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Research Article

Hopf Bifurcation of a Delayed Ecoepidemic Model with Ratio-Dependent Transmission Rate

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A delayed ecoepidemic model with ratio-dependent transmission rate has been proposed in this paper. Effects of the time delay due to the gestation of the predator are the main focus of our work. Sufficient conditions for local stability and existence of a Hopf bifurcation of the model are derived by regarding the time delay as the bifurcation parameter. Furthermore, properties of the Hopf bifurcation are investigated by using the normal form theory and the center manifold theorem. Finally, numerical simulations are carried out in order to validate our obtained theoretical results.

1. Introduction

In recent years, many dynamical models characterizing the propagation of infectious disease [1–3], spread of computer viruses [4–6], and dynamics of some other systems [7–10] are studied by scholars. Ecoepidemiological research deals with the study of the spread of diseases among interacting populations, where the epidemic and demographic aspects are merged within one model. And they have been investigated by many scholars at home and abroad since the pioneer work of Kermack and McKendrick [11], and the interests in investigating the dynamics of ecoepidemic models will be increasing steadily due to its importance from both the mathematical and the ecological points of view.

Many scholars studied different predator-prey models with disease infection in the prey. Chakraborty et al. [12] studied a ratio-dependent ecoepidemic model with prey harvesting and they assumed that both the susceptible and infected prey are subjected to combined harvesting. Upadhyay and Roy [13] proposed an ecoepidemic model with simple law of mass action and modified Holling type II functional response based on the model in [14]. They analyzed stability (linear and nonlinear) of the model. Zhang et al. [15] proposed a three species ecoepidemic model perturbed by white noise and they studied stochastic stability and longtime behavior of the model. Zhou et al. [16] studied local and global stability of a modified Leslie-Gower predator-prey model with prey infection. Some delayed ecoepidemic models with disease infection in the prey have been proposed, and the effect of the delay on the models has been investigated [17–19]. Similarly, some scholars proposed and investigated the ecoepidemic models with disease in predators. Sarwardi et al. [20] and Shaikh et al. [21] studied a Leslie-Gower Holling type II predator-prey model with disease in predator and Leslie-Gower Holling type III predator-prey model with disease in predator, respectively. Some other ecoepidemic models with disease in predators one can refer to include [22–29].

Clearly, most of the epidemic models above are formulated based on the bilinear transmission rate, which is based on the law of mass action. As stated in [30], transmission rate plays an important role in the modelling of epidemic dynamics and the infection probability per contact is likely influenced by the number of infective individuals. Thus, it can be concluded that nonlinear transmission rate seems more reasonable than the bilinear one. To study the effect of a nonlinear incidence rate on the dynamics of an ecoepidemic model, Maji et al. [31] proposed the following ecoepidemic model based the work of Morozov [32]:

$$\begin{aligned}
\frac{dS(t)}{dt} &= S(t)\left[r\left(1-\frac{S(t)+I(t)}{K}\right)\right.\\
&\quad\left.-\left(\lambda_0+\frac{aP(t)}{1+bP(t)}\right)\frac{I(t)}{S(t)+I(t)}\right],\\
\frac{dI(t)}{dt} &= \left(\lambda_0+\frac{aP(t)}{1+bP(t)}\right)\frac{S(t)I(t)}{S(t)+I(t)}-dI(t)\\
&\quad-\frac{\alpha_1 I(t)P(t)}{1+\beta I(t)},\\
\frac{dP(t)}{dt} &= \frac{\alpha_2 I(t)P(t)}{1+\beta I(t)}-\delta P(t),
\end{aligned} \tag{1}$$

where $S(t) > 0$, $I(t) \geq 0$, and $P(t) > 0$ present the densities of the healthy prey, the infected prey, and the predator population, respectively. More parameters are listed in Table 1. They studied stability and persistence of system (1).

As we know, delay differential equations exhibit much more complicated dynamics than ordinary differential equations, and delays can make a dynamical system lose its stability and can induce various oscillations and periodic solutions [17, 23, 26, 33–38]. It is interesting to study the effect of time delay on system (1). To this end, and considering the time required for the gestation of the predator, we incorporate time delay due to the gestation of the predator into system (1) and get the following delayed ecoepidemic system:

$$\begin{aligned}
\frac{dS(t)}{dt} &= S(t)\left[r\left(1-\frac{S(t)+I(t)}{K}\right)\right.\\
&\quad\left.-\left(\lambda_0+\frac{aP(t)}{1+bP(t)}\right)\frac{I(t)}{S(t)+I(t)}\right],\\
\frac{dI(t)}{dt} &= \left(\lambda_0+\frac{aP(t)}{1+bP(t)}\right)\frac{S(t)I(t)}{S(t)+I(t)}-dI(t)\\
&\quad-\frac{\alpha_1 I(t)P(t)}{1+\beta I(t)},\\
\frac{dP(t)}{dt} &= \frac{\alpha_2 I(t-\tau)P(t-\tau)}{1+\beta I(t-\tau)}-\delta P(t),
\end{aligned} \tag{2}$$

subjected to the initial condition:

$$\begin{aligned}
S(\theta) &= \phi_1(\theta) > 0,\\
I(\theta) &= \phi_2(\theta) > 0,\\
P(\theta) &= \phi_3(\theta) > 0, \quad \theta \in [-\tau, 0)
\end{aligned} \tag{3}$$

where τ is the time delay due to the gestation of the predator.

This paper is organized as follows. Section 2 deals with local stability and existence of the Hopf bifurcation. In Section 3, direction and stability of the Hopf bifurcation are obtained by using center manifold and normal form theory. In Section 4, some numerical simulations are presented in order to verify the analytical findings. Conclusions and discussions are presented in Section 5.

