)| + 2|S \setminus N_G(S, 2)|$. Moreover, $\gamma_{th}(G) \leq 2\gamma_h(G)$.*

Proof. Let S be a hop dominating set of G. If S is a total hop dominating set of G (possible by Lemma 1), then $S \cap N_G(S, 2) = S$ and $S \setminus N_G(S, 2) = \varnothing$. Hence, the inequality holds. Suppose now that S is not a total hop dominating set. Then $S \setminus N_G(S, 2) \neq \varnothing$. Let $x \in S \setminus N_G(S, 2)$. Then, since $\gamma(G) \neq 1$, there exists $v_x \in V(G) \setminus S$ such that $d_G(x, v_x) = 2$. Let $D_S = \{v_x : x \in S \setminus N_G(S, 2)\}$. Then, clearly, $|D_S| \leq |S \setminus N_G(S, 2)|$ and $S^* = S \cup D_S$ is a total hop dominating set of G. Thus,

$$\gamma_{th}(G) \leq |S^*| \leq |S \cap N_G(S, 2)| + 2|S \setminus N_G(S, 2)|.$$

In particular, $\gamma_{th}(G) \leq 2\gamma_h(G)$. □

In what follows, $\rho_H(G) = \min\{|S \cap N_G(S, 2)| + pnd(H)|S \setminus N_G(S, 2)| : S \text{ is a hop dominating set of } G\}$.

Corollary 3. *Let G and H be non-trivial connected graphs of orders m and n, respectively. Then*

(i) $\gamma_h(G[H]) = \rho_H(G)$ *if* $\gamma(G) = 1$;

(ii) $\gamma_h(G[H]) = \gamma_{th}(G)$ *if* $\gamma(G) \neq 1$; *and*

(iii) $\gamma_h(G[H]) = m[pnd(H)]$ *if* $G = K_m$.

Proof. (i) Suppose first that $\gamma(G) = 1$. Then, by Lemma 1, G does not admit a total hop dominating set (hence, $\gamma_h(G[H]) \neq \gamma_{th}(G)$). Now let S' be a hop dominating set of G such that $\rho_H(G) = |S' \cap N_G(S', 2)| + pnd(H)|S' \setminus N_G(S', 2)|$, and let D' be a *pnd*-set of H. Set $Q_x = D'$ for each $x \in S' \setminus N_G(S', 2)$ and $Q_y = \{q\}$, where $q \in V(H)$, for each $y \in S' \cap N_G(S', 2)$. Then $C' = \cup_{x \in S'}[\{x\} \times Q_x]$ is a hop dominating set of $G[H]$ by Theorem 3. Hence,

$$\gamma_h(G[H]) \leq |C'| = \sum_{x \in S' \cap N_G(S',2)} |Q_x| + \sum_{x \in S' \setminus N_G(S',2)} |Q_x| = \rho_H(G).$$

Next, suppose that $C_0 = \cup_{x \in S_0}[\{x\} \times T_x]$ is a γ_h-set of $G[H]$. By Theorem 3, S_0 is a hop dominating set of G and T_x is a *pnd*-set of H for each $x \in S_0 \setminus N_G(S_0, 2)$. Clearly, $|T_x| = 1$ for all $x \in S_0 \cap N_G(S_0, 2)$. Hence,

$$\gamma_h(G[H]) = |C_0| = |S_0 \cap N_G(S_0, 2)| + pnd(H)|S_0 \setminus N_G(S_0, 2)| \geq \rho_H(G),$$

showing that equality in (i) holds.

