Hopf Bifurcation Analysis and Approximation of Limit Cycle in Coupled Van Der Pol and Duffing Oscillators

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Abstract: Using Routh-Hurwitz criterion and Hopf bifurcation theorem, stability of the equilibrium points and Hopf bifurcation of coupled van der Pol and Duffing oscillators are investigated. The parametric curve corresponding to Hopf bifurcation and the existence region for the limit cycle in the parameter space are derived. The method of multiple time scales is used to deduce the analytical approximation of the limit cycle, and the accuracy of the analytical approximation is verified by direct numerical integration.

INTRODUCTION

Duffing oscillator and van der Pol oscillator are two classical nonlinear oscillators. Their dynamics are very rich in terms of stability, bifurcation and chaos. Since the last decade, much attention has been paid to the study of coupled oscillators, such as two coupled Duffing oscillators [1, 2] and two coupled van der Pol oscillators [3-5]. The reason for this study is twofold. On the one hand, various physical, electromechanical and biological systems can be described as a collection of coupled oscillators [6]. On the other hand, dynamical systems consisting of a collection of coupled oscillators are higher-dimensional and their dynamics is more complex and hence need further investigation. But, as pointed out by Chedjou *et al*. [7], less effort had been done in a system consisting of a self-excited oscillator and Duffing oscillator.

In this paper, a dynamical system consisting of coupled van der Pol and Duffing oscillators is discussed as follows

$$\begin{cases} x'' + \omega_1^2 x + \mu_1(1 - x^2)x' + fy'' = 0 & (1) \\ y'' + \omega_2^2 y + \mu_2 y' + cy^3 - dx = 0 & (2) \end{cases}$$

where the prime denotes the derivative with respective to time and $\omega_i, \mu_i (i = 1, 2), c, d$ and f are the parameters. Equations (1)-(2) can be used to model a self-sustained electromechanical transducer consisting of an electrical part and a mechanical part coupled through Laplace force and Lenz electromotive voltage, which was proposed by Chedjou *et al*. [7]. In the absence of the coupling, i.e., $f = d = 0$, equations (1) and (2) reduce to the classical van der Pol oscillator and the damped Duffing oscillator without excitation respectively. Chedjou *et al*. [7] analyzed stability of the equilibriums of this equations using Routh-Hurwitz criterion and investigated the oscillatory state using the averaging method.

Shilnikov chaos of equations (1)-(2) was also studied by applying Shilnikov theorem. However, Hopf bifurcation and the parametric curve in the parametric space corresponding to Hopf bifurcation were not discussed in [7].

Our attention is addressed to Hopf bifurcation analysis of equations (1)-(2) and the existence and approximation of the limit cycle. The paper is arranged as follows. In section 2, stability of the equilibriums of equations (1)-(2) is analyzed. In section 3, Hopf bifurcation of equations (1)-(2) is analyzed and the parametric curve for Hopf bifurcation and the existence region for the limit cycle in the parameter space are obtained. In section 4, the method of multiple scales is used to deduce the analytical approximation for the limit cycle. The accuracy of the analytical approximation is compared with the results of direct numerical integration. The final section is devoted to some conclusions.

STABILITY OF EQUILIBRIUM POINTS

Equations (1)-(2) can be rewritten in the form

$$x_1' = x_2, \tag{3a}$$

$$\begin{aligned} x_2' = &-\mu_1(1 - x_1^2)x_2 - (\omega_1^2 + fd)x_1 \\ &+ f\omega_2^2 y_1 + fcy_1^3 + f\mu_2 y_2, \end{aligned} \tag{3b}$$

$$y_1' = y_2, \tag{3c}$$

$$y_2' = -\mu_2 y_2 - \omega_2^2 y_1 - cy_1^3 + dx_1. \tag{3d}$$

From equations (3), it is easy to verify that

1. For $c \geq 0$, $(x_{10}, x_{20}, y_{10}, y_{20}) = (0, 0, 0, 0)$ is the only equilibrium point of equations (1)-(2),

2. For $c < 0$, besides $(0, 0, 0, 0)$,

    $(x_{10}, x_{20}, y_{10}, y_{20}) = (0, 0, \pm\dfrac{\omega_2}{\sqrt{-c}}, 0)$ are also the equilibrium points of equations (1)-(2).

