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HOPF HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS WITH LIE PARALLEL NORMAL JACOBI OPERATOR

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Abstract. In this paper we give some non-existence theorems for Hopf hypersurfaces in the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator $\bar{R}_N$ and totally geodesic $\mathfrak{D}$ and $\mathfrak{D}^\perp$ components of the Reeb flow.

0. Introduction

The Jacobi fields along geodesics of a given Riemannian manifold $(\bar{M}, \bar{g})$ play an important role in the study of differential geometry. It satisfies a very well-known differential equation. This classical differential equation naturally inspires the so-called Jacobi operators. That is, if $\bar{R}$ is the curvature operator of $\bar{M}$ and X is any vector field tangent to $\bar{M}$, the Jacobi operator with respect to X at $x \in \bar{M}$, $\bar{R}_X \in \mathrm{End}(T_x\bar{M})$, is defined as $\bar{R}_X(Y)(x) = (\bar{R}(Y,X)X)(x)$ for all $Y \in T_x\bar{M}$, being a self-adjoint endomorphism of the tangent bundle $T\bar{M}$ of $\bar{M}$. Clearly, each vector field X tangent to $\bar{M}$ provides a Jacobi operator with respect to X (See [7] and [9]).

If the structure vector field $\xi = -JN$ of a real hypersurface M in complex projective space $P_n(\mathbb{C})$ is invariant under the shape operator, ξ is said to be *Hopf*, where J denotes a Kähler structure of $P_n(\mathbb{C})$, and N is a unit normal vector field of M in $P_n(\mathbb{C})$.

In the quaternionic projective space $\mathbb{H}P^m$ Pérez and Suh [10] classified the real hypersurfaces in $\mathbb{H}P^m$ with $\mathfrak{D}^\perp$-parallel curvature tensor $\nabla_{\xi_\nu} R = 0$ for $\nu = 1, 2, 3$, where R denotes the curvature tensor of M in $\mathbb{H}P^m$ and $\mathfrak{D}^\perp$ is a distribution defined by $\mathfrak{D}^\perp = \mathrm{Span}\,\{\xi_1, \xi_2, \xi_3\}$. In this case they are congruent to a tube of radius $\frac{\pi}{4}$ over a totally geodesic quaternionic submanifold $\mathbb{H}P^k$ in $\mathbb{H}P^m$, $2 \le k \le m-2$.

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The vector fields $\{\xi_1, \xi_2, \xi_3\}$ mentioned above, which are said to be *almost contact structure*, are defined by $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$, where $\{J_1, J_2, J_3\}$ denote a local basis of a quaternionic Kähler structure of $\mathbb{H}P^m$ and N is a unit normal vector field of M in $\mathbb{H}P^m$.

In quaternionic space forms, Berndt [1] introduced the notion of *normal Jacobi operator*

$$\bar{R}_N X = \bar{R}(X, N)N \in \mathrm{End}\,(T_x M)\,, \quad x \in M$$

for real hypersurfaces M in a quaternionic projective space $\mathbb{H}P^m$ or in a quaternionic hyperbolic space $\mathbb{H}H^m$, where $\bar{R}$ denotes the curvature tensor of $\mathbb{H}P^m$ and $\mathbb{H}H^m$ respectively. Berndt [1] also showed that "*the curvature adaptedness*", when the normal Jacobi operator $\bar{R}_N$ commutes with the shape operator A, is equivalent to the fact that the distributions $\mathfrak{D}$ and $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator A of M, where $T_x M = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$.

Now let us consider a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ which consists of all complex 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The situation for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+1})$ with parallel normal Jacobi operator $\bar{R}_N$ is not so simple and will be quite different from the cases in $\mathbb{H}P^m$.

In this paper the present authors consider a real hypersurface M in the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator, that is, $\mathcal{L}_X \bar{R}_N = 0$ for any $X \in T_x M$, $x \in M$, where $\bar{R}$ and N respectively denote the curvature tensor of the ambient space $G_2(\mathbb{C}^{m+2})$ and a unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. The curvature tensor $\bar{R}(X, Y)Z$ for any vector fields X, Y and Z on $G_2(\mathbb{C}^{m+2})$ is explicitly defined in Section 1. Then the normal Jacobi operator $\bar{R}_N$ for the unit normal vector field N can be defined from the curvature tensor $\bar{R}(X, N)N$ by putting $Y = Z = N$.

The ambient space $G_2(\mathbb{C}^{m+2})$ is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J (See Berndt [2]). From these two structures J and $\mathfrak{J}$, we have geometric conditions naturally induced on a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ such that $[\xi] = \mathrm{Span}\ \{\xi\}$ or $\mathfrak{D}^\perp = \mathrm{Span}\ \{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator. By these two conditions, Berndt and Suh [3] proved the following:

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

The structure vector field ξ of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be a *Reeb* vector field. Moreover, the Reeb vector field ξ is said to be *Hopf* if it is invariant under the shape operator A. The 1-dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. By the formulas in section 2 it can be easily checked that M is Hopf if and only if the Reeb vector field ξ is Hopf. The flow generated by the integral curves of the Reeb vector field is said to be a *geodesic* Reeb flow if M becomes a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$.

We say that the Reeb vector field is *Killing* if the Lie derivative of the Riemannian metric g for M in $G_2(\mathbb{C}^{m+2})$ along the Reeb direction vanishes, that is, $\mathcal{L}_\xi g = 0$. This means that the Reeb flow is isometric. Using such a notion, Berndt and Suh [4] proved that a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$ with isometric Reeb flow becomes an open part of a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. In [15], Suh also gave a characterization for this kind of hypersurfaces in terms of another geometric *Lie invariant*. Namely, he characterized them as the hypersurfaces in $G_2(\mathbb{C}^{m+2})$ such that the shape operator A is invariant under the Reeb flow.