2. Local Stability of the Positive Equilibrium

By direct computation, we can conclude that if $\alpha_2 > \delta\beta$, then system (2) has positive equilibrium $E_*(S_*, I_*, P_*)$, where

$$\begin{aligned}
I_* &= \frac{\delta}{\alpha_2-\delta\beta},\\
P_* &= \frac{C_2S_*^2+C_1S_*+C_0}{D_2S_*^2+D_1S_*+D_0},
\end{aligned} \tag{4}$$

where S_* is the positive root of (5)

$$K_5S^5+K_4S^4+K_3S^3+K_2S^2+K_1S+K_0=0, \tag{5}$$

with

$$\begin{aligned}
K_0 &= -\left(A_2C_0^2+A_1C_0D_0+A_0D_0^2\right),\\
K_1 &= B_2C_0^2+B_1C_0D_0+B_0D_0^2-2A_2C_0C_1-A_1C_1D_0\\
&\quad-A_1C_0D_1-2A_0D_0D_1,\\
K_2 &= 2B_2C_0C_1+B_1C_1D_0+B_1C_0D_1+2B_0D_0D_1\\
&\quad-A_2C_1^2-2A_2C_0C_2-A_1C_2D_0-A_1C_1D_1\\
&\quad-A_1C_0D_2-A_0D_1^2-2A_0D_0D_2,\\
K_3 &= B_2C_1^2+2B_2C_0C_2+B_1C_2D_0+B_1C_1D_1\\
&\quad+B_1C_0D_2+B_0D_1^2+2B_0D_0D_2-2A_2C_1C_2\\
&\quad-A_1C_2D_1-A_1C_1D_2-2A_0D_1D_2,\\
K_4 &= 2B_2C_1C_2+B_1C_2D_1+B_1C_1D_2+2B_0D_1D_2\\
&\quad-A_2C_2^2-A_1C_2D_2-A_0D_2^2,\\
K_5 &= B_2C_2^2+B_1C_2D_2,
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
A_0 &= dI_*(1+\beta I_*),\\
A_1 &= \alpha_1 I_*+bdI_*(1+\beta I_*),\\
A_2 &= b\alpha_1 I_*,\\
B_0 &= (\lambda_0-dI_*)(1+\beta I_*),\\
B_1 &= [a+b(\lambda_0-dI_*)](1+\beta I_*)-\alpha_1,\\
B_2 &= -b\alpha_1,\\
C_0 &= r(K-I_*)I_*-k\lambda_0 I_*,\\
C_1 &= R(K-2I_*),\\
C_2 &= -r,\\
D_0 &= KI_*(b\lambda_0+a),\\
D_1 &= brI_*,\\
D_2 &= br.
\end{aligned} \tag{7}$$

TABLE 1: Parameters and their meanings in this paper.

Parameter	Description
K	The carrying capacity of the environment
r	The maximal per capita growth rate of the healthy prey
λ_0	The transmission rate in the absence of predator
a	The predator density mediated additional disease transmission rate
b	The inhibitory effect
d	The death rate of the infected prey population
α_1	The per capita predator consumption rate
α_2	The conversion efficiency of the predator
β	The encounter rate between the predator and the infected prey

The Jacobian matrix of system (2) at $E_*(S_*, I_*, P_*)$ is

$$J(E_*) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & b_{32}e^{-\lambda\tau} & a_{33} + b_{33}e^{-\lambda\tau} \end{pmatrix}, \qquad (8)$$

where

$$\begin{aligned}
a_{11} &= \frac{S_* I_*}{(S_* + I_*)^2}\left(\lambda_0 + \frac{aP_*}{1 + bP_*}\right) - \frac{rS_*}{K}, \\
a_{12} &= -\left(\lambda_0 + \frac{aP_*}{1 + bP_*}\right)\frac{S_*^2}{(S_* + I_*)^2} - \frac{rS_*}{K}, \\
a_{13} &= -\frac{aS_* I_*}{(S_* + I_*)(1 + bP_*)^2}, \\
a_{21} &= \left(\lambda_0 + \frac{aP_*}{1 + bP_*}\right)\frac{I_*^2}{(S_* + I_*)^2}, \\
a_{22} &= \frac{\alpha_1 \beta I_* P_*}{(1 + \beta I_*)^2} - \left(\lambda_0 + \frac{aP_*}{1 + bP_*}\right)\frac{S_* I_*}{(S_* + I_*)^2}, \\
a_{23} &= \frac{aS_* I_*}{(S_* + I_*)(1 + bP_*)^2} - \frac{\alpha_1 I_*}{1 + \beta I_*}, \\
a_{33} &= -\delta, \\
b_{32} &= \frac{\alpha_2 P_*}{(1 + \beta I_*)^2}, \\
b_{33} &= \frac{\alpha_2 I_*}{1 + \beta I_*}.
\end{aligned} \qquad (9)$$

Thus, the characteristic equation of $J(E_*)$ about the positive equilibrium E_* is given by

$$\begin{aligned} &\lambda^3 + A_{02}\lambda^2 + A_{01}\lambda + A_{00} \\ &\quad + \left(B_{02}\lambda^2 + B_{01}\lambda + B_{00}\right)e^{-\lambda\tau} = 0, \end{aligned} \qquad (10)$$

with

$$\begin{aligned}
A_{00} &= a_{33}\left(a_{12}a_{21} - a_{11}a_{22}\right), \\
A_{01} &= a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21}, \\
A_{02} &= -\left(a_{11} + a_{22} + a_{33}\right), \\
B_{00} &= b_{32}\left(a_{11}a_{23} - a_{13}a_{21}\right) + b_{33}\left(a_{12}a_{21} - a_{11}a_{22}\right), \\
B_{01} &= b_{33}\left(a_{11} + a_{22}\right) - a_{23}b_{32}, \\
B_{02} &= -b_{33}.
\end{aligned} \qquad (11)$$