(ii) Suppose that $\gamma(G) \neq 1$. Then G admits a total hop dominating set by Lemma 1. Let S be a γ_{th}-set of G and let $D = \{a\}$, where $a \in V(H)$. Set $T_x = D$ for each $x \in S$. Then $C = \cup_{x \in S}[\{x\} \times T_x] = S \times D$ is a hop dominating set of $G[H]$ by Theorem 3. Hence,

$$\gamma_h(G[H]) \leq |S||D| = \gamma_{th}(G).$$

Next, suppose that $C^* = \cup_{x \in S^*}[\{x\} \times R_x]$ is a γ_h-set of $G[H]$. By Theorem 3, S^* is a hop dominating set of G and R_x is a *pnd*-set of H for each $x \in S^* \setminus N_G(S^*, 2)$. Since C^* is a γ_h-set, $|R_x| = 1$ for all $x \in S^* \cap N_G(S^*, 2)$. Moreover, since H is a non-trivial connected graph, $|R_x| = pnd(H) \geq 2$ for each $x \in S^* \setminus N_G(S^*, 2)$ by Proposition 1(ii). Thus, by Theorem 4,

$$\gamma_h(G[H]) = |C^*| \geq |S^* \cap N_G(S^*, 2)| + 2|S^* \setminus N_G(S^*, 2)| \geq \gamma_{th}(G).$$

This establishes the desired equality in (ii).

(iii) Suppose that $G = K_m$. Since $\gamma(G) = 1$, $\gamma_h(G[H]) = \rho_H(G)$. Now, since $S = V(K_m)$ is the only hop dominating set of G, it follows that

$$\gamma_h(G[H]) = \rho_H(G) = m[pnd(H)].$$

This proves the assertion in (iii). □

Corollary 4. *Let G be a non-trivial connected graph and let H be any non-trivial graph. If H has an isolated vertex, then $\gamma_h(G[H]) = \gamma_h(G)$.*

Proof. Since H has an isolated vertex, $pnd(H) = 1$ by Proposition 1(ii). Let $C = \cup_{x \in S}[\{x\} \times T_x]$ be a γ_h-set of $G[H]$. By Theorem 3, S is a hop dominating set of G and T_x is a *pnd*-set of H for each $x \in S \setminus N_G(S, 2)$. Further, since C is γ_h-set, $|T_x| = 1$ for all $x \in S \cap N_G(S, 2)$. Hence,

$$\gamma_h(G[H]) = |C| = |S \cap N_G(S, 2)| + |S \setminus N_G(S, 2)| = |S| \geq \gamma_h(G).$$

Now if S_0 is a γ_h-set of G and D_0 is a *pnd*-set of H, then $C_0 = S_0 \times D_0$ is a γ_h-set of $G[H]$ by Theorem 3. Thus, $\gamma_h(G[H]) \leq |C_0| = |S_0||D_0| = |S_0| = \gamma_h(G)$. This establishes the desired equality. □

The *Cartesian product* of graphs G and H, denoted by $G\Box H$, is the graph with vertex set $V(G\Box H) = V(G) \times V(H)$ such that $(v,p)(u,q) \in E(G\Box H)$ if and only if $uv \in E(G)$ and $p = q \in E(H)]$ or $u = v$ and $pq \in E(H)$.

Theorem 5. *Let G and H be connected non-trivial graphs. A subset $C = \cup_{x\in S}[\{x\} \times T_x]$ of $V(G\Box H)$ is a hop dominating set of $G\Box H$ if and only if the following conditions hold:*

(i) *For each $x \in V(G)\backslash S$ and for each $p \in V(H)$, at least one of the following statements is satisfied:*

- (a) *There exists $y \in S \cap N_G(x)$ such that $T_y \cap N_H(p) \neq \varnothing$.*
- (b) *There exists $z \in S \cap N_G(x,2)$ such that $p \in T_z$.*

(ii) *For each $v \in S$ and for each $p \in V(H) \setminus T_v$, at least one of the following statements is satisfied:*

- (c) *$N_H(p,2) \cap T_v \neq \varnothing$.*
- (d) *There exists $y \in S \cap N_G(v)$ such that $T_y \cap N_H(p) \neq \varnothing$.*
- (e) *There exists $z \in S \cap N_G(v,2)$ such that $p \in T_z$.*

Proof. Suppose C is a hop dominating set of $G\Box H$. Let $x \in V(G) \setminus S$ and let $p \in V(H)$. Since C is a hop dominating set and $(x,p) \notin C$, there exists $(y,q) \in C$ such that $d_{G\Box H}((x,p)(y,q)) = 2$. Since $y \in S$, $x \neq y$. If $xy \in E(G)$, then $pq \in E(H)$. Hence, $q \in T_y \cap N_H(p)$, showing that (a) holds. So suppose that $y \notin N_G(x)$. Since $d_{G\Box H}((x,p)(y,q)) = 2$, it follows that $y \in N_G(x,2)$ and $p = q$. Hence, $p \in T_y$, showing that (b) holds.