Now by putting a unit normal vector field N into the curvature tensor $\bar{R}$ of the ambient space $G_2(\mathbb{C}^{m+2})$, the normal Jacobi operator $\bar{R}_N$ can be defined in such a way that

$$\begin{aligned}\bar{R}_N X &= \bar{R}(X,N)N \\ &= X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\ &\quad - \sum_{\nu=1}^{3}\big\{\eta_\nu(\xi)(\phi_\nu\phi X - \eta(X)\xi_\nu) - \eta_\nu(\phi X)\phi_\nu\xi\big\}\end{aligned}$$

for any tangent vector field X on M in $G_2(\mathbb{C}^{m+2})$.

In the paper [8] due to Jeong, Pérez and Suh, we classified real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with commuting normal Jacobi operator, that is, $\bar{R}_N\circ\phi = \phi\circ\bar{R}_N$ or $\bar{R}_N\circ A = A\circ\bar{R}_N$. The fact that the normal Jacobi operator $\bar{R}_N$ *commutes* with the shape operator A (or the structure tensor ϕ) of M in $G_2(\mathbb{C}^{m+2})$ means that the eigenspaces of the normal Jacobi operator are invariant under the shape operator A (or the structure tensor ϕ). Also, in [5], Jeong, Kim and Suh introduced the notion of *parallel* normal Jacobi operator for real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$. Such an operator is said to be parallel if $\nabla_X\bar{R}_N = 0$ for any tangent vector field X on M. This means that the eigenspaces of the normal Jacobi operator $\bar{R}_N$ are parallel along any curve γ in M. Here the eigenspaces of the normal Jacobi operator $\bar{R}_N$ are said to be *parallel* along γ if they are invariant with respect to any parallel displacement along γ. Using this notion, they gave a non-existence theorem for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel normal Jacobi operator.

Related to such a parallel normal Jacobi operator, in this paper the authors give a theorem for real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ with *Lie parallel* normal Jacobi operator, that is, $\mathcal{L}_X\bar{R}_N = 0$ for any $X\in T_xM$, $x\in M$. This means that all the eigenspaces of the normal Jacobi operator $\bar{R}_N$ are invariant under any parallel displacement ϕ_t^* generated from the flow ϕ_t such that $\phi_t(x) = \gamma(t)$ and $\gamma(0) = x$ for the integral curve γ of X in T_xM, $x \in M$.

Then the authors prove the following for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operators:

Theorem 1. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator. If the integral curves of $\mathfrak{D}$ and $\mathfrak{D}^\perp$ components of the Reeb vector field ξ are totally geodesic, then ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

On the other hand, in the paper [6] of Jeong and Suh, we gave non-existence theorems for real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ with *Lie ξ-parallel* normal Jacobi operator, that is, $\mathcal{L}_\xi \bar{R}_N = 0$ as follows:

Theorem B. *There does not exist any real hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\mathcal{L}_\xi \bar{R}_N = 0$ if the Reeb vector field $\xi \in \mathfrak{D}^\perp$.*

Theorem C. *There does not exist any real hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\mathcal{L}_\xi \bar{R}_N = 0$ if the Reeb vector field $\xi \in \mathfrak{D}$.*

Then as an application of Theorem 1 to Theorems B and C the authors can assert the following:

Theorem 2. *There does not exist any Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator if the integral curves of $\mathfrak{D}$ and $\mathfrak{D}^\perp$ components of the Reeb vector field are totally geodesic.*

1. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details refer to [2], [3], and [4]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. The space $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, negative B restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximum sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight.

When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight. When $m = 2$, we note that the isomorphism $Spin(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces in $\mathbb{R}^6$. From now on, in this paper we will assume $m \geq 3$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition, that is, a Cartan decomposition

$$\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R},$$

where $\mathfrak{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_ν, $\nu = 1, 2, 3$, is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_\nu = J_\nu J$, and JJ_ν is a symmetric endomorphism with $(JJ_\nu)^2 = I$ and $\mathrm{tr}(JJ_\nu) = 0$.

A canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index ν is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\bar{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{1.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\bar{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ & - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3}\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\} \\ & + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},\end{aligned} \tag{1.2}$$

where J_1, J_2, J_3 is any canonical local basis of $\mathfrak{J}$.

2. Some fundamental formulas for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

Now in this section we want to derive some fundamental formulas which will be used in the proof of our theorems and the equation of Codazzi for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ (See [3], [4], [12], [13], and [14]).

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a submanifold of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal field of M and A the shape operator of M with respect to N. The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ, ξ, η, g). More explicitly, we can define a tensor field ϕ of type $(1,1)$, a vector field ξ and its dual 1-form η on M by $g(\phi X, Y) = g(JX, Y)$

and $\eta(X) = g(X, \xi)$ for any tangent vector fields X and Y on M. Then they satisfy the following

$$\phi^2 X = -X + \eta(X)\xi\,, \quad \phi\xi = 0\,, \quad \eta(\phi X) = 0 \quad \text{and} \quad \eta(\xi) = 1$$

for any tangent vector field X.

Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M in such a way that a tensor filed ϕ_ν of type $(1,1)$, a vector field ξ_ν and its dual 1-form η_ν on M defined by $g(\phi_\nu X, Y) = g(J_\nu X, Y)$ and $\eta_\nu(X) = g(\xi_\nu, X)$ for any tangent vector fields X and Y on M. Then they also satisfy the following

$$\phi_\nu^2 X = -X + \eta_\nu(X)\xi\,, \quad \phi_\nu\xi_\nu = 0\,, \quad \eta_\nu(\phi_\nu X) = 0 \quad \text{and} \quad \eta_\nu(\xi_\nu) = 1$$

for any vector field X tangent to M and $\nu = 1, 2, 3$.

