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Research Article

Hopf Bifurcation and Hybrid Control of a Delayed Ecoepidemiological Model with Nonlinear Incidence Rate and Holling Type II Functional Response

Miao Peng,[1] **Zhengdi Zhang**,[1] **C. W. Lim**,[2] **and Xuedi Wang**[1]

[1]*Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu, 212013, China*
[2]*Department of Architecture and Civil Engineering, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong, China*

Correspondence should be addressed to Zhengdi Zhang; dyzhang@ujs.edu.cn

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Hopf bifurcation analysis of a delayed ecoepidemiological model with nonlinear incidence rate and Holling type II functional response is investigated. By analyzing the corresponding characteristic equations, the conditions for the stability and existence of Hopf bifurcation for the system are obtained. In addition, a hybrid control strategy is proposed to postpone the onset of an inherent bifurcation of the system. By utilizing normal form method and center manifold theorem, the explicit formulas that determine the direction of Hopf bifurcation and the stability of bifurcating period solutions of the controlled system are derived. Finally, some numerical simulation examples confirm that the hybrid controller is efficient in controlling Hopf bifurcation.

1. Introduction

In the natural world, transmissible diseases in ecological environment cannot be ignored. Since the pioneering study of Anderson and May [1], great and interesting predator-prey models with disease were discussed by researchers recently [2–11]. In [2], Liu and Wang considered a predator-prey model with disease in the prey; the Bogdanov-Takens bifurcation and the Hopf bifurcation were analyzed. However, referring to diseases that are transmissible in different populations, Guo et al. [12] studied an ecoepidemiological model with disease spreading within the predator population as follows:

$$\begin{aligned}
\dot{x}(t) &= x(t)\left(r - a_{11}x(t)\right) - \frac{a_{12}x(t)S(t)}{1+mx(t)}, \\
\dot{S}(t) &= \frac{a_{21}x(t)S(t)}{1+mx(t)} - r_1 S(t) - \beta S(t) I(t), \\
\dot{I}(t) &= \beta S(t) I(t) - r_2 I(t),
\end{aligned} \tag{1}$$

where $x(t)$, $S(t)$, and $I(t)$ denote the densities of the prey, the susceptible predator, and the infected predator population at time t, respectively. The parameters r, a_{11}, a_{12}, a_{21}, m, r_1, r_2, and β in model (1) are all positive constants in which r is the intrinsic growth rate of prey and r/a_{11} represents the carrying capacity of prey; only the susceptible predators have the ability to capture the prey with capturing rate a_{12}; a_{21}/a_{12} is the conversion rate of the susceptible predators; m is the half-capturing saturation constant, r_1 is the natural death rate of the susceptible predator, r_2 is the natural and disease-related mortality rate of the infected predator, and $\beta > 0$ is called the disease transmission coefficient. Guo et al. discussed the sufficient conditions for the Hopf bifurcation analysis of model (1).

The term βSI in model (1) is called the bilinear incidence rate. In the ecoepidemiological model, it is generally assumed that the average perinfected individual is effectively connected to the other members of the βN population at the same time (N represents the total scale of population), but the activity ability about them at the same time is always limited. Therefore, as the scale of population increases infinitely, the contact rate does not increase, but it gradually tends to a saturated state. This saturation contact rate is also known as the nonlinear incidence rate. Capasso and Serio [13] considered the cholera epidemic spread in Bari in 1973.

They introduced a nonlinear incidence rate $\beta IS/(1+\alpha I)$. This incidence rate seems more reasonable than the bilinear incidence rate βSI, because the nonlinear incidence rate is faster than the linear growth. For instance, before the outbreak of an infectious disease, there would be many contacts with the infected individuals. It includes the behavioral change and crowding effect of the infective individuals and prevents the unboundedness of the contact rate by choosing suitable parameters.

Motivated by the works of Guo et al. [12] and Capasso and Serio [13] and based on the influence about the time delay, in this paper, we consider the following ecoepidemiological model with nonlinear incidence rate, time delay, and Holling type II functional response:

$$\begin{aligned}\dot{x}(t) &= x(t)\left(r - a_{11}x(t)\right) - \frac{a_{12}x(t)S(t)}{1+mx(t)},\\ \dot{S}(t) &= \frac{a_{21}x(t-\tau)S(t-\tau)}{1+mx(t-\tau)} - r_1S(t) - \frac{\beta S(t)I(t)}{1+\alpha I(t)},\\ \dot{I}(t) &= \frac{\beta S(t)I(t)}{1+\alpha I(t)} - r_2I(t),\end{aligned} \tag{2}$$

where the parameters $a_{11}, a_{12}, a_{21}, r, r_1, r_2$, and β are defined in system (1). In this model, only the susceptible predators have ability to capture prey with Holling type II functional response $x/(1+mx)$ and $\tau > 0$ is the time delay due to the gestation of the susceptible predator.

The initial conditions for system (2) take the form of

$$\begin{aligned} x(0) &> 0,\\ S(0) &> 0,\\ I(0) &> 0. \end{aligned} \tag{3}$$

According to the fundamental theory of functional differential equations [14], system (2) has a unique solution $(x(t), S(t), I(t))$ that satisfies the initial conditions (3). It is easy to show that all solutions of system (2) with initial conditions (3) are defined on $[0,+\infty)$ and remain positive for all $t \geq 0$.

In recent years, bifurcation control has been extensively concerned by researchers from various disciplines. The aim of bifurcation control is to design a controller to modify the bifurcation properties of a given nonlinear system, thereby achieving some desirable dynamical behaviors. From the control theory point of view, many effective control methods have been proposed, such as the state feedback control [15, 16] and hybrid control strategy [17–24]. Especially, the hybrid control has also been widely used recently. Cheng and Cao [20] considered Hopf bifurcation control for a complex network model with time delays, and they presented a hybrid control strategy to control the model. Kiani et al. [21] used the hybrid control method for a three-pole active magnetic bearing (AMB), and the method showed that the power usage decreased in the hybrid control method comparing to a simple linear control. Peng et al. [24] studied the Hopf bifurcation control for a Lotka-Volterra predator-prey model with two delays by using a hybrid control strategy.

From the viewpoint of an ecological model, the corresponding complex bifurcation behavior means that the system changes from a stable state to an unstable one. It even causes the system to explode, which may be harmful to the ecological balance. Based on this point, a hybrid control strategy by combining the state feedback control and perturbation parameter is used in order to postpone the onset of an inherent bifurcation and enlarge the stable range in model (2).

The rest of this paper is organized as follows. In Section 2, the local stability of the positive equilibrium and the existence of Hopf bifurcation for system (2) are discussed. In Section 3, we propose a hybrid control strategy in which the state feedback and parameter perturbation are combined into system (2) and it is used to control the Hopf bifurcation. The formulas for determining the direction of Hopf bifurcation and the stability of bifurcating period solutions of the controlled system are derived. In Section 4, some numerical simulation examples are carried out to illustrate the validity of the main results. A brief conclusion is given in the last section to conclude this work.

2. A Delayed Ecoepidemiological Model without Control

It is easy to see that system (2) has a unique positive equilibrium $E^*(x^*, S^*, I^*)$, where

$$\begin{aligned} x^* &= \frac{-\left(a_{21}r\alpha - rr_1\alpha m - rm\beta + a_{11}r_1\alpha + a_{11}\beta\right) + \sqrt{\Delta_1}}{2\left(-a_{11}a_{21}\alpha + a_{11}r_1m\alpha + a_{11}m\beta\right)},\\ S^* &= \frac{-r_2\left(1+mx^*\right)}{x^*\left(a_{21}\alpha - r_1m\alpha - m\beta\right) - r_1\alpha - \beta},\\ I^* &= \frac{\beta S^* - r_2}{r_2\alpha}, \end{aligned} \tag{4}$$

with

$$\begin{aligned} \Delta_1 = {} & \left(a_{21}r\alpha - rr_1m\alpha - rm\beta + a_{11}r\alpha + a_{11}\beta\right)^2\\ & - 4\left(-a_{11}a_{21}\alpha + a_{11}r_1m\alpha + a_{11}m\beta\right)\\ & \cdot \left(a_{12}r_2 - r\beta - rr_1\alpha\right) \end{aligned} \tag{5}$$

if the following condition holds:

$$\begin{aligned} H(1):\quad & a_{11}r_1m\alpha + a_{11}m\beta > a_{11}a_{21}\alpha,\\ & a_{12}r_2 < r\beta + rr_1\alpha. \end{aligned} \tag{6}$$

In this part, we shall study the local stability of linearized system at the positive equilibrium and the existence of Hopf bifurcations for system (2).