Next, let $v \in S$ and let $p \in V(H) \setminus T_v$. Since C is a hop dominating set and $(v,p) \notin C$, there exists $(w,q) \in C$ such that $d_{G\Box H}((v,p)(w,q)) = 2$. Suppose that ($d$) and ($e$) do not hold. Then, since $d_{G\Box H}((v,p)(w,q)) = 2$, $v = w$ and $d_H(p,q) = 2$. Thus, $q \in T_v \cap N_H(p,2)$, showing that (c) holds.

For the converse, suppose that C satisfies properties (i) and (ii). Let $(v,t) \in V(G[H]) \setminus C$ and consider the following cases:

Case 1. $v \notin S$

If (a) of (i) holds, then there exist $y \in S \cap N_G(v)$ and $h \in T_y \cap N_H(p)$. Hence, $(y,h) \in C \cap N_{G\Box H}((v,t),2)$. If ($b$) of ($i$) holds, then there exists $z \in S \cap N_G(v,2)$ such that $t \in T_z$. It follows that $(z,t) \in C \cap N_{G\Box H}((v,t),2)$.

Case 2. $v \in S$

Then $t \notin T_v$. If (c) of (ii) holds, then we may take any $q \in N_H(t,2) \cap T_v$. Clearly, $(v,q) \in C \cap N_{G\Box H}((v,t),2)$. As in the first case, if (d) or (e) of (ii) holds, then there exists $(w,h) \in C \cap N_{G\Box H}((v,t),2)$.

Accordingly, C is a hop dominating set of $G\Box H$. □

Corollary 5. *Let G and H be non-trivial connected graphs. Then*

$$\gamma_h(G\square H) \leq \min\{\gamma(G)\gamma^{*t}_{1,2}(H), \gamma(H)\gamma^{*t}_{1,2}(G)\}.$$

Proof. Let S be a γ-set of G and let D be a $\gamma^{*t}_{1,2}$-set of H. Set $T_x = D$ for each $x \in S$ and let $C = \cup_{x\in S}[\{x\} \times T_x] = S \times D$. Let $x \in V(G) \setminus S$ and let $p \in V(H)$. Since S is a dominating set of G, there exists $y \in S \cap N_G(x)$. Now, since $T_y = D$ is a total dominating set of H, there exists $q \in T_y \cap N_H(p)$. Thus, (a) of property (i) of Theorem 5 holds. Next, let $v \in S$ and let $t \in V(H) \setminus T_v$. Since $T_v = D$ is a hop dominating set of H, $T_v \cap N_H(t, 2) \neq \varnothing$. Hence, (c) of property (ii) of Theorem 5 holds. Therefore, by Theorem 5, C is a hop dominating set of $G\square H$. Thus, $\gamma_h(G\square H) \leq |C| = \gamma(G)\gamma^{*t}_{1,2}(H)$. This proves the assertion. □

Remark 1. *The bound given in Corollary* 5 *is tight. Moreover, the inequality is also attainable.*

To see this, consider $P_3\square P_4$ and $P_4\square P_4$. It can easily be verified that $\gamma_h(P_3\square P_4) = 2 = \gamma(P_3)\gamma^{*t}_{1,2}(P_4)$ and $\gamma_h(P_4\square P_4) = 4 = \gamma(P_4)\gamma^{*t}_{1,2}(P_4)$. The inequality is attainable since $\gamma_h(K_4\square K_4) = 3 < 4 = \gamma(K_4)\gamma^{*t}_{1,2}(K_4)$.

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