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Linearizing equations (3) around a considered equilibrium point $(x_{10}, x_{20}, y_{10}, y_{20})$ results in the following Jocabian matrix

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(2\mu_1 x_{10} x_{20} + \omega_1^2 + fd) & \mu_1(1 - x_{10}^2) & f(\omega_2^2 + 3cy_{10}^2) & \mu_2 f \\ 0 & 0 & 0 & 1 \\ d & 0 & -(\omega_2^2 + 3cy_{10}^2) & -\mu_2 \end{bmatrix}. \quad (4)$$

The eigenvalues of M are the roots of the following characteristic polynomial

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \quad (5)$$

in which

$$a_1 = \mu_2 - \mu_1(1 - x_{10}^2), \quad (6a)$$

$$a_2 = \omega_1^2 + \omega_2^2 + fd - \mu_2\mu_1(1 - x_{10}^2)$$
$$+ 3cy_1^2 + 2\mu_1 x_{10} x_{20}, \quad (6b)$$

$$a_3 = \mu_2(2\mu_1 x_{10} x_{20} + \omega_1^2)$$
$$- \mu_1(1 - x_{10}^2)(\omega_2^2 + 3cy_{10}), \quad (6c)$$

$$a_4 = (\omega_2^2 + 3cy_{10}^2)(\omega_1^2 + 2\mu_1 x_{10} x_{20}). \quad (6d)$$

According to Routh-Hurwitz criterion, M is a Hurwitz matrix and the considered equilibrium point $(x_{10}, x_{20}, y_{10}, y_{20})$ is stable if the following inequalities hold,

$$a_i (i = 1, 2, 3, 4) > 0,$$

$$\Delta_2 = a_1 a_2 - a_3 > 0,$$

and

$$\Delta_3 = a_1 a_2 a_3 - a_3^2 - a_1^2 a_4 > 0.$$

HOPF BIRURCATION ANALYSIS

In what follows, the parameter $c \geq 0$ is pre-assumed, consequently, equation (1)-(2) process only an equilibrium point $(0,0,0,0)$. For the equilibrium point $(0,0,0,0)$, equations (6) reduce to

$$a_1 = \mu_2 - \mu_1, \quad (7a)$$

$$a_2 = \omega_1^2 + \omega_2^2 + fd - \mu_2\mu_1, \quad (7b)$$

$$a_3 = \mu_2\omega_1^2 - \mu_1\omega_2^2, \quad (7c)$$

$$a_4 = \omega_2^2\omega_1^2. \quad (7d)$$

Thus, according to Routh-Hurwitz criterion, the equilibrium point $(0,0,0,0)$ is stable if

$$\mu_2 > \mu_1 \text{ and } \mu_2\mu_1 < fd. \quad (8)$$

Denote $a = fd - \mu_2\mu_1$ and suppose that $\omega_1 = \omega_2$ such that the coupled oscillators (1)-(2) are in resonant state. Thus, for the equilibrium point $(0,0,0,0)$, equation (5) turns out to be

$$\lambda^4 + (\mu_2 - \mu_1)\lambda^3 + (2\omega^2 + a)\lambda^2 + (\mu_2 - \mu_1)\omega^2\lambda + \omega^4 = 0. \quad (9)$$

The characteristic roots of equation (9) is given by

$$\lambda_{1,2} = -\frac{\mu_2 - \mu_1}{4} + \frac{1}{2}\sqrt{-a + \frac{(\mu_2 - \mu_1)^2}{4}} \pm \sqrt{\Delta_1}, \quad (10a)$$

$$\lambda_{3,4} = -\frac{\mu_2 - \mu_1}{4} - \frac{1}{2}\sqrt{-a + \frac{(\mu_2 - \mu_1)^2}{4}} \pm \sqrt{\Delta_2}, \quad (10b)$$

where

$$\Delta_1 = \frac{1}{2}(\mu_2 - \mu_1)^2 - \frac{1}{2}(\mu_2 - \mu_1)\sqrt{(\mu_2 - \mu_1)^2 - 4a} - a - 4\omega^2, \quad (11a)$$

and

$$\Delta_2 = \frac{1}{2}(\mu_2 - \mu_1)^2 + \frac{1}{2}(\mu_2 - \mu_1)\sqrt{(\mu_2 - \mu_1)^2 - 4a} - a - 4\omega^2. \quad (11b)$$

From equation (10a) we can find that

1) if $a = 0$, i.e.

$$\mu_2\mu_1 = fd, \quad (12)$$

Equation (10) turns out to be

$$\lambda_{1,2} = \pm i\omega \text{ and } \lambda_{3,4} = -\frac{\mu_2 - \mu_1}{2} \pm \sqrt{(\mu_2 - \mu_1)^2 - 4\omega^2}.$$

2) if $a > 0$, i.e.

$$fd > \mu_2\mu_1, \quad (13)$$

Equation (9) process two complex roots and two real roots with negative real parts.