Consider the linearized system of system (2) at the positive equilibrium $E^*(x^*, S^*, I^*)$,

$$\begin{aligned} \dot{x}(t) &= \tilde{a}_{11}x(t) + \tilde{a}_{12}S(t),\\ \dot{S}(t) &= \tilde{a}_{22}S(t) + \tilde{a}_{23}I(t) + \tilde{b}_{21}x(t-\tau) + \tilde{b}_{22}S(t-\tau),\\ \dot{I}(t) &= \tilde{a}_{32}S(t) + \tilde{a}_{33}I(t), \end{aligned} \tag{7}$$

where

$$\begin{aligned}
\tilde{a}_{11} &= r - 2a_{11}x^* - \frac{a_{12}S^*}{(1+mx^*)^2},\\
\tilde{a}_{12} &= \frac{-a_{12}x^*}{1+mx^*},\\
\tilde{a}_{22} &= -r_1 - \frac{\beta I^*}{1+\alpha I^*},\\
\tilde{a}_{23} &= -\frac{\beta S^*}{(1+\alpha I^*)^2},\\
\tilde{a}_{32} &= \frac{\beta I^*}{1+\alpha I^*},\\
\tilde{a}_{33} &= \frac{\beta S^*}{(1+\alpha I^*)^2} - r_2,\\
\tilde{b}_{21} &= \frac{a_{21}S^*}{(1+mx^*)^2},\\
\tilde{b}_{22} &= \frac{a_{21}x^*}{1+mx^*}.
\end{aligned} \tag{8}$$

The characteristic equation of the linearized system (7) is

$$\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 + \left(B_2\lambda^2 + B_1\lambda + B_0\right)e^{-\lambda\tau} = 0, \tag{9}$$

where

$$\begin{aligned}
A_0 &= \tilde{a}_{11}\tilde{a}_{23}\tilde{a}_{32} - \tilde{a}_{11}\tilde{a}_{22}\tilde{a}_{33},\\
A_1 &= \tilde{a}_{11}\tilde{a}_{22} + \tilde{a}_{22}\tilde{a}_{33} + \tilde{a}_{11}\tilde{a}_{33} - \tilde{a}_{23}\tilde{a}_{32},\\
A_2 &= -\left(\tilde{a}_{11} + \tilde{a}_{22} + \tilde{a}_{33}\right),\\
B_0 &= \tilde{a}_{12}\tilde{a}_{33}\tilde{b}_{21} - \tilde{a}_{11}\tilde{a}_{33}\tilde{b}_{22},\\
B_1 &= \tilde{a}_{11}\tilde{b}_{22} + \tilde{a}_{33}\tilde{b}_{22} - \tilde{a}_{12}\tilde{b}_{21},\\
B_2 &= -\tilde{b}_{22}.
\end{aligned} \tag{10}$$

For $\tau = 0$, (9) reduces to

$$\lambda^3 + \left(A_2 + B_2\right)\lambda^2 + \left(A_1 + B_1\right)\lambda + \left(A_0 + B_0\right) = 0. \tag{11}$$

It is not difficult to verify that $A_0 + B_0 > 0,\ A_2 + B_2 > 0$. According to the Routh-Hurwitz criteria, the necessary and sufficient condition for all roots of (11) to have a negative real part is given in the following form:

$$H(2): \left(A_2 + B_2\right)\left(A_1 + B_1\right) - \left(A_0 + B_0\right) > 0. \tag{12}$$

Namely, the equilibrium $E^*(x^*, S^*, I^*)$ is locally asymptotically stable when the condition $H(2)$ is satisfied.

For $\tau \neq 0$, substituting $\lambda = i\omega$ $(\omega > 0)$ into (9) and separating real and imaginary parts, we obtain

$$\begin{aligned}
-\omega^3 + A_1\omega &= \left(-B_2\omega^2 + B_0\right)\sin\omega\tau - B_1\omega\cos\omega\tau,\\
-A_2\omega^2 + A_0 &= \left(B_2\omega^2 - B_0\right)\cos\omega\tau - B_1\omega\sin\omega\tau.
\end{aligned} \tag{13}$$

Squaring and adding the two equations of (13), it follows that

$$\begin{aligned}
-\omega^3 + A_1\omega &= \left(-B_2\omega^2 + B_0\right)\sin\omega\tau - B_1\omega\cos\omega\tau,\\
-A_2\omega^2 + A_0 &= \left(B_2\omega^2 - B_0\right)\cos\omega\tau - B_1\omega\sin\omega\tau,
\end{aligned} \tag{14}$$

where $e_{20} = A_0^2 - B_0^2$, $e_{21} = A_1^2 - B_1^2 - 2A_0A_2 + 2B_0B_2$, and $e_{22} = A_2^2 - 2A_1 - B_2^2$.

Let $z = \omega^2$. Equation (14) can be written as

$$z^3 + e_{22}z^2 + e_{21}z + e_{20} = 0. \tag{15}$$

Denote

$$h_1(z) = z^3 + e_{22}z^2 + e_{21}z + e_{20}. \tag{16}$$

Since $h_1(0) = e_{20}$, $\lim_{z\to+\infty}h_1(z) = +\infty$, and from (16), we have

$$h_1'(z) = 3z^2 + 2e_{22}z + e_{21}. \tag{17}$$

After discussion about the roots of (17) that are similar to those in [25], we have the following lemma.

Lemma 1. *For the polynomial equation (15), we have the following results.*

(1) If (H21) $e_{20} \geq 0$ and $\Delta = e_{22}^2 - 3e_{21} \leq 0$ holds, then (15) has no positive root.

(2) If (H22) $e_{20} \geq 0$, $\Delta = e_{22}^2 - 3e_{21} > 0$, $z^* = (-e_{21} + \sqrt{\Delta})/3 > 0$, $h_1(z^*) \leq 0$, or (H23) $e_{20} < 0$ holds, then (15) has positive roots.

Suppose that (15) has positive roots. Without loss of generality, we assume that it has three positive roots, denoted by z_1, z_2, and z_3. Then (14) has three positive roots $\omega_k = \sqrt{k}$, $k = 1, 2, 3$. The corresponding critical value of time delay $\tau_k^{(j)}$ is

$$\tau_k^{(j)} = \frac{1}{\omega_k}\arccos\left\{\frac{\left(-B_1 + A_2B_2\right)\omega_k^4 + \left(A_1B_1 - A_2B_0 - A_0B_2\right)\omega_k^2 + A_0B_0}{-\left[B_2^2\omega_k^4 + \left(B_1^2 - 2B_0B_2\right)\omega_k^2 + B_0^2\right]}\right\} + \frac{2\pi j}{\omega_k}, \tag{18}$$

where $j = 0, 1, 2 \ldots$.

Thus $\pm\omega_k$ is a pair of purely imaginary roots of (9) with $\tau = \tau_k^{(j)}$, and let $\tau_0 = \min_{k\in\{1,2,3\}}\{\tau_k^{(0)}\}$, $\omega_0 = \omega_{k_0}$.

