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REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS WITH LIE ξ-PARALLEL NORMAL JACOBI OPERATOR

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Abstract. In this paper we give some non-existence theorems for real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with Lie ξ-parallel normal Jacobi operator $\bar{R}_N$ and another geometric conditions.

0. Introduction

In the geometry of real hypersurfaces in complex space forms $M_n(c)$ or in quaternionic space forms $\mathbb{Q}_n(c)$ Kimura [7] (resp. Pérez [10]) has classified real hypersurfaces in $M_n(c)$ and (resp. in $\mathbb{Q}_n(c)$) with commuting Ricci tensor, that is, $S\phi = \phi S$, (resp. $S\phi_i = \phi_i S$, $i = 1, 2, 3$) where S and ϕ (resp. S and ϕ_i, $i = 1, 2, 3$) denote the Ricci tensor and the structure tensor of a real hypersurface M in $M_n(c)$ (resp. in $\mathbb{Q}_n(c)$).

In particular, Kimura and Maeda [8] have considered a real hypersurface M in a complex projective space $P_n(\mathbb{C})$ with Lie ξ-parallel Ricci tensor and classified that M is locally congruent to of type (A), a tube over a totally geodesic $P_k(\mathbb{C})$, of type (B), a tube over a complex quadric Q_{n-1}, $\cot^2 2r = n-2$, of type (C), a tube over $P_1(\mathbb{C})\times P_{(n-1)/2}(\mathbb{C})$, $\cot^2 2r = \frac{1}{n-2}$ and n is odd, of type (D), a tube over a complex two-plane Grassmannian $G_2(\mathbb{C}^5)$, $\cot^2 2r = \frac{3}{5}$ and $n = 9$, of type (E), a tube over a Hermitian symmetric space $SO(10)/U(5)$, $\cot^2 2r = \frac{5}{9}$ and $n = 15$. Then it turns out that all of them mentioned above are Hopf hypersurfaces and have commuting Ricci tensors.

If the structure vector $\xi = -JN$ of a real hypersurface M in $P_n(\mathbb{C})$ is invariant by the shape operator, M is said to be a Hopf hypersurface, where J denotes a Kaehler structure of $P_n(\mathbb{C})$, N a unit normal vector of M in $P_n(\mathbb{C})$.

In a quaternionic projective space $\mathbb{Q}P^m$ Pérez and the second author [11] have classified real hypersurfaces in $\mathbb{Q}P^m$ with $\mathfrak{D}^{\perp}$-parallel curvature tensor $\nabla_{\xi_i} R = 0$, $i = 1, 2, 3$, where R denotes the curvature tensor of M in $\mathbb{Q}P^m$

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and $\mathfrak{D}^{\perp}$ a distribution defined by $\mathfrak{D}^{\perp} = \text{Span } \{\xi_1, \xi_2, \xi_3\}$. In such a case they are congruent to a tube of radius $\frac{\pi}{4}$ over a totally geodesic $\mathbb{Q}P^k$ in $\mathbb{Q}P^m$, $2 \leq k \leq m-2$.

The almost contact structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ mentioned above are defined by $\xi_i = -J_i N$, $i = 1, 2, 3$, where $\{J_1, J_2, J_3\}$ denote a quaternionic Kähler structure of $\mathbb{Q}P^m$ and N a unit normal field of M in $\mathbb{Q}P^m$.

In quaternionic space forms Berndt [2] has introduced the notion of normal Jacobi operator $\bar{R}_N = \bar{R}(X, N)N \in \text{End } T_x M$, $x \in M$ for real hypersurfaces M in a quaternionic projective space $\mathbb{Q}P^m$ or in a quaternionic hyperbolic space $\mathbb{Q}H^m$, where $\bar{R}$ denotes the curvature tensor of $\mathbb{Q}P^m$ and $\mathbb{Q}H^m$ respectively. He [2] has also shown that the curvature adaptedness, that is, the normal Jacobi operator $\bar{R}_N$ commutes with the shape operator A, is equivalent to the fact that the distributions $\mathfrak{D}$ and $\mathfrak{D}^{\perp} = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant by the shape operator A of M, where $T_x M = \mathfrak{D} \oplus \mathfrak{D}^{\perp}$, $x \in M$.

Now let us consider a complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ which consists of all complex 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. Then the situation for real hypersurfaces in $G_2(\mathbb{C}^{m+1})$ with parallel normal Jacobi operator is not so simple and will be quite different from the cases mentioned above.

Now in this paper we consider a real hypersurface M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with Lie ξ-parallel normal Jacobi operator, $\mathcal{L}_{\xi} \bar{R}_N = 0$, where $\bar{R}$ and N respectively denotes the curvature tensor of the ambient space $G_2(\mathbb{C}^{m+2})$ and a unit normal vector of M in $G_2(\mathbb{C}^{m+2})$. The curvature tensor $\bar{R}(X, Y)Z$ for any vector fields X, Y and Z on $G_2(\mathbb{C}^{m+2})$ is explicitly defined in section 1. Then the normal Jacobi operator $\bar{R}_N$ for the unit normal vector N can be defined from the curvature tensor $\bar{R}(X, N)N$ by putting $Y = Z = N$.

The ambient space $G_2(\mathbb{C}^{m+2})$ is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J (See Berndt [3]). So, in $G_2(\mathbb{C}^{m+2})$ we have the two natural geometric conditions for real hypersurfaces that $[\xi] = \text{Span } \{\xi\}$ or $\mathfrak{D}^{\perp} = \text{Span } \{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator. By using such kinds of conditions Berndt and the second author [4] have proved the following:

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^{\perp}$ are invariant under the shape operator of M if and only if*

- (A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*
- (B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

If the structure vector field ξ of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is invariant by the shape operator, M is said to be a *Hopf hypersurface*. In such a case the integral curves of the structure vector field ξ are geodesics (See

Berndt and Suh [5]). Moreover, the flow generated by the integral curves of the structure vector field ξ for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ is said to be *geodesic Reeb flow*. Moreover, we say that the Reeb vector field is Killing, that is, $\mathcal{L}_\xi g = 0$ for the Lie derivative along the direction of the structure vector field ξ, where g denotes the Riemannian metric induced from $G_2(\mathbb{C}^{m+2})$. Then this is equivalent to the fact that the structure tensor ϕ commutes with the shape operator A of M in $G_2(\mathbb{C}^{m+2})$. This condition also has the geometric meaning that the flow of Reeb vector field is isometric. Moreover, Berndt and the second author [5] have proved that real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with isometric flow is of a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Now by putting a unit normal vector N into the curvature tensor $\bar{R}$ of the ambient space $G_2(\mathbb{C}^{m+2})$, we calculate the normal Jacobi operator $\bar{R}_N$ in such a way that

$$\begin{aligned}\bar{R}(X,N)N &= X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\ &\quad - \sum_{\nu=1}^{3}\{\eta_\nu(\xi)J_\nu(\phi X + \eta(X)N) - \eta_\nu(\phi X)(\phi_\nu\xi + \eta_\nu(\xi)N)\} \\ &= X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\ &\quad - \sum_{\nu=1}^{3}\{\eta_\nu(\xi)(\phi_\nu\phi X - \eta(X)\xi_\nu) - \eta_\nu(\phi X)\phi_\nu\xi\}\end{aligned}$$

for any tangent vector field X on M in $G_2(\mathbb{C}^{m+2})$.