When $\tau = 0$, (10) becomes

$$\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 = 0, \qquad (12)$$

where

$$\begin{aligned} p_0 &= A_{00} + B_{00}, \\ p_1 &= A_{01} + B_{01}, \\ p_2 &= A_{02} + B_{02}. \end{aligned} \qquad (13)$$

Based on the Routh-Hurwitz criterion and the discussion in [31], it follows that the positive equilibrium E_* is locally asymptotically stable if the following condition holds: (H_1): $p_0 > 0$, $p_1 > 0$ and $p_1 p_2 > p_0$.

For $\tau > 0$, let $\lambda = i\omega(\omega > 0)$ be the root of (10); then

$$\begin{aligned} B_{01}\sin\tau\omega + \left(B_{00} - B_{02}\omega^2\right)\cos\tau\omega &= A_{02}\omega^2 - A_{00}, \\ B_{01}\cos\tau\omega - \left(B_{00} - B_{02}\omega^2\right)\sin\tau\omega &= \omega^3 - A_{01}\omega. \end{aligned} \qquad (14)$$

Thus,

$$\omega^6 + l_2\omega^4 + l_1\omega^2 + l_0, \qquad (15)$$

where

$$\begin{aligned} l_0 &= A_{00}^2 - B_{00}^2, \\ l_1 &= A_{01}^2 - B_{01}^2 - 2A_{00}A_{02} + 2B_{00}B_{02}, \\ l_2 &= A_{02}^2 - B_{02}^2 - 2A_{01}. \end{aligned} \qquad (16)$$

Suppose that

(H_2) (15) has at least one positive root ω_0.

For ω_0, from (14)

$$\tau_0 = \frac{1}{\omega_0} \times \arccos\left\{\frac{(B_{01} - A_{02}B_{02})\,\omega_0^4 + (A_{00}B_{02} + A_{02}B_{22} - A_{01}B_{01})\,\omega_0^2 - A_{00}B_{00}}{B_{01}^2\omega_0^2 + (B_{00} - B_{02}\omega_0^2)^2}\right\}. \quad (17)$$

Differentiating both sides of (10) with respect to τ yields

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = -\frac{3\lambda^2 + 2A_{02}\lambda + A_{01}}{\lambda(\lambda^3 + A_{02}\lambda^2 + A_{01}\lambda + A_{00})} + \frac{2B_{02}\lambda + B_{01}}{\lambda(B_{02}\lambda^2 + B_{01}\lambda + B_{00})} - \frac{\tau}{\lambda}. \quad (18)$$

Further, we have

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} = \frac{f'(v_{**})}{(\beta_1\omega_0 - \beta_3\omega_0^3)^2 + (\beta_0 - \beta_2\omega_0^2)^2}, \quad (19)$$

where $f(v) = v^3 + l_2v^2 + l_1v + l_0$ and $v = \omega^2$, $v_{**} = \omega_0^2$.

Obviously, if the condition $(H_3)\, f'(\omega_0^2) \neq 0$ holds, then $\operatorname{Re}[d\lambda/d\tau]^{-1}_{\tau=\tau_0} \neq 0$. Therefore, based on the Hopf bifurcation theorem in [39], we can obtain the following results.

Theorem 1. *Suppose that the conditions (H_1)-(H_3) hold for system (2). The positive equilibrium $E_*(S_*, I_*, P_*)$ is locally asymptotically stable when $\tau \in [0, \tau_0)$ and a Hopf bifurcation occurs at the positive equilibrium $E_*(S_*, I_*, P_*)$ when $\tau = \tau_0$.*

3. Property of the Hopf Bifurcation

Let $\tau = \tau_0 + \mu$, $\mu \in R$; then $\mu = 0$ is the Hopf bifurcation value of system (2). Rescaling the time delay $t \longrightarrow (t/\tau)$, then system (2) can be transformed into a functional differential equation in $C = C([-1, 0], R^3)$ as

$$\dot{u}(t) = L_\mu u_t + F(\mu, u_t) \quad (20)$$

where

$$L_\mu\phi = (\tau_0 + \mu)(M_1\phi(0) + M_2\phi(-1)) \quad (21)$$

and

$$F(\mu, \phi) = (\tau_0 + \mu)(F_1, F_2, F_3)^T, \quad (22)$$

with

$$M_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b_{32} & b_{33} \end{pmatrix}, \quad (23)$$

and

$$\begin{aligned} F_1 &= g_1\phi_1^2(0) + g_2\phi_1(0)\phi_2(0) + g_3\phi_1(0)\phi_3(0) \\ &\quad + g_4\phi_2(0)\phi_3(0) + g_5\phi_2^2(0) + g_6\phi_3^2(0) \\ &\quad + g_7\phi_1^3(0) + g_8\phi_2^3(0) + g_9\phi_3^3(0) \\ &\quad + g_{10}\phi_1(0)\phi_2^2(0) + \cdots, \\ F_2 &= h_1\phi_1^2(0) + h_2\phi_1(0)\phi_2(0) + h_3\phi_1(0)\phi_3(0) \\ &\quad + h_4\phi_2(0)\phi_3(0) + h_5\phi_2^2(0) + h_6\phi_3^2(0) \\ &\quad + h_7\phi_1^3(0) + h_8\phi_2^3(0) + h_9\phi_3^3(0) \\ &\quad + h_{10}\phi_1(0)\phi_2^2(0) + \cdots, \\ F_3 &= k_1\phi_2^2(-1) + k_2\phi_2(-1)\phi_3(-1) + k_3\phi_2^3(-1) \\ &\quad + k_4\phi_2^2(-1)\phi_3(-1) + \cdots, \end{aligned} \quad (24)$$