According to the Hopf Bifurcation Theorem [26], we need to verify the transversality condition. Differentiating the two sides of (9) with respect to τ, we obtain

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = -\frac{3\lambda^2 + 2A_2\lambda + A_1}{\lambda\left(\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0\right)} + \frac{2B_2\lambda + B_1}{\lambda\left(B_2\lambda^2 + B_1\lambda + B_0\right)} - \frac{\tau}{\lambda}, \tag{19}$$

Then

$$\begin{aligned}\operatorname{sign}\left\{\frac{d(\operatorname{Re}\lambda)}{d\tau}\right\}_{\lambda=i\omega_0} &= \operatorname{sign}\left\{\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\lambda=i\omega_0} \\ &= \operatorname{sign}\left\{\frac{3\omega_0^4 + 2\left(A_2^2 - 2A_1\right)\omega_0^2 + \left(A_1^2 - 2A_0A_2\right)}{\omega_0^6 + \left(-2A_1 + A_2^2\right)\omega_0^4 + \left(A_1^2 - 2A_0A_2\right)\omega_0^2 + A_0^2} - \frac{2B_2^2\omega_0^2 + B_1^2 - 2B_0B_2}{B_2^2\omega_0^4 + \left(B_1^2 - 2B_0B_2\right)\omega_0^2 + B_0^2}\right\}.\end{aligned} \tag{20}$$

We derive from (13) that

$$\omega_0^6 + \left(-2A_1 + A_2^2\right)\omega_0^4 + \left(A_1^2 - 2A_0A_2 + 2B_0B_2\right) + A_0^2 = B_2^2\omega_0^4 + \left(B_1^2 - 2B_0B_2\right)\omega_0^2 + B_0^2. \tag{21}$$

Then, it follows that

$$\begin{aligned}\left\{\frac{d(\operatorname{Re}\lambda)}{d\tau}\right\}_{\lambda=i\omega_0} &= \operatorname{sign}\left\{\frac{3\omega_0^4 + 2\left(A_2^2 - 2A_1 - B_2^2\right)\omega_0^2 + \left(A_1^2 - 2A_0A_2 - B_1^2 + 2B_0B_2\right)}{B_2^2\omega_0^4 + \left(B_1^2 - 2B_0B_2\right)\omega_0^2 + B_0^2}\right\} \\ &= \operatorname{sign}\left\{\frac{h_1'(z_0)}{B_2^2\omega_0^4 + \left(B_1^2 - 2B_0B_2\right)\omega_0^2 + B_0^2}\right\}.\end{aligned} \tag{22}$$

Therefore, $\{d(\operatorname{Re}\lambda)/d\tau\}_{\lambda=i\omega_0} \neq 0$ if the following condition holds:

$$H(24): h_1'\left(\omega_0^2\right) \neq 0. \tag{23}$$

According to the analysis above, we have the following results.

Theorem 2. *For system (2),*

(1) If (H21) holds, then the positive equilibrium $E^*(x^*, S^*, I^*)$ is asymptotically stable for all $\tau \geq 0$.

(2) If (H22) or (H23) and (H24) hold, then the positive equilibrium $E^*(x^*, S^*, I^*)$ is asymptotically stable for all $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$. Furthermore, system (2) undergoes a Hopf bifurcation at the positive equilibrium $E^*(x^*, S^*, I^*)$ when $\tau = \tau_0$.

3. A Delayed Ecoepidemiological Model with Hybrid Control

In this part, a hybrid control strategy is proposed, in which the state feedback and parameter perturbation are combined in an effort to postpone the occurrence of Hopf bifurcation in system (2). Here, a controlled model as follows is considered:

$$\begin{aligned}\dot{x}(t) &= p\left[x(t)\left(r - a_{11}x(t)\right) - \frac{a_{12}x(t)S(t)}{1 + mx(t)}\right] + qx(t), \\ \dot{S}(t) &= p\left[\frac{a_{21}x(t-\tau)S(t-\tau)}{1 + mx(t-\tau)} - r_1S(t) - \frac{\beta S(t)I(t)}{1 + \alpha I(t)}\right] + qS(t), \\ \dot{I}(t) &= p\left[\frac{\beta S(t)I(t)}{1 + \alpha I(t)} - r_2I(t)\right],\end{aligned} \tag{24}$$

where $p > 0$ and $q \in R$ are control parameters. The parameters a_{11}, a_{12}, a_{21}, m, r, r_1, r_2, β, and τ are defined in system (2), $qx(t)$ and $qS(t)$ affect the densities of prey and susceptible predator at time t, respectively, and $q > 0$ denotes increase in the quantity, while $q < 0$ otherwise.

Similar to the discussion in Section 2, model (24) has a unique positive equilibrium $E^\circ(x^\circ, S^\circ, I^\circ)$, where

$$\begin{aligned}x^\circ &= \frac{-C_2 + \sqrt{\Delta_2}}{2C_1}, \\ S^\circ &= \frac{-pr_2\left(1 + mx^\circ\right)}{x^\circ\left(\alpha pa_{21} - \alpha pr_1m - p\beta m + \alpha qm\right) - \alpha pr_1 - p\beta + \alpha q}, \\ I^\circ &= \frac{\beta S^\circ - r_2}{r_2\alpha},\end{aligned} \tag{25}$$

with

$$\begin{aligned}C_1 &= -a_{11}p\left(\alpha pa_{21} - \alpha pr_1m - p\beta m + \alpha qm\right), \\ C_2 &= \alpha p^2ra_{21} - \alpha p^2rr_1m - p^2\beta rm + pqr\alpha m \\ &\quad + \alpha p^2r_1a_{11} + p^2a_{11}\beta - \alpha pqa_{11} + \alpha pqa_{21} \\ &\quad - \alpha pqr_1m - pq\beta m + \alpha q^2m,\end{aligned}$$

$$\Delta_2 = C_2^2 - 4C_1\left(-\alpha p^2 r r_1 + \alpha pqr - p^2\beta r - \alpha pqr_1 - pq\beta + \alpha q^2 + p^2 a_{12} r_2\right), \tag{26}$$

if the following condition holds:

$$(H3): \quad -a_{11}p\left(\alpha p a_{21} - \alpha p r_1 m - p\beta m + \alpha q m\right) > 0, \quad \left(-\alpha p^2 r r_1 + \alpha pqr - p^2\beta r - \alpha pqr_1 - pq\beta + \alpha q^2 + p^2 a_{12} r_2\right) < 0. \tag{27}$$

Let $\overline{x}(t) = x(t) - x^\circ$, $\overline{S}(t) = S(t) - S^\circ$, and $\overline{I}(t) = I(t) - I^\circ$ and still denote $\overline{x}(t), \overline{S}(t), \overline{I}(t)$, respectively. Using Taylor expansion to expand system (24) at the positive equilibrium $E^\circ(x^\circ, S^\circ, I^\circ)$, we have

$$\begin{aligned} \dot{x}(t) &= a'_{11}x(t) + a'_{12}S(t) + f'_1, \\ \dot{S}(t) &= a'_{22}S(t) + a'_{23}I(t) + b'_{21}x(t-\tau) + b'_{22}S(t-\tau) + f'_2, \\ \dot{I}(t) &= a'_{32}S(t) + a'_{33}I(t) + f'_3, \end{aligned} \tag{28}$$

where

$$\begin{aligned} a'_{11} &= pr - 2pa_{11}x^\circ - \frac{a_{12}pS^\circ}{(1+mx^\circ)^2} + q, \\ a'_{12} &= \frac{-a_{12}px^\circ}{1+mx^\circ}, \\ a'_{22} &= -pr_1 - \frac{p\beta I^\circ}{1+\alpha I^\circ} + q, \\ a'_{23} &= \frac{-p\beta S^\circ}{(1+\alpha I^\circ)}, \\ a'_{32} &= \frac{p\beta I^\circ}{1+\alpha I^\circ}, \\ a'_{33} &= \frac{p\beta S^\circ}{(1+\alpha I^\circ)^2} - r_2 p, \\ b'_{21} &= \frac{pa_{21}S^\circ}{(1+mx^\circ)^\circ}, \\ b'_{22} &= \frac{pa_{21}x^\circ}{1+mx^\circ}, \\ f'_1 &= a'_{13}x^2 + a'_{14}xS + a'_{15}x^2S + a'_{16}x^3, \\ f'_2 &= a'_{24}x^2(t-\tau) + a'_{25}x(t-\tau)S(t-\tau) \\ &\quad + a'_{26}x^2(t-\tau)S(t-\tau) + a'_{27}x^3(t-\tau) \\ &\quad + a'_{28}SI + a'_{29}I^2 + a'_{30}I^2S + a'_{31}I^3, \\ f'_3 &= a'_{34}SI + a'_{35}I^2 + a'_{36}I^2S + a'_{37}I^3, \end{aligned} \tag{29}$$