On the other hand, we introduce the following theorem due to Pérez and the present authors [6] as follows:

Theorem B. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If the normal Jacobi and the structure operators both commute with the shape operator, then M is congruent to one of the following:*

(*A*) *an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(*B*) *an open part of a tube around a totally geodesic and totally real $\mathbb{Q}P^n$, $m = 2n$, in $G_2(\mathbb{C}^{m+2})$.*

But related to the normal Jacobi operator $\bar{R}_N$, in this paper we want to give some non-existence theorems for real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ with Lie ξ-parallel normal Jacobi operator, that is, $\mathcal{L}_\xi \bar{R}_N = 0$ as follows:

Theorem 1. *There do not exist any real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying $\mathcal{L}_\xi \bar{R}_N = 0$ and $\xi\in\mathfrak{D}^\perp$.*

Theorem 2. *There do not exist any real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying $\mathcal{L}_\xi \bar{R}_N = 0$ and $\xi\in\mathfrak{D}$.*

On the other hand, we say that a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ has *commuting shape operator* on the distribution $\mathfrak{D}^\perp$ if the shape operator A of M commutes with the structure tensor ϕ on $\mathfrak{D}^\perp$, that is, $A\phi\xi_\nu = \phi A\xi_\nu$, $\nu = 1, 2, 3$.

Now in the final section, as an application of Theorems 1 and 2 we consider a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with Lie ξ-parallel and commuting shape operator on the distribution $\mathfrak{D}^\perp$. Then by virtue of Theorems 1 and 2 we assert the following:

Theorem 3. *There do not exist any Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\mathcal{L}_\xi \bar{R}_N = 0$ and commuting shape operator on the distribution $\mathfrak{D}^\perp$.*

1. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [3], [4], and [5]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_o G_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight. Since $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight we will assume $m \geq 2$ from now on. Note that the isomorphism $\mathrm{Spin}(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces of $\mathbb{R}^6$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_1 is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_1 = J_1 J$, and JJ_1 is a symmetric endomorphism with $(JJ_1)^2 = I$ and $\mathrm{tr}(JJ_1) = 0$. This fact will be used frequently throughout this paper.

A canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$, where the index is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\bar{\nabla}_X J_\nu = q_{\nu+2}(X) J_{\nu+1} - q_{\nu+1}(X) J_{\nu+2} \tag{1.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

Let $p \in G_2(\mathbb{C}^{m+2})$ and W a subspace of $T_pG_2(\mathbb{C}^{m+2})$. We say that W is a quaternionic subspace of $T_pG_2(\mathbb{C}^{m+2})$ if $JW \subset W$ for all $J \in \mathfrak{J}_p$. And we say that W is a totally complex subspace of $T_pG_2(\mathbb{C}^{m+2})$ if there exists a one-dimensional subspace $\mathfrak{V}$ of $\mathfrak{J}_p$ such that $JW \subset W$ for all $J \in \mathfrak{V}$ and $JW \perp W$ for all $J \in \mathfrak{V}^{\perp} \subset \mathfrak{J}_p$. Here, the orthogonal complement of $\mathfrak{V}$ in $\mathfrak{J}_p$ is taken with respect to the bundle metric and orientation on $\mathfrak{J}$ for which any local oriented orthonormal frame field of $\mathfrak{J}$ is a canonical local basis of $\mathfrak{J}$. A quaternionic (resp. totally complex) submanifold of $G_2(\mathbb{C}^{m+2})$ is a submanifold all of whose tangent spaces are quaternionic (resp. totally complex) subspaces of the corresponding tangent spaces of $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$
\begin{aligned}
&\bar{R}(X,Y)Z \\
&= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\
&\quad - g(JX,Z)JY - 2g(JX,Y)JZ \\
&\quad + \sum_{\nu=1}^{3}\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\} \\
&\quad + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},
\end{aligned}
\tag{1.2}
$$

where J_1, J_2, J_3 is any canonical local basis of $\mathfrak{J}$.

2. Some fundamental formulas for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

Now in this section we want to derive the normal Jacobi operator from the curvature tensor of complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ given in (1.2) and the equation of Gauss. Moreover, in this section we derive some basic formulae from the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (See [4], [5], [13], [14], and [15]).

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M,g). Let N be a local unit normal field of M and A the shape operator of M with respect to N. The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ,ξ,η,g). Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu,\xi_\nu,\eta_\nu,g)$ on M. Using the above expression for $\bar{R}$, the Codazzi equation becomes

$$
\begin{aligned}
&(\nabla_X A)Y - (\nabla_Y A)X \\
&= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi
\end{aligned}
$$

$$+\sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y-\eta_\nu(Y)\phi_\nu X-2g(\phi_\nu X,Y)\xi_\nu\}$$
$$+\sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi Y-\eta_\nu(\phi Y)\phi_\nu\phi X\}$$
$$+\sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y)-\eta(Y)\eta_\nu(\phi X)\}\xi_\nu.$$

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

$$\begin{aligned}\phi_{\nu+1}\xi_\nu &= -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1}=\xi_{\nu+2},\\ \phi\xi_\nu &= \phi_\nu\xi, \quad \eta_\nu(\phi X)=\eta(\phi_\nu X),\\ \phi_\nu\phi_{\nu+1}X &= \phi_{\nu+2}X+\eta_{\nu+1}(X)\xi_\nu,\\ \phi_{\nu+1}\phi_\nu X &= -\phi_{\nu+2}X+\eta_\nu(X)\xi_{\nu+1}.\end{aligned} \tag{2.1}$$

Now let us put

$$JX=\phi X+\eta(X)N, \quad J_\nu X=\phi_\nu X+\eta_\nu(X)N \tag{2.2}$$

for any tangent vector X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a normal vector of M in $G_2(\mathbb{C}^{m+2})$. Then from this and the formulas (1.1) and (2.1) we have that

$$(\nabla_X\phi)Y=\eta(Y)AX-g(AX,Y)\xi, \quad \nabla_X\xi=\phi AX, \tag{2.3}$$

$$\nabla_X\xi_\nu=q_{\nu+2}(X)\xi_{\nu+1}-q_{\nu+1}(X)\xi_{\nu+2}+\phi_\nu AX, \tag{2.4}$$

$$\begin{aligned}(\nabla_X\phi_\nu)Y=&-q_{\nu+1}(X)\phi_{\nu+2}Y+q_{\nu+2}(X)\phi_{\nu+1}Y+\eta_\nu(Y)AX\\ &-g(AX,Y)\xi_\nu.\end{aligned} \tag{2.5}$$

Summing up these formulas, we find the following

$$\begin{aligned}\nabla_X(\phi_\nu\xi)&=\nabla_X(\phi\xi_\nu)\\ &=(\nabla_X\phi)\xi_\nu+\phi(\nabla_X\xi_\nu)\\ &=q_{\nu+2}(X)\phi_{\nu+1}\xi-q_{\nu+1}(X)\phi_{\nu+2}\xi+\phi_\nu\phi AX\\ &\quad -g(AX,\xi)\xi_\nu+\eta(\xi_\nu)AX.\end{aligned} \tag{2.6}$$

Moreover, from $JJ_\nu=J_\nu J$, $\nu=1,2,3$, it follows that

$$\phi\phi_\nu X=\phi_\nu\phi X+\eta_\nu(X)\xi-\eta(X)\xi_\nu. \tag{2.7}$$

3. Lie ξ-parallel normal Jacobi operator

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with Lie ξ-parallel normal Jacobi operator, that is, $\mathfrak{L}_\xi \bar{R}_N = 0$. Then first of all, we write the normal Jacobi operator $\bar{R}_N$, which is given by