with

$$g_1 = \left(\lambda_0 + \frac{aP_*}{1 + bP_*}\right)\frac{I_*(I_* - S_*)}{2(S_* + I_*)^3} - \frac{r}{2K},$$

$$g_2 = \left(\lambda_0 + \frac{aP_*}{1 + bP_*}\right)\frac{I_*(S_* - I_*)}{2(S_* + I_*)^3},$$

$$g_3 = \frac{aS_*I_*}{(S_* + I_*)^2(1 + bP_*)^2},$$

$$g_4 = \frac{aS_*^2}{(S_* + I_*)^2(1 + bP_*)^2},$$

$$g_5 = \left(\lambda_0 + \frac{aP_*}{1 + bP_*}\right)\frac{S_*^2}{(S_* + I_*)^3},$$

$$g_6 = \frac{abS_*I_*}{(S_* + I_*)(1 + bP_*)^3},$$

$$g_7 = \left(\lambda_0 + \frac{aP_*}{1 + bP_*}\right)\frac{I_*(2S_* - I_*)}{6(S_* + I_*)^4},$$

$$g_8 = -\left(\lambda_0 + \frac{aP_*}{1 + bP_*}\right)\frac{S_*^2}{6(S_* + I_*)^4},$$

$$g_9 = \left(\lambda_0 + \frac{aP_*}{1 + bP_*}\right)\frac{S_*(2I_* - S_*)}{6(S_* + I_*)^4},$$

$$g_{10} = \frac{2I_*(S_* - I_*)}{2(S_* + I_*)^4},$$

$$h_1 = -\left(\lambda_0 + \frac{aP_*}{1+bP_*}\right)\frac{I_*^2}{(S_*+I_*)^3},$$

$$h_2 = \left(\lambda_0 + \frac{aP_*}{1+bP_*}\right)\frac{2S_*I_*}{(S_*+I_*)^3},$$

$$h_3 = \frac{aI_*^2}{2(1+bP_*)^2(S_*+I_*)^2},$$

$$h_4 = \frac{\alpha_1\beta I_*}{(1+\beta I_*)^2} - \frac{aS_*I_*}{(1+bP_*)^2(S_*+I_*)^2},$$

$$h_5 = \frac{\alpha_1\beta P_*(1-\beta)}{2(1+\beta I_*)^3} + \left(\lambda_0 + \frac{aP_*}{1+bP_*}\right)\frac{S_*(I_*-S_*)}{2(1+\beta I_*)^3},$$

$$h_6 = -\frac{abS_*I_*}{(S_*+I_*)(1+bP_*)^3},$$

$$h_7 = \left(\lambda_0 + \frac{aP_*}{1+bP_*}\right)\frac{I_*^2}{(S_*+I_*)^4},$$

$$h_8 = \frac{2\alpha_1\beta^3 I_*P_*}{(1+\beta I_*)^4} + \frac{2S_*(S_*-I_*)}{(S_*+I_*)^4},$$

$$h_9 = \frac{ab^2S_*I_*}{(S_*+I_*)(1+bP_*)^4},$$

$$h_{10} = -\left(\lambda_0 + \frac{aP_*}{1+bP_*}\right)\frac{3S_*I_*}{(S_*+I_*)^4},$$

$$k_1 = -\frac{\alpha_2\beta P_*}{(1+\beta I_*)^3},$$

$$k_2 = \frac{\alpha_2}{(1+\beta I_*)^2},$$

$$k_3 = \frac{\alpha_2\beta^2 P_*}{(1+\beta I_*)^4},$$

$$k_4 = -\frac{\alpha_2\beta}{(1+\beta I_*)^3}. \quad (25)$$

Thus, there exists a 3×3 matrix function $\eta(\theta,\mu)$, $\theta \in [-1,0]$, such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta,\mu)\,\phi(\theta), \quad \phi \in C. \quad (26)$$

In view of (21), we choose

$$\eta(\theta,\mu) = (\tau_0+\mu)\left(M_1\delta(\theta) + M_2\delta(\theta+1)\right), \quad (27)$$

where δ is the Dirac delta function.

For $\phi \in C([-1,0], R^3)$, define

$$A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \displaystyle\int_{-1}^{0} d\eta(\theta,\mu)\,\phi(\theta), & \theta = 0, \end{cases} \quad (28)$$

and

$$R(\mu)\phi = \begin{cases} 0, & -1 \le \theta < 0, \\ F(\mu,\phi), & \theta = 0. \end{cases} \quad (29)$$

Then system (20) is equivalent to

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t. \quad (30)$$

where $u_t(\theta) = u(t+\theta)$ for $\theta \in [-1,0]$.

For $\varphi \in C^1([0,1], (R^3)^*)$, define

$$A^*(\varphi) = \begin{cases} -\dfrac{d\varphi(s)}{ds}, & 0 < s \le 1, \\ \displaystyle\int_{-1}^{0} d\eta^T(s,0)\,\varphi(-s), & s = 0, \end{cases} \quad (31)$$

and a bilinear inner product

$$\langle\varphi(s),\phi(\theta)\rangle = \overline{\varphi}(0)\phi(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta}\overline{\varphi}(\xi-\theta)\,d\eta(\theta)\,\phi(\xi)\,d\xi, \quad (32)$$

where $\eta(\theta) = \eta(\theta,0)$. Then $A(0)$ and A^* are adjoint operators.