Using the above expression (1.2) for the curvature tensor $\bar{R}$ of the ambient space $G_2(\mathbb{C}^{m+2})$, the equation of Codazzi becomes

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X = {} & \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\ & + \sum_{\nu=1}^{3} \big\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\big\} \\ & + \sum_{\nu=1}^{3} \big\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\big\} \\ & + \sum_{\nu=1}^{3} \big\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\big\}\xi_\nu\,.\end{aligned}$$

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

$$\begin{aligned}&\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \\ &\phi\xi_\nu = \phi_\nu\xi, \quad \eta_\nu(\phi X) = \eta(\phi_\nu X), \\ &\phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu, \\ &\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}.\end{aligned} \tag{2.1}$$

Now let us note that

$$JX = \phi X + \eta(X)N\,, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{2.2}$$

for any vector field X tangent to M in $G_2(\mathbb{C}^{m+2})$, where N denotes a unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. Then from this and the formulas (1.1) and (2.1) we have that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{2.3}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{2.4}$$

$$(2.5)\qquad \begin{aligned}(\nabla_X\phi_\nu)Y = & -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX \\ & - g(AX,Y)\xi_\nu.\end{aligned}$$

Summing up these formulas, we find the following

$$(2.6)\qquad \begin{aligned}\nabla_X(\phi_\nu\xi) &= \nabla_X(\phi\xi_\nu) \\ &= (\nabla_X\phi)\xi_\nu + \phi(\nabla_X\xi_\nu) \\ &= q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu\phi AX \\ &\quad - g(AX,\xi)\xi_\nu + \eta(\xi_\nu)AX.\end{aligned}$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1,2,3$, it follows that

$$(2.7)\qquad \phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu.$$

3. Lie parallel normal Jacobi operator

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator, that is, $\mathcal{L}_X\bar{R}_N = 0$ for any vector field X tangent to M. Then first of all, we write the normal Jacobi operator $\bar{R}_N$, which is given by

(3.1)

$$\begin{aligned}\bar{R}_N(X) = \bar{R}(X,N)N &= X + 3\eta(X)\xi + 3\sum_{\nu=1}^3\eta_\nu(X)\xi_\nu \\ &\quad - \sum_{\nu=1}^3\Big\{\eta_\nu(\xi)J_\nu(\phi X + \eta(X)N) - \eta_\nu(\phi X)(\phi_\nu\xi + \eta_\nu(\xi)N)\Big\} \\ &= X + 3\eta(X)\xi + 3\sum_{\nu=1}^3\eta_\nu(X)\xi_\nu \\ &\quad - \sum_{\nu=1}^3\Big\{\eta_\nu(\xi)(\phi_\nu\phi X - \eta(X)\xi_\nu) - \eta_\nu(\phi X)\phi_\nu\xi\Big\}\end{aligned}$$

where we have used the following

$$\begin{aligned}g(J_\nu JN,N) &= -g(JN,J_\nu N) = -g(\xi,\xi_\nu) = -\eta_\nu(\xi), \\ g(J_\nu JX,N) &= g(X,JJ_\nu N) = -g(X,J\xi_\nu) \\ &= -g(X,\phi\xi_\nu + \eta(\xi_\nu)N) = -g(X,\phi\xi_\nu),\end{aligned}$$

and

$$J_\nu JN = -J_\nu\xi = -\phi_\nu\xi - \eta_\nu(\xi)N.$$

Of course, by (2.7) we know that the normal Jacobi operator $\bar{R}_N$ is a symmetric endomorphism of T_xM, $x\in M$.

Now let us consider the Lie derivative of the normal Jacobi operator along any direction. Then for any vector fields X and Y tangent to M it is given by

$$(3.2)\qquad \begin{aligned}(\mathcal{L}_X\bar{R}_N)Y &= \mathcal{L}_X(\bar{R}_NY) - \bar{R}_N(\mathcal{L}_XY) \\ &= [X,\bar{R}_NY] - \bar{R}_N[X,Y] \\ &= (\nabla_X\bar{R}_N)Y - \nabla_{\bar{R}_NY}X + \bar{R}_N(\nabla_YX)\end{aligned}$$

where the terms in the right side can be given respectively as follows:

$$(\nabla_X\bar{R}_N)Y = 3(\nabla_X\eta)(Y)\xi + 3\eta(Y)\nabla_X\xi + 3\sum_{\nu=1}^3(\nabla_X\eta_\nu)(Y)\xi_\nu$$

$$+ 3\sum_{\nu=1}^{3}\eta_\nu(Y)\nabla_X\xi_\nu - \sum_{\nu=1}^{3}\Big[X(\eta_\nu(\xi))(\phi_\nu\phi Y - \eta(Y)\xi_\nu)$$
$$+ \eta_\nu(\xi)\{(\nabla_X\phi_\nu\phi)Y - (\nabla_X\eta)(Y)\xi_\nu - \eta(Y)\nabla_X\xi_\nu\}$$
$$- (\nabla_X\eta_\nu)(\phi Y)\phi_\nu\xi - \eta_\nu((\nabla_X\phi)Y)\phi_\nu\xi - \eta_\nu(\phi Y)\nabla_X(\phi_\nu\xi)\Big],$$

$$\nabla_{\bar{R}_N Y}X = \nabla_Y X + 3\eta(Y)\nabla_\xi X + 3\sum_{\nu=1}^{3}\eta_\nu(Y)\nabla_{\xi_\nu}X - \sum_{\nu=1}^{3}\eta_\nu(\xi)\nabla_{\phi_\nu\phi Y}X$$
$$+ \sum_{\nu=1}^{3}\eta_\nu(\xi)\eta(Y)\nabla_{\xi_\nu}X + \sum_{\nu=1}^{3}\eta_\nu(\phi Y)\nabla_{\phi_\nu\xi}X$$

and

$$\bar{R}_N(\nabla_Y X) = \nabla_Y X + 3\eta(\nabla_Y X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(\nabla_Y X)\xi_\nu$$
$$- \sum_{\nu=1}^{3}\{\eta_\nu(\xi)(\phi_\nu\phi\nabla_Y X - \eta(\nabla_Y X)\xi_\nu) - \eta_\nu(\phi\nabla_Y X)\phi_\nu\xi\}.$$