(3.1)
$$\begin{aligned}\bar{R}_N(X) &= \bar{R}(X,N)N \\ &= X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\ &\quad - \sum_{\nu=1}^{3}\Big\{\eta_\nu(\xi)J_\nu(\phi X + \eta(X)N) - \eta_\nu(\phi X)(\phi_\nu\xi + \eta_\nu(\xi)N)\Big\} \\ &= X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\ &\quad - \sum_{\nu=1}^{3}\Big\{\eta_\nu(\xi)(\phi_\nu\phi X - \eta(X)\xi_\nu) - \eta_\nu(\phi X)\phi_\nu\xi\Big\},\end{aligned}$$

where we have used the following

$$\begin{aligned}g(J_\nu JN, N) &= -g(JN, J_\nu N) = -g(\xi,\xi_\nu) = -\eta_\nu(\xi), \\ g(J_\nu JX, N) &= g(X, JJ_\nu N) = -g(X, J\xi_\nu) \\ &= -\,g(X, \phi\xi_\nu + \eta(\xi_\nu)N) = -g(X,\phi\xi_\nu),\end{aligned}$$

and

$$J_\nu JN = -J_\nu\xi = -\phi_\nu\xi - \eta_\nu(\xi)N.$$

Of course, by (2.7) we know that the normal Jacobi operator $\bar{R}_N$ could be symmetric endomorphism of T_xM, $x\in M$.

Now let us consider a Lie derivative of the normal Jacobi operator along the direction ξ. Then it is given by

$$\begin{aligned}(\mathcal{L}_\xi\bar{R}_N)X &= \mathcal{L}_\xi(\bar{R}_N X) - \bar{R}_N(\mathcal{L}_\xi X) \\ &= [\xi, \bar{R}_N X] - \bar{R}_N[\xi, X] \\ &= (\nabla_\xi\bar{R}_N)X - \phi A\bar{R}_N X + \bar{R}_N\phi AX,\end{aligned} \tag{3.2}$$

where the terms in the right side can be given respectively as follows:

$$\begin{aligned}(\nabla_\xi\bar{R}_N)X &= 3(\nabla_\xi\eta)(X)\xi + 3\eta(X)\nabla_\xi\xi + 3\sum_{\nu=1}^{3}(\nabla_\xi\eta_\nu)(X)\xi_\nu \\ &\quad + 3\sum_{\nu=1}^{3}\eta_\nu(X)\nabla_\xi\xi_\nu - \sum_{\nu=1}^{3}\Big[\xi(\eta_\nu(\xi))(\phi_\nu\phi X - \eta(X)\xi_\nu) \\ &\quad + \eta_\nu(\xi)\{(\nabla_\xi\phi_\nu\phi)X - (\nabla_\xi\eta)(X)\xi_\nu - \eta(X)\nabla_\xi\xi_\nu\} \\ &\quad - (\nabla_\xi\eta_\nu)(\phi X)\phi_\nu\xi - \eta_\nu((\nabla_\xi\phi)X)\phi_\nu\xi - \eta_\nu(\phi X)\nabla_\xi(\phi_\nu\xi)\Big],\end{aligned}$$

$$\begin{aligned}\phi A\bar{R}_N X &= \phi AX + 3\eta(X)\phi A\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\phi A\xi_\nu \\ &\quad - \sum_{\nu=1}^{3}\Big[\eta_\nu(\xi)(\phi A\phi_\nu\phi X - \eta(X)\phi A\xi_\nu) - \eta_\nu(\phi X)\phi A\phi_\nu\xi\Big]\end{aligned}$$

and

$$\bar{R}_N \phi AX = \phi AX + 3\sum_{\nu=1}^{3} \eta_\nu(\phi AX)\xi_\nu - \sum_{\nu=1}^{3} \{\eta_\nu(\xi)\phi_\nu \phi^2 AX - \eta_\nu(\phi^2 AX)\phi_\nu \xi\}.$$

Then by the formulas given in section 2, a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with Lie ξ-parallel of $\bar{R}_N$ along the direction of ξ and satisfies the following

(3.3)

$$\begin{aligned}(\mathcal{L}_\xi \bar{R}_N)X &= (\nabla_\xi \bar{R}_N)X - \phi A \bar{R}_N X + \bar{R}_N \phi AX \\ &= 3g(\phi A\xi, X)\xi + 3\sum_{\nu=1}^{3} g(\phi_\nu A\xi, X)\xi_\nu + 3\sum_{\nu=1}^{3} \eta_\nu(X)\phi_\nu A\xi \\ &\quad - \sum_{\nu=1}^{3} \Big[\xi(\eta_\nu(\xi))(\phi_\nu \phi X - \eta(X)\xi_\nu) \\ &\quad + \eta_\nu(\xi)\{-q_{\nu+1}(\xi)\phi_{\nu+2}\phi X + q_{\nu+2}(\xi)\phi_{\nu+1}\phi X \\ &\quad + \eta_\nu(\phi X)A\xi - g(A\xi, \phi X)\xi_\nu + \eta(X)\phi_\nu A\xi \\ &\quad - g(A\xi, X)\phi_\nu \xi - g(\phi A\xi, X)\xi_\nu \\ &\quad - \eta(X)(q_{\nu+2}(\xi)\xi_{\nu+1} - q_{\nu+1}(\xi)\xi_{\nu+2} + \phi_\nu A\xi)\} \\ &\quad - g(\phi_\nu A\xi, \phi X)\phi_\nu \xi - \eta(X)\eta_\nu(A\xi)\phi_\nu \xi + g(A\xi, X)\eta_\nu(\xi)\phi_\nu \xi \\ &\quad - \eta_\nu(\phi X)\{\eta_\nu(\xi)A\xi - g(A\xi, \xi)\xi_\nu + \phi_\nu \phi A\xi\}\Big] \\ &\quad - 3\sum_{\nu=1}^{3} \eta_\nu(X)\phi A\xi_\nu + \sum_{\nu=1}^{3} \{\eta_\nu(\xi)(\phi A\phi_\nu \phi X - \eta(X)\phi A\xi_\nu) \\ &\quad - \eta_\nu(\phi X)\phi A\phi_\nu \xi\} + 3\sum_{\nu=1}^{3} \eta_\nu(\phi AX)\xi_\nu \\ &\quad + \sum_{\nu=1}^{3} \{\eta_\nu(\xi)\phi_\nu AX - \eta_\nu(AX)\phi_\nu \xi\} = 0,\end{aligned}$$

where in the second equality we have used the following formulas

$$3\sum_{\nu=1}^{3} g(q_{\nu+2}(\xi)\xi_{\nu+1} - q_{\nu+1}(\xi)\xi_{\nu+2}, X)\xi_\nu + 3\sum_{\nu=1}^{3} \eta_\nu(X)\{q_{\nu+2}(\xi)\xi_{\nu+1} - q_{\nu+1}(\xi)\xi_{\nu+2}\} = 0$$

and

$$\sum_{\nu=1}^{3} \{\eta_{\nu+1}(\phi X)q_{\nu+2}(\xi)\phi_\nu \xi - \eta_{\nu+2}(\phi X)q_{\nu+1}(\xi)\phi_\nu \xi - \eta_\nu(\phi X)q_{\nu+1}(\xi)\phi_{\nu+2}\xi + \eta_\nu(\phi X)q_{\nu+2}(\xi)\phi_{\nu+1}\xi\} = 0.$$