Next, we suppose that $\rho(\theta) = (1,\rho_2,\rho_3)^T e^{i\omega_0\tau_0\theta}$ is the eigenvector of $A(0)$ belonging to $+i\omega_0\tau_0$ and $\rho^*(s) = D(1,\rho_2^*,\rho_3^*)e^{i\omega_0\tau_0 s}$ is the eigenvector of $A^*(0)$ belonging to $-i\omega_0\tau_0$. According to the definition of $A(0)$ and A^*, we can obtain

$$\rho_2 = \frac{a_{21}+a_{23}\rho_3}{i\omega_0 - a_{22}},$$

$$\rho_3 = \frac{(i\omega_0-a_{11})(i\omega_0-a_{22}) - a_{12}a_{21}}{a_{13}(i\omega_0-a_{22}) - a_{12}a_{23}},$$

$$\rho_2^* = -\frac{i\omega_0+a_{22}}{a_{21}},$$

$$\rho_3^* = \frac{(i\omega_0+a_{11})(i\omega_0+a_{22}) - a_{12}a_{21}}{b_{32}e^{i\tau_0\omega_0}}. \quad (33)$$

From (32), we can get

$$\overline{D} = \left[1 + \rho_2\overline{\rho}_2^* + \rho_3\overline{\rho}_3^* + \tau_0 e^{-i\tau_0\omega_0}\left(b_{32}\rho_2\overline{\rho}_2^* + b_{33}\rho_3\overline{\rho}_3^*\right)\right]^{-1} \quad (34)$$

such that $\langle\rho^*,\rho\rangle = 1$.

Following the method in [39] and using similar computation process in [40], we can get the following coefficients:

$$g_{20}=2\tau_0\overline{D}\left[g_1+g_2\rho_2+g_3\rho_3+g_4\rho_2\rho_3+g_5\rho_2^2+g_6\rho_3^2+\overline{\rho}_2^*\left(h_1+h_2\rho_2+h_3\rho_3+h_4\rho_2\rho_3+h_5\rho_2^2+h_6\rho_3^2\right)+\overline{\rho}_3^*\left(k_1\rho_2^2e^{-2i\tau_0\omega_0}+k_2\rho_2\rho_3e^{-2i\tau_0\omega_0}\right)\right],$$

$$g_{11}=\tau_0\overline{D}\left[2g_1+g_2\left(\rho_2+\overline{\rho}_2\right)+g_3\left(\rho_3+\overline{\rho}_3\right)+g_4\left(\rho_2\overline{\rho}_3+\overline{\rho}_2\right)+2g_5\rho_2\overline{\rho}_2+2g_6\rho_3\overline{\rho}_3+\rho_2^*\left(2h_1+h_2\left(\rho_2 \mid \overline{\rho}_2\right)+h_3\left(\rho_3+\overline{\rho}_3\right)+h_4\left(\rho_2\overline{\rho}_3+\overline{\rho}_2\right)+2h_5\rho_2\overline{\rho}_2+2h_6\rho_3\overline{\rho}_3\right)+\overline{\rho}_3^*\left(2k_1\rho_2\overline{\rho}_2+k_2\left(\rho_2\overline{\rho}_3+\overline{\rho}_2\rho_3\right)\right)\right],$$

$$g_{02}=2\tau_0\overline{D}\left[g_1+g_2\overline{\rho}_2+g_3\overline{\rho}_3+g_4\overline{\rho}_2\overline{\rho}_3+g_5\overline{\rho}_2^2+g_6\overline{\rho}_3^2+\overline{\rho}_2^*\left(h_1+h_2\overline{\rho}_2+h_3\overline{\rho}_3+h_4\overline{\rho}_2\overline{\rho}_3+h_5\overline{\rho}_2^2+h_6\overline{\rho}_3^2\right)+\overline{\rho}_3^*\left(k_1\overline{\rho}_2^2e^{2i\tau_0\omega_0}+k_2\overline{\rho}_2\overline{\rho}_3e^{2i\tau_0\omega_0}\right)\right],$$