Then by the formulas given in section 2, (3.2) gives the following for a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator $\bar{R}_N$:

$$
\begin{aligned}
(\mathcal{L}_X\bar{R}_N)Y = {} & 3g(\phi AX, Y)\xi + 3\eta(Y)\phi AX + 3\sum_{\nu=1}^{3}g(\phi_\nu AX, Y)\xi_\nu \\
& + 3\sum_{\nu=1}^{3}\eta_\nu(Y)\phi_\nu AX \\
& - \sum_{\nu=1}^{3}\Bigg[X(\eta_\nu(\xi))(\phi_\nu\phi Y - \eta(Y)\xi_\nu) \\
& + \eta_\nu(\xi)\Big\{ - q_{\nu+1}(X)\phi_{\nu+2}\phi Y + q_{\nu+2}(X)\phi_{\nu+1}\phi Y \\
& + \eta_\nu(\phi Y)AX - g(AX, \phi Y)\xi_\nu \\
& + \eta(Y)\phi_\nu AX - g(AX, Y)\phi_\nu\xi - g(\phi AX, Y)\xi_\nu \\
& - \eta(Y)(q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX)\Big\} \\
& - g(\phi_\nu AX, \phi Y)\phi_\nu\xi - \eta(Y)\eta_\nu(AX)\phi_\nu\xi + g(AX, Y)\eta_\nu(\xi)\phi_\nu\xi \\
& - \eta_\nu(\phi Y)\Big\{\eta_\nu(\xi)AX - g(AX, \xi)\xi_\nu + \phi_\nu\phi AX\Big\}\Bigg] \\
& - 3\eta(Y)\nabla_\xi X - 3\sum_{\nu=1}^{3}\eta_\nu(Y)\nabla_{\xi_\nu}X \\
& + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)(\nabla_{\phi_\nu\phi Y}X - \eta(Y)\nabla_{\xi_\nu}X) - \eta_\nu(\phi Y)\nabla_{\phi_\nu\xi}X\} \\
& + 3\eta(\nabla_Y X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(\nabla_Y X)\xi_\nu \\
& - \sum_{\nu=1}^{3}\Big\{\eta_\nu(\xi)(\phi_\nu\phi\nabla_Y X - \eta(\nabla_Y X)\xi_\nu) - \eta_\nu(\phi\nabla_Y X)\phi_\nu\xi\Big\} \\
= {} & 0,
\end{aligned}
\tag{3.3}
$$

where in the first equality we have used the following formulas

$$3\sum_{\nu=1}^{3} g(q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2}, Y)\xi_\nu + 3\sum_{\nu=1}^{3} \eta_\nu(Y)\{q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2}\} = 0$$

and

$$\sum_{\nu=1}^{3} \{\eta_{\nu+1}(\phi Y)q_{\nu+2}(X)\phi_\nu\xi - \eta_{\nu+2}(\phi Y)q_{\nu+1}(X)\phi_\nu\xi - \eta_\nu(\phi Y)q_{\nu+1}(X)\phi_{\nu+2}\xi + \eta_\nu(\phi Y)q_{\nu+2}(X)\phi_{\nu+1}\xi\} = 0.$$

In particular by putting $X = \xi$ in (3.3) we have the following

$$\begin{aligned}
(\mathcal{L}_\xi \bar{R}_N)Y = {} & 3g(\phi A\xi, Y)\xi + 3\sum_{\nu=1}^{3} g(\phi_\nu A\xi, Y)\xi_\nu \\
& + 3\sum_{\nu=1}^{3} \eta_\nu(Y)\phi_\nu A\xi \\
& - \sum_{\nu=1}^{3} \Big[\xi(\eta_\nu(\xi))(\phi_\nu\phi Y - \eta(Y)\xi_\nu) \\
& + \eta_\nu(\xi)\Big\{ - q_{\nu+1}(\xi)\phi_{\nu+2}\phi Y + q_{\nu+2}(\xi)\phi_{\nu+1}\phi Y \\
& + \eta_\nu(\phi Y)A\xi - g(A\xi, \phi Y)\xi_\nu \\
& + \eta(Y)\phi_\nu A\xi - g(A\xi, Y)\phi_\nu\xi - g(\phi A\xi, Y)\xi_\nu \\
& - \eta(Y)(q_{\nu+2}(\xi)\xi_{\nu+1} - q_{\nu+1}(\xi)\xi_{\nu+2} + \phi_\nu A\xi)\Big\} \\
& - g(\phi_\nu A\xi, \phi Y)\phi_\nu\xi - \eta(Y)\eta_\nu(A\xi)\phi_\nu\xi + g(A\xi, Y)\eta_\nu(\xi)\phi_\nu\xi \\
& - \eta_\nu(\phi Y)\Big\{\eta_\nu(\xi)A\xi - g(A\xi, \xi)\xi_\nu + \phi_\nu\phi A\xi\Big\}\Big] \\
& - 3\sum_{\nu=1}^{3} \eta_\nu(Y)\phi A\xi_\nu + 3\sum_{\nu=1}^{3} \eta_\nu(\phi AY)\xi_\nu \\
& + \sum_{\nu=1}^{3} \Big[\eta_\nu(\xi)\Big\{\phi A\phi_\nu\phi Y - \eta(Y)\phi A\xi_\nu\Big\} - \eta_\nu(\phi Y)\phi A\phi_\nu\xi\Big] \\
& + \sum_{\nu=1}^{3} \Big[\eta_\nu(\xi)\{\phi_\nu AY - \eta(AY)\phi_\nu\xi\} - \eta_\nu(AY)\phi_\nu\xi \\
& + \eta(AY)\eta_\nu(\xi)\phi_\nu\xi\Big] \\
= {} & 0,
\end{aligned} \tag{3.4}$$

where in the first equality we have used the second formula of (2.3). From this, by putting $Y = \xi$ in (3.4) we have the following

$$(\mathcal{L}_\xi \bar{R}_N)\xi = 6\sum_{\nu=1}^{3} g(\phi_\nu A\xi, \xi)\xi_\nu + 4\sum_{\nu=1}^{3} \eta_\nu(\xi)\phi_\nu A\xi$$

$$
\begin{aligned}
&+\sum_{\nu=1}^{3}\Big[\xi(\eta_\nu(\xi))\xi_\nu+\eta_\nu(\xi)\big\{q_{\nu+2}(\xi)\xi_{\nu+1}-q_{\nu+1}(\xi)\xi_{\nu+2}\big\}\Big] \\
&-4\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi A\xi_\nu \\
=\ &0.
\end{aligned} \tag{3.5}
$$