From this, by putting $X = \xi$ and using the formulas in Section 2 we have the following

$$\begin{aligned}(\mathcal{L}_\xi \bar{R}_N)\xi &= 6\sum_{\nu=1}^{3} g(\phi_\nu A\xi, \xi)\xi_\nu + 4\sum_{\nu=1}^{3} \eta_\nu(\xi)\phi_\nu A\xi \\ &\quad + \sum_{\nu=1}^{3} \Big[\xi(\eta_\nu(\xi))\xi_\nu + \eta_\nu(\xi)\{q_{\nu+2}(\xi)\xi_{\nu+1} - q_{\nu+1}(\xi)\xi_{\nu+2}\}\Big] \\ &\quad - 4\sum_{\nu=1}^{3} \eta_\nu(\xi)\phi A\xi_\nu = 0.\end{aligned} \tag{3.4}$$

4. Lie ξ-parallel normal Jacobi operator for $\xi\in\mathfrak{D}^\perp$

In this section we want to give a complete proof of Theorem 1. In order to do this, we consider the case that $\xi\in\mathfrak{D}^\perp$. Accordingly, we may put $\xi = \xi_1$. Then (3.1) implies the following for any X on M

$$
\begin{aligned}
0 = {} & 3g(\phi A\xi, X)\xi + 3\sum_{\nu=1}^{3} g(\phi_\nu A\xi, X)\xi_\nu + 3\sum_{\nu=1}^{3}\eta_\nu(X)\phi_\nu A\xi \\
& + q_2(\xi)\phi_3\phi X - q_3(\xi)\phi_2\phi X + \eta(X)\{q_3(\xi)\xi_2 - q_2(\xi)\xi_3\} \\
& - g(\phi_2 A\xi, \phi X)\xi_3 - \eta(X)\eta_2(A\xi)\xi_3 + g(\phi_3 A\xi, \phi X)\xi_2 \\
& + \eta(X)\eta_3(A\xi)\xi_2 + \alpha\{\eta_2(X)\xi_3 - \eta_3(X)\xi_2\} \\
& + \eta_3(X)\phi_2\phi A\xi - \eta_2(X)\phi_3\phi A\xi \\
& - 3\sum_{\nu=1}^{3}\eta_\nu(X)\phi A\xi_\nu + \phi A\phi_1\phi X - \eta(X)\phi A\xi_1 \\
& + \eta_3(X)\phi A\xi_3 + \eta_2(X)\phi A\xi_2 + 3\{\eta_3(AX)\xi_2 - \eta_2(AX)\xi_3\} \\
& + \phi_1 AX + \eta_2(AX)\xi_3 - \eta_3(AX)\xi_2,
\end{aligned} \tag{4.1}
$$

where α denotes $g(A\xi, \xi)$.

On the other hand, from $\nabla_X\xi_1 = \nabla_X\xi$ we know that

$$
q_2(\xi) = 2g(A\xi, \xi_2), \quad q_3(\xi) = 2g(A\xi, \xi_3). \tag{4.2}
$$

By putting $X = \xi_2$ in (4.1), we have

$$
\begin{aligned}
0 &= (\mathcal{L}_\xi \bar{R}_N)\xi_2 \\
&= 3g(A\xi, \xi_1)\xi_3 + 3\phi_2 A\xi + q_3(\xi)\xi_1 - \phi_3\phi A\xi - \phi A\xi_2 \\
&\quad + 2\{\eta_3(A\xi_2)\xi_2 - \eta_2(A\xi_2)\xi_3\} + \phi_1 A\xi_2.
\end{aligned}
$$

From this, taking an inner product with ξ_1, we have

$$
0 = 3g(A\xi, \xi_3) + q_3(\xi) + g(A\xi, \xi_3).
$$

Then from this, together with (4.2), it follows that

$$
q_3(\xi) = 0 \quad \text{and} \quad g(A\xi, \xi_3) = 0.
$$

Similarly, by putting $X = \xi_3$ in (4.1) we have

$$
\begin{aligned}
0 &= (\mathcal{L}_\xi \bar{R}_N)\xi_3 \\
&= -3g(A\xi, \xi)\xi_2 + 3\phi_3 A\xi - q_2(\xi)\xi_1 \\
&\quad + \phi_2\phi A\xi - \phi A\xi_3 + \phi_1 A\xi_3 \\
&\quad + 2g(A\xi_3, \xi_3)\xi_2 - 2g(A\xi_3, \xi_2)\xi_3.
\end{aligned}
$$

From this, by taking an inner product with ξ_1 and using (4.2) we have

$$
q_2(\xi) = 0 \quad \text{and} \quad g(A\xi, \xi_2) = 0.
$$

Then we may summarize such a fact as follows:

Lemma 4.1. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying Lie ξ-parallel normal Jacobi operator and $\xi\in\mathfrak{D}^\perp$. Then $A\xi = \alpha\xi + \beta U$, where U is a unit vector field orthogonal to ξ and belongs to $\mathfrak{D}$.*

From Lemma 4.1 we can prove the following

Lemma 4.2. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying Lie ξ-parallel normal Jacobi operator and $\xi\in\mathfrak{D}^\perp$. Then β identically vanishes, that is, the structure vector ξ is principal.*

Proof. By Lemma 4.1 we may put

$$A\xi = \alpha\xi + \beta U \tag{4.3}$$

for some unit normal U orthogonal to the structure vector ξ. Now let us construct an open set $\mathfrak{V}$ in such a way that $\mathfrak{V} = \{p\in M|\beta(p)\neq 0\}$. Then on such an open $\mathfrak{V}$ we proceed our assertion. Now substituting (4.3) into (4.1), we have the following

$$\begin{aligned} 0 = {} & 3\beta\phi_2 U - \beta\phi_3\phi U - \phi A\xi_2 \\ & + 2g(A\xi_2,\xi_3)\xi_2 - 2g(A\xi_2,\xi_2)\xi_3 + \phi_1 A\xi_2. \end{aligned}$$

From this, by taking an inner product with $\phi_2 U$ we have

$$\begin{aligned} 0 = {} & -3\beta g(\phi_2 U,\phi_2 U) - \beta g(\phi_3\phi U,\phi_2 U) - g(\phi A\xi_2,\phi_2 U) \\ & + g(\phi_1 A\xi_2,\phi_2 U) \\ = {} & 3\beta + \beta g(\phi U,\phi_3\phi_2 U) \\ = {} & 3\beta + \beta g(\phi U, -\phi_1 U + \eta_2(U)\xi_3) \\ = {} & 3\beta - \beta g(\phi U,\phi_1 U) \\ = {} & 2\beta, \end{aligned}$$

where in the second equality we have used $\nabla_{\xi_2}\xi = \nabla_{\xi_2}\xi_1$ and in the final equality we have used the formula $\nabla_\xi\xi = \nabla_\xi\xi_1$. But this is impossible on the open subset $\mathfrak{V}$. Accordingly, such an open $\mathfrak{V}$ can not exit on M. So we have our assertion. □

Lemma 4.3. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying Lie ξ-parallel normal Jacobi operator and $\xi\in\mathfrak{D}^\perp$. Then $g(A\mathfrak{D},\mathfrak{D}^\perp) = 0$.*