$$\begin{aligned}g_{21}={}&2\tau_0\overline{D}\Big[g_1\left(2W_{11}^{(1)}(0)+W_{20}^{(1)}(0)\right)+g_2\Big(W_{11}^{(1)}(0)\cdot\rho_2+\frac{1}{2}W_{20}^{(1)}(0)\overline{\rho}_2+W_{11}^{(2)}(0)+\frac{1}{2}W_{20}^{(2)}(0)\Big)\\&+g_3\Big(W_{11}^{(1)}(0)\rho_3+\frac{1}{2}W_{20}^{(1)}(0)\overline{\rho}_3+W_{11}^{(2)}(0)+\frac{1}{2}\cdot W_{20}^{(2)}(0)\Big)+g_4\Big(W_{11}^{(2)}(0)\rho_3+\frac{1}{2}W_{20}^{(2)}(0)\overline{\rho}_3\\&+W_{11}^{(3)}(0)\rho_2+\frac{1}{2}W_{20}^{(3)}(0)\overline{\rho}_2\Big)+g_5\left(2W_{11}^{(2)}(0)\rho_2+W_{20}^{(2)}(0)\overline{\rho}_2\right)+g_6\left(2W_{11}^{(3)}(0)\rho_3+W_{20}^{(3)}(0)\overline{\rho}_3\right)\\&+3g_7+3g_8\rho_2^2\overline{\rho}_2+3g_9\rho_3^2\overline{\rho}_3+g_{10}\left(\overline{\rho}_2+2\rho_2\overline{\rho}_2\right)\\&+\overline{\rho}_2^*\Big(h_1\left(2W_{11}^{(1)}(0)+W_{20}^{(1)}(0)\right)+h_2\Big(W_{11}^{(1)}(0)\rho_2+\frac{1}{2}W_{20}^{(1)}(0)\overline{\rho}_2+W_{11}^{(2)}(0)+\frac{1}{2}W_{20}^{(2)}(0)\Big)\\&+h_3\Big(W_{11}^{(1)}(0)\rho_3+\frac{1}{2}W_{20}^{(1)}(0)\overline{\rho}_3+W_{11}^{(2)}(0)+\frac{1}{2}W_{20}^{(2)}(0)\Big)+h_4\Big(W_{11}^{(2)}(0)\rho_3+\frac{1}{2}W_{20}^{(2)}(0)\overline{\rho}_3\\&+W_{11}^{(3)}(0)\rho_2+\frac{1}{2}W_{20}^{(3)}(0)\overline{\rho}_2\Big)+h_5\left(2W_{11}^{(2)}(0)\rho_2+W_{20}^{(2)}(0)\overline{\rho}_2\right)+g_6\left(2W_{11}^{(3)}(0)\rho_3+W_{20}^{(3)}(0)\overline{\rho}_3\right)\\&+3h_7+3h_8\rho_2^2\overline{\rho}_2+3h_9\rho_3^2\overline{\rho}_3+h_{10}\left(\overline{\rho}_2+2\rho_2\overline{\rho}_2\right)\Big)\\&+\overline{\rho}_3^*\Big(k_1\left(2W_{11}^{(2)}(-1)\rho_2e^{-i\tau_0\omega_0}\right)+k_2\Big(W_{11}^{(2)}(-1)\rho_3e^{-i\tau_0\omega_0}+\frac{1}{2}W_{20}^{(2)}(-1)\overline{\rho}_3e^{i\tau_0\omega_0}\\&+W_{11}^{(3)}(-1)\rho_2e^{-i\tau_0\omega_0}+\frac{1}{2}W_{20}^{(3)}(-1)\overline{\rho}_2e^{i\tau_0\omega_0}\Big)+3k_3\rho_2e^{-2i\tau_0\omega_0}+k_4\Big(\rho_2e^{-i\tau_0\omega_0}\overline{\rho}_3\\&+2\rho_2\overline{\rho}_2\rho_3e^{-i\tau_0\omega_0}\Big)\Big)\Big],\end{aligned} \tag{35}$$

with

$$\begin{aligned}W_{20}(\theta)&=\frac{ig_{20}\rho(0)}{\tau_0\omega_0}e^{i\tau_0\omega_0\theta}+\frac{i\overline{g}_{02}\overline{\rho}(0)}{3\tau_0\omega_0}e^{-i\tau_0\omega_0\theta}+E_1e^{2i\tau_0\omega_0\theta},\\W_{11}(\theta)&=-\frac{ig_{11}\rho(0)}{\tau_0\omega_0}e^{i\tau_0\omega_0\theta}+\frac{i\overline{g}_{11}\overline{\rho}(0)}{\tau_0\omega_0}e^{-i\tau_0\omega_0\theta}+E_2,\end{aligned} \tag{36}$$

where E_1 and E_2 can be determined by the following two equations:

$$\begin{aligned}&\begin{pmatrix}2i\omega_0-a_{11} & -a_{12} & -a_{13}\\ -a_{21} & 2i\omega_0-a_{22} & -a_{23}\\ 0 & -b_{32}e^{-2i\tau_0\omega_0} & 2i\omega_0-a_{33}-b_{33}e^{-2i\tau_0\omega_0}\end{pmatrix}E_1=2\begin{pmatrix}E_{11}\\E_{12}\\E_{13}\end{pmatrix},\\&\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ 0 & b_{32} & a_{33}+b_{33}\end{pmatrix}E_2=-\begin{pmatrix}E_{21}\\E_{22}\\E_{23}\end{pmatrix},\end{aligned} \tag{37}$$

and

$$\begin{aligned}E_{11}&=g_1+g_2\rho_2+g_3\rho_3+g_4\rho_2\rho_3+g_5\rho_2^2+g_6\rho_3^2,\\E_{12}&=h_1+h_2\rho_2+h_3\rho_3+h_4\rho_2\rho_3+h_5\rho_2^2+h_6\rho_3^2,\\E_{13}&=k_1\rho_2^2e^{-2i\tau_0\omega_0}+k_2\rho_2\rho_3e^{-2i\tau_0\omega_0},\\E_{21}&=2g_1+g_2\left(\rho_2+\overline{\rho}_2\right)+g_3\left(\rho_3+\overline{\rho}_3\right)+g_4\left(\rho_2\overline{\rho}_3+\overline{\rho}_2\right)+2g_5\rho_2\overline{\rho}_2+2g_6\rho_3\overline{\rho}_3,\\E_{22}&=2h_1+h_2\left(\rho_2 \mid \overline{\rho}_2\right)+h_3\left(\rho_3+\overline{\rho}_3\right)+h_4\left(\rho_2\overline{\rho}_3+\overline{\rho}_2\right)+2h_5\rho_2\overline{\rho}_2+2h_6\rho_3\overline{\rho}_3,\\E_{23}&=2k_1\rho_2\overline{\rho}_2+k_2\left(\rho_2\overline{\rho}_3+\overline{\rho}_2\rho_3\right).\end{aligned} \tag{38}$$

Then, we can get the following coefficients which determine the properties of the Hopf bifurcation:

$$C_1(0)=\frac{i}{2\tau_0\omega_0}\left(g_{11}g_{20}-2\left|g_{11}\right|^2-\frac{\left|g_{02}\right|^2}{3}\right)+\frac{g_{21}}{2},$$

$$\mu_2=-\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_0)\}},$$

$$\beta_2 = 2\mathrm{Re}\{C_1(0)\},$$
$$T_2 = -\frac{\mathrm{Im}\{C_1(0)\} + \mu_2 \mathrm{Im}\{\lambda'(\tau_0)\}}{\tau_0\omega_0}. \tag{39}$$

In conclusion, we have the following results.