with

$$\begin{aligned} a'_{13} &= -a_{11}p + \frac{pa_{12}mS^\circ}{(1+mx^\circ)^3}, \\ a'_{14} &= -\frac{pa_{12}}{(1+mx^\circ)^2}, \\ a'_{15} &= \frac{pa_{12}m}{(1+mx^\circ)^3}, \\ a'_{16} &= -\frac{pa_{12}m^2S^\circ}{(1+mx^\circ)^4}, \\ a'_{24} &= \frac{-pa_{21}mS^\circ}{(1+mx^\circ)^3}, \\ a'_{25} &= \frac{pa_{21}}{(1+mx^\circ)^2}, \\ a'_{26} &= -\frac{pa_{21}m}{(1+mx^\circ)^3}, \\ a'_{27} &= \frac{pa_{21}m^2S^\circ}{(1+mx^\circ)^4}, \\ a'_{28} &= \frac{-p\beta}{(1+\alpha I^\circ)^2}, \\ a'_{29} &= \frac{p\alpha\beta S^\circ}{(1+\alpha I^\circ)^3}, \\ a'_{30} &= \frac{p\alpha\beta}{(1+\alpha I^\circ)^3}, \\ a'_{31} &= -\frac{p\alpha^2\beta S^\circ}{(1+\alpha I^\circ)^4}, \\ a'_{34} &= \frac{p\beta}{(1+\alpha I^\circ)^2}, \\ a'_{35} &= \frac{-p\alpha\beta S^\circ}{(1+\alpha I^\circ)^3}, \\ a'_{36} &= \frac{-p\alpha\beta}{(1+\alpha I^\circ)^3}, \\ a'_{37} &= \frac{p\alpha^2\beta S^\circ}{(1+\alpha I^\circ)^4}. \end{aligned} \tag{30}$$

Then we obtain the linearized system of system (24) as follows:

$$\begin{aligned}\dot{x}(t) &= a'_{11}x(t) + a'_{12}S(t),\\ \dot{S}(t) &= a'_{22}S(t) + a'_{23}I(t) + b'_{21}x(t-\tau) + b'_{22}S(t-\tau),\\ \dot{I}(t) &= a'_{32}S(t) + a'_{33}I(t).\end{aligned} \tag{31}$$

Therefore, the corresponding characteristic equation of system (31) is given by

$$\lambda^3 + A'_2\lambda^2 + A'_1\lambda + A'_0 + \left(B'_2\lambda^2 + B'_1\lambda + B'_0\right)e^{-\lambda\tau'} = 0, \tag{32}$$

where

$$\begin{aligned}A'_0 &= a'_{11}a'_{23}a'_{32} - a'_{11}a'_{22}a'_{33},\\ A'_1 &= a'_{11}a'_{22} + a'_{22}a'_{33} + a'_{11}a'_{33} - a'_{23}a'_{32},\\ A'_2 &= -\left(a'_{11} + a'_{22} + a'_{33}\right),\\ B'_0 &= a'_{12}a'_{33}b'_{21} - a'_{11}a'_{33}b'_{22},\\ B'_1 &= a'_{11}b'_{22} + a'_{33}b'_{22} - a'_{12}b'_{21},\\ B'_2 &= -b'_{22}.\end{aligned} \tag{33}$$

Obviously, the characteristic equation of system (24) is similar to (9). Therefore, the analysis method is very similar to Section 2; we will omit the local stability and Hopf bifurcation analysis of system (24). We obtain the corresponding critical value of time delay $(\tau'_k)^{(j)}$ as

$$\left(\tau'_k\right)^{(j)} = \frac{1}{\omega'_k}\arccos\left\{\frac{\left(-B'_1 + A'_2B'_2\right)\left(\omega'_k\right)^4 + \left(A'_1B'_1 - A'_2B'_0 - A'_0B'_2\right)\left(\omega'_k\right)^2 + A'_0B'_0}{-\left[\left(B'_2\right)^2\left(\omega'_k\right)^4 + \left(\left(B'_1\right)^2 - 2B'_0B'_2\right)\left(\omega'_k\right)^2 + \left(B'_0\right)^2\right]}\right\} + \frac{2\pi j}{\omega'_k}, \tag{34}$$

where $k = 1, 2, 3$; $j = 0, 1, 2, \ldots, \omega'_k$ is a positive root of

$$\left(\omega'\right)^6 + e_{32}\left(\omega'\right)^4 + e_{31}\left(\omega'\right)^2 + e_{30} = 0, \tag{35}$$

with

$$\begin{aligned}e_{30} &= \left(A'_0\right)^2 - \left(B'_0\right)^2,\\ e_{31} &= \left(A'_1\right)^2 - \left(B'_1\right)^2 - 2A'_0A'_2 + 2B'_0B'_2,\\ e_{32} &= \left(A'_2\right)^2 - 2A'_1 - \left(B'_2\right)^2.\end{aligned} \tag{36}$$

Thus $\pm\omega'_k$ is a pair of purely imaginary roots of (32) with $\tau' = (\tau'_k)^{(j)}$, and let $\tau'_0 = \min_{k\in\{1,2,3\}}\{(\tau'_k)^{(0)}\}$, $\omega'_0 = \omega'_{k_0}$.

In the following, we will investigate the direction of Hopf bifurcation and the stability of bifurcating periodic solutions of the controlled system (24) at τ'_0. The theoretical approach we will apply is based on the normal form theory and center manifold theorem [26].

Let $\tau = \tau'_0 + \mu$, $\mu \in R$, $t = s\tau$, $x(s\tau) = \widehat{x}(s)$, $S(s\tau) = \widehat{S}(s)$, and $I(s\tau) = \widehat{I}(s)$. Then $\mu = 0$ is the Hopf bifurcation value of the controlled system (24). Denote $x = \widehat{x}$, $S = \widehat{S}$, $I = \widehat{I}$, and $t = s$, then system (24) can be written as a functional differential equation (FDE) in $C = C([-1, 0], R^3)$:

$$u'(t) = L_\mu\left(u_t\right) + F\left(\mu, u_t\right), \tag{37}$$

where $u(t) = (x(t), S(t), I(t))^T \in C$, $u_t(\theta) = u(t+\theta) = (x(t+\theta), S(t+\theta), I(t+\theta))^T \in C$, and $L_\mu : C \to R^3$, $F : R\times C \to R^3$ are given by

$$\begin{aligned}L_\mu\phi &= \left(\tau'_0 + \mu\right)\left[A'\phi(0) + B'\phi(-1)\right],\\ F(\mu,\phi) &= \left(\tau'_0 + \mu\right)\left(F_1, F_2, F_3\right)^T,\end{aligned} \tag{38}$$

where

$$\begin{aligned}\phi(\theta) &= \left(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)\right)^T \in C,\\ A' &= \begin{pmatrix} a'_{11} & a'_{12} & 0\\ 0 & a'_{22} & a'_{23}\\ 0 & a'_{32} & a'_{33}\end{pmatrix},\\ B' &= \begin{pmatrix} 0 & 0 & 0\\ b'_{21} & b'_{22} & 0\\ 0 & 0 & 0\end{pmatrix},\\ F_1 &= a'_{13}\phi_1^2(0) + a'_{14}\phi_1(0)\phi_2(0) + a'_{15}\phi_1^2(0)\phi_2(0) + a'_{16}\phi_1^3(0),\\ F_2 &= a'_{24}\phi_1^2(-1) + a'_{25}\phi_1(-1)\phi_2(-1) + a'_{26}\phi_1^2(-1)\phi_2(-1) + a'_{27}\phi_1^3(-1)\\ &\quad + a'_{28}\phi_2(0)\phi_3(0) + a'_{29}\phi_3^2(0) + a'_{30}\phi_2(0)\phi_3^2(0) + a'_{31}\phi_3^3(0),\\ F_3 &= a'_{34}\phi_2(0)\phi_3(0) + a'_{35}\phi_3^2(0) + a'_{36}\phi_2(0)\phi_3^2(0) + a'_{37}\phi_3^3(0).\end{aligned} \tag{39}$$

Hence, by the Riesz representation theorem, there exists a 3×3 matrix function $\eta(\theta,\mu)$ of bounded variation for $\theta \in [-1, 0]$, such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta,\mu)\,\phi(\theta), \quad \text{for } \phi \in C. \tag{40}$$

In fact, we can choose

$$\eta(\theta,\mu) = (\tau_0' + \mu) A' \delta(\theta) - (\tau_0' + \mu) B' \delta(\theta + 1), \quad (41)$$

where $\delta(\theta)$ is the Dirac function.