4. Lie parallel normal Jacobi operator

In this section we want to prove the following:

Proposition 4.1. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator. If the integral curves of $\mathfrak{D}$ and $\mathfrak{D}^\perp$ components of the Reeb vector field ξ are totally geodesic, then ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

Proof. When the function $\alpha = g(A\xi,\xi)$ identically vanishes, the proposition was proved directly by Pérez and Suh [11]. Thus we consider only the case that the function α is non-vanishing in this proof.
By putting $A\xi = \alpha\xi$ into (3.5) we have

$$\sum_{\nu=1}^{3}\eta_\nu(\xi)(\alpha\phi_\nu\xi-\phi A\xi_\nu)=0, \tag{4.1}$$

where we have used the following formula

$$\sum_{\nu=1}^{3}\Big[\xi(\eta_\nu(\xi))\xi_\nu+\eta_\nu(\xi)\{q_{\nu+2}(\xi)\xi_{\nu+1}-q_{\nu+1}(\xi)\xi_{\nu+2}\}\Big]=0.$$

Now let us put $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ for some unit $X_0 \in \mathfrak{D}$ and $\xi_1 \in \mathfrak{D}^\perp$. Then naturally we know that $\eta(\xi_2) = \eta(\xi_3) = 0$. Hereafter, unless otherwise stated, let us assume $\eta(X_0)\eta(\xi_1) \neq 0$.
Then (4.1) reduces to

$$\alpha\phi_1\xi-\phi A\xi_1=0.$$

From this, by taking the structure tensor ϕ and also using that ξ is principal, we have

$$A\xi_1=\alpha\xi_1 \quad\text{and}\quad AX_0=\alpha X_0. \tag{4.2}$$

Then putting $X = X_0$ and $Y = \xi$ into (3.3) and using (4.2) gives

$$
\begin{aligned}
0 &= (\mathcal{L}_{X_0}\bar{R}_N)\xi \\
&= 3\alpha\phi X_0+3\alpha\sum_{\nu=1}^{3}g(\phi_\nu X_0,\xi)\xi_\nu+3\alpha\eta_1(\xi)\phi_1X_0 \\
&\quad+\eta_1(\xi)\{q_3(X_0)\xi_2-q_2(X_0)\xi_3\}-3\nabla_\xi X_0-4\eta_1(\xi)\nabla_{\xi_1}X_0 \\
&\quad+3\eta(\nabla_\xi X_0)\xi+3\sum_{\nu=1}^{3}\eta_\nu(\nabla_\xi X_0)\xi_\nu-\eta_1(\xi)\phi_1\phi\nabla_\xi X_0 \\
&\quad+\eta_1(\xi)\eta(\nabla_\xi X_0)\xi_1+\sum_{\nu=1}^{3}\eta_\nu(\phi\nabla_\xi X_0)\phi_\nu\xi,
\end{aligned}
$$

where we have used

$$X_0(\eta_1(\xi))\xi_1=g(\nabla_{X_0}\xi_1,\xi)\xi_1+g(\xi_1,\nabla_{X_0}\xi)\xi_1$$

$$
\begin{aligned}
&= g(\phi_1 AX_0, \xi)\xi_1 + g(\xi_1, \phi AX_0)\xi_1 \\
&= -\alpha g(X_0, \phi_1\xi)\xi_1 - \alpha g(\phi_1\xi, X_0)\xi_1 \\
&= -2\alpha g(X_0, \phi_1(\eta(X_0)X_0 + \eta(\xi_1)\xi_1)) \\
&= -2\alpha\eta(X_0)g(X_0, \phi_1 X_0) \\
&= 0\,.
\end{aligned}
$$

From this, together with (2.3) and (2.4), and using $\phi X_0 \in \mathfrak{D}$, $\nabla_\xi X_0 \in \mathfrak{D}$ and $\eta(\nabla_\xi X_0) = 0$, we have

$$
\begin{aligned}
0 &= (\mathcal{L}_{X_0}\bar{R}_N)\xi \\
&= 3\alpha(\phi X_0 + \eta_1(\xi)\phi_1 X_0) + \eta_1(\xi)\{q_3(X_0)\xi_2 - q_2(X_0)\xi_3\} \\
&\quad - 3\nabla_\xi X_0 - 4\eta_1(\xi)\nabla_{\xi_1}X_0 - \eta_1(\xi)\phi_1\phi\nabla_\xi X_0 \\
&\quad + \sum\nolimits_{\nu=1}^{3}\eta_\nu(\phi\nabla_\xi X_0)\phi_\nu\xi,
\end{aligned}
\tag{4.3}
$$

because we know the following

$$
g(\phi X_0, \xi_\nu) = -g(X_0, \phi\xi_\nu) = -g(X_0, \phi_\nu\xi) = 0\,,
$$

$$
\eta(\nabla_\xi X_0) = g(\nabla_\xi X_0, \xi) = g(\nabla_\xi X_0, \eta(X_0)X_0 + \eta(\xi_1)\xi_1) = 0
$$

and

$$
\begin{aligned}
g(\nabla_\xi X_0, \xi_\nu) &= -g(X_0, \nabla_\xi\xi_\nu) \\
&= -\alpha g(X_0, \phi_\nu\xi) \\
&= -\alpha g(X_0, \phi\xi_\nu) \\
&= \alpha g(\phi X_0, \xi_\nu) \\
&= 0
\end{aligned}
$$

for any $\nu = 1, 2, 3$.