3) if $a < 0$, i.e.
$$fd < \mu_2\mu_1, \quad (14)$$

Equation (9) process two complex roots with positive real parts.

According to Hopf bifurcation theorem [8], under the parametric condition (12), Hopf bifurcation of equations (1)-(2) occurs. Hence, equation (12) represents the parametric curve corresponding to Hopf bifurcation and inequality (13) determines the existence region of limit cycle in the parametric space.

Fix the parameters of equations (1)-(2) as follows: $\mu_1 = 0.33, d = 0.66, f = 0.33, c = 0.01, \omega_1 = \omega_2 = 1.0$. According to the discussions above, it can be concluded that for the control parameter $\mu_2 \in (0.33, 0.66)$, the equilibrium point $(0,0,0,0)$ is stable and $\mu_{2c} = 0.66$ is the critical parameter value corresponding to Hopf bifurcation. Direct numerical simulations shown in Figs. (**1**,**2**) verify these conclusions.

THE APPROXIMATION OF THE LIMIT CYCLE

Next, we seek analytical approximation of the limit cycle nearing the equilibrium point by using the method of multiple scales. Supposing that the damp and the nonlinearity in the van der Pol oscillator and Duffing oscillator and the coupling between them are relatively small, equations (1)-(2) can be rescaled in the following form

$$x'' + \omega_1^2 x + \varepsilon\mu_1(1 - x^2)x' + \varepsilon f y'' = 0, \quad (15)$$

$$y'' + \omega_2^2 y + \varepsilon \mu_2 y' + \varepsilon c y^3 - \varepsilon d x = 0, \tag{16}$$

where ε formally stands for a dimensionless small parameter but it will be set to unit in the final analysis.
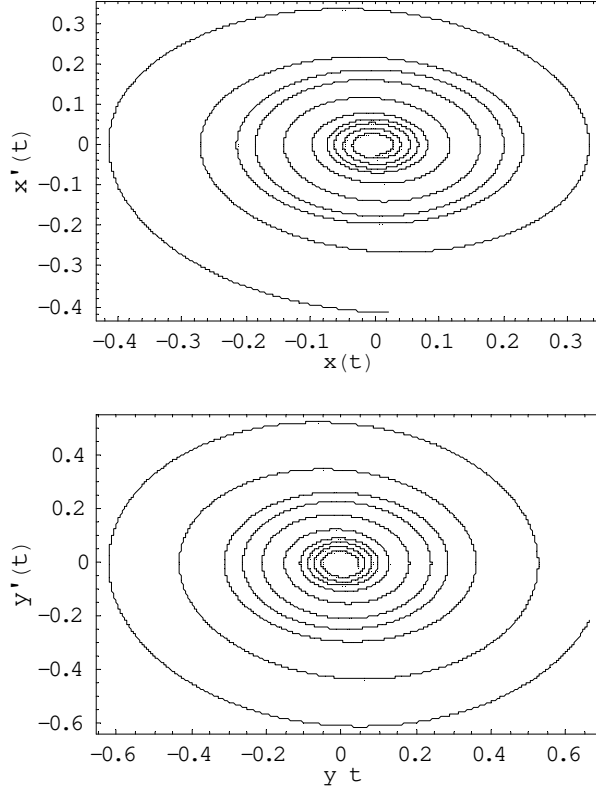

Fig. (1). Simulations of the orbit of equations (1)-(2) starting from an arbitrary initial condition when $\mu_2 = 0.5$.

We begin by introducing the time scales

$$T_0 = t, T_1 = \varepsilon t, \cdots, T_k = \varepsilon^k t, \cdots, \ k = 0,1,2 \cdots, \tag{17}$$

which yields

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots, \tag{18}$$

$$\frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2D_0 D_2) + \cdots. \tag{19}$$

Next, assume that the analytical formulation for the limit cycle is

$$x(t,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k x_k(T_0, T_1, T_2, \cdots), \tag{20}$$

$$y(t,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k y_k(T_0, T_1, T_2, \cdots). \tag{21}$$