Theorem 2. *For system (2), If $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical). If $\beta_2 < 0$ ($\beta_2 > 0$), then the bifurcating periodic solutions are stable (unstable). If $T_2 > 0$ ($T_2 < 00$), then the bifurcating periodic solutions increase (decrease).*

4. Numerical Simulation

We choose the same parameters of system (2) as those in [21]: $r = 3$, $K = 5$, $\lambda_0 = 1.5$, $a = 1$, $b = 1$, $d = 0.5$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta = 1$, and $\delta = 0.5$, while setting τ as the bifurcation parameter. Then, we get the specific case of system (2) as follows:

$$\begin{aligned}\frac{dS(t)}{dt} &= S(t)\left[3\left(1 - \frac{S(t)+I(t)}{5}\right) - \left(1.5 + \frac{P(t)}{1+P(t)}\right)\frac{I(t)}{S(t)+I(t)}\right],\\ \frac{dI(t)}{dt} &= \left(1.5 + \frac{P(t)}{1+P(t)}\right)\frac{S(t)I(t)}{S(t)+I(t)} - 0.5I(t) - \frac{I(t)P(t)}{1+I(t)},\\ \frac{dP(t)}{dt} &= \frac{I(t-\tau)P(t-\tau)}{1+I(t-\tau)} - 0.5P(t),\end{aligned} \tag{40}$$

from which we can obtain the unique positive equilibrium $E_*(3.107, 1, 2.328)$. Numerically for $\tau = 0$ we have drawn the figure of Lyapunov exponents (Figure 1). Since all the LEs are negative, the system is stable for $\tau = 0$. Further, we can obtain $\omega_0 = 0.0042$ and the critical value $\tau_0 = 0.3408$ at which a Hopf bifurcation occurs. As is shown in Figure 2, E_* is locally asymptotically stable when $\tau = 0.265 < \tau_0$. In this case, the three species in system (40) can coexist in an ideal stable state. However, E_* loses its stability and a family of periodic solutions bifurcate from E_* when $\tau = 0.405 > \tau_0$, which can be illustrated by Figure 3.

On the other hand, by some complex calculations, we can obtain $\lambda'(\tau_0) = 0.002582 + 0.102144i$ and $C - 1(0) = -0.005236 + 0.000094i$. And further we have $\mu_2 = 2.0279 > 0$, $\beta_2 = -0.0105 < 0$ and $T_2 = -144.7797 < 0$. Thus, based on the Theorem 2, we can conclude that the Hopf bifurcation is supercritical and the bifurcating periodic solutions are stable and decrease. Since the bifurcating periodic solutions are stable, the three species in system (40) can coexist in an oscillatory mode under some given conditions. This is valuable from the viewpoint of biology.

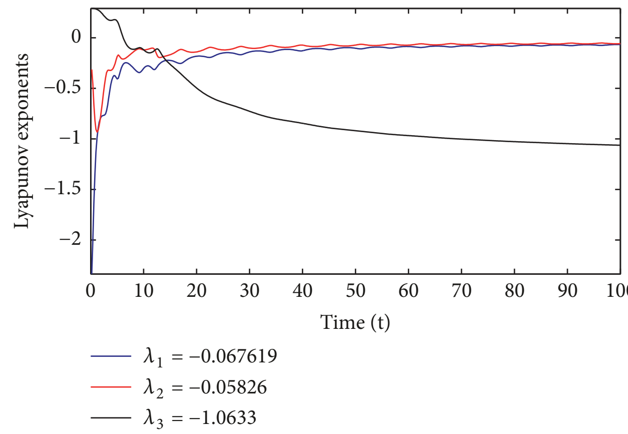

Figure 1: Lyapunov exponents for $\tau = 0$, depicting a stable system.

5. Conclusions

In the present paper, we propose a delayed ecoepidemic model by incorporating the time delay due to the gestation of the predator in the model studied in [31]. Compared with the work in [31], we mainly consider the effect of the time delay on the stability of system (2). The model investigated in our paper is more general since the time required for the gestation of the predator and the results we obtained are suitable complements to the literature [31]. By regarding the time delay due to the gestation of the predator as the bifurcation parameter, sufficient conditions for the local stability of the model and the critical value τ_0 at which a Hopf bifurcation occurs are derived. It is found that when the value of the time delay is suitablely small, system (2) is locally asymptotically stable. In this case, the densities of the healthy prey, the infected prey, and the predator population will tend to stabilization. Namely, the densities of the three species will be in ideal stable state and the disease spreading among the prey can be controlled. Once the value of the time delay passes through the critical value τ_0, system (2) loses stability and a family of periodic solutions bifurcate from the positive equilibrium E_*, which shows that the delay due to the gestation of the predator plays a very complicated role in destabilizing the stability of system (2). In this case, the densities of the three species may coexist in an oscillatory and the disease spreading among the prey will be out of control. In addition, the explicit formulae determining stability and direction of the Hopf bifurcation are derived by using the normal form theory and then center manifold theorem for the further investigation.