For $\phi \in C([-1,0], R^3)$, define

$$A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \displaystyle\int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta), & \theta = 0, \end{cases}$$

$$R_\mu(\phi) = \begin{cases} 0, & -1 \le \theta < 0, \\ F(\mu,\phi), & \theta = 0. \end{cases} \quad (42)$$

Then (37) can be transformed into the following operator equation:

$$u_t' = A(\mu) u_t + R(\mu) u_t. \quad (43)$$

The adjoint operator A^* of $A(0)$ is defined by

$$A^*\varphi(s) = \begin{cases} -\dfrac{d\varphi(s)}{ds}, & s \in (0,1], \\ \displaystyle\int_{-1}^{0} d\eta^T(t,0)\varphi(-t), & s = 0. \end{cases} \quad (44)$$

For $\phi \in C([-1,0], R^3)$ and $\varphi \in C([-1,0], (R^3)^*)$, define the bilinear form:

$$\langle \varphi(s), \phi(s) \rangle = \overline{\varphi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{\varphi}(\xi-\theta)\, d\eta(\theta)\,\phi(\xi)\, d\xi, \quad (45)$$

where $\eta(\theta) = \eta(\theta, 0)$, $A = A(0)$ and A^* are adjoint operators.

Referring to the previous discussion, we know that $\pm i\omega_0'\tau_0'$ are the eigenvalues of $A(0)$; thus they are also the eigenvalues of A^*. Suppose that $q(\theta) = (1, q_1, q_2)^T e^{i\omega_0'\tau_0'\theta}$ is the eigenvector of $A(0)$ corresponding to $i\omega_0'\tau_0'$ and $q^*(s) = D(1, q_1^*, q_2^*)e^{i\omega_0'\tau_0's}$ is the eigenvector of A^* corresponding to $-i\omega_0'\tau_0'$. By direct computation, we obtain

$$\begin{aligned} q_1 &= \frac{i\omega_0' - a_{11}'}{a_{12}'}, \\ q_2 &= \frac{a_{32}'(i\omega_0' - a_{11}')}{a_{12}'(i\omega_0' - a_{33}')}, \\ q_1^* &= \frac{-i\omega' - a_{11}'}{b_{21}' e^{i\omega_0'\tau_0'}}, \\ q_2^* &= \frac{a_{23}'(i\omega_0' + a_{11}')}{b_{21}' e^{i\omega_0'\tau_0'}(i\omega_0' + a_{33}')}, \end{aligned} \quad (46)$$

$\langle q^*(s), q(\theta)\rangle = 1$, and $\langle q^*(s), \overline{q}(\theta)\rangle = 0$, where

$$\overline{D} = \frac{1}{1 + q_1\overline{q}_1^* + q_2\overline{q}_2^* + e^{-i\omega_0'\tau_0'}\tau_0' q_1^*(b_{21}' + b_{22}' q_1)}. \quad (47)$$

Next, we can obtain the coefficients used in determining the direction of Hopf bifurcation and the stability of the bifurcation periodic solutions by the algorithms given in [26].

$$\begin{aligned} g_{20} &= 2\overline{D}\tau_0' \Big[a_{13}' + a_{14}' q_1 + \overline{q}_1^* \Big(a_{24}' e^{-2i\omega_0'\tau_0'} \\ &\quad + a_{25}' q_1 e^{-2i\omega_0'\tau_0'} + a_{28}' q_1 q_2 + a_{29}' q_2^2\Big) + \overline{q}_2^* \Big(a_{34}' q_1 q_2 \\ &\quad + a_{35}' q_2^2\Big)\Big], \\ g_{11} &= \overline{D}\tau_0' \Big[2a_{13}' + a_{14}'(q_1 + \overline{q}_1) + \overline{q}_1^* \Big(2a_{24}' + a_{25}'(q_1 \\ &\quad + \overline{q}_1) + a_{28}'(q_1\overline{q}_2 + \overline{q}_1 q_2) + 2a_{29}' q_2\overline{q}_2\Big) \\ &\quad + \overline{q}_2^* \Big(a_{34}'(q_1\overline{q}_2 + \overline{q}_1 q_2) + 2a_{35}' q_2\overline{q}_2\Big)\Big], \\ g_{02} &= 2\overline{D}\tau_0' \Big[a_{13}' + a_{14}'\overline{q}_1 + \overline{q}_1^* \Big(a_{24}' e^{2i\omega_0'\tau_0'} \\ &\quad + a_{25}'\overline{q}_1 e^{2i\omega_0'\tau_0'} + a_{28}'\overline{q}_1\overline{q}_2 + a_{29}'\overline{q}_2^2\Big) + \overline{q}_2^* \Big(a_{34}'\overline{q}_1\overline{q}_2 \\ &\quad + a_{35}'\overline{q}_2^2\Big)\Big], \\ g_{21} &= 2\overline{D}\tau_0' \Big[a_{13}'\Big(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)\Big) \\ &\quad + a_{14}'\Big(W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0) + \frac{1}{2}\overline{q}_1 W_{20}^{(1)}(0) \\ &\quad + q_1 W_{11}^{(1)}(0)\Big) + a_{15}'(2q_1 + \overline{q}_1) + 3a_{16}' \\ &\quad + \overline{q}_1^* \Big(a_{24}'\Big(2W_{11}^{(1)}(-1)e^{-i\omega_0'\tau_0'} + W_{20}^{(1)}(-1)e^{-i\omega_0'\tau_0'}\Big) \\ &\quad + a_{25}'\Big(W_{11}^{(2)}(-1)e^{-i\omega_0'\tau_0'} + \frac{1}{2}W_{20}^{(2)}(-1)e^{i\omega_0'\tau_0'} \\ &\quad + \frac{1}{2}W_{20}^{(1)}(-1)\overline{q}_1 e^{i\omega_0'\tau_0'} + W_{11}^{(1)}(-1) q_1 e^{-i\omega_0'\tau_0'}\Big) \\ &\quad + a_{26}'\Big(2q_1 e^{-i\omega_0'\tau_0'} + \overline{q}_1 e^{-i\omega_0'\tau_0'}\Big) + 3a_{27}' e^{-i\omega_0'\tau_0'} \\ &\quad + a_{28}'\Big(q_1 W_{11}^{(3)}(0) + \frac{1}{2}\overline{q}_1 W_{20}^{(3)}(0) + \frac{1}{2}\overline{q}_2 W_{20}^{(2)}(0) \\ &\quad + q_2 W_{11}^{(2)}(0)\Big) + a_{29}'\Big(2q_2 W_{11}^{(3)}(0) + \overline{q}_2 W_{20}^{(3)}(0)\Big) \\ &\quad + a_{30}'\Big(2q_1 q_2\overline{q}_2 + \overline{q}_1 q_2^2\Big) + 3a_{31}' q_2^2\overline{q}_2\Big) \\ &\quad + \overline{q}_2^* \Big(a_{34}'\Big(q_1 W_{11}^{(3)}(0) + \frac{1}{2}\overline{q}_1 W_{20}^{(3)}(0) \\ &\quad + \frac{1}{2}\overline{q}_2 W_{20}^{(2)}(0) + q_2 W_{11}^{(2)}(0)\Big) + a_{35}'\Big(2q_2 W_{11}^{(3)}(0) \\ &\quad + \overline{q}_2 W_{20}^{(3)}(0)\Big) + a_{36}'\Big(2q_1 q_2\overline{q}_2 + \overline{q}_1 q_2^2\Big) \\ &\quad + 2a_{37}' q_2^2\overline{q}_2\Big)\Big]. \end{aligned} \quad (48)$$