Substituting equations (20)-(21) into equations (15)-(16) and balancing coefficient of the same power of ε yields the ordered perturbation equations:

$$\varepsilon^0: \ D_0^2 x_0 + \omega_1^2 x_0 = 0, \tag{22}$$

$$\quad\ \ D_0^2 y_0 + \omega_2^2 y_0 = 0, \tag{23}$$

$$\varepsilon^1: \ D_0^2 x_1 + \omega_1^2 x_1$$

$$= -2D_0 D_1 x_0 + \mu_1 (1 - x_0^2) D_0 x_0 + f \omega_2^2 y_0, \tag{24}$$

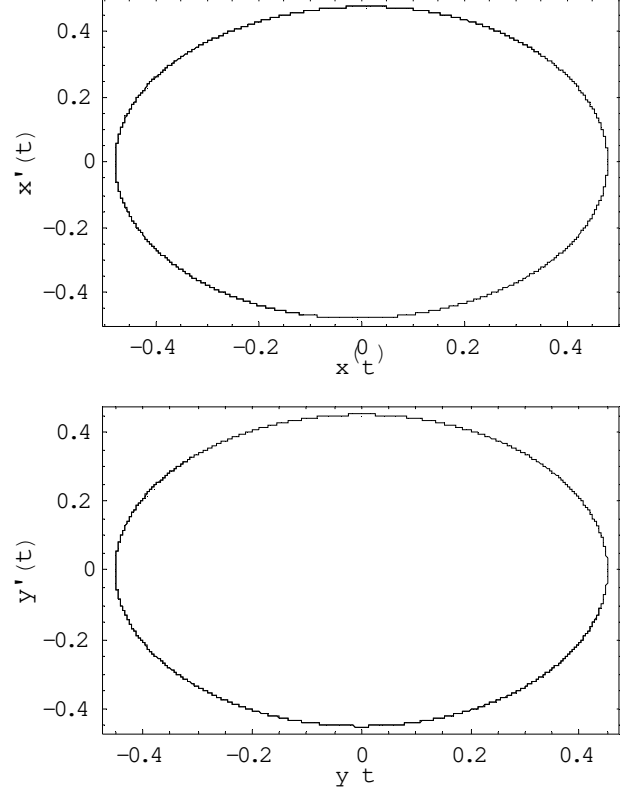**Fig. (2).** Phase portraits of the limit cycle of equations (1)-(2) on the $x - x'$ and $y - y'$ planes when $\mu_2 = 0.7$.

$$D_0^2 y_1 + \omega_1^2 y_1$$

$$= -2D_0 D_1 y_0 + \mu_2 D_0 y_0 - c y_0^3 + d x_0, \tag{25}$$

The solutions of equations (22)-(23) can be given by

$$x_0 = A \cos(T_0 + \varphi_1) \text{ and } y_0 = B \cos(T_0 + \varphi_2), \tag{26}$$

where

$$A = A(T_1), \ \varphi_1 = \varphi_1(T_1), B = B(T_1), \ \varphi_2 = \varphi_2(T_1), \tag{27}$$

which in fact means that $k = 1$ in equation (17). Substituting equations (26) into the right-hand side of equations (24)-(25) yields

$$D_0^2 x_1 + \omega_1^2 x_1 = (2\omega_1 D_1 A - \mu_1 \omega_1 A + \frac{\mu_1 \omega_1 A^3}{4}) \sin(\omega_1 T_0 + \varphi_1)$$

$$+ 2\omega_1 A D_1 \varphi_1 \cos(\omega_1 T_0 + \varphi_1) + f \omega_2^2 B \cos(\omega_2 T_0 + \varphi_2)$$

$$+ \frac{3\mu_1 \omega_1 A^3}{4} \sin(3\omega_1 T_0 + 3\varphi_1), \tag{28}$$

$$D_0^2 y_1 + \omega_1^2 y_1$$

$$= (2\omega_2 D_1 B + \mu_2 \omega_2 B) \sin(\omega_2 T_0 + \varphi_2)$$

$$+(2\omega_2 BD_1\varphi_2 - \frac{3cB^3}{4})\cos(\omega_2 T_0 + \varphi_2)$$

$$+dA\cos(\omega_1 T_0 + \varphi_1) - \frac{cB^3}{4}\sin(3\omega_2 T_0 + 3\varphi_2). \tag{29}$$