On the other hand, we know that

$$
\nabla_{\xi_1} X_0 \in \mathfrak{D}\,, \tag{4.4}
$$

because

$$
\begin{aligned}
g(\nabla_{\xi_1}X_0, \xi_\nu) &= -g(X_0, \nabla_{\xi_1}\xi_\nu) \\
&= -g(X_0, q_{\nu+2}(\xi_1)\xi_{\nu+1} - q_{\nu+1}(\xi_1)\xi_{\nu+2} + \phi_\nu A\xi_1) \\
&= -\alpha g(X_0, \phi_\nu\xi_1) \\
&= 0\,.
\end{aligned}
$$

Moreover, the following formulas hold

$$
g(\phi\nabla_\xi X_0, \xi_2) = 0 \quad \text{and} \quad g(\phi\nabla_\xi X_0, \xi_3) = 0\,. \tag{4.5}
$$

In fact, differentiating $g(\phi X_0, \xi_2) = 0$ gives

$$
\begin{aligned}
0 &= g((\nabla_\xi\phi)X_0, \xi_2) + g(\phi\nabla_\xi X_0, \xi_2) + g(\phi X_0, \nabla_\xi\xi_2) \\
&= g(\phi\nabla_\xi X_0, \xi_2) + \alpha g(\phi X_0, \phi\xi_2)
\end{aligned}
$$

$$= g(\phi\nabla_\xi X_0, \xi_2)$$

and similarly the latter term comes from $g(\phi X_0, \xi_3) = 0$.

By taking the inner product (4.3) with ξ_3, and using the facts that ϕX_0, $\phi_1 X_0$, $\nabla_\xi X_0$ and $\nabla_{\xi_1} X_0$ belong to the distribution $\mathfrak{D}$, we have

$$\begin{aligned} 0 &= -\,\eta_1(\xi)q_2(X_0) - \eta_1(\xi)g(\phi_1\phi\nabla_\xi X_0, \xi_3) + \eta_1(\phi\nabla_\xi X_0)g(\phi_1\xi, \xi_3) \\ &= -\,\eta_1(\xi)q_2(X_0). \end{aligned}$$

Similarly, by taking the inner product with ξ_2 to (4.3), we have the following relations

$$q_2(X_0) = 0 \quad \text{and} \quad q_3(X_0) = 0 \tag{4.6}$$

under the assumption of $\eta_1(\xi)\neq 0$. Then (4.4), (4.5) and (4.6) give

$$\begin{aligned} 0 &= (\mathcal{L}_{X_0}\bar{R}_N)\xi \\ &= 3\alpha(\phi X_0 + \eta_1(\xi)\phi_1 X_0) - 3\nabla_\xi X_0 - 4\eta_1(\xi)\nabla_{\xi_1} X_0 \\ &\quad - \eta_1(\xi)\phi_1\phi\nabla_\xi X_0 + \eta_1(\phi\nabla_\xi X_0)\phi_1\xi. \end{aligned} \tag{4.7}$$

On the other hand, by the assumption of M being Hopf and using (4.2), we have

$$\begin{aligned} \nabla_\xi\xi &= \phi A\xi \\ &= \phi A(\eta(X_0)X_0 + \eta(\xi_1)\xi_1) \\ &= \alpha(\eta(X_0)\phi X_0 + \eta(\xi_1)\eta(X_0)\phi_1 X_0) \\ &= \alpha\eta(X_0)(\phi X_0 + \eta(\xi_1)\phi_1 X_0) \\ &= 0. \end{aligned}$$

Consequently, we see

$$\phi X_0 + \eta(\xi_1)\phi_1 X_0 = 0. \tag{4.8}$$

from the assumption of $\alpha \neq 0$ and $\eta(X_0) \neq 0$.

Substituting (4.8) into (4.7), we have

$$\begin{aligned} 0 &= (\mathcal{L}_{X_0}\bar{R}_N)\xi \\ &= -3\nabla_\xi X_0 - 4\eta_1(\xi)\nabla_{\xi_1} X_0 - \eta_1(\xi)\phi_1\phi\nabla_\xi X_0 + \eta_1(\phi\nabla_\xi X_0)\phi_1\xi. \end{aligned}$$

Now, by applying the operator ϕ_1 to (4.8) we have

$$\phi_1\phi X_0 = \eta(\xi_1)X_0. \tag{4.9}$$

Then by differentiating (4.9) along the direction of the Reeb vector field ξ and using (2.1), (2.3), (2.4), (2.5) and (4.8), we have

$$q_2(\xi)\eta(\xi_1)\phi_2 X_0 + q_3(\xi)\eta(\xi_1)\phi_3 X_0 + \phi_1\phi\nabla_\xi X_0 = \eta(\xi_1)\nabla_\xi X_0. \tag{4.10}$$

By taking the inner product (4.10) with ξ_2 and ξ_3 respectively and using the fact that $\nabla_\xi X_0$, $\phi_\nu X_0 \in \mathfrak{D}$, $\nu = 1, 2, 3$, we have the following respectively

$$g(\nabla_\xi X_0, \phi_3 X_0) = 0 \quad \text{and} \quad g(\nabla_\xi X_0, \phi_2 X_0) = 0. \tag{4.11}$$

On the other hand, the assumption that $\mathfrak{D}^\perp$-component of ξ is totally geodesic and (4.2) give

$$q_2(\xi_1) = q_3(\xi_1) = 0\,. \tag{4.12}$$

Let us differentiate the formula (4.9) along the direction of ξ_1. Then by virtue of the formulas (2.3), (2.4), (2.5) and (4.12), we have

$$\phi_1\phi\nabla_{\xi_1}X_0 = \eta(\xi_1)\nabla_{\xi_1}X_0\,. \tag{4.13}$$

On the other hand, by taking the inner product (4.10) with ϕ_2X_0, ϕ_3X_0 respectively and using (2.1), (2.7) and (4.11) respectively we have

$$q_2(\xi) = 0 \quad \text{and} \quad q_3(\xi) = 0. \tag{4.14}$$

Then (4.10) implies that

$$\phi_1\phi\nabla_\xi X_0 = \eta(\xi_1)\nabla_\xi X_0. \tag{4.15}$$

Moreover, by differentiating (4.8) along the direction of ξ and using (2.3), (2.4), (2.5) and (4.14), we have

$$\phi\nabla_\xi X_0 = \alpha\eta_1(\xi)\eta(X_0)\xi_1 - \eta_1(\xi)\phi_1\nabla_\xi X_0.$$