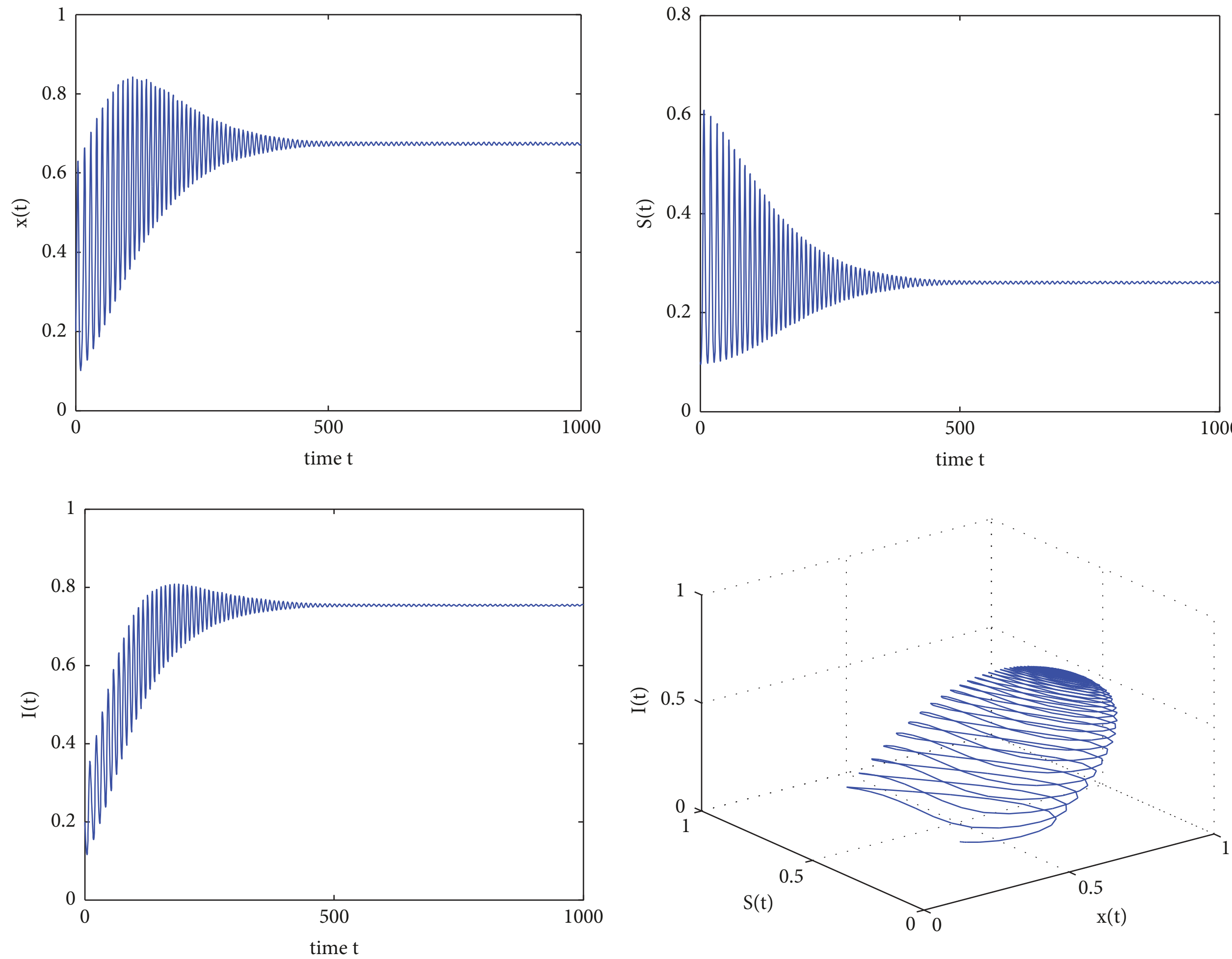

FIGURE 1: Behavior and phase portrait of the uncontrolled system (2) with $\tau = 0.1 < \tau_0$. The positive equilibrium E^* is asymptotically stable.

However,

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0'\tau_0'} q(0) e^{i\omega_0'\tau_0'\theta} + \frac{i\overline{g}_{02}}{3\omega_0'\tau_0'} \overline{q}(0) e^{-i\omega_{10}'\tau_0'\theta} + E_1 e^{2i\omega_{10}'\tau_{10}'\theta},$$
$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0'\tau_0'} q(0) e^{i\omega_0'\tau_0'\theta} + \frac{i\overline{g}_{11}}{\omega_{10}'\tau_{10}'} \overline{q}(0) e^{-i\omega_0'\tau_0'\theta} + E_2, \quad (49)$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)})^T \in R^3$ and $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)})^T \in R^3$ are also constant vectors and they can be determined, respectively, by

$$\begin{pmatrix} 2i\omega_0' - a_{11}' & -a_{12}' & 0 \\ -b_{21}' e^{-i\omega_0'\tau_0'} & 2i\omega_{10}' - a_{22}' - b_{22}' e^{-i\omega_0'\tau_0'} & -a_{23}' \\ 0 & -a_{32}' & 2i\omega_{10}' - a_{33}' \end{pmatrix} E_1 = 2\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix},$$
$$\begin{pmatrix} -a_{11}' & -a_{12}' & 0 \\ -b_{21}' & -a_{22}' - b_{22}' & -a_{23}' \\ 0 & -a_{32}' & -a_{33}' \end{pmatrix} E_2 = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}, \quad (50)$$

with

$$H_1 = a_{13}' + a_{14}' q_1,$$
$$H_2 = a_{24}' e^{-2i\omega_0'\tau_0'} + a_{25}' q_1 e^{-2i\omega_0'\tau_0'} + a_{28}' q_1 q_2 + a_{29}' q_2^2,$$
$$H_3 = a_{34}' q_1 q_2 + a_{35}' q_2^2,$$
$$P_1 = 2a_{13}' + a_{14}' (q_1 + \overline{q}_1),$$
$$P_2 = 2a_{24}' + a_{25}' (q_1 + \overline{q}_1) + a_{28}' (q_1 \overline{q}_2 + \overline{q}_1 q_2)$$