Proof. Now we consider (4.1) when the structure vector ξ is principal. Then it follows that

$$\begin{aligned} 0 = {} & -2\eta_2(X)\phi A\xi_2 - 2\eta_3(X)\phi A\xi_3 + \phi A\phi_1\phi X \\ & + 2\eta_3(AX)\xi_2 - 2\eta_2(AX)\xi_3 + \phi_1 AX. \end{aligned} \tag{4.4}$$

Now let us take an inner product (4.4) with ξ_2. Then it follows that

$$\begin{aligned} 0 = {} & -2\eta_2(X)g(A\xi_2,\xi_3) - 2\eta_3(X)g(A\xi_3,\xi_3) \\ & + g(\phi A\phi_1\phi X,\xi_2) + 2\eta_3(AX) + g(\phi_1 AX,\xi_2) \\ = {} & -2\eta_2(X)g(A\xi_2,\xi_3) - 2\eta_3(X)g(A\xi_3,\xi_3), \end{aligned} \tag{4.5}$$

where in the first equality we have used the following formula

$$\begin{aligned}g(\phi A\phi_1\phi X,\xi_2) &= -g(A\phi_1\phi X,\phi\xi_2)\\ &= g(A\phi_1\phi X,\xi_3)\\ &= g(A\phi\phi_1 X,\xi_3)\\ &= -g(\phi_1 X,\nabla_{\xi_3}\xi)\\ &= g(\nabla_{\xi_3}(\phi X),\xi)\\ &= g(\eta(X)A\xi_3 - g(A\xi_3,X)\xi,\xi)\\ &= -g(A\xi_3,X).\end{aligned}$$

From this, by putting $X=\xi_2$ and $X=\xi_3$ we have

$$g(A\xi_3,\xi_3)=g(A\xi_2,\xi_3)=0.$$

On the other hand, by taking an inner product (4.4) with ξ_3 we have

$$2\eta_2(X)g(A\xi_2,\xi_2)+2\eta_3(X)g(A\xi_3,\xi_2)=0.$$

Then from this, by putting $X=\xi_2$ and $X=\xi_3$ we have respectively

$$g(A\xi_2,\xi_2)=g(A\xi_3,\xi_2)=0.$$

Summing up these formulas, we conclude that $g(A\xi_i,\xi_j)=0$ for any i and j except $i=j=1$. Then we may put $A\xi_2=X_2$ and $A\xi_3=X_3$ for some X_2, $X_3\in\mathfrak{D}$.

Now substituting these one into (4.4), we get the following

$$\begin{aligned}0 &= g(\phi A\phi_1 X,\xi_2)+2\eta_3(AX)+g(\phi_1 AX,\xi_2)\\ &= -g(A\phi_1 X,\phi\xi_2)+2g(X_3,X)-g(AX,\xi_3)\\ &= g(\phi_1 X,X_3)+g(X_3,X)\end{aligned}\tag{4.6}$$

for any tangent vector field X on M. Then from this, by replacing X by $\phi_1 X$ we have

$$\begin{aligned}0 &= g({\phi_1}^2 X,X_3)+g(X_3,\phi_1 X)\\ &= -g(X,X_3)+g(X_3,\phi_1 X).\end{aligned}\tag{4.7}$$

Then (4.6) and (4.7) gives X_2 and X_3 identically vanishing. That is, $A\xi_2=0$ and $A\xi_3=0$. Accordingly, we have our assertion in Lemma 4.2. □

Before going to give the proof of Theorem 1 in the introduction let us check that "What kind of model hypersurfaces given in Theorem A satisfy Lie ξ-parallel normal Jacobi operator." In other words, it will be an interesting problem to know whether there exist any real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying the condition $\mathcal{L}_\xi\bar{R}_N=0$ for $\xi\in\mathfrak{D}^\perp$.

Then by virtue of Lemmas 4.1 and 4.2, we are able to recall a proposition given by Berndt and the second author [4] as follows:

For a tube of type A in Theorem A we have the following

Proposition A. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1 N$. Then M has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r),\ \beta = \sqrt{2}\cot(\sqrt{2}r),\ \lambda = -\sqrt{2}\tan(\sqrt{2}r),\ \mu = 0$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1,\ m(\beta) = 2,\ m(\lambda) = 2m - 2 = m(\mu),$$

and the corresponding eigenspaces we have

$$\begin{aligned} T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1, \\ T_\beta &= \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3, \\ T_\lambda &= \{X | X \perp \mathbb{H}\xi, JX = J_1X\}, \\ T_\mu &= \{X | X \perp \mathbb{H}\xi, JX = -J_1X\}, \end{aligned}$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$ and $\mathbb{Q}\xi$ respectively denotes real, complex and quaternionic span of the structure vector ξ and $\mathbb{C}^\perp\xi$ denotes the orthogonal complement of $\mathbb{C}\xi$ in $\mathbb{H}\xi$.

Then in the proof of Lemma 4.3 we have asserted that $A\xi_2 = 0$ and $A\xi_3 = 0$. But the principal curvature $\beta = \sqrt{2}\cot(\sqrt{2}r)$ given in Proposition A is never vanishing for any $r \in (0, \frac{\pi}{4})$. So this makes a contradiction. Accordingly, we completed the proof of our Theorem 1.

5. Lie ξ-parallel normal Jacobi operator for $\xi \in \mathfrak{D}$

In this section, in order to prove our Theorem 2 in the introduction we will give several lemmas. Now we consider for the case that $\xi \in \mathfrak{D}$. Then using $\xi \in \mathfrak{D}$ in (3.3) we have the following

$$\begin{aligned} (\mathcal{L}_\xi \bar{R}_N)X &= (\nabla_\xi \bar{R}_N)X - \phi A\bar{R}_N X + \bar{R}_N\phi AX \\ &= 3g(\phi A\xi, X)\xi + 3\sum_{\nu=1}^{3} g(\phi_\nu A\xi, X)\xi_\nu \\ &\quad + 3\sum_{\nu=1}^{3}\eta_\nu(X)\phi_\nu A\xi \\ &\quad + \sum_{\nu=1}^{3}\Big[g(\phi_\nu A\xi, \phi X)\phi_\nu\xi + \eta(X)\eta_\nu(A\xi)\phi_\nu\xi \\ &\quad + \eta_\nu(\phi X)\{-g(A\xi,\xi)\xi_\nu + \phi_\nu\phi A\xi\}\Big] \\ &\quad - 3\sum_{\nu=1}^{3}\eta_\nu(X)\phi A\xi_\nu - \sum_{\nu=1}^{3}\eta_\nu(\phi X)\phi A\phi_\nu\xi \\ &\quad + 3\sum_{\nu=1}^{3}\eta_\nu(\phi AX)\xi_\nu - \sum_{\nu=1}^{3}\eta_\nu(AX)\phi_\nu\xi = 0. \end{aligned} \tag{5.1}$$

Then we assert the following

Lemma 5.1. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying Lie ξ-parallel normal Jacobi operator and $\xi \in \mathfrak{D}$. Then the structure vector ξ is principal.*

Proof. Now let us put $X = \xi$ in (5.1) and use $\xi \in \mathfrak{D}$, we have

$$\begin{aligned} 0 &= 3\sum_{\nu=1}^{3} g(\phi_\nu A\xi, \xi)\xi_\nu + \sum_{\nu=1}^{3} \eta_\nu(A\xi)\phi_\nu\xi + 3\sum_{\nu=1}^{3} \eta_\nu(\phi A\xi)\xi_\nu \\ &\quad - \sum_{\nu=1}^{3} \eta_\nu(A\xi)\phi_\nu\xi \\ &= 6\sum_{\nu=1}^{3} g(\phi_\nu A\xi, \xi)\xi_\nu. \end{aligned}$$