From this, by applying ϕ and using (4.15) we have

$$\nabla_\xi X_0 = -\alpha\eta(\xi_1)\phi_1X_0. \tag{4.16}$$

Now differentiating (4.8) along the direction ξ_1 and using (2.3) and (2.5), we have

$$\alpha\eta(X_0)\xi_1 + \phi\nabla_{\xi_1}X_0 = -\eta_1(\xi)\phi_1\nabla_{\xi_1}X_0.$$

Similarly, by applying ϕ to above equation and using (4.13) we have

$$\nabla_{\xi_1}X_0 = \alpha\phi_1X_0. \tag{4.17}$$

Then (4.16) and (4.17) give

$$\nabla_\xi X_0 = -\eta(\xi_1)\nabla_{\xi_1}X_0. \tag{4.18}$$

On the other hand, we know that

$$\begin{aligned}\nabla_\xi X_0 &= \nabla_{\eta(X_0)X_0+\eta(\xi_1)\xi_1}X_0\\ &= \eta(X_0)\nabla_{X_0}X_0 + \eta(\xi_1)\nabla_{\xi_1}X_0\\ &= \eta(\xi_1)\nabla_{\xi_1}X_0,\end{aligned} \tag{4.19}$$

because the $\mathfrak{D}$-component of the Reeb vector field ξ is totally geodesic. From (4.18) and (4.19) we see that $\eta(\xi_1)\nabla_{\xi_1}X_0 = 0$. This means that $\nabla_{\xi_1}X_0 = 0$. From this together with (4.17), it follows that $\phi_1X_0 = 0$. This gives a contradiction. So we only have $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$. □

5. Lie parallel normal Jacobi operator for $\xi \in \mathfrak{D}^\perp$

In order to give a complete proof of Theorem 2, first we consider the case that the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$. Now in this direction we introduce some lemmas given in Jeong and Suh [6] as follows:

Lemma 5.A. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying Lie ξ-parallel normal Jacobi operator and $\xi \in \mathfrak{D}^\perp$. Then $A\xi = \alpha\xi + \beta U$, where U is a unit vector field orthogonal to ξ and belongs to $\mathfrak{D}$.*

Moreover, from Lemma 5.A, they proved the following lemmas:

Lemma 5.B. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying Lie ξ-parallel normal Jacobi operator and $\xi \in \mathfrak{D}^\perp$. Then β identically vanishes, that is, the Reeb vector field ξ is principal.*

Lemma 5.C. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying Lie ξ-parallel normal Jacobi operator and $\xi \in \mathfrak{D}^\perp$. Then $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.*

From these lemmas we assert:

Lemma 5.1. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying Lie parallel normal Jacobi operator and $\xi \in \mathfrak{D}^\perp$. Then the Reeb vector ξ is principal and $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.*

Before going to give our proof of Theorem 2 in the introduction, let us check "What kind of model hypersurfaces given in Theorem A satisfy Lie parallel normal Jacobi operator." In other words, it will be an interesting problem to know whether there exist real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying the condition $\mathcal{L}_X \bar{R}_N = 0$ for $\xi \in \mathfrak{D}^\perp$.
Then by virtue of Lemmas 5.1, we are able to recall the proposition given by Berndt and Suh [3] as follows:
For a tube of type (A) in Theorem A we have the following:

Proposition A. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1 N$. Then M has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r),\ \beta = \sqrt{2}\cot(\sqrt{2}r),\ \lambda = -\sqrt{2}\tan(\sqrt{2}r),\ \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are

$$m(\alpha) = 1,\ m(\beta) = 2,\ m(\lambda) = 2m-2 = m(\mu),$$

and the corresponding eigenspaces we have

$$\begin{aligned} T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1, \\ T_\beta &= \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3, \\ T_\lambda &= \{X | X \perp \mathbb{H}\xi, JX = J_1 X\}, \\ T_\mu &= \{X | X \perp \mathbb{H}\xi, JX = -J_1 X\}, \end{aligned}$$

where $\mathbb{R}\xi$*,* $\mathbb{C}\xi$ *and* $\mathbb{H}\xi$ *respectively denotes real, complex and quaternionic span of the structure vector* ξ *and* $\mathbb{C}^{\perp}\xi$ *denotes the orthogonal complement of* $\mathbb{C}\xi$ *in* $\mathbb{H}\xi$*.*

In the proof of Lemma 5.C (See Section 4 in [6]) we have asserted that $A\xi_2 = 0$ and $A\xi_3 = 0$. But the principal curvature $\beta = \sqrt{2}\cot(\sqrt{2}r)$ given in Proposition A is never vanishing for any $r \in (0, \frac{\pi}{4})$. So this gives a contradiction. Accordingly, we completed the proof of our Theorem 2 for the case $\xi \in \mathfrak{D}^{\perp}$.

6. Lie parallel normal Jacobi operator for $\boldsymbol{\xi} \in \mathfrak{D}$

Next we consider the case that the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$. Then in this section we introduce the following lemmas due to Jeong and Suh [6] for hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with Lie ξ-parallel normal Jacobi operator.