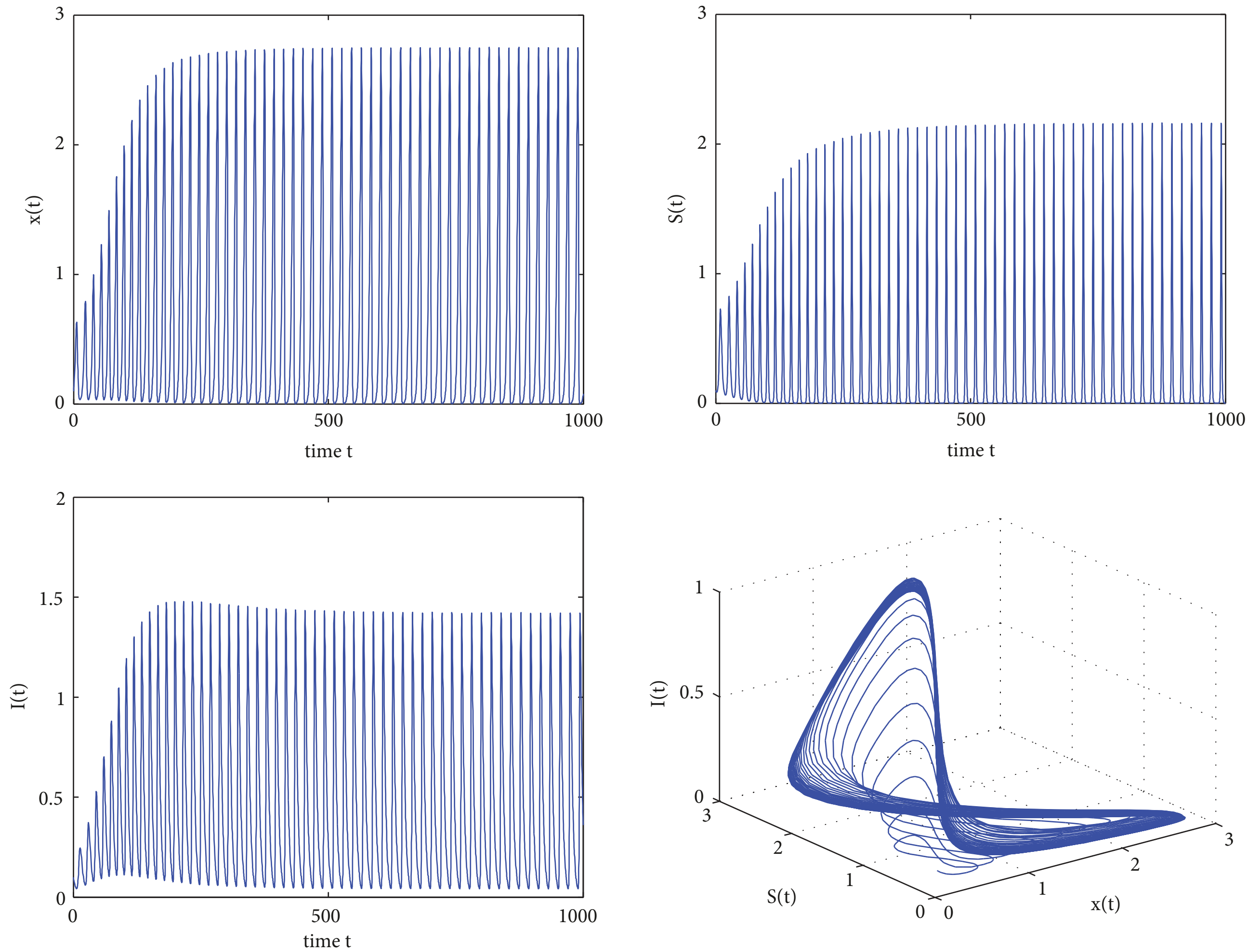

Figure 2: Behavior and phase portrait of the uncontrolled system (2) with $\tau = 0.3 > \tau_0$. Hopf bifurcation occurs from the positive equilibrium E^*.

$$+ 2a'_{29}q_2\overline{q}_2,$$
$$P_3 = a'_{34}\left(q_1\overline{q}_2 + \overline{q}_1 q_2\right) + 2a'_{35}q_2\overline{q}_2. \tag{51}$$

Therefore, we can determine g_{21} and derive the expressions

$$\begin{aligned} c_1(0) &= \frac{i}{2\omega'_0\tau'_0}\left(g_{20}g_{11} - 2\left|g_{11}\right|^2 - \frac{\left|g_{02}\right|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\{\lambda'(\tau'_0)\}}, \\ \beta_2 &= 2\operatorname{Re}\left(c_1(0)\right), \\ T_2 &= -\frac{\operatorname{Im}\{c_1(0)\} + \mu_2 \operatorname{Im}\left\{\lambda'(\tau_0)\right\}}{\omega'_0\tau'_0}, \end{aligned} \tag{52}$$

which describe the properties of bifurcation period solutions at $\tau = \tau'_0$ on the center manifold. From the discussion above, we have the following result.

Theorem 3. *For system (24), the direction of Hopf bifurcation is determined by the sign of μ_2: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical). The stability of the bifurcating periodic solutions is determined by the sign of β_2: if $\beta_2 < 0$ ($\beta_2 > 0$), the bifurcating periodic solutions are stable (unstable). The period of the bifurcating periodic solutions is determined by the sign of T_2: if $T_2 > 0$ ($T_2 < 0$), the bifurcating periodic solutions increase (decrease).*

4. Numerical Examples

In this section, we present some numerical examples by using Matlab to verify the analytical predictions obtained in the previous sections. The hybrid control strategy to gain control of the Hopf bifurcation in model (2) is applied.

Let $a_{11} = 0.125$, $a_{12} = 1.8$, $a_{21} = 1.35$, $r = 0.55$, $r_1 = 0.18$, $r_2 = 0.25$, $m = 0.01$, $\alpha = 0.0005$, and $\beta = 0.96$. Then, we have the following particular example of system (2):

$$\dot{x}(t) = x(t)\left(0.55 - 0.125x(t)\right) - \frac{1.8x(t)S(t)}{1 + 0.01x(t)},$$
$$\dot{S}(t) = \frac{1.35x(t-\tau)S(t-\tau)}{1 + 0.01x(t-\tau)} - 0.18S(t)$$

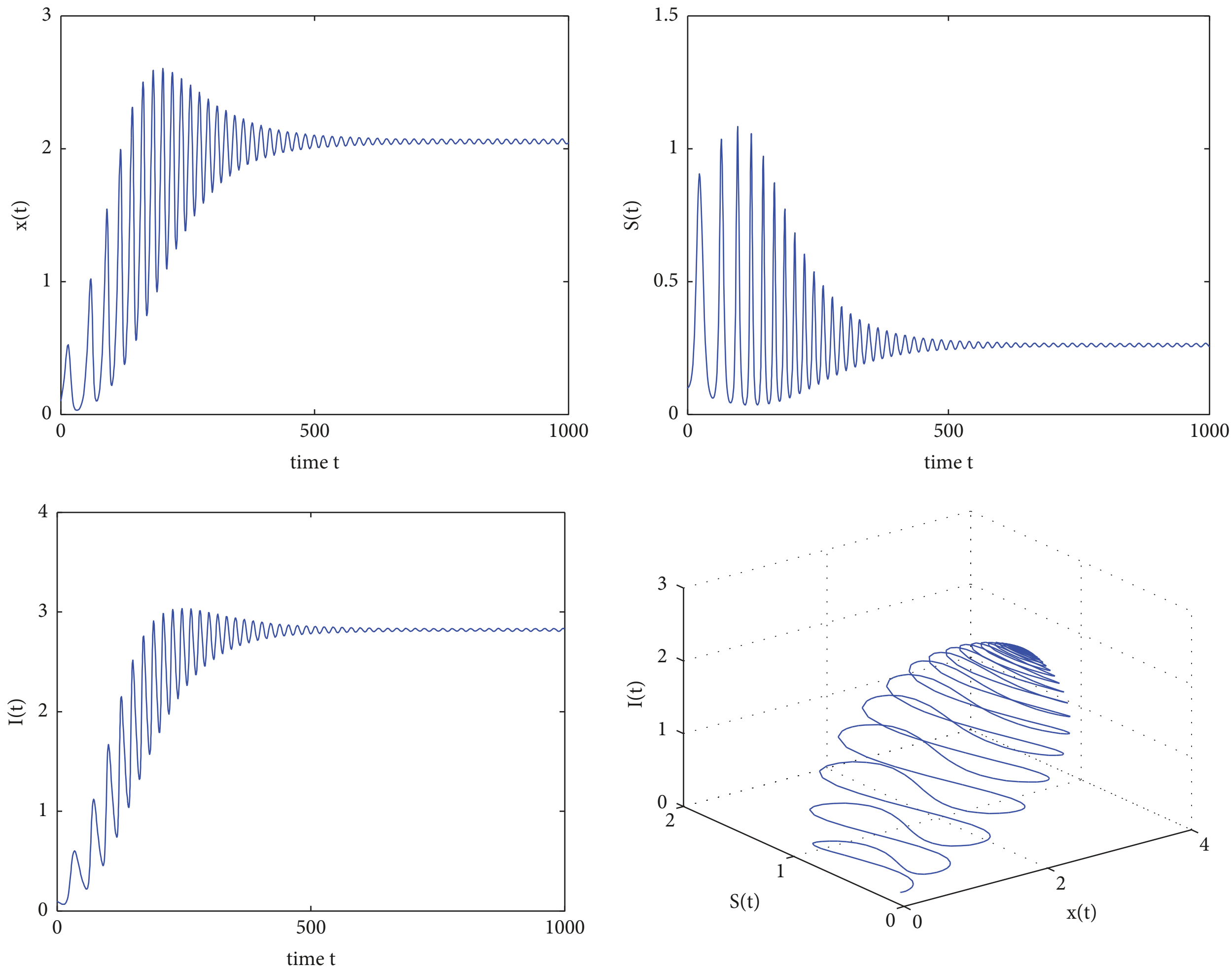

FIGURE 3: Behavior and phase portrait of the controlled system (24) with $\tau' = 0.3 < \tau_0'$. The positive equilibrium E° is asymptotically stable.

$$- \frac{0.96S(t)I(t)}{1+0.0005I(t)},$$
$$\dot{I}(t) = \frac{0.96S(t)I(t)}{1+0.0005I(t)} - 0.25I(t). \tag{53}$$

It is easy to show that if $(H1)$ holds, system (53) has a unique coexistence equilibrium $E^*(0.6723, 0.2605, 0.7519)$. For $\tau = 0$, $(H2)$ is satisfied and then the equilibrium is locally asymptotically stable. For $\tau \neq 0$, we obtain $\omega_0 = 0.7176$, $\tau_0 = 0.1718$, and $h_1' = 0.3476 \neq 0$; that is, the transversal condition is satisfied. From Theorem 2, the coexistence equilibrium $E^*(0.6723, 0.2605, 0.7519)$ is asymptotically stable for $\tau \in (0, \tau_0]$. For $\tau = 0.1 < \tau_0$, which can be shown in Figure 1, the positive equilibrium $E^*(0.6723, 0.2605, 0.7519)$ is unstable for $\tau > \tau_0$. For $\tau = 0.3 > \tau_0$, this property can be illustrated in Figure 2.