From this we assert the following for any $\nu = 1, 2, 3$

$$g(A\xi, \phi_\nu\xi) = 0. \tag{5.2}$$

On the other hand, let us take an inner product (5.1) with the structure vector ξ and use the fact $\xi \in \mathfrak{D}$ and (5.2). Then it follows

$$\begin{aligned} 0 &= 3g(\phi A\xi, X) + 3\sum_{\nu=1}^{3} \eta_\nu(X) g(\phi_\nu A\xi, \xi) \\ &\quad + \sum_{\nu=1}^{3} \eta_\nu(\phi X) g(\phi_\nu \phi A\xi, \xi) \\ &= 3g(\phi A\xi, X) - \sum_{\nu=1}^{3} \eta_\nu(\phi X)\eta_\nu(A\xi). \end{aligned} \tag{5.3}$$

Now by putting $X = \phi\xi_\mu$ into (5.3) we have

$$g(A\xi, \xi_\mu) = 0 \tag{5.4}$$

for any $\mu = 1, 2, 3$. Then by virtue of (5.2) and (5.4) we may put

$$A\xi = \alpha\xi + X_0 \tag{5.5}$$

for some $X_0 \in \mathfrak{D}$ orthogonal to $\xi, \phi_1\xi, \phi_2\xi, \phi_3\xi$. Then by putting $X = \phi X_0$ in (5.3) we have $g(A\xi, X_0) = 0$. From this, together with (5.5), we have our assertion. □

Then by using Lemma 5.1 we want to verify $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$. In order to do this, first of all, we should verify the following

Lemma 5.2. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying Lie ξ-parallel normal Jacobi operator and $\xi \in \mathfrak{D}$. Then $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.*

Proof. From the results of Lemma 5.1, we have the following

$$
\begin{aligned}
(\mathcal{L}_\xi \bar{R}_N)X = {} & 4\alpha\sum_{\nu=1}^{3} g(\phi_\nu\xi, X)\xi_\nu + 3\alpha\sum_{\nu=1}^{3}\eta_\nu(X)\phi_\nu\xi \\
& - \sum_{\nu=1}^{3}\Big[\xi(\eta_\nu(\xi))(\phi_\nu\phi X - \eta(X)\xi_\nu) \\
& + \eta_\nu(\xi)\{-q_{\nu+1}(\xi)\phi_{\nu+2}\phi X + q_{\nu+2}(\xi)\phi_{\nu+1}\phi X \\
& - \eta(X)(q_{\nu+2}(\xi)\xi_{\nu+1} - q_{\nu+1}(\xi)\xi_{\nu+2} + \alpha\phi_\nu\xi)\} \\
& - \alpha g(\phi_\nu\xi, \phi X)\phi_\nu\xi\Big] \\
& - 3\sum_{\nu=1}^{3}\eta_\nu(X)\phi A\xi_\nu \\
& + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)(\phi A\phi_\nu\phi X - \eta(X)\phi A\xi_\nu) \\
& - \eta_\nu(\phi X)\phi A\phi_\nu\xi\} + 3\sum_{\nu=1}^{3}\eta_\nu(\phi AX)\xi_\nu \\
& + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu AX - \eta_\nu(AX)\phi_\nu\xi\} = 0.
\end{aligned} \tag{5.6}
$$

Since ξ is principal and $\xi\in\mathfrak{D}$, we have

$$g(A\xi, \mathfrak{D}^\perp) = 0. \tag{5.7}$$

From the formula (5.6) and $\xi\in\mathfrak{D}$, we have the following

$$
\begin{aligned}
(\mathcal{L}_\xi \bar{R}_N)X = {} & 4\alpha\sum_{\nu=1}^{3} g(\phi_\nu\xi, X)\xi_\nu + 4\alpha\sum_{\nu=1}^{3}\eta_\nu(X)\phi_\nu\xi \\
& - 3\sum_{\nu=1}^{3}\eta_\nu(X)\phi A\xi_\nu - \sum_{\nu=1}^{3}\eta_\nu(\phi X)\phi A\phi_\nu\xi \\
& + 3\sum_{\nu=1}^{3}\eta_\nu(\phi AX)\xi_\nu - \sum_{\nu=1}^{3}\eta_\nu(AX)\phi_\nu\xi = 0.
\end{aligned} \tag{5.8}
$$

Now let us put $\mathfrak{D}_0(x) = \{X\in\mathfrak{D} | X\perp\xi\}$. From this, for $X\in\mathfrak{D}_0$, we have

$$
\begin{aligned}
0 = {} & 4\alpha\sum_{\nu=1}^{3} g(\phi_\nu\xi, X)\xi_\nu - \sum_{\nu=1}^{3}\eta_\nu(\phi X)\phi A\phi_\nu\xi \\
& + 3\sum_{\nu=1}^{3}\eta_\nu(\phi AX)\xi_\nu - \sum_{\nu=1}^{3}\eta_\nu(AX)\phi_\nu\xi.
\end{aligned} \tag{5.9}
$$

Let us take an inner product the above equation with $\phi_i\xi$. Then we have

$$0 = \sum_{\nu=1}^{3}\eta_\nu(\phi X) g(\phi A\phi_\nu\xi, \phi_i\xi) + g(AX, \xi_i). \tag{5.10}$$

By the formula (5.10), for $X\in\mathfrak{D}_1$, we have

$$g(AX, \xi_i) = 0, \quad i = 1, 2, 3, \tag{5.11}$$

where the distribution $\mathfrak{D}_1$ is given by $\mathfrak{D}_1 = \{X\in\mathfrak{D}_0 | X\perp\phi_i\xi,\ i = 1,2,3\}$. On the other hand, by (2.3) and (2.4), we have the following

$$\begin{aligned} g(A\phi_i\xi, \xi_\mu) &= g(A\xi_\mu, \phi_i\xi) \\ &= g(A\xi_\mu, \phi\xi_i) \\ &= -g(\phi A\xi_\mu, \xi_i) \\ &= -g(\nabla_{\xi_\mu}\xi, \xi_i) \\ &= g(\xi, \nabla_{\xi_\mu}\xi_i) \\ &= g(\xi, \phi_i A\xi_\mu) \\ &= -g(A\phi_i\xi, \xi_\mu). \end{aligned}$$

From the above equation, we have

$$g(A\phi_i\xi, \xi_\mu) = 0 \tag{5.12}$$

for any $i, \mu = 1,2,3$. Hence, by (5.7), (5.11) and (5.12), we know that

$$g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0.$$

□

Now by virtue of these Lemmas 5.1 and 5.2 we are able to use Theorem A due to Berndt and the second author [4]. That is, M is locally a tube over a totally geodesic and totally real quaternionic projective space $\mathbb{Q}P^n$, $m = 2n$. So for the geometrical structure for such a tube we recall the following proposition