Lemma 6.A. *Let* M *be a real hypersurface in* $G_2(\mathbb{C}^{m+2})$ *satisfying Lie* ξ*-parallel normal Jacobi operator and* $\xi \in \mathfrak{D}$*. Then the Reeb vector* ξ *is principal.*

Then by using Lemma 6.A, Jeong and Suh [6] also verified the following:

Lemma 6.B. *Let* M *be a real hypersurface in* $G_2(\mathbb{C}^{m+2})$ *satisfying Lie* ξ*-parallel normal Jacobi operator and* $\xi \in \mathfrak{D}$*. Then* $g(A\mathfrak{D}, \mathfrak{D}^{\perp}) = 0$*.*

By virtue of these Lemmas 6.A and 6.B we have

Lemma 6.C. *Let* M *be a real hypersurface in* $G_2(\mathbb{C}^{m+2})$ *satisfying Lie parallel normal Jacobi operator and* $\xi \in \mathfrak{D}$*. Then the Reeb vector field* ξ *is principal and* $g(A\mathfrak{D}, \mathfrak{D}^{\perp}) = 0$*.*

From this Lemma 6.1, together with Theorem A due to Berndt and Suh [3], we have that M is locally a tube over a totally geodesic and totally real quaternionic projective space $\mathbb{H}P^n$, $m = 2n$. So for the geometrical structure of such a tube we recall the following proposition.

Proposition B. *Let* M *be a connected real hypersurface of* $G_2(\mathbb{C}^{m+2})$*. Suppose that* $A\mathfrak{D} \subset \mathfrak{D}$*,* $A\xi = \alpha\xi$*, and* ξ *is tangent to* $\mathfrak{D}$*. Then the quaternionic dimension* m *of* $G_2(\mathbb{C}^{m+2})$ *is even, say* $m = 2n$*, and* M *has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r),\ \beta = 2\cot(2r),\ \gamma = 0,\ \lambda = \cot(r),\ \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$*. The corresponding multiplicities are*

$$m(\alpha) = 1,\ m(\beta) = 3 = m(\gamma),\ m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi,\ T_\beta = \mathfrak{J}J\xi,\ T_\gamma = \mathfrak{J}\xi,\ T_\lambda,\ T_\mu,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^{\perp},\ \mathfrak{J}T_\lambda = T_\lambda,\ \mathfrak{J}T_\mu = T_\mu,\ JT_\lambda = T_\mu.$$

Now, using the assumption that M is Hopf in (3.4), we have the following

$$\begin{aligned}&(\mathcal{L}_\xi \bar{R}_N)Y\\ &= 4\alpha\sum_{\nu=1}^{3} g(\phi_\nu\xi, Y)\xi_\nu + 4\alpha\sum_{\nu=1}^{3}\eta_\nu(Y)\phi_\nu\xi\\ &\quad - 3\sum_{\nu=1}^{3}\eta_\nu(Y)\phi A\xi_\nu + 3\sum_{\nu=1}^{3}\eta_\nu(\phi AY)\xi_\nu\\ &\quad + \sum_{\nu=1}^{3}\Big\{\eta_\nu(\xi)\big(\phi A\phi_\nu\phi Y - \eta(Y)\phi A\xi_\nu\big) - \eta_\nu(\phi Y)\phi A\phi_\nu\xi\Big\}\\ &\quad + \sum_{\nu=1}^{3}\Big\{\eta_\nu(\xi)\big(\phi_\nu AY - \alpha\eta(Y)\phi_\nu\xi\big) - \eta_\nu(AY)\phi_\nu\xi + \eta(AY)\eta_\nu(\xi)\phi_\nu\xi\Big\}\\ &= 0\,.\end{aligned}$$

Moreover, using the fact that the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$, we have

$$\begin{aligned}(\mathcal{L}_\xi \bar{R}_N)Y &= 4\alpha\sum_{\nu=1}^{3} g(\phi_\nu\xi, Y)\xi_\nu + 4\alpha\sum_{\nu=1}^{3}\eta_\nu(Y)\phi_\nu\xi\\ &\quad - 3\sum_{\nu=1}^{3}\eta_\nu(Y)\phi A\xi_\nu + 3\sum_{\nu=1}^{3}\eta_\nu(\phi AY)\xi_\nu\\ &\quad - \sum_{\nu=1}^{3}\eta_\nu(\phi Y)\phi A\phi_\nu\xi - \sum_{\nu=1}^{3}\eta_\nu(AY)\phi_\nu\xi\\ &= 0\end{aligned} \tag{6.1}$$

for any $Y \in T_xM$, $x \in M$.

Let us construct a sub-distribution $\mathfrak{D}_0$ of the distribution $\mathfrak{D}$ in such a way that

$$[\xi] \oplus \mathfrak{D}_0 = \mathfrak{D},$$

where $[\xi]$ denotes an one-dimensional vector subspace spanned by the Reeb vector field ξ. Then $\mathfrak{D}_0$ becomes $\mathfrak{D}_0 = \{Y \in \mathfrak{D}\,|\,Y \perp \xi\}$. Here, if we substitute any $Y \in \mathfrak{D}_0$ in (6.1) and use $\xi \in \mathfrak{D}$, the left side of (6.1) becomes

$$\begin{aligned}(\mathcal{L}_\xi \bar{R}_N)Y &= 4\alpha\sum_{\nu=1}^{3} g(\phi_\nu\xi, Y)\xi_\nu + 3\sum_{\nu=1}^{3}\eta_\nu(\phi AY)\xi_\nu\\ &\quad - \sum_{\nu=1}^{3}\eta_\nu(\phi Y)\phi A\phi_\nu\xi - \sum_{\nu=1}^{3}\eta_\nu(AY)\phi_\nu\xi.\end{aligned}$$

From this, putting $Y = \phi_\mu\xi \in T_\gamma$, and using $A\phi_\mu\xi = 0$, $\mu = 1, 2, 3$, given in Proposition B, it becomes

$$(\mathcal{L}_\xi \bar{R}_N)\phi\xi_\mu = 4\alpha\xi_\mu.$$

From this, with the assumption of $\mathcal{L}_\xi \bar{R}_N = 0$, we have $\alpha = 0$. But the principal curvature $\alpha = -2\tan(2r)$ in Proposition B never vanishes for $r \in (0, \frac{\pi}{4})$. This gives a contradiction for the case $\xi \in \mathfrak{D}$. Accordingly, we complete the proof of our Theorem 2 for $\xi \in \mathfrak{D}$ in the introduction.

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