It should be pointed out that predator-prey models involving delays and also spatial diffusion are increasingly applied to the study of a variety of situations. Based on this consideration, we will investigate the dynamics of the eco-epidemic model with diffusion based on the delayed model in our present paper in the near future.

Data Availability

All the data can be accessed in our manuscript in the Numerical Simulation.

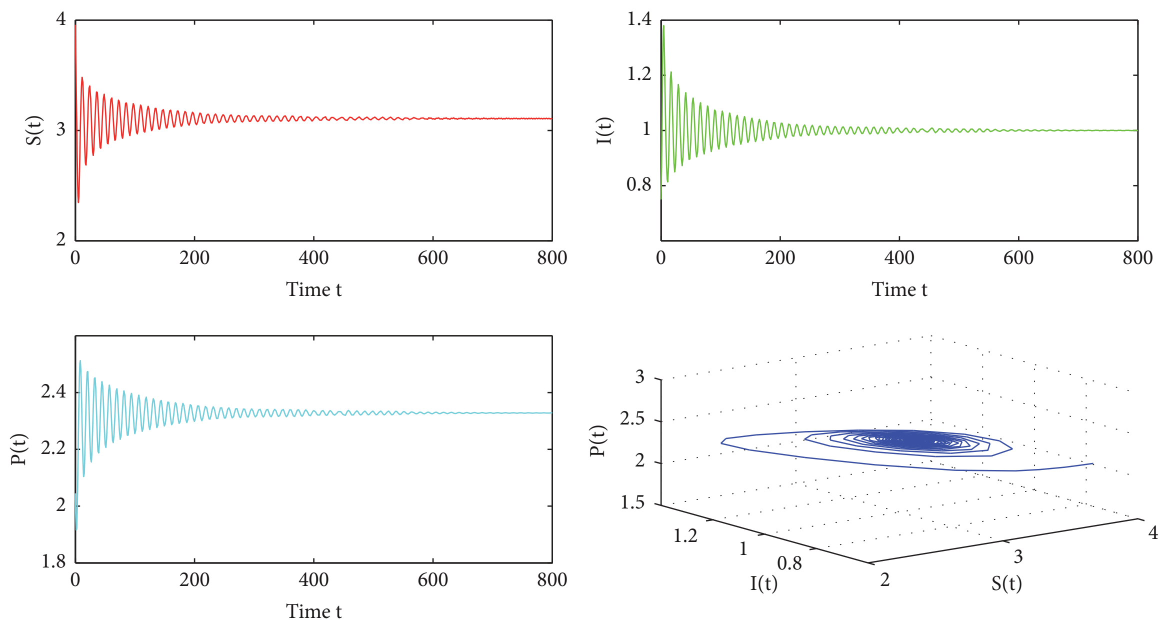

Figure 2: E_* is locally asymptotically stable when $\tau = 0.265 < \tau_0$.

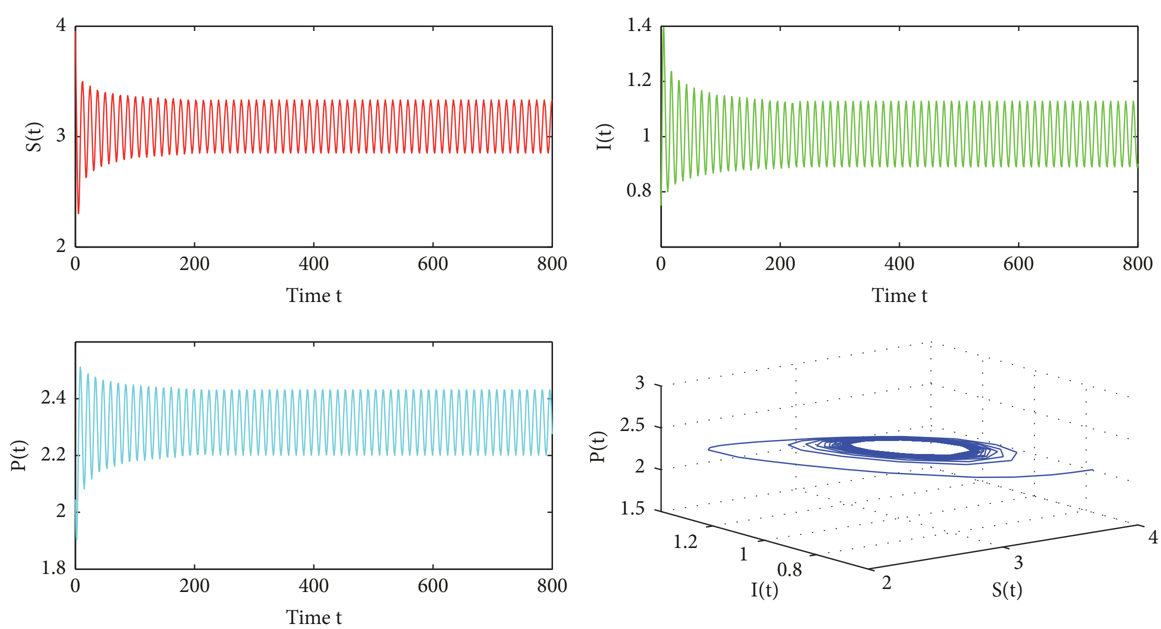

Figure 3: E_* loses its stability when $\tau = 0.405 > \tau_0$.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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