Next, we choose appropriate values of p, q to control system (2). Let us consider the following system with hybrid control strategy:

$$\dot{x}(t) = 0.3\left[x(t)(0.55 - 0.125x(t)) - \frac{1.8x(t)S(t)}{1+0.01x(t)}\right] + 0.05x(t),$$
$$\dot{S}(t) = 0.3\left[\frac{1.35x(t-\tau)S(t-\tau)}{1+0.01x(t-\tau)} - 0.18S(t) - \frac{0.96S(t)I(t)}{1+0.0005I(t)}\right] + 0.05S(t),$$
$$\dot{I}(t) = 0.3\left[\frac{0.96S(t)I(t)}{1+0.0005I(t)} - 0.25I(t)\right]. \tag{54}$$

It is not difficult to verify that if $(H3)$ holds, we obtain the positive equilibrium $E^\circ(2.0434, 0.2606, 1.4080)$. From the analysis in Section 3, we obtain $\omega_0' = 0.3192$ and $\tau_0' = 0.9075$. By choosing $\tau' = 0.3$ and $\tau' = 1.3$, the dynamical behavior of this controlled model (54) is illustrated in Figures 3 and 4. That is, for $\tau' = 0.3 < \tau_0'$, the positive equilibrium E° is asymptotically stable. However, when the time delay τ' passes through the critical value τ_0', the positive equilibrium E° will lose its stability, a Hopf bifurcation occurs, and a family of periodic solutions bifurcate from the positive equilibrium E°.

Comparing Figures 3 and 4 with Figures 1 and 2, we can easily make the Hopf bifurcation of the uncontrolled system (2) disappear. It is shown that the onset of Hopf bifurcation is delayed when the hybrid controller has been incorporated

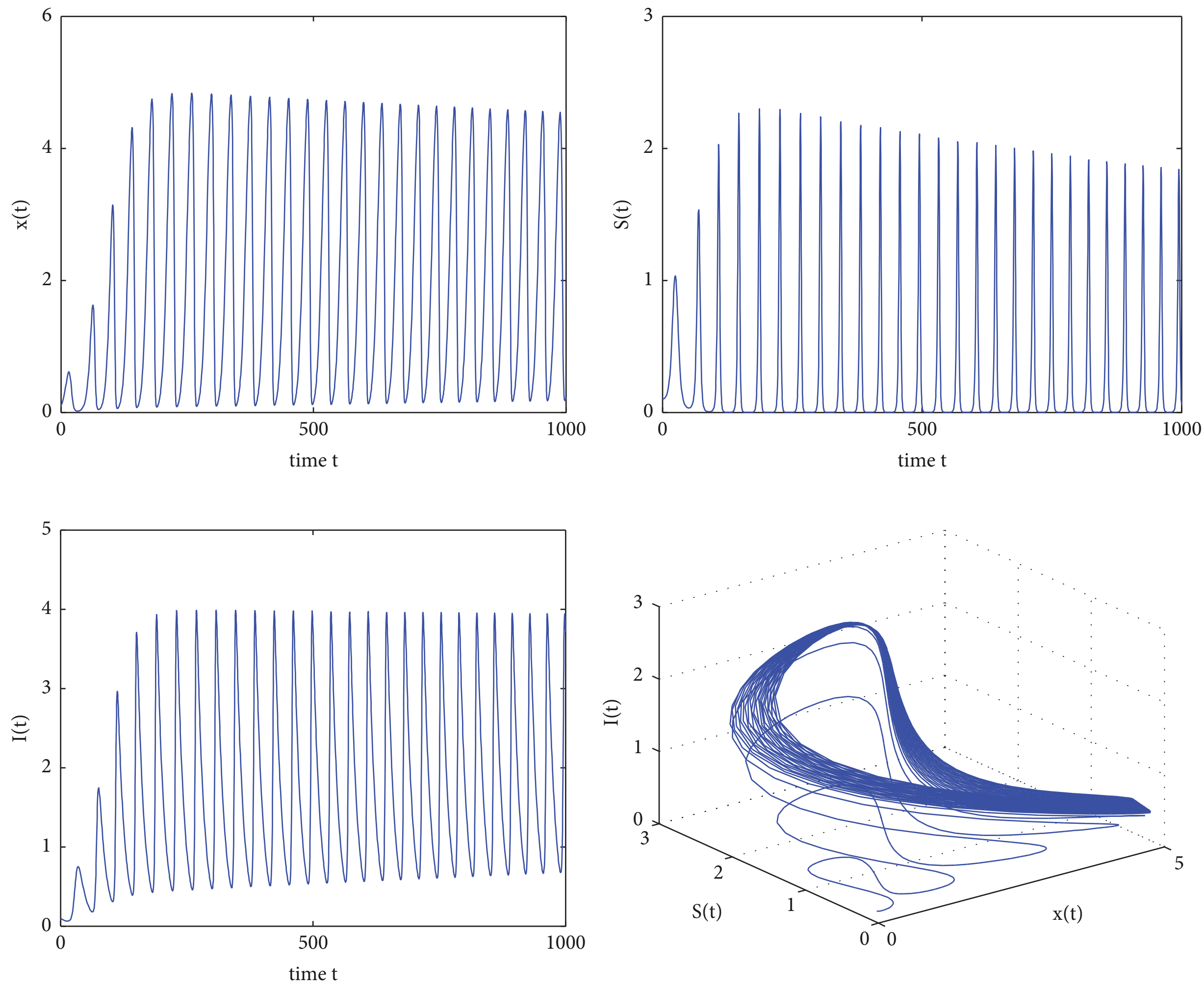

Figure 4: Behavior and phase portrait of the controlled system (24) with $\tau' = 1.3 > \tau_0'$. Hopf bifurcation occurs from the positive equilibrium E°.

into the model and the critical value of delay increases from $\tau_0 = 0.1718$ to $\tau_0' = 0.9075$. After computation of (52), we obtain $c_1(0) = -8.5035 + 6.0113i$, $T_2 = 68.0768$, $\mu_2 = 368.1169$, and $\beta_2 = -17.0070$. From Theorem 3, the Hopf bifurcation is supercritical, the bifurcation period solutions are stable, and the bifurcating periodic solutions increase.

These numerical simulation results illustrate excellent validations of the new theoretical analysis presented in this paper. Because the bifurcation periodic solutions are stable, the species in model (24) imply coexistence in an oscillatory mode from the viewpoint of biology.

5. Conclusions

In this paper, we have incorporated nonlinear incidence rate and time delay into an ecoepidemiological model. By analyzing the associated characteristic equation, its local stability and the existence of Hopf bifurcation with respect to time delay are established. To postpone the onset of the Hopf bifurcation, we use a hybrid controller in system (2). It has been shown that the critical value of the delay increases from $\tau_0 = 0.1718$ to $\tau_0' = 0.9075$. By the normal form theory and the center manifold theorem, we analyze the stability and direction of the bifurcating periodic solutions. The hybrid control strategy is closely related to the continuous survival of the population. The unstable equilibrium point of a uncontrolled system becomes an asymptotically stable equilibrium point in the controlled system after using hybrid control by combining the state feedback control and perturbation parameter. That is, the number of predators and preys eventually implies stability and coexistence. The numerical results are in accord with theoretical analysis. It has certain ecological significance and provides a theoretical basis for the continuous survival of the population in nature.

In addition, the model with the disease spreading in predator population is investigated in this paper. If diseasees spreading in prey and predator population coexist together, the influence will be analyzed in a future work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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