Denote $\upsilon(T_1) = \varphi_2(T_1) - \varphi_1(T_1)$ and let us recall the fact of $\omega_2 = \omega_1$. Consequently, $\cos(\omega_2 T_0 + \varphi_2)$ in equation (28) can be rewritten as $\cos(\omega_1 T_0 + \varphi_1 + \upsilon)$ and similarly $\cos(\omega_1 T_0 + \varphi_1)$ in equation (29) can be rewritten as $\cos(\omega_2 T_0 + \varphi_2 - \upsilon)$. Thus, elimination of the secular terms in equation (28)-(29) requires

$$\begin{cases} (2\omega_1 D_1 A - \mu_1\omega_1 A + \frac{\mu_1\omega_1 A^3}{4}) - f\omega_2^2 B\sin\upsilon = 0, \\ (2\omega_2 D_1 B + \mu_2\omega_2 B) + dA\sin\upsilon = 0, \\ 2\omega_1 AD_1\varphi_1 + f\omega_2^2 B\cos\upsilon = 0, \\ 2\omega_2 BD_1\varphi_2 - \frac{3cB^3}{4} + dA\cos\upsilon = 0 \end{cases} \tag{30}$$

where $\upsilon(T_1) = \varphi_2(T_1) - \varphi_1(T_1)$.

Equations (30) can be transformed into

$$\begin{cases} D_1 A = \frac{\mu_1 A}{2}(1 - \frac{A^2}{4}) + \frac{f\omega_2^2 B}{2\omega_1}\sin\upsilon, \\ D_1 B = -\frac{\mu_2 B}{2} - \frac{dA}{2\omega_2}\sin\upsilon, \\ D_1\upsilon = -\frac{3cB^3}{8\omega_2} - (\frac{dA}{2\omega_2 B} - \frac{f\omega_2^2 B}{2\omega_1 A})\cos\upsilon. \end{cases} \tag{31}$$

Equations (31) are identical with that of Chedjou *et al.* [6] derived by the averaging method.

Due to the fact that limit cycles are the steady-state solutions of dynamical systems, the amplitude and frequency of the fundamental solution (26) of equations (1)-(2) can be solved from the following nonlinear algebraic equations by setting the right-hand side of equations (31) to zero.

$$\begin{cases} \frac{\mu_1 A}{2}(1 - \frac{A^2}{4}) + \frac{f\omega_2^2 B}{2\omega_1}\sin\upsilon = 0, \\ -\frac{\mu_2 B}{2} - \frac{dA}{2\omega_2}\sin\upsilon = 0, \\ -\frac{3cB^3}{8\omega_2} - (\frac{dA}{2\omega_2 B} - \frac{f\omega_2^2 B}{2\omega_1 A})\cos\upsilon = 0. \end{cases} \tag{32}$$

It can be deduced from equations (32) that

$$\begin{cases} B^2 = \dfrac{d\omega_1\mu_1 A^2(1 - \dfrac{A^2}{4})}{f\omega_2^3\mu_2} \\ \dfrac{2cB^3}{8\omega_2} = \dfrac{1}{2}(\dfrac{dA}{\omega_2} - \dfrac{f\omega_2^2 B^2}{\omega_1 A})\sqrt{1 - B^2(\dfrac{\mu_2\omega_2}{dA})^2} \quad . \end{cases} \tag{33}$$

Hence, A and B can be solved from equations (33). Till now, the first-order approximation for the limit cycle of equations (1)- (2) is given by

$$x(t) = A\cos(t + \varphi_1) - \frac{\mu_1 A^3}{32\omega_1}\sin(3\omega_1 + 3\varphi_1), \tag{34}$$

$$y(t) = B\cos(t + \varphi_2) - \frac{cB^3}{32\omega_2^2}\cos(3\omega_2 + 3\varphi_2), \tag{35}$$

where the constants A and B are determined by equations (33).

If the parameters $\mu_1 = 0.33, d = 0.66, f = 0.33, c = 0.01$, and $\omega_1 = \omega_2 = 1.0$ are fixed and $\mu_2 = 0.7$ is chosen, it can be deduced from equations (33) that $A = 0.478152$ and $B = 0.450826$. Fig. (**3**) shows the phase portraits of the limit cycle of equations (1)-(2) on the $x - x'$ and $y - y'$ planes, from which we know that the analytical and numerical integrated results fit very well.