Proposition B. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r),\ \beta = 2\cot(2r),\ \gamma = 0,\ \lambda = \cot(r),\ \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1,\ m(\beta) = 3 = m(\gamma),\ m(\lambda) = 4n-4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi,\ T_\beta = \mathfrak{J}J\xi,\ T_\gamma = \mathfrak{J}\xi,\ T_\lambda,\ T_\mu,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{HC}\xi)^\perp,\ \mathfrak{J}T_\lambda = T_\lambda,\ \mathfrak{J}T_\mu = T_\mu,\ JT_\lambda = T_\mu.$$

Now let us construct a subdistribution $\mathfrak{D}_0$ in such a way that

$$[\xi]\oplus\mathfrak{D}_0 = \mathfrak{D},$$

where $[\xi]$ denotes a one-dimensional vector subspace spanned by the structure vector ξ. Then $\mathfrak{D}_0$ becomes $\mathfrak{D}_0 = \{X\in\mathfrak{D} | X\perp\xi\}$. Now we substitute any $X\in\mathfrak{D}_0$

in (5.17) and use $\xi\in\mathfrak{D}$ we have

$$(\mathcal{L}_\xi \bar{R}_N)X = 4\alpha\sum_{\nu=1}^{3} g(X,\phi_\nu\xi)\xi_\nu - \sum_{\nu=1}^{3}\eta_\nu(\phi X)\phi A\phi_\nu\xi + 3\sum_{\nu=1}^{3}\eta_\nu(\phi AX)\xi_\nu - \sum_{\nu=1}^{3}\eta_\nu(AX)\phi_\nu\xi.$$

From this, putting $X=\phi_\mu\xi$ and using $A\phi_\mu\xi=0$, $\mu=1,2,3$ in Proposition B, we have

$$(\mathcal{L}_\xi \bar{R}_N)\phi\xi_\mu = 4\alpha\xi_\mu.$$

But we have assumed that $\mathcal{L}_\xi \bar{R}_N = 0$. Then this gives $\alpha = 0$. But the constant principal curvature $\alpha = -2\tan(2r)$ in Proposition B never vanishing for $r\in(0,\frac{\pi}{4})$. This makes a contradiction for this case $\xi\in\mathfrak{D}$. So we complete the proof of Theorem 2 in the introduction.

6. Hopf hypersurfaces with ξ-parallel normal Jacobi operator

A real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be a *Hopf* if the structure vector ξ of M is principal. This means that $A\xi=\alpha\xi$, $\alpha=g(A\xi,\xi)$, for the shape operator A of M in $G_2(\mathbb{C}^{m+2})$. Of course, all of hypersurfaces in $G_2(\mathbb{C}^{m+2})$ mentioned in Theorem A are Hopf hypersurfaces. Moreover, by Propositions A and B we have known that the structure vector ξ for real hypersurfaces of type (A) and of type (B) in Theorem A belongs to the distribution $\mathfrak{D}^\perp$ and the distribution $\mathfrak{D}$ respectively.

In this section we consider a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with Lie ξ-parallel normal Jacobi operator $\bar{R}_N$. Then it will be an interesting fact to check whether Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with Lie ξ-parallel normal Jacobi operator can exist or not.

In order to do this, we prove the following lemma which will be useful in the proof of our Theorem 3 given in the introduction.

Lemma 6.1. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator along the direction of ξ. Then the directional derivative of the principal curvature α is given by*

$$Y\alpha = -4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y)$$

for any vector field Y on M.

Proof. Now we assume that M is Hopf. So we may put $A\xi=\alpha\xi$. Then the formula (3.4) implies that

$$\alpha\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu\xi = \sum_{\nu=1}^{3}\eta_\nu(\xi)\phi A\xi_\nu. \tag{6.1}$$

Now let us consider a vector U defined in such a way that

$$U=\sum_{\nu=1}^{3}\eta_\nu(\xi)\xi_\nu. \tag{6.2}$$

If we put $\xi = X_1 + X_2$ for some vector X_1 in the distribution $\mathfrak{D}^\perp$ and some vector X_2 in $\mathfrak{D}$, then we know that X_1 becomes the vector U. Now hereafter, unless otherwise stated, let us decompose the structure vector ξ by $\xi = U + X_2$. Then (6.1) can be written as follows

$$\phi AU = \alpha\phi U. \tag{6.3}$$

Now differentiating (6.2) covariantly and using the formulas given in Section 2, we have

$$\begin{aligned}\nabla_X U &= \sum_{\nu=1}^{3}\{g(\nabla_X\xi_\nu,\xi)\xi_\nu + g(\xi_\nu,\nabla_X\xi)\xi_\nu + \eta_\nu(\xi)\nabla_X\xi_\nu\}\\ &= 2\sum_{\nu=1}^{3} g(\xi_\nu,\phi AX)\xi_\nu + \sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu AX.\end{aligned}$$

On the other hand, by applying the structure tensor ϕ to (6.1) we know the following

$$AU = \alpha U \quad \text{and} \quad AX_2 = \alpha X_2. \tag{6.4}$$

Now differentiating the first formula of (6.4) and using the above formula, we have the following

$$(\nabla_X A)U + A\nabla_X U = (X\alpha)U + \alpha\nabla_X U.$$

Then it follows that

$$\begin{aligned}&g(U,(\nabla_X A)Y)\\ &= g((\nabla_X A)U,Y)\\ &= (X\alpha)g(U,Y) + \alpha g(\nabla_X U,Y) - g(A\nabla_X U,Y)\\ &= (X\alpha)\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(Y) + \alpha\{2\sum_{\nu=1}^{3} g(\xi_\nu,\phi AX)\eta_\nu(Y)\\ &\quad + \sum_{\nu=1}^{3}\eta_\nu(\xi)g(\phi_\nu AX,Y)\}\\ &\quad - g(2\sum_{\nu=1}^{3} g(\xi_\nu,\phi AX)A\xi_\nu + \sum_{\nu=1}^{3}\eta_\nu(\xi)A\phi_\nu AX,Y).\end{aligned} \tag{6.5}$$

From this, let us take a skew-symmetric part of (6.5), then by virtue of the equation of Codazzi the left side becomes

$$\begin{aligned}&g((\nabla_X A)Y - (\nabla_Y A)X,U)\\ &= \sum_{\nu=1}^{3}\{\eta(X)g(\phi Y,\xi_\nu)\eta_\nu(\xi) - \eta(Y)g(\phi X,\xi_\nu)\eta_\nu(\xi)\}\\ &\quad - 2\sum_{\nu=1}^{3} g(\phi X,Y)\eta_\nu(\xi)^2 - 2\sum_{\nu=1}^{3} g(\phi_\nu X,Y)\eta_\nu(\xi)\\ &\quad + 2\sum_{\nu=1}^{3}\big[\eta_\nu(X)\{-\eta_{\nu+2}(Y)\eta_{\nu+1}(\xi) + \eta_{\nu+1}(Y)\eta_{\nu+2}(\xi)\}\\ &\quad + \eta_\nu(\phi X)\{-\eta_{\nu+2}(\phi Y)\eta_{\nu+1}(\xi) + \eta_{\nu+1}(\phi Y)\eta_{\nu+2}(\xi)\}\big]\\ &\quad + \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\eta_\nu(\xi),\end{aligned} \tag{6.6}$$

where we have used the following

$$\begin{aligned} g(\phi_\nu\phi Y, U) &= -g(\phi Y, \phi_\nu U) \\ &= -g(\phi Y, \eta_{\nu+1}(\xi)\phi_\nu\xi_{\nu+1} + \eta_{\nu+2}(\xi)\phi_\nu\xi_{\nu+2}) \\ &= -\eta_{\nu+2}(\phi Y)\eta_{\nu+1}(\xi) + \eta_{\nu+1}(\phi Y)\eta_{\nu+2}(\xi) \end{aligned}$$

and

$$g(\phi_\nu Y, U) = -\eta_{\nu+2}(Y)\eta_{\nu+1}(\xi) + \eta_{\nu+1}(Y)\eta_{\nu+2}(\xi).$$