With the control parameter μ_2 increasing from the Hopf bifurcation point $\mu_{2c} = 0.66$, for example, when it reaches $\mu_2 = 2.0$, the phase portraits of the analytical and numerical integrated results are illustrated in Fig. (**4**). We find that on the $x - x'$ plane, the analytical result agrees well with that of numerical integration. However, on the $y - y'$ plane, the amplitude derived from numerical integration is larger than that of analytically approximate method. The reason for this phenomenon is clear if we note that the parameter μ_2 is the damped parameter of equation (2).
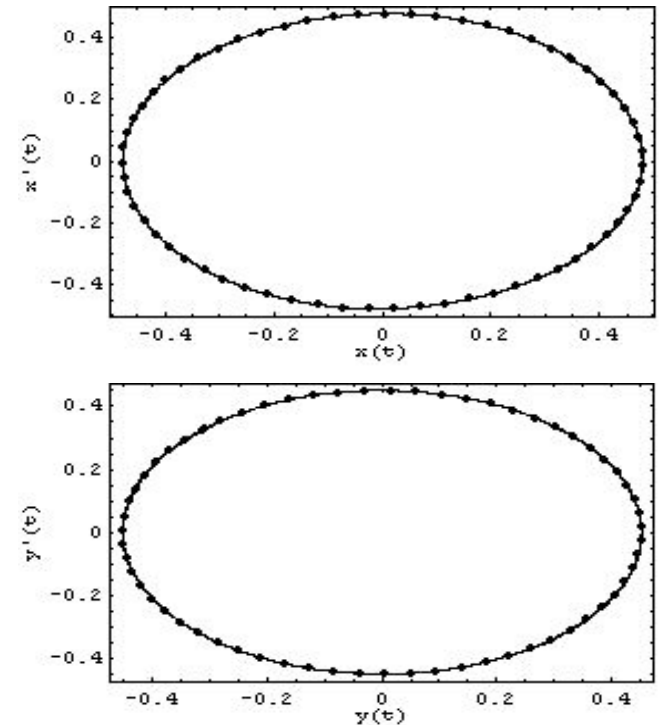

Fig. (3). Phase portraits of the limit cycle on the $x - x'$ and $y - y'$ planes when $\mu_2 = 0.7$, dot line for numerically integrated result and solid line for analytical result.

CONCLUSIONS

Stability of the equilibrium point of coupled van der Pol and Duffing oscillators is investigated by using the Routh-Hurwitz criterion. Utilizing Hopf bifurcation theorem, existence of limit cycle is proved. The critical parametric curve corresponding to Hopf bifurcation and the existence region for the limit cycle in the parameter space are also obtained. The method of multiple scales is applied to deduce the analytical approximation of the limit cycle. If the control parameter is closed to the Hopf bifurcation point, the analytical and numerical integrated results are well agreed. However, as the control parameter increases gradually and deviates from the Hopf bifurcation point, the analytical approximations do not match well with numerical integrated results.
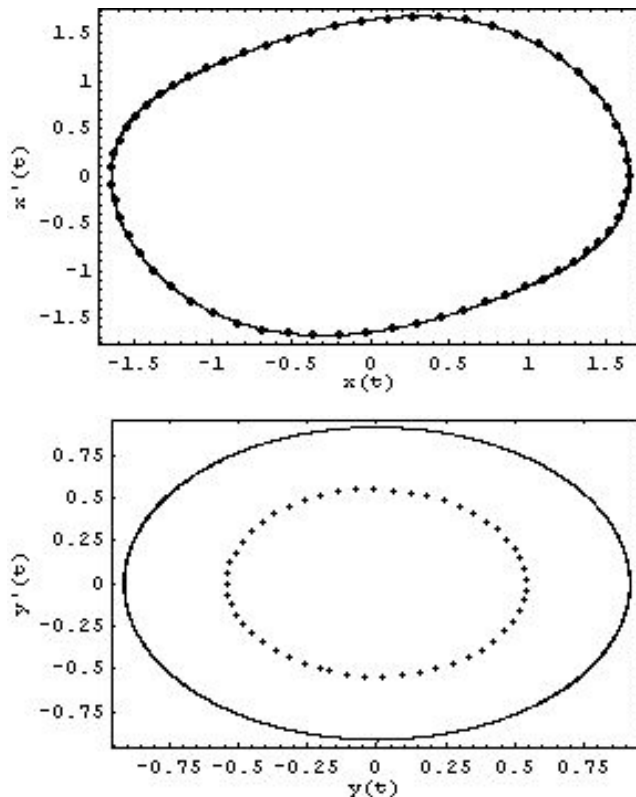
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Fig. (4). Phase portraits of the limit cycle on the $x - x'$ and $y - y'$ planes when $\mu_2 = 2.0$, dot line for numerically integrated result and solid line for analytical result.