Moreover, the skew-symmetric part in the right side of (6.5) becomes

$$\begin{aligned} &(X\alpha)\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(Y) - (Y\alpha)\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(X) \\ &\quad + 2\alpha\sum_{\nu=1}^{3}\{g(\xi_\nu, \phi AX)\eta_\nu(Y) - g(\xi_\nu, \phi AY)\eta_\nu(X)\} \\ &\quad + \alpha\sum_{\nu=1}^{3}\eta_\nu(\xi)g((\phi_\nu A + A\phi_\nu)X, Y) - 2\sum_{\nu=1}^{3}\{g(\xi_\nu, \phi AX)g(A\xi_\nu, Y) \\ &\quad - g(\xi_\nu, \phi AY)g(A\xi_\nu, X)\} - 2\sum_{\nu=1}^{3}\eta_\nu(\xi)g(A\phi_\nu AX, Y). \end{aligned}$$

Then by putting $X = \xi$ into the both sides of the above formulas and using $A\xi = \alpha\xi$, we have

(6.7)

$$\begin{aligned} 4\sum_{\nu=1}^{3}\eta_\nu(\phi Y)\eta_\nu(\xi) &= (\xi\alpha)\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(Y) - (Y\alpha)\sum_{\nu=1}^{3}\eta_\nu(\xi)^2 \\ &\quad + \alpha^2\sum_{\nu=1}^{3}\eta_\nu(\xi)g(\phi_\nu\xi, Y) + \alpha\sum_{\nu=1}^{3}g(\xi_\nu, \phi AY)\eta_\nu(\xi). \end{aligned}$$

On the other hand, if we differentiate $A\xi = \alpha\xi$ and take an inner product with ξ, then the Codazzi equation gives the following

$$\begin{aligned} &-2g(\phi X, Y) + 2\sum_{\nu=1}^{3}\{\eta_\nu(X)\eta_\nu(\phi Y) - \eta_\nu(Y)\eta_\nu(\phi X) - g(\phi_\nu X, Y)\eta_\nu(\xi)\} \\ &= g((\nabla_X A)Y - (\nabla_Y A)X, \xi) \\ &= g((\nabla_X A)\xi, Y) - g((\nabla_Y A)\xi, X) \\ &= (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((A\phi + \phi A)X, Y) - 2g(A\phi AX, Y). \end{aligned}$$

From this, if we put $X = \xi$, then

$$Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y). \tag{6.8}$$

Then by putting $Y = X_2$ in (6.8) we have

$$X_2\alpha = \|X_2\|^2(\xi\alpha), \tag{6.9}$$

where we have used

$$\eta_\nu(\phi X_2) = -g(\phi_\nu\xi, X_2) = -g(\phi_\nu U + \phi_\nu X_2, X_2) = 0. \tag{6.10}$$

Now we put $Y = X_2$ in (6.7), and use (6.8), (6.10) and $AX_2 = \alpha X_2$ in the obtained equation. Then it follows that

$$(X_2\alpha)(1 - \sum_{\nu=1}^{3}\eta_\nu(\xi)^2) = (\xi\alpha)\eta(X_2). \tag{6.11}$$

Then from (6.9) and (6.11) we have

$$||X_2||^2(1 - \sum_{\nu=1}^{3} \eta_\nu(\xi)^2)(\xi\alpha) = (\xi\alpha)\eta(X_2) = (\xi\alpha)||X_2||^2,$$

which gives that

$$(\sum_{\nu=1}^{3} \eta_\nu(\xi)^2)||X_2||^2(\xi\alpha) = 0. \tag{6.12}$$

From this, together with the decomposition of the structure vector ξ in the assumption, we have $\xi\alpha = 0$. Then (6.8) completes the proof of Lemma 6.1. □

Now let us show that the structure vector ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$ when a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ has commuting shape operator, that is $A\phi = \phi A$ on the distribution $\mathfrak{D}^\perp$. In order to do this we also assumed that the structure vector ξ is decomposed into two distributions $\mathfrak{D}$ and $\mathfrak{D}^\perp$. That is, ξ is decomposed into $\xi = U + X_2$

Now, by using $\xi\alpha = 0$ in (6.7) and (6.8), we have

$$(Y\alpha)(\sum_{\nu=1}^{3} \eta_\nu(\xi)^2 - 1) = \alpha^2 g(\phi U, Y) - \alpha g(\phi U, AY). \tag{6.13}$$

Moreover, from (6.8) together with $\xi\alpha = 0$ we have

$$Y\alpha = 4g(\phi U, Y) \tag{6.14}$$

for any tangent vector field Y on M. So (6.14) gives $Y\alpha = 0$ for any Y orthogonal to ϕU. Then from this together with (6.13) we have

$$\alpha g(A\phi U, Y) = 0 \tag{6.15}$$

for any Y orthogonal to ϕU.

For the case where $\alpha = 0$, by (6.14) we can make a contradiction, because $\phi U = -\phi X_2$ never vanishing under the decomposition. So we assume that the function $\alpha \neq 0$. Then (6.15) gives that $g(A\phi U, Y) = 0$ for any Y orthogonal to ϕU. So we may put

$$A\phi U = \beta\phi U. \tag{6.16}$$

Now by putting $Y = \phi U$ in (6.13) and (6.14), and using (6.16), we have

$$\begin{aligned} -4||\phi U||^2||X_2||^2 &= -(\phi U\alpha)||X_2||^2 \\ &= (\alpha^2 - \alpha\beta)||\phi U||^2 \\ &= \alpha(\alpha - \beta)||\phi U||^2. \end{aligned}$$

This gives

$$\alpha(\alpha - \beta) = -4||X_2||^2. \tag{6.17}$$

But we have asserted that M has commuting shape operator on the distribution $\mathfrak{D}^\perp$. This means that $\phi AU = A\phi U = \alpha\phi U$ for $U = \sum_{\nu=1}^{3} \eta_\nu(\xi)\xi_\nu \in \mathfrak{D}^\perp$. From this together with (6.17), we can make a contradiction. Then summing up these process and Lemma 6.1 we can assert the following

Lemma 6.2. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator along the direction of ξ. If M has commuting shape operator on the distribution $\mathfrak{D}^\perp$, then the structure vector ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

Accordingly, by Lemma 6.2 and together with Theorem 1 and Theorem 2 for each case $\xi \in \mathfrak{D}^\perp$ and $\xi \in \mathfrak{D}$ respectively, we give the complete proof of our Theorem 3 mentioned in the introduction.

Remark 6.1. A tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ in Theorem A has commuting shape operator on the distribution $\mathfrak{D}^\perp$. Of course, it is Hopf. But, in section 4 we have asserted that such a hypersurface can not satisfy $\mathcal{L}_\xi \bar{R}_N = 0$.

Remark 6.2. A tube over a totally real totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$ has not commuting shape operator on the distribution $\mathfrak{D}^\perp$. In section 5 we have also proved that such a hypersurface is Hopf but can not satisfy $\mathcal{L}_\xi \bar{R}_N = 